

ISOMETRIES BETWEEN FINITE GROUPS

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ABSTRACT. We prove that if H is a subgroup of index n of any cyclic group G then G can be isometrically embedded in (H^n, d_{Ham}^n) , thus generalizing previous results of Carlet (1998) for $G = \mathbb{Z}_{2^k}$ and Yildiz and Ödemiş Özger (2012) for $G = \mathbb{Z}_{p^k}$ with p prime. Next, for any positive integer q we define the q -adic metric d_q in \mathbb{Z}_{q^n} and prove that (\mathbb{Z}_{q^n}, d_q) is isometric to (\mathbb{Z}_q^n, d_{RT}) for every n , where d_{RT} is the Rosenbloom–Tsfasman metric. More generally, we then demonstrate that any pair of finite groups of the same cardinality are isometric to each other for some metrics that can be explicitly constructed. Finally, we consider a chain \mathcal{C} of subgroups of a given group and define the chain metric $d_{\mathcal{C}}$ and chain isometries between two chains. Let G, K be groups with $|G| = q^n$, $|K| = q$ and let $H < G$. Using chains, we prove that under certain conditions, $(G, d_{\mathcal{C}}) \simeq (K^n, d_{RT})$ and $(G, d_{\mathcal{C}}) \simeq (H^{[G:H]}, d_{BRT})$ where d_{BRT} is the block Rosenbloom–Tsfasman metric which generalizes d_{RT} .

1. INTRODUCTION

Historical background. The Hamming metric d_{Ham} is the most classic and commonly used metric in coding theory, typically in codes defined over finite fields. Since the 90s, the Lee metric d_{Lee} was also considered on the rings \mathbb{Z}_m . The Gray map is an isometry between (\mathbb{Z}_4, d_{Lee}) and $(\mathbb{Z}_2 \times \mathbb{Z}_2, d_{Ham})$. This map naturally extends to an isometry from \mathbb{Z}_4^n to \mathbb{Z}_2^{2n} . In a famous paper from 1994, Hammons et al ([8]) used the Gray isometry to explain the formal duality exhibited by some pairs of binary non-linear codes such as Kerdock and Preparata codes and Goethals and Goethals–Delsarte codes (previously, Nechaev obtained some similar results in [11]).

Few years later, Salagean–Mandache ([16]) proved that, except for the known case $p = n = 2$, it is not possible to construct a metric d induced by a weight in \mathbb{Z}_{p^n} such that (\mathbb{Z}_{p^n}, d) is isometric to $(\mathbb{Z}_p^n, d_{Ham})$ for any prime p . This result was then extended by Sueli Costa and collaborators showing the non-existence of isometries from \mathbb{Z}_{m^n} to a Hamming space X^n , $|X| = m$ (see [14] for $m = p$ prime, [10] for arbitrary m).

In another direction, Carlet ([2]) generalized the Gray map to an embedding between \mathbb{Z}_{2^k} and $\mathbb{Z}_2^{2^{k-1}}$ preserving distances. This map naturally extends coordinatewise to $(\mathbb{Z}_{2^k})^n$ and $\mathbb{Z}_2^{2^{k-1}n}$. A couple of years later, Yildiz–Özger ([17]) proved that \mathbb{Z}_{p^k} , with p an odd prime, can be isometrically embedded into $\mathbb{Z}_p^{2^{k-1}}$ with the Hamming metric for any $k > 1$ (see Remark 4.5). From a more general point of view, Greferath and Schmidt ([7]) further generalized the Gray map to an embedding from an arbitrary finite chain ring R with the homogeneous metric to the residue field $F = R/\mathfrak{m}$ with de Hamming metric. More precisely, $(R, d_{Hom}) \hookrightarrow (F^{q^{m-1}}, d_{Ham})$ where $q = |F|$ and $m = \text{length}(R)$. They used their map to

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construct interesting non-linear binary codes. More recently, D'Oliveira and Firer ([4], [5]) showed that up to a decoding equivalence, any metric space can be isometrically embedded into a hypercube with the Hamming metric.

The goal of this work is to better understand isometries and isometric embeddings between finite groups (typically finite fields or finite rings for their applications in coding theory). We will give new explicit isometries and isometric embeddings from cyclic groups and also provide a general procedure to obtain isometries between arbitrary groups.

Outline and results. Now, we briefly summarize the structure of, and the results in, the paper. In Section 2, we first recall some basic preliminaries on metric spaces, G -invariant metrics and isometries. If G is a group acting on a metric space (X, d) , we define the associated symmetry group and their G -representations. In Proposition 2.4 we show that given (X, d) with a G -representation, there is a bijection $\varphi : X \rightarrow G$ inducing a group structure on X and a metric d_G on G such that φ is a group isomorphism and $\varphi : (X, d) \rightarrow (G, d_G)$ is an isometry.

In the next section we give a simple group-theoretical proof of the known result that there are no cyclic representations of a Hamming space X^n for X of prime cardinality (see Proposition 3.3). In particular, there is no isometry between \mathbb{Z}_{p^n} and $(\mathbb{Z}_p^n, d_{Ham})$.

In Section 4 we consider isometric embeddings, i.e. injective maps between metric spaces preserving distances. We generalize the result of Yildiz-Özger asserting that \mathbb{Z}_{p^k} , p prime, can be isometrically embedded into \mathbb{Z}_p^{k-1} for any $k > 1$ with the Hamming metric. In Theorem 4.4 we generalize this result by proving that for any m and any subgroup H of \mathbb{Z}_m of index n , \mathbb{Z}_m can be isometrically embedded into H^n with the Hamming metric. This allows to isometrically embed a ring into rings of different characteristics as noted in Remark 4.7. In Example 4.8 we consider the subgroups of \mathbb{Z}_{12} . In Remark 4.9 we show that the isometric embedding of \mathbb{Z}_{2n} into \mathbb{Z}_2^n with the Hamming metric recovers the Lee metric on \mathbb{Z}_{2n} .

