SOME NEW POSITIVE OBSERVATIONS

ALEXANDER BERKOVICH

Dedicated to the memory of Dora Bitman

ABSTRACT. We revisit Bressoud's generalized Borwein conjecture. Making use of the new positivity-preserving transformations for q-binomial coefficients we establish the truth of infinitely many cases of the Bressoud conjecture. In addition, we prove new bounded versions of Lebesgue's identity and of Euler's Pentagonal Number Theorem. Finally, we discuss new companions to Andrews-Gordon *mod* 21 and Bressoud *mod* 20 identities.

1. INTRODUCTION

Bressoud [10] considered the following polynomials

$$G(N,M,\alpha,\beta,K,q)=\sum_{j=-\infty}^{\infty}(-1)^j q^{Kj\frac{(\alpha+\beta)j+(\alpha-\beta)}{2}}\begin{bmatrix}N+M\\N-Kj\end{bmatrix}_q, \tag{1.1}$$

where

$$\begin{bmatrix}m+n\\m\end{bmatrix}_q:=\begin{cases}\frac{(q)_{m+n}}{(q)_m(q)_n}, & \text{for } m,n\in\mathbf{N},\\ 0, & \text{otherwise,}\end{cases} \tag{1.2}$$

and

$$(q)_m=\prod_{j=1}^{m}(1-q^j), \text{for } m\in\mathbf{N},$$

where $\mathbf{N}$ denotes the set of nonnegative integers. More generally, for $m\in\mathbf{N}$ we define

$$(a)_m=(a;q)_m=\prod_{j=0}^{m-1}(1-aq^j),$$
$$(a_1,a_2,\ldots,a_k;q)_m=(a_1;q)_m(a_2;q)_m\ldots(a_k;q)_m. \tag{1.3}$$

Here and throughout we assume that $|q|<1$. We note that $(a)_0=1$.

In 1996, Bressoud [10] conjectured that

Conjecture 1.1. *Let $K\in\mathbb{Z}_{>1}$, $N,M,\alpha K,\beta K\in\mathbb{Z}_{\geq 0}$ such that*

$$1\leq\alpha+\beta\leq 2K-1,$$
$$\beta-K\leq N-M\leq K-\alpha, \tag{1.4}$$

(strict inequality when $K=2$). Then $G(N,M,\alpha,\beta,K,q)\geq 0$.

Date: June 23, 2020.

2010 *Mathematics Subject Classification.* 11B65, 11P84, 05A30, 33D15.

Key words and phrases. Bressoud's conjectures, Positivity-preserving transformations, Rogers-Ramanujan type identities, Terminating q-Series.

Research is partly supported by the Simons Foundation, Award ID: 308929.

Here, and everywhere, $P(q) \geq 0$ means that $P(q)$ is a polynomial in q with nonnegative coefficients.
We remark that

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q \geq 0.$$

The famous conjecture of Peter Borwein (Theorem since 2019 [14]) can be stated as

$$\begin{aligned} A_n(q) &= G(n,n,\frac{5}{3},\frac{4}{3},3,q) \geq 0, \\ B_n(q) &= G(n-1,n+1,\frac{7}{3},\frac{2}{3},3,q) \geq 0, \\ C_n(q) &= G(n-1,n+1,\frac{8}{3},\frac{1}{3},3,q) \geq 0, \end{aligned} \tag{1.5}$$

and

$$\prod_{k=1}^{n}(1-q^{3k-1})(1-q^{3k-2}) = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3). \tag{1.6}$$

When $\alpha, \beta \in \mathbb{Z}$, $G(N,M,\alpha,\beta,K,q)$ is a generating function for the so-called partitions with prescribed hook differences [4]. Bressoud's conjecture is nontrivial when α, β assume fractional values.
Many cases of Bressoud's conjecture were settled in the literature [9], [12], [15], [16], [7], [14].
In the next section, we will show how to settle new infinite family of cases.

Theorem 1.2. *For $L \in \mathbf{N}$, $\nu \in \mathbb{Z}_{>0}$, $s = 0,1,2,\ldots,\nu-1$*

$$G(L, L+1+2s, (\nu+1)(1+\frac{1+2s}{2\nu+1}), (\nu+1)(1-\frac{1+2s}{2\nu+1}), 2\nu+1, q) \geq 0. \tag{1.7}$$

Also, in Section 2, we discuss new bounded versions of Lebesgue's identity and of Euler's Pentagonal Number Theorem. In Section 3, we establish and prove some additional isolated positivity results and introduce new companions to Andrews-Gordon *mod* 21 and Bressoud *mod* 20 identities.
We conclude this section with a list of seven useful formulas, which can be found in [2]:

$$\lim_{L\to\infty} \begin{bmatrix} L \\ m \end{bmatrix}_q = \frac{1}{(q)_m}, \tag{1.8}$$

$$\lim_{L,M\to\infty} \begin{bmatrix} L+M \\ L \end{bmatrix}_q = \frac{1}{(q)_\infty}, \tag{1.9}$$

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_{q^{-1}} = q^{-nm} \begin{bmatrix} n+m \\ n \end{bmatrix}_q, \tag{1.10}$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q + q^m \begin{bmatrix} n-1 \\ m \end{bmatrix}_q = \begin{bmatrix} n-1 \\ m \end{bmatrix}_q + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q, \tag{1.11}$$

$$\sum_{n\geq 0} q^{\binom{n}{2}} z^n \begin{bmatrix} L \\ n \end{bmatrix}_q = (-z;q)_L, \tag{1.12}$$

$$\sum_{j=-\infty}^{\infty} (-1)^j z^j q^{j^2} = \left(q^2, \frac{q}{z}, zq; q^2\right)_\infty, \tag{1.13}$$

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} z^j \begin{bmatrix} L+M \\ L-j \end{bmatrix}_{q^2} = \left(\frac{q}{z}; q^2\right)_M \left(zq; q^2\right)_L, \tag{1.14}$$

with $L, M, m, n \in \mathbf{N}$.

2. Positivity-preserving Transformations

We start with the following summation formula

Theorem 2.1. *For $L \in \mathbf{N}$, $a \in \mathbf{Z}$*

$$\sum_{k\geq 0} C_{L,k}(q) \begin{bmatrix} k \\ \lfloor \frac{k-a}{2} \rfloor \end{bmatrix}_q = q^{T(a)} \begin{bmatrix} 2L+1 \\ L-a \end{bmatrix}_q, \tag{2.1}$$

where

$$T(j) := \binom{j+1}{2}$$

and

$$C_{L,k}(q) = \sum_{m=0}^{L} q^{T(m)+T(m+k)} \begin{bmatrix} L \\ m,k \end{bmatrix}_q, \tag{2.2}$$

with

$$\begin{bmatrix} L \\ m,k \end{bmatrix}_q = \begin{bmatrix} L \\ m \end{bmatrix}_q \begin{bmatrix} L-m \\ k \end{bmatrix}_q = \begin{bmatrix} L \\ k \end{bmatrix}_q \begin{bmatrix} L-k \\ m \end{bmatrix}_q \geq 0. \tag{2.3}$$

Observe that $C_{L,k}(q) \geq 0$. Using transformation (2.1) it is easy to check that identity of the form

$$F(L,q) = \sum_{j=-\infty}^{\infty} \alpha(j,q) \begin{bmatrix} L \\ \lfloor \frac{L-j}{2} \rfloor \end{bmatrix}_q, \tag{2.4}$$

implies that the following identity holds

$$\sum_{k\geq 0} C_{L,k}(q) F(k,q) = \sum_{j=-\infty}^{\infty} \alpha(j,q) \sum_{k\geq 0} C_{L,k}(q) \begin{bmatrix} k \\ \lfloor \frac{k-j}{2} \rfloor \end{bmatrix}_q = \sum_{j=-\infty}^{\infty} \alpha(j,q) q^{T(j)} \begin{bmatrix} 2L+1 \\ L-j \end{bmatrix}_q. \tag{2.5}$$

Hence, if $F(L,q) \geq 0$ then

$$\sum_{j=-\infty}^{\infty} \alpha(j,q) q^{T(j)} \begin{bmatrix} 2L+1 \\ L-j \end{bmatrix}_q \geq 0. \tag{2.6}$$

For that reason, we say that (2.1) is positivity-preserving.
Transformation (2.1) is an easy corollary of the theorem proven in [6].