In Section 5, for any $q, n \in \mathbb{N}$ with $q \geq 2$, we define the q -adic metric d_q on \mathbb{Z}_{q^n} . The RT -metric was introduced by Rosenbloom and Tsfasman in [15] and has since then proven to be a quite useful metric in coding theory. In Theorem 5.2 we give a short and direct proof that \mathbb{Z}_{q^n} with the q -adic metric is isometric to (\mathbb{Z}_q^n, d_{RT}) , that is

$$(\mathbb{Z}_{q^n}, d_q) \simeq (\mathbb{Z}_q^n, d_{RT}).$$

In the next section we show that any isometry between subgroups can be extended to the ambient groups (see Theorem 6.1). This implies that any pair of groups of the same size (and hence all) are isometric (see Corollary 6.3). So, for instance, \mathbb{Z}_2^3 , $\mathbb{Z}_2 \times \mathbb{Z}_4$, \mathbb{Z}_8 , \mathbb{D}_4 and \mathbb{Q}_8 are all mutually isometric.

In Section 7, we consider metrics on chain of subgroups and chain isometries. If G has a chain \mathcal{C} of subgroups, in Definition 7.1 we introduce the associated chain metric $d_{\mathcal{C}}$. In Remark 7.2 we show how the q -adic metric and the RT -metric can be naturally considered as chain metrics. In Definition 7.5 we define the notion of chain isometry, that is when two chains of subgroups of the same length of two groups of the same size are isometric.

To say that two groups are chain isometric gives more information than merely saying that they are isometric, since this implies that every step of the chains are isometric to each other (see (4.4)). In Theorem 7.11, using geometric chains (see (7.9)) we generalize Theorem 5.2

to groups not necessarily cyclic. More precisely, if $H < G$, with $|G| = q^n$ and $|H| = q$ then $(G, d_C) \simeq (H^n, d_{RT})$ where d_C is the chain metric associated to some chain of length n with initial term H . The most general result will be obtained in the next section.

Finally, in Section 8, we consider the block Rosenbloom–Tsfasman metric d_{BRT} which generalizes the RT -metric (see Definition 8.1). In Theorem 8.2 we prove that given a proper subgroup H of a group G and a chain \mathcal{C} with initial term H we have that G with the metric d_C induced by the chain is isometric to $H^{[G:H]}$ with the block RT -metric, i.e.

$$(G, d_C) \simeq (H^{[G:H]}, d_{BRT}).$$

2. INVARIANT METRICS ON GROUPS

In this paper X will always denote a finite set and G a finite group. We begin by recalling some standard definitions. A function $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ is called a *metric* on X if it is definite positive, symmetric, and satisfies the triangle inequality. That is, (a) $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$, (b) $d(x, y) = d(y, x)$, and (c) $d(x, y) \leq d(x, z) + d(z, y)$ hold for all $x, y, z \in X$. The pair (X, d) is called a *metric space*. If d takes values in \mathbb{N}_0 and $\text{Im}(d) \subset \llbracket 0, n \rrbracket$ with $n = |X|$ we say that d is *integral* and that (X, d) is an *integral metric space*.

Given an injective function $f : X \rightarrow Y$ between sets and a metric d on Y , one can define the *pullback metric* of f on X by

$$(2.1) \quad d_f(x, x') = d(f(x), f(x')), \quad x, x' \in X.$$

Two metric spaces (X_1, d_1) and (X_2, d_2) are said to be *isometric*, denoted by $(X_1, d_1) \simeq (X_2, d_2)$, if there is an *isometry* between X_1 and X_2 . That is, there is a bijection $\varphi : X_1 \rightarrow X_2$ such that for every $x, y \in X_1$ we have

$$(2.2) \quad d_1(x, y) = d_2(\varphi(x), \varphi(y)).$$

In other words, d_1 is the pullback metric of d_2 .

A metric d on X can be naturally extended to the metric d^n on X^n in the following way

$$(2.3) \quad d^n(x, y) = \sum_{i=1}^n d(x_i, y_i)$$

with $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in X^n$. For instance, the Hamming metric d_{Ham} on X extends to X^n giving the most popular metric in coding theory

$$d_{Ham}^n(x, y) = \sum_{i=1}^n d_{Ham}(x_i, y_i) = |\{1 \leq i \leq n : x_i \neq y_i\}|.$$

Sometimes, the extended metric d^n is also called d . This is a particular case of the product metric. If (X_i, d_i) , $i = 1, \dots, n$ are metric spaces then the *product metric* $d_\pi = d_1 \times \dots \times d_n$ on $X = X_1 \times \dots \times X_n$ is given by

$$d_\pi(x, y) = d_1(x_1, y_1) + \dots + d_n(x_n, y_n)$$

for $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$.

A map $w : X \rightarrow \mathbb{R}$ is called a *weight function* if $w(x) \geq 0$ for all $x \in X$ and $w(x) = 0$ for exactly one element x of X . If w takes integral values and moreover $\text{Im}(w) \subset \llbracket 0, N \rrbracket$ for some $N \in \mathbb{N}$ we will say that w is an *integral weight*. The pair (X, w) is called a *weight space*

or integral weight space if w is integral. If $(X, +)$ is a group, then w must also satisfy the subadditive property, that is $w(x + y) \leq w(x) + w(y)$ for every $x, y \in X$. Given a metric space (X, d) and $a \in X$ we can canonically define a weight function w_a by

$$w_a(x) = d(x, a), \quad x \in X.$$

If $|X| = n$, there are n different weight functions as above. For instance, if X is a finite set and x_0 is a fixed element, the *Hamming weight relative to x_0* is given by

$$(2.4) \quad w_{x_0}(x) = d_{Ham}(x, x_0) = \begin{cases} 1 & \text{if } x \neq x_0, \\ 0 & \text{if } x = x_0. \end{cases}$$