Theorem 2.2 (Berkovich–Uncu)**.**

$$\sum_{k\geq 0} q^{T(k)} \begin{bmatrix} L \\ k \end{bmatrix}_q \left\{ T_{-1}\left(\begin{matrix} k \\ a \end{matrix} ; q \right) + T_{-1}\left(\begin{matrix} k \\ a+1 \end{matrix} ; q \right) \right\} = q^{T(a)} \begin{bmatrix} 2L+1 \\ L-a \end{bmatrix}_q. \tag{2.7}$$

The Andrews-Baxter q-trinomial coefficients [3] can be defined as

$$T_{-1}\left(\begin{matrix} k \\ a \end{matrix} ; q \right) = \sum_{\substack{m\geq 0, \\ m\equiv k+a (mod 2)}} q^{T(m)} \begin{bmatrix} k \\ m \end{bmatrix}_q \begin{bmatrix} k-m \\ \frac{k-m-a}{2} \end{bmatrix}_q. \tag{2.8}$$

It is easy to check that

$$T_{-1}\left(\begin{matrix} k \\ a \end{matrix} ; q \right) + T_{-1}\left(\begin{matrix} k \\ a+1 \end{matrix} ; q \right) = \sum_{m\geq 0} q^{T(m)} \begin{bmatrix} k \\ m \end{bmatrix}_q \begin{bmatrix} k-m \\ \lfloor \frac{k-m-a}{2} \rfloor \end{bmatrix}_q. \tag{2.9}$$

Substituting (2.9) into left hand side of (2.7) and changing $k \to k+m$ we complete the proof of (2.1).

It is instructive to compare (2.1) with the Corollary (2.6) in [16].

Theorem 2.3 (Warnaar)**.** *For $L \in \mathbf{N}$, $a \in \mathbf{Z}$*

$$\sum_{k\geq 0} W_{L,k}(q) \begin{bmatrix} 2k \\ k-a \end{bmatrix}_q = q^{2a^2} \begin{bmatrix} 2L \\ L-2a \end{bmatrix}_q, \tag{2.10}$$

where

$$W_{L,k}(q) = \sum_{m=0}^{L} q^{(m+k)^2+k^2} \begin{bmatrix} L \\ m, 2k \end{bmatrix}_q \geq 0. \tag{2.11}$$

Observe that unlike (2.10), transformation (2.1) can not be iterated. Interestingly enough, there exists an *odd* companion to Theorem 2.3.

Theorem 2.4. *For $L \in \mathbf{N}$, $a \in \mathbf{Z}$*

$$\sum_{k\geq 0} O_{L,k}(q) \begin{bmatrix} 2k+1 \\ k-a \end{bmatrix}_q = q^{4T(a)} \begin{bmatrix} 2L \\ L-2a-1 \end{bmatrix}_q, \tag{2.12}$$

where

$$O_{L,k}(q) = \sum_{m=0}^{L} q^{2T(m+k)+2T(k)} \begin{bmatrix} L \\ m, 2k+1 \end{bmatrix}_q \geq 0. \tag{2.13}$$

We remark that while Theorem 2.4 is not explicitly stated in [16], it is a special case of an identity on page 222 there.

Schur's bounded version of Euler's Pentagonal Number Theory states

$$1 = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{3j+1}{2}j} \begin{bmatrix} L \\ \lfloor \frac{L-3j}{2} \rfloor \end{bmatrix}_q. \tag{2.14}$$

With the aid of (2.5) we can convert (2.14) into

$$0 \leq \sum_{k=0}^{L} C_{L,k}(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{2j(3j+1)} \begin{bmatrix} 2L+1 \\ L-3j \end{bmatrix}_q. \tag{2.15}$$

Hence,

$$G(L, L+1, \frac{8}{3}, \frac{4}{3}, 3, q) \geq 0.$$

Making use of (1.12), it is easy to check that

$$\sum_{k=0}^{L} C_{L,k}(q) = \sum_{k=0}^{L} q^{T(k)} \begin{bmatrix} L \\ k \end{bmatrix}_q (-q)_k. \tag{2.16}$$

And so identity (2.15) can be rewritten as

$$\sum_{k=0}^{L} q^{T(k)} \begin{bmatrix} L \\ k \end{bmatrix}_q (-q)_k = \sum_{j=-\infty}^{\infty} (-1)^j q^{2j(3j+1)} \begin{bmatrix} 2L+1 \\ L-3j \end{bmatrix}_q. \tag{2.17}$$