If (X, d) is a metric space with integral weight function w , the *weight distribution* of (X, d) is the set of weight frequencies $\{A_0, A_1, \dots, A_N\}$ where $A_i = \#\{x \in X : w(x) = i\}$. The *weight enumerator polynomial* of (X, d) is defined by

$$(2.5) \quad \mathcal{W}_{(X,d)}(t) = \sum_{x \in X} t^{w(x)} = \sum_{i=0}^N A_i t^i.$$

Let $(X_1, d_1), (X_2, d_2)$ be two metric spaces such that $0 \in X_1, X_2$ and consider the product space $X = X_1 \times X_2$ with the product metric $d_1 \times d_2$. Notice that we get

$$\mathcal{W}_{(X,d)}(t) = \mathcal{W}_{(X_1,d_1)}(t) + \mathcal{W}_{(X_2,d_2)}(t) + \sum_{(x_1,x_2) \in X_1^* \times X_2^*} t^{w((x_1,x_2))}$$

where X_i^* denotes $X_i \setminus \{0\}$ for $i = 1, 2$.

All metrics and weights considered in this paper will be integral.

G-invariant metrics. We are interested in the particular case in which $X = G$ is a group. The metric d is called *right (resp. left) translation invariant* if for any g, g', h in G we have

$$d(gh, g'h) = d(g, g')$$

(resp. $d(hg, hg') = d(g, g')$). If G is abelian both notions coincide and d is called *translation invariant*. There is a distinguished weight function $w(x) = d(x, e)$, where e is the identity element of G . Also, if (G, w) is a weight space, one can define a metric d on G by

$$d(x, y) = w(x - y)$$

for every $x, y \in G$, provided that $w(-x) = w(x)$ holds for every $x \in G$ (or, in multiplicative notation, requiring that $d(x, y) = w(xy^{-1})$ with $w(x^{-1}) = w(x)$ for every $x \in G$), which is automatic for elementary 2-groups.

Let \mathbb{S}_X denote the permutation group of X . If G acts on X we have $G \leq \mathbb{S}_X$.

Definition 2.1. Let (X, d) be a metric space and $G \leq \mathbb{S}_X$. We say that (X, d) is σ -invariant for $\sigma \in G$, denoted $d^\sigma = d$, if

$$d(\sigma(x), \sigma(y)) = d(x, y)$$

for all $x, y \in X$. Further, (X, d) is called G -invariant if d is σ -invariant for every $\sigma \in G$. The *symmetry group* of (X, d) is defined by

$$(2.6) \quad \Gamma(X, d) = \{\sigma \in \mathbb{S}_X : d^\sigma = d\}.$$

We will say that (X, d) has a G -representation if there is a group $G \leq \Gamma(X, d)$ which is *regular* (or *simply transitive*); that is, $|G| = |X|$ and the action of G is transitive.

Notice that if $X = G$ and d is right translation invariant then d is G_R -invariant, where $G_R : G \rightarrow \mathbb{S}_G$ is the right regular representation given by $g \mapsto R_g$ for $g \in G$ with $R_g(x) = xg$ for any $x \in G$. Furthermore, (X, d) is G_R -invariant if and only if $G_R \leq \Gamma(X, d)$. Similarly, the above facts hold for d a left translation invariant metric and the left regular representation G_L .

Remark 2.2. Let $f : X \rightarrow Y$ be a bijective map from X to a G -invariant metric space (Y, d) . Then, the action σ_Y of G on Y can be transferred to X in such a way that the pullback metric d_f becomes G -invariant. In fact, defining the action of G on X by $\sigma_X = f^{-1} \circ \sigma_Y \circ f$ we have

$$\begin{aligned} d_f(\sigma_X(x), \sigma_X(x')) &= d_f(f^{-1}(\sigma_Y(f(x))), f^{-1}(\sigma_Y(f(x')))) \\ &= d(\sigma_Y(f(x)), \sigma_Y(f(x'))) = d(f(x), f(x')) = d_f(x, x') \end{aligned}$$

for $x, x' \in X$, where we have used that d is G -invariant.

Example 2.3. Let (X, d) be a metric space with $|X| = n$ and d the discrete metric, that is $d(x, x) = 0$ and $d(x, y) = 1$ for every $x \neq y$. Then $\Gamma(X, d) \simeq \mathbb{S}_n$ and, hence, (X, d) has a G -representation for every group of order n , as a consequence of Cayley's Theorem.

We now show that given a G -representation on a metric space (X, d) , the group G inherits a metric and the set X inherits a group structure.

Proposition 2.4. *Suppose that the metric space (X, d) has a G -representation. Then, there is a bijection $\varphi : X \rightarrow G$ which induces a group structure on X and a metric d_G on G such that φ is a group isomorphism and $\varphi : (X, d) \rightarrow (G, d_G)$ is an isometry. Moreover, $d_G = d_{\varphi^{-1}}$ is translation invariant, that is $d_G(g_1h, g_2h) = d_G(g_1, g_2)$ for every $g_1, g_2, h \in G$.*

Proof. Fix an element $x_0 \in X$. Since G acts regularly on X , G acts transitively on X and $|G| = |X|$. Thus, for each $x \in X$ there is a unique $g = g_x \in G$ such that $g(x_0) = x$. Hence, we can define the map

$$\varphi : X \rightarrow G, \quad x \mapsto g_x.$$

This gives a group structure on X by considering the product

$$xy = g_y(x).$$

To check associativity, note that $x(yz) = g_{yz}(x)$ and $(xy)z = g_z(xy) = g_zg_y(x)$. Since $g_zg_y(x_0) = zy = g_{yz}(x_0)$ we have that $g_zg_y = g_{yz}$ and hence $x(yz) = (xy)z$ for any $x, y, z \in G$. It is easy to see that x_0 is the identity element in X and

$$x^{-1} = \varphi^{-1}(g_x^{-1}) = g_x^{-1}(x_0).$$

Therefore, φ is a group homomorphism and hence an isomorphism.