Letting $L \to \infty$ and using the Jacobi triple product identity (1.13) yields a special case of the Lebesgue identity [13]

$$\sum_{m\geq 0} \frac{q^{T(m)}}{(q)_m} (-q)_m = \frac{(q^4; q^4)_\infty}{(q)_\infty}, \tag{2.18}$$

and so, (2.17) is a new bounded version of the Lebesgue identity. Perform $q \to \frac{1}{q}$ in (2.17) and use (1.10) together with

$$(-q^{-1}; q^{-1})_n = (-q)_n q^{-T(n)}, \ n \in \mathbf{N} \tag{2.19}$$

to obtain after simplification a new polynomial version of Euler's Pentagonal Number Theorem

$$\sum_{k=0}^{L}(-q)_{L-k}q^{(L+1)k}\begin{bmatrix}L\\k\end{bmatrix}_q=\sum_{j=-\infty}^{\infty}(-1)^jq^{3j^2+j}\begin{bmatrix}2L+1\\L-3j\end{bmatrix}_q. \tag{2.20}$$

It proves that

$$G(L,L+1,\frac{4}{3},\frac{2}{3},3,q)\geq 0. \tag{2.21}$$

We now move on to prove Theorem 1.2. We start with the finite analogue of the Andrews-Gordon identity due to Foda-Quano [11].

For $L\in\mathbf{N}$, $\nu\in\mathbb{Z}_{>0}$, $s=0,1,\ldots,\nu-1$

$$\sum_{n_2,\ldots,n_\nu\geq 0}q^{N_2^2+\ldots+N_\nu^2+N_{\nu+1-s}+\ldots+N_\nu}\prod_{i=2}^{\nu}\begin{bmatrix}n_i+L-2\sum_{j=2}^{i}N_j-E_{i,s}^{\nu}\\n_i\end{bmatrix}_q=$$
$$\sum_{j=-\infty}^{\infty}(-1)^jq^{\frac{(2\nu+1)j^2+j(1+2s)}{2}}\begin{bmatrix}L\\\lfloor\frac{L-(2\nu+1)j-s}{2}\rfloor\end{bmatrix}_q. \tag{2.22}$$

Here, $N_j=n_j+n_{j+1}+\ldots+n_\nu$, $j=2,\ldots,\nu$ and $E_{i,s}^{\nu}=max(i+s-\nu,0)$. Observe that (2.14) is the case $\nu=1$ of (2.22).

With the aid of (2.5) we obtain

$$0\leq\sum_{k,n_2,\ldots,n_\nu\geq 0}C_{L,k}(q)q^{N_2^2+\ldots+N_\nu^2+N_{\nu+1-s}+\ldots+N_\nu}\prod_{i=2}^{\nu}\begin{bmatrix}n_i+k-2\sum_{j=2}^{i}N_j-E_{i,s}^{\nu}\\n_i\end{bmatrix}_q=$$
$$q^{T(s)}\sum_{j=-\infty}^{\infty}(-1)^jq^{(\nu+1)(2\nu+1)j^2+(\nu+1)(2s+1)j}\begin{bmatrix}2L+1\\L-s-(2\nu+1)j\end{bmatrix}_q. \tag{2.23}$$

Hence,

$$G(L,L+1+2s,(\nu+1)(1+\frac{1+2s}{2\nu+1}),(\nu+1)(1-\frac{1+2s}{2\nu+1}),2\nu+1,q)\geq 0,$$

for all $L\in\mathbf{N}$, $\nu\in\mathbb{Z}_{>0}$, $s=0,1,2,\ldots,\nu-1$. This completes the proof of Theorem 1.2.

3. Further Observations

We replace q^2 by q in (1.14) and then set $M=L,L+1$, $z=q^{\frac{1}{2}}$ to find that for $L\in\mathbf{N}$

$$\sum_{j=-\infty}^{\infty}(-1)^jq^{T(j)}\begin{bmatrix}2L\\L-j\end{bmatrix}_q=\delta_{L,0} \tag{3.1}$$

and

$$\sum_{j=-\infty}^{\infty}(-1)^jq^{T(j)}\begin{bmatrix}2L+1\\L-j\end{bmatrix}_q=0, \tag{3.2}$$

where $\delta_{L,0}=1$ if $L=0$ and $\delta_{L,0}=0$ if $L>0$. The formulas (3.1) and (3.2) can be combined into

$$\sum_{j=-\infty}^{\infty}(-1)^jq^{T(j)}\begin{bmatrix}L\\\lfloor\frac{L-2j}{2}\rfloor\end{bmatrix}_q=\delta_{L,0}. \tag{3.3}$$

Applying Theorem 2.3 to (3.1) yields

$$W_{L,0}(q)=\sum_{n\geq 0}q^{n^2}\begin{bmatrix}L\\n\end{bmatrix}_q=\sum_{j=-\infty}^{\infty}(-1)^jq^{\frac{5j+1}{2}j}\begin{bmatrix}2L\\L-2j\end{bmatrix}_q, \tag{3.4}$$

which is Bressoud's bounded version of the first Rogers-Ramanujan identity [9]. Analogously, applying (2.5) to (3.3) yields

$$C_{L,0}(q) = \sum_{n\geq 0} q^{n^2+n} \begin{bmatrix} L \\ n \end{bmatrix}_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{5j^2+3j}{2}} \begin{bmatrix} 2L+1 \\ L-2j \end{bmatrix}_q, \tag{3.5}$$

which can be recognized as Warnaar's bounded version of the second Rogers-Ramanujan identity [15]. Next, we perform the change of summation variables below

$$\begin{aligned} &\sum_{j=-\infty}^{\infty} (-1)^j q^{5T(j)} \begin{bmatrix} 2L \\ L-2j-1 \end{bmatrix}_q = \\ &- \sum_{j=-\infty}^{\infty} (-1)^j q^{5T(-1-j)} \begin{bmatrix} 2L \\ L+2j+1 \end{bmatrix}_q = \\ &- \sum_{j=-\infty}^{\infty} (-1)^j q^{5T(j)} \begin{bmatrix} 2L \\ L-2j-1 \end{bmatrix}_q \end{aligned} \tag{3.6}$$

to conclude that

$$q^{L+1} \sum_{j=-\infty}^{\infty} (-1)^j q^{5T(j)} \begin{bmatrix} 2L \\ L-2j-1 \end{bmatrix}_q = 0. \tag{3.7}$$

Adding (3.4) and (3.7) and employing recursion relation (1.11) we obtain

$$\sum_{n\geq 0} q^{n^2} \begin{bmatrix} L \\ n \end{bmatrix}_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{5j^2+j}{2}} \begin{bmatrix} 2L+1 \\ L-2j \end{bmatrix}_q. \tag{3.8}$$

Observe that (3.4) and (3.8) imply that for $k \in \mathbf{N}$

$$\sum_{n\geq 0} q^{n^2} \begin{bmatrix} \lfloor \frac{k}{2} \rfloor \\ n \end{bmatrix}_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{5j^2+j}{2}} \begin{bmatrix} k \\ \lfloor \frac{k-4j}{2} \rfloor \end{bmatrix}_q. \tag{3.9}$$

Apply Theorem 2.1 to (3.9) to obtain

$$\sum_{k,n\geq 0} C_{L,k} q^{n^2} \begin{bmatrix} \lfloor \frac{k}{2} \rfloor \\ n \end{bmatrix}_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{21j^2+5j}{2}} \begin{bmatrix} 2L+1 \\ L-4j \end{bmatrix}_q, \tag{3.10}$$

which proves that

$$G(L, L+1, \frac{13}{4}, 2, 4, q) \geq 0.$$

In the limit as $L \to \infty$ (3.10) becomes

$$\sum_{m,k,n\geq 0} \frac{q^{T(m)+T(m+k)+n^2} \begin{bmatrix} \lfloor \frac{k}{2} \rfloor \\ n \end{bmatrix}_q}{(q)_m (q)_k} = \frac{(q^{21}, q^8, q^{13}; q^{21})_n}{(q)_\infty}. \tag{3.11}$$