Now, d induces the metric d_G in G by

$$d_G(g_x, g_y) = d(x, y).$$

Clearly, (X, d) and (G, d_G) are isometric since $d_G(\varphi(x), \varphi(y)) = d(x, y)$, by definition. It only remains to show that d_G is translation invariant. For $g_x, g_y, h \in G$ we have

$$\begin{aligned} d_G(g_x \cdot h, g_y \cdot h) &= d_G((g_x \cdot h)(x_0), (g_y \cdot h)(x_0)) = d(h(g_x(x_0)), h(g_y(x_0))) \\ &= d(h(x), h(y)) = d(x, y) = d_G(g_x, g_y) \end{aligned}$$

as we wanted to see. \square

Remark 2.5. A similar idea as in the previous proposition was established by Forney in ([6]) in the context of geometrically uniform signal sets in \mathbb{R}^n with the Euclidean metric.

We now illustrate the above proposition, showing that the G -representations of a set strongly depend on the chosen metric and on the symmetry of the group. For clarity we will sometimes use the graph of distances of a finite metric space (X, d) . If $|X| = n$, the graph of distances of X is the weighted complete graph K_n where each edge xy has weight $d(x, y)$.

Example 2.6. Consider the set $X = \{x, y, w, z\}$. Since X has 4 elements, any G -representation of X has only two possibilities: $G \simeq \mathbb{Z}_4$ or $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Consider some metrics on X given by the following graphs of distances:

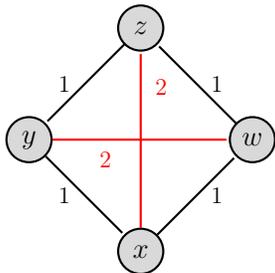


FIGURE 1. d_1

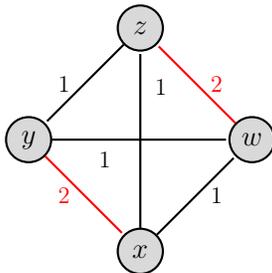


FIGURE 2. d_2

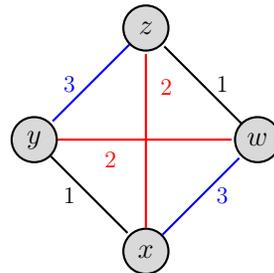


FIGURE 3. d_3

We now show that the number of G -representations of (X, d) depends strongly on the chosen metric. Let

$$G_1 = \langle \rho = (xywz) \rangle \quad \text{and} \quad G_2 = \langle \tau_1 = (xy)(wz), \tau_2 = (xz)(yw) \rangle$$

be the groups defined by the permutations ρ and τ_1, τ_2 , respectively. Note that $G_1 \simeq \mathbb{Z}_4$, $G_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, and that they act transitively on X .

(i) We have that (X, d_1) is G_i -invariant, that is $G_i \leq \Gamma(X, d_1)$, for $i = 1, 2$. Fix x as the identity element in X and define $\varphi_1 : X \rightarrow G_1$ as follows

$$x \mapsto e, \quad y \mapsto \rho = (xywz), \quad w \mapsto \rho^2 = (xw)(yz), \quad z \mapsto \rho^3 = (xzyw).$$

Also, define $\varphi_2 : X \rightarrow G_2$ by

$$x \mapsto e, \quad y \mapsto \tau_1 = (xy)(wz), \quad w \mapsto \tau_1\tau_2 = (xw)(yz), \quad z \mapsto \tau_2 = (xz)(yw).$$

By Proposition 2.4, there are isometries $(X, d_1) \simeq (G_1, d_{G_1})$ and $(X, d_1) \simeq (G_2, d_{G_2})$. Note that under the isomorphisms $G_1 \simeq \mathbb{Z}_4$ and $G_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ the metrics d_{G_1} and d_{G_2} correspond to the Lee metric d_{Lee} on \mathbb{Z}_4 and to the Hamming metric d_{Ham}^2 on $\mathbb{Z}_2 \times \mathbb{Z}_2$. That is

$$(X, d_1) \simeq (\mathbb{Z}_4, d_{Lee}) \quad \text{and} \quad (X, d_1) \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2, d_{Ham}^2).$$

In particular, by transitivity, we have recovered the known isometry between \mathbb{Z}_4 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ given by the Gray map.

(ii) Consider now the metric d_2 . Notice that (X, d_2) is G_2 -invariant but it is not G_1 -invariant. In this case, (X, d_2) has only one G -representation with $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

(iii) Finally, observe that when the metric d_3 is considered, the metric space (X, d_3) has no G -representations at all because the group of symmetries of (X, d) is trivial (none of the groups can preserve the distances). \diamond

Remark 2.7. From now on, if G is a group, (G, d) will denote a metric space where the distance d is G -invariant and G acts by right translations, i.e. we identify G with its right regular representation $G_R \leq \mathbb{S}_G$.

3. HAMMING SPACES ARE NOT ISOMETRIC TO CYCLIC GROUPS

Due to the relevance shown by the Gray map $\mathcal{G} : \mathbb{Z}_4 \rightarrow \mathbb{Z}_2^2$ in coding theory, people was concerned whether there is a generalization of this isometry sending \mathbb{Z}_{p^n} to $(\mathbb{Z}_p)^n$, with p prime. As already mentioned in the Introduction, Salagean-Mandache proved ([16]) that, except for the known case $p = n = 2$, it is impossible to construct a metric d in \mathbb{Z}_{p^n} such that (\mathbb{Z}_{p^n}, d) is isometric to $(\mathbb{Z}_p^n, d_{Ham})$. Sueli Costa and collaborators deal with the existence of isometries of a Hamming space X^n , $|X| = m$ (see [14] for $m = p$ prime, [10] for arbitrary m). They proved that there are no G -representations of the Hamming space X^n with G a cyclic group, except for the Gray map and the trivial case $n = 1$; that is, we have

Theorem 3.1 ([10]). *Let (X^n, d_{Ham}) be a Hamming space, with $|X| = m$. If $(m, n) \neq (2, 2)$ and $n > 1$, there does not exist any cyclic group G and any metric d on G such that (G, d) is isometric with (X^n, d_{Ham}) .*

We will give an alternative simple proof of this result, using group theory, in the case that $|X| = p$ is prime.

Lemma 3.2. *If G is a finite group containing two subgroups $H \simeq \mathbb{Z}_p^k$ and $K \simeq \mathbb{Z}_{p^\ell}$, with p prime and $k, \ell \in \mathbb{N}$, then the order of G is divisible by $p^{k+\ell-1}$.*

Proof. By Sylow's theorems it is enough to consider only the case when $G = P$ is a p -group. Therefore, P contains subgroups H and K isomorphic to \mathbb{Z}_p^k and \mathbb{Z}_{p^ℓ} respectively. Then we have that

$$|P| \geq |HK| = \frac{|H||K|}{|H \cap K|} \geq \frac{p^k \cdot p^\ell}{p} = p^{k+\ell-1},$$

where we have used that $|H \cap K| = 1$ or p , since $H \cap K$ is cyclic of order p . Since $|P|$ is a power of p and $|P| \geq p^{k+\ell-1}$, then $|P|$ is divisible by $p^{k+\ell-1}$. \square

We now restate Theorem 3.1 in terms of representations for spaces of prime cardinality.

Proposition 3.3. *Let (X^n, d_{Ham}) be a Hamming space, with $|X| = p$ prime. If $(p, n) \neq (2, 2)$ and $n > 1$, then there is no cyclic representation of (X^n, d_{Ham}) . In particular, there is no isometry between \mathbb{Z}_{p^n} and $(\mathbb{Z}_p^n, d_{Ham})$.*

Proof. The Hamming space X^n has a cyclic representation if and only if the symmetry group has an element of order p^n , such that the subgroup generated by this element acts regularly. It is known that the symmetry group of the Hamming space is (see for instance [1] or [13])

$$\Gamma(X^n, d_{Ham}) \simeq \mathbb{S}_p \wr \mathbb{S}_n = (\mathbb{S}_p)^n \rtimes \mathbb{S}_n,$$

where \wr denotes wreath product. Assume that there exists a cyclic representation of (X^n, d_{Ham}) . We have that $\mathbb{Z}_{p^n} \subsetneq \Gamma(X^n, d_{Ham})$ and also that $\mathbb{Z}_p^n \subsetneq \Gamma(X^n, d_{Ham})$. Thus, by Lemma 3.2, p^{n+n-1} must divide $|\Gamma(X^n, d_{Ham})|$, that is

$$p^{2n-1} \mid (p!)^n n!$$

On the other hand, note that if ν_p denotes the p -adic valuation, we have

$$\nu_p((p!)^n n!) = n\nu_p(p!) + \nu_p(n!) = n\nu_p(p) + \nu_p(n!) = n + \nu_p(n!).$$

Now, suppose that $n = n_0 + n_1p + n_2p^2 + \cdots + n_rp^r$ is the p -adic expansion of n , and let $s_p(n) = n_0 + n_1 + \cdots + n_r$. Then, the Legendre formula for the p -adic valuation of $n!$ implies that $\nu_p(n!) = \frac{n - s_p(n)}{p-1}$. Then we have that

$$(3.1) \quad \nu_p((p!)^n n!) = n + \frac{n - s_p(n)}{p-1} \leq n + n - 1 = 2n - 1.$$

Moreover, the equality holds in (3.1) if and only if $p = 2$ and $n = 2^k$ for some k .

It only remains to prove the case $p = 2$. It is enough to show that if

$$g \in \Gamma(X^n, d_{Ham}^n) \simeq \mathbb{Z}_2^n \rtimes \mathbb{S}_n$$

then its order satisfies $|g| < 2^n$. We recall Landau's function $G(n) = \max\{ord(\sigma) : \sigma \in \mathbb{S}_n\}$ and the known bound $G(n) \leq e^{\frac{n}{e}}$. Now, if $g = (t, s)$ with $t \in \mathbb{Z}_2^n$ and $s \in \mathbb{S}_n$ then we have $|g| \leq |t||s| \leq 2e^{\frac{n}{e}}$, by the bound on Landau's function. In particular if $n > 2$,

$$|g| \leq 2e^{\frac{n}{e}} < 2^n,$$

and hence there is no element of order 2^n in $\Gamma(X^n, d_{Ham}^n)$. \square

4. ISOMETRIC EMBEDDINGS

We begin with the following definition.

Definition 4.1. A map $\varphi : (X_1, d_1) \rightarrow (X_2, d_2)$ between metric spaces is an *isometric embedding* if it is injective and preserves distances. That is, for every $x, y \in X_1$ we have

$$d_1(x, y) = d_2(\varphi(x), \varphi(y)).$$

As we previously mentioned, for any fixed m , the cyclic group \mathbb{Z}_m cannot be isometric to any Hamming space (X^n, d_{Ham}^n) where $m = |X^n|$. However, there are isometric embeddings of \mathbb{Z}_{p^k} into the Hamming space $\mathbb{Z}_p^{p^{k-1}}$ with p prime due to Carlet ([2], $p = 2$), Greferath and Yildiz-Özger ([7] and [17], any prime), thus generalizing the Gray map. Namely, we have the following result.