This is to be contrasted with Andrews-Gordon identity mod 21 [1]

$$\sum_{n_1,n_2,\ldots,n_9\geq 0} \frac{q^{N_1^2+N_2^2+\ldots+N_9^2+N_8+N_9}}{(q)_{n_1}(q)_{n_2}\ldots(q)_{n_9}} = \frac{(q^{21}, q^8, q^{13}; q^{21})_\infty}{(q)_\infty}, \tag{3.12}$$

with $N_i = n_i + \ldots + n_9$, $i = 1, \ldots, 9$. On the left of (3.11) one has 3-fold sum, while on the left side of (3.12) one has 9-fold sum. Analogously, applying Theorem 2.3 to (3.4) and Theorem 2.4 to (3.5) and

(3.8), we prove that

$$G(L,L,\frac{11}{4},\frac{5}{2},4,q)\geq 0,$$
$$G(L-1,L+1,4,\frac{5}{4},4,q)\geq 0,$$
$$G(L-1,L+1,\frac{15}{4},\frac{3}{2},4,q)\geq 0,$$

and obtain, as $L\to\infty$

$$\sum_{m,k,n\geq 0}\frac{q^{k^2+(m+k)^2+n^2}}{(q)_m(q)_{2k}}\begin{bmatrix}k\\n\end{bmatrix}_q=\frac{(q^{21},q^{10},q^{11};q^{21})_\infty}{(q)_\infty}, \tag{3.13}$$

$$\sum_{m,k,n\geq 0}\frac{q^{2T(k)+2T(m+n)+2T(n)}}{(q)_m(q)_{2k+1}}\begin{bmatrix}k\\n\end{bmatrix}_q=\frac{(q^{21},q^{5},q^{16};q^{21})_\infty}{(q)_\infty}, \tag{3.14}$$

and

$$\sum_{m,k,n\geq 0}\frac{q^{2T(k)+2T(m+n)+n^2}}{(q)_m(q)_{2k+1}}\begin{bmatrix}k\\n\end{bmatrix}_q=\frac{(q^{21},q^{6},q^{15};q^{21})_\infty}{(q)_\infty}, \tag{3.15}$$

respectively.

In [7, p. 2332] the following identity was derived

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2}\begin{bmatrix}2L\\L-2j\end{bmatrix}_q=(-q;q^2)_L. \tag{3.16}$$

We now follow a well-trodden path and check that

$$q^{L+1}\sum(-1)^j q^{2j^2+2j}\begin{bmatrix}2L\\L-2j-1\end{bmatrix}_q=0. \tag{3.17}$$

Adding (3.16) and (3.17) we derive, with the aid of (1.11), that

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2}\begin{bmatrix}2L+1\\L-2j\end{bmatrix}_q=(-q;q^2)_L. \tag{3.18}$$

Equations (3.16) and (3.18) imply that for $k\in\mathbf{N}$

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2}\begin{bmatrix}k\\ \lfloor\frac{k-4j}{2}\rfloor\end{bmatrix}_q=(-q;q^2)_{\lfloor\frac{k}{2}\rfloor}. \tag{3.19}$$

Applying Theorem 2.1 to (3.19) and letting $L\to\infty$ we obtain

$$\sum_{m,k\geq 0}\frac{q^{T(m)+T(m+k)}(-q;q^2)_{\lfloor\frac{k}{2}\rfloor}}{(q)_m(q)_k}=\frac{(q^{20},q^{8},q^{12};q^{20})_\infty}{(q)_\infty}. \tag{3.20}$$

Compare it with the Bressoud formula in [8]

$$\sum_{n_1,\ldots,n_9\geq 0}\frac{q^{N_1^2+\ldots+N_9^2+N_8+N_9}}{(q)_{n_1}\ldots(q)_{n_8}(q^2;q^2)_{n_9}}=\frac{(q^{20},q^{8},q^{12};q^{20})_\infty}{(q)_\infty}, \tag{3.21}$$

where $N_i=n_i+\ldots+n_9$, $i=1,\ldots,9$.

Analogously, applying Theorem 2.3 to (3.16) and Theorem 2.4 to (3.18) we get as $L\to\infty$

$$\sum_{m,k\geq 0}\frac{q^{k^2+(m+k)^2}}{(q)_m(q)_{2k}}(-q;q^2)_k=\frac{(q^{20},q^{10},q^{10};q^{20})_\infty}{(q)_\infty} \tag{3.22}$$

and

$$\sum_{m,k\geq 0}\frac{q^{2T(k)+2T(m+k)}}{(q)_m(q)_{2k+1}}(-q;q^2)_k=\frac{(q^{20},q^6,q^{14};q^{20})_\infty}{(q)_\infty}, \tag{3.23}$$

respectively.