Theorem 4.2 ([2], [7], [17]). *Let p be a prime and $k > 1$, then there exists an isometric embedding from (\mathbb{Z}_{p^k}, d) to $(\mathbb{Z}_p^{p^{k-1}}, d_{Ham}^{p^{k-1}})$.*

In this section we will generalize the previous result to any cyclic group. More precisely, we will see that, for any $m \in \mathbb{N}$, it is always possible to isometrically embed \mathbb{Z}_m into a Hamming space X^n with $m < |X^n|$ for some n . For simplicity we will denote this by

$$\mathbb{Z}_m \hookrightarrow (X^n, d_{Ham}^n).$$

Let $G = \mathbb{Z}_m = \{0, 1, \dots, m-1\}$ and H be a subgroup of index n , i.e. $n = [G : H]$. Consider $v \in H^n$ and $\rho \in \mathbb{S}_n$. We define the map (well-defined since H is a group)

$$\Psi_{v, \rho} : \mathbb{Z}_m \rightarrow H^n, \quad t \mapsto \Psi_t(\rho) \cdot v$$

where $\rho \cdot v$ is the action $\rho(v_1, \dots, v_n) = (v_{\rho(1)}, \dots, v_{\rho(n)})$ and

$$\Psi_t(x) = \frac{x^t - 1}{x - 1} = x^{t-1} + \cdots + x + 1.$$

In the previous notations, we have the following.

Lemma 4.3. *Let H be a subgroup of $G = \mathbb{Z}_m$ of index n . Suppose $H = \langle h \rangle$ and $e_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}_m^n$ with 1 in the i -th coordinate. If $v = he_i$ and ρ is an n -cycle then $\Psi_{v,\rho}$ is injective.*

Proof. For $0 \leq t < m$, if $t = qn + r$ with $0 \leq r < n$, then using that $\rho(e_i) = e_{\rho(i)}$ we have

$$\Psi_{v,\rho}(t) = \sum_{k=0}^{t-1} \rho^k(he_i) = (q+1) \sum_{k=0}^{r-1} h e_{\rho^k(i)} + q \sum_{k=r}^{n-1} h e_{\rho^k(i)}.$$

Now, for $0 \leq s < m$, if $s = q'n + r'$ with $0 \leq r' < n$, we can see that $\Psi_{v,\rho}(s) = \Psi_{v,\rho}(t)$ if and only if $q = q'$, and $r = r'$, that is, only if $s = t$. Therefore $\Psi_{v,\rho}$ is injective. \square

We now generalize Theorem 4.2 to \mathbb{Z}_m , with m any positive integer.

Theorem 4.4. *Let H be a subgroup of $G = \mathbb{Z}_m$ of index n . Consider $v = he_i \in H^n$ with $H = \langle h \rangle$, $1 \leq i \leq n$, and $\rho \in \mathbb{S}_n$ an n -cycle. Then we have the isometric embedding*

$$(4.1) \quad \Psi_{v,\rho} : (\mathbb{Z}_m, d_n) \hookrightarrow (H^n, d_{Ham}^n),$$

where d_n is the translation invariant metric with associated weight given by

$$(4.2) \quad w_n(t) = \begin{cases} t & \text{if } t \leq n, \\ n & \text{if } n \leq t \leq m - n, \\ m - t & \text{if } m - n \leq t \leq m - 1. \end{cases}$$

Proof. Consider $\tilde{\Psi}_{v,\rho} : \mathbb{Z}_m \rightarrow H^n \times \mathbb{S}_n$ given by $t \mapsto (\Psi_t(\rho) \cdot v, \rho^t)$ and notice that it is a homomorphism. In fact, given $t, s \in \mathbb{Z}_m$ we have

$$(4.3) \quad \begin{aligned} \tilde{\Psi}_{v,\rho}(t) \tilde{\Psi}_{v,\rho}(s) &= (\Psi_t(\rho)v, \rho^t) (\Psi_s(\rho)v, \rho^s) \\ &= (\rho^s \Psi_t(\rho)v + \Psi_s(\rho)v, \rho^t \rho^s) \\ &= (\rho^s(\rho^{t-1} + \dots + \rho + 1)v + (\rho^{s-1} + \dots + \rho + 1)v, \rho^{t+s}) \\ &= ((\rho^{t+s-1} + \dots + \rho + 1)v, \rho^{t+s}) \\ &= (\Psi_{t+s}(\rho)v, \rho^{t+s}) = \tilde{\Psi}_{v,\rho}(t+s). \end{aligned}$$

Further, if $t + s \equiv u \pmod{m}$, then $\Psi_{v,\rho}(t + s) = \Psi_{v,\rho}(u)$ and $\rho^{t+s} = \rho^u$, and hence we get $\tilde{\Psi}_{v,\rho}(s + t) = \tilde{\Psi}_{v,\rho}(u)$.

Note that $\Psi_{v,\rho} = \pi \circ \tilde{\Psi}_{v,\rho}$, so we have the following diagram

$$(4.4) \quad \begin{array}{ccc} \mathbb{Z}_m & \xrightarrow{\tilde{\Psi}_{v,\rho}} & H^n \times \mathbb{S}_n \\ & \searrow \Psi_{v,\rho} & \downarrow \pi \\ & & H^n. \end{array}$$

Now, $\Psi_{v,\rho}$ is 1-1 by Lemma 4.3 and hence $\tilde{\Psi}_{v,\rho}$ is also injective.

Denote $\Psi_{v,\rho}$ by Ψ and let d_n be the pull-back metric in \mathbb{Z}_m of the Hamming metric in H^n , that is

$$d_n(a, b) = d_{Ham}^n(\Psi(a), \Psi(b)).$$

Hence Ψ preserves the metric d_n by definition.

We now prove that d_n is translation invariant. Note that $H^n \rtimes \mathbb{S}_n \subsetneq \mathbb{S}_H^n \rtimes \mathbb{S}_n$, since $H \subsetneq \mathbb{S}_H$ by Cayley's Theorem, and $\Gamma(H^n, d_{Ham}^n) \simeq \mathbb{S}_H^n \rtimes \mathbb{S}_n$. Thus, we have

$$(4.5) \quad H^n \rtimes \mathbb{S}_n \subsetneq \Gamma(H^n, d_{Ham}^n).$$

In this way, for every $a, b, c \in \mathbb{Z}_m$ we have

$$\begin{aligned} d_n(a+c, b+c) &= d_{Ham}^n(\Psi(a+c), \Psi(b+c)) \\ &= d_{Ham}^n(\rho^c \Psi(a) + \Psi(c), \rho^c \Psi(b) + \Psi(c)) \\ &= d_{Ham}^n(\Psi(a), \Psi(b)) = d_n(a, b), \end{aligned}$$

where in the second equality we have used (4.5) and that $\Psi(a+c) = \rho^c \Psi(a) + \Psi(c)$, deduced from (4.3).

Finally, we check the weights. For $t \in \mathbb{Z}_m$ we have

$$w_n(t) = d_{Ham}^n(\Psi(t), 0) = w_{Ham}(\Psi(t)) = w_{Ham}((\rho^{t-1} + \dots + 1) \cdot h e_i).$$

Thus, considering $t = qn + r$, with $0 \leq r < n$, we arrive at

$$w_n(t) = w_{Ham}((q+1) \sum_{k=0}^{r-1} h e_{\rho^k(i)} + q \sum_{k=r}^{n-1} h e_{\rho^k(i)}),$$

from which (4.2) readily follows. \square

In the situation of the previous theorem, there are $\phi(\frac{m}{n})n!$ different isometric embeddings $\Psi_{he_i, \rho}$, where ϕ is the Euler totient function. Indeed, there are n vectors e_i , $(n-1)!$ different n -cycles ρ and $\phi(\frac{m}{n})$ different generators h of H of index n in \mathbb{Z}_m . However, all these maps have the same associated metric d_n .

Remark 4.5. Let $G = \mathbb{Z}_{p^k}$ with p prime and for $1 \leq i \leq k-1$ consider the subgroup $H_i = \mathbb{Z}_{p^i}$ of index $n_i = p^{k-i}$. By the previous theorem, there is an isometric embedding

$$(\mathbb{Z}_{p^k}, d_{n_i}) \hookrightarrow ((\mathbb{Z}_{p^i})^{p^{k-i}}, d_{Ham}^{p^{k-i}})$$

determined by $\Psi_{e_j, \rho}$ for any $e_j \in H_i^{n_i}$ and any n_i -cycle ρ in \mathbb{S}_{n_i} . In particular, if we take $H_1 = \mathbb{Z}_p$, $v = e_1 = (1, 0, \dots, 0)$ and $\rho = (12 \dots n_1)$ then the weight w_{n_1} becomes the extended Lee weight over \mathbb{Z}_{p^k}

$$w_L(x) = \begin{cases} x & \text{if } x \leq p^{k-1}, \\ p^{k-1} & \text{if } p^{k-1} \leq x \leq p^k - p^{k-1}, \\ p^k - x & \text{if } p^k - p^{k-1} \leq x \leq p^k - p^k - 1. \end{cases}$$

and thus we recover the isometric embedding $(\mathbb{Z}_{p^k}, d_L) \hookrightarrow (\mathbb{Z}_p^{p^{k-1}}, d_{Ham}^{p^{k-1}})$ previously given by Yildiz-Özger ([17], Theorem 2.1).

Remark 4.6. Consider $G = \mathbb{Z}_m$. One could want H^n to be of the least possible size such that Ψ is close to be a bijective embedding (i.e. an isometry). In this case, we must choose H minimizing $|H|^n$. On the other hand, if we want to minimize the dimension of the Hamming space of the embedding, we should choose H to be the subgroup of maximum cardinality.

For instance, let $G = \mathbb{Z}_{p_1^{k_1} p_2^{k_2}} = \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}}$ where $p_1 < p_2$ are different primes. By Theorem 4.4 and choosing $H = \mathbb{Z}_{p_1^{k_1-1} p_2^{k_2}}$ in order to minimize the size of the embedding space, we have the isometric embedding

$$\mathbb{Z}_{p_1^{k_1} p_2^{k_2}} \hookrightarrow ((\mathbb{Z}_{p_1^{k_1-1} p_2^{k_2}})^{p_1}, d_{Ham}^{p_1}).$$

On the other hand, we can apply the same theorem to $\mathbb{Z}_{p_1^{k_1}}$ and $\mathbb{Z}_{p_2^{k_2}}$ separately and then concatenate the spaces obtaining the isometric embedding

$$\mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \hookrightarrow ((\mathbb{Z}_{p_1^{k_1-1}})^{p_1} \times (\mathbb{Z}_{p_2^{k_2-1}})^{p_2}, d_{Ham}^{p_1} \times d_{Ham}^{p_2}).$$

However, note that although in general the second group is smaller, we must pay the price that the coordinates have different alphabets.

Remark 4.7. It is possible to isometrically embed a ring into rings of different characteristics. In fact, if $G = \mathbb{Z}_{pq}$ with p, q primes, by Theorem 4.4 we have $\mathbb{Z}_{pq} \hookrightarrow ((\mathbb{Z}_p)^q, d_{Ham}^q)$ and $\mathbb{Z}_{pq} \hookrightarrow ((\mathbb{Z}_q)^p, d_{Ham}^p)$.

Example 4.8. Let $G = \mathbb{Z}_{12}$, we can consider the four subgroups $H_1 \simeq \mathbb{Z}_2$, $H_2 \simeq \mathbb{Z}_3$, $H_3 \simeq \mathbb{Z}_4$ and $H_4 \simeq \mathbb{Z}_6$ with corresponding indices $n_1 = 6$, $n_2 = 4$, $n_3 = 3$ and $n_4 = 2$. Thus, by Theorem 4.4 we have the four isometric embeddings $\mathbb{Z}_{12} \hookrightarrow (H_i^{n_i}, d_{Ham}^{n_i})$ for $i = 1, 2, 3, 4$. That is

$$\mathbb{Z}_{12} \hookrightarrow (\mathbb{Z}_2^6, d_{Ham}^6), \quad \mathbb{Z}_{12} \hookrightarrow (\mathbb{Z}_3^4, d_{Ham}^4), \quad \mathbb{Z}_{12} \hookrightarrow (\mathbb{Z}_4^3, d_{Ham}^3), \quad \mathbb{Z}_{12} \hookrightarrow (\mathbb{Z}_6^2, d_{Ham}^2).$$