For our final example, we employ Dyson's identity [5], [7, p. 2330]

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{T(3j)}\begin{bmatrix}2L+1\\L-3j\end{bmatrix}_q=\frac{(q^3;q^3)_L}{(q)_L}. \tag{3.24}$$

Applying Theorem 2.4 to (3.24) yields

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{5T(3j)}\begin{bmatrix}2L\\L-1-6j\end{bmatrix}_q=\sum_{k\geq 0}O_{L,k}\frac{(q^3;q^3)_k}{(q)_k}\geq 0. \tag{3.25}$$

This proves that

$$G(L-1,L+1,5,\frac{5}{2},6,q)\geq 0. \tag{3.26}$$

Letting $L\to\infty$ in (3.25) and using (1.13) we arrive at a new elegant result

$$\sum_{m,k\geq 0}\frac{q^{2T(k)+2T(m+k)}}{(q)_m(q)_{2k+1}}\frac{(q^3;q^3)_k}{(q)_k}=\frac{(q^{15};q^{15})_\infty}{(q)_\infty}. \tag{3.27}$$

4. Acknowledgement

I would like to thank George Andrews, James Mc Laughlin and Ali Uncu for their kind interest.

References

[1] G. E. Andrews, *An analytic generalization of the Rogers-Ramanujan identities for odd moduli*, Proc. Nat. Acad. Sci. USA **71** (1974), 4082–4085.

[2] G. E. Andrews, *The theory of partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998 Reprint of the 1976 original. MR1634067 (99c:11126)

[3] G. E. Andrews, and R. J. Baxter, *Lattice gas generalization of the hard hexagon model. III. q-Trinomial coefficients*, J. Statist. Phys. **47** (1987), no: 3-4, 297–330.

[4] G. E. Andrews, R. J. Baxter, D. M. Bressoud, W. H. Burge, P. J. Forrester, G. Viennot, *Partitions with prescribed hook differences*, Europe J. Comb. **8** (1987) no. 4, 341–350.

[5] W. N. Bailey, *Some identities in combinatorial analysis*, Proc. London Math. Soc. **2** (1947), 421–435.

[6] A. Berkovich, A. K. Uncu, *Elementary polynomial identities involving q-trinomial coefficients*, Ann. Comb. **23** (2019), 549–560.

[7] A. Berkovich, S. O. Warnaar, *Positivity preserving transformations for q-binomial coefficients*, Trans. Amer. Math. Soc. **357** (2005), no. 6, 2291–2351.

[8] D. M. Bressoud, *An analytic generalization of Rogers-Ramanujan identities with interpretation*, Quart. J. Math. Oxford (2) **31** (1980), 389–399.

[9] D. M. Bressoud, *Some identities for terminating q-series*, Math. Proc. Camb. Phil. Soc. **89** (1981), 211–223.

[10] D. M. Bressoud, *The Borwein conjecture and partitions with prescribed hook differences*, Electron. J. Combin. **3** (1996) no. 2, Research Paper 4.

[11] O. Foda, Y. H. Quano, *Polynomial identities of the Rogers-Ramanujan type*, Int. J. Mod. Phys. A **10** (1995), 2291–2315.

[12] M. E. H. Ismail, D. Kim, D. Stanton, *Lattice paths and positive trigonometric sums*, Const. Approx. **15** (1999), 69–81.

[13] V. A. Lebesgue, *Sommation de quelques series*, J. Math. Pure. Appl. **5** (1840), 42–71.

[14] Chen Wang, *Analytic proof of the Borwein conjecture*, arXiv:1901.10886v1. [math.CO].

[15] S. O. Warnaar, *The generalized Borwein conjecture. I.The Burge transform, in: B.C. Berndt, K. Ono (Eds.), q-Series with Applications to Combinatorics, Number Theory and Physics*, Contemp. Math. **291**, AMS, Providence, RI (2001), 243–267.

[16] S. O. Warnaar, *The generalized Borwein conjecture. II. Refined q-trinomial coefficients*, Discrete Math. **272** (2003), no. 2-3, 215–258.

Department of Mathematics, University of Florida, 358 Little Hall, Gainesville FL 32611, USA
E-mail address: `alexb@ufl.edu`