By (4.2), we have the following weight distributions

t	0	1	2	3	4	5	6	7	8	9	10	11
$w_1(t)$	0	1	2	3	4	5	6	5	4	3	2	1
$w_2(t)$	0	1	2	3	4	4	4	4	4	3	2	1
$w_3(t)$	0	1	2	3	3	3	3	3	3	3	2	1
$w_4(t)$	0	1	2	2	2	2	2	2	2	2	2	1

The corresponding weight enumerators are

$$(4.6) \quad \begin{aligned} \mathcal{W}_{(\mathbb{Z}_{12}, d_1)}(t) &= t^6 + 2t^5 + 2t^4 + 2t^3 + 2t^2 + 2t + 1, \\ \mathcal{W}_{(\mathbb{Z}_{12}, d_2)}(t) &= 5t^4 + 2t^3 + 2t^2 + 2t + 1, \\ \mathcal{W}_{(\mathbb{Z}_{12}, d_3)}(t) &= 7t^3 + 2t^2 + 2t + 1, \\ \mathcal{W}_{(\mathbb{Z}_{12}, d_4)}(t) &= 9t^2 + 2t + 1. \end{aligned}$$

Note that the associated metrics d_i obtained are all different and that the metric d_1 is just the Lee metric.

Remark 4.9. In general, considering different subgroups H of G , the isometric embeddings provided by Theorem 4.4 give rise to different metrics. In the particular case that $G = \mathbb{Z}_{2n}$, we have $H = \mathbb{Z}_n$, and the isometric embedding

$$\mathbb{Z}_{2n} \hookrightarrow ((\mathbb{Z}_2)^n, d_{Ham}^n)$$

recovers the Lee metric on \mathbb{Z}_{2n} since the associated weight function w is given by

$$(w(i))_{i=0}^{2n-1} = (0, 1, 2, \dots, n-1, n, n-1, \dots, 2, 1).$$

5. ISOMETRIES BETWEEN \mathbb{Z}_{q^n} AND \mathbb{Z}_q^n

Here we will prove that the groups \mathbb{Z}_{q^n} and \mathbb{Z}_q^n are isometric for positive integers n and q with $q \geq 2$ by using metrics different from the Hamming metric. Namely, the *RT*-metric in \mathbb{Z}_q^n and the q -adic metric on \mathbb{Z}_{q^n} that we now define.

Definition 5.1. Let $n, q \in \mathbb{N}$ with $q \geq 2$. The q -adic metric d_q in \mathbb{Z}_{q^n} is given by

$$(5.1) \quad d_q(x, y) = \min_{0 \leq i \leq n} \{i : q^{n-i} \mid x - y\}$$

for any $x, y \in \mathbb{Z}_{q^n}$.

Indeed, d_q is a translation invariant metric. To check that it is a metric it is enough to show the triangle inequality, the other conditions being straightforward. Let $x, y, z \in \mathbb{Z}_{q^n}$ and suppose that

$$i = d(x, z), \quad j = d(z, y) \quad \text{and} \quad k = d(x, y).$$

Then, $q^{n-i} \mid x - z$ and $q^{n-j} \mid z - y$, and thus we have that

$$q^{n-\max\{i, j\}} \mid (x - z) + (z - y) = x - y.$$

Hence we have $k \leq \max\{i, j\}$ and therefore $d(x, y) \leq d(x, z) + d(z, y)$. In fact, d_q is an ultrametric. Finally, we have $d_q(x + z, y + z) = d_q(x, y)$ by definition, hence d_q is translation invariant. Notice that alternatively we have

$$d_q(x, y) = \lceil \log_q(\text{ord}(x - y)) \rceil,$$

where *ord* denotes the order of an element in the group. In particular, if $q = p$ is prime we simply get $d_p(x, y) = \log_p(\text{ord}(x - y))$.

We recall that the *Rosenbloom–Tsfasman metric* (or *RT*-metric) was originally defined over \mathbb{F}_q^n ([15]), hence for q a prime power. However, this metric can be defined over G^n for any group G . Thus, we now define the *RT*-metric on \mathbb{Z}_q^n for any pair of integers n, q with $q \geq 2$ as follows:

$$(5.2) \quad d_{RT}(x, y) = \max_{1 \leq i \leq n} \{i : x_i - y_i \neq 0\}.$$

Note that d_{RT} is translation invariant by definition. It is known that it coincides with the poset metric d_P on \mathbb{Z}_q^n given by the chain poset P defined by $1 \preceq 2 \preceq \dots \preceq n$ (see [12]).

Now, we construct an explicit isometry between the groups \mathbb{Z}_{q^n} and \mathbb{Z}_q^n with the previous metrics. Let $q \geq 2$ and n be positive integers and consider the function

$$(5.3) \quad \begin{aligned} \varphi : \mathbb{Z}_q^n &\rightarrow \mathbb{Z}_{q^n} \\ \varphi(a_1, a_2, \dots, a_n) &\mapsto a_1 q^{n-1} + a_2 q^{n-2} + \dots + a_{n-1} q + a_n \pmod{q^n}. \end{aligned}$$

One can check that its inverse

$$(5.4) \quad \varphi^{-1} : \mathbb{Z}_{q^n} \rightarrow \mathbb{Z}_q^n$$

is given by the q -base expansion, namely

$$(5.5) \quad \begin{array}{ll} 0 & \mapsto 0000 \cdots 000 \\ 1 & \mapsto 0000 \cdots 001 \\ & \vdots \\ q-1 & \mapsto 0000 \cdots 00(q-1) \\ q & \mapsto 0000 \cdots 010 \\ q+1 & \mapsto 0000 \cdots 011 \\ & \vdots \\ q^2-1 & \mapsto 0000 \cdots 0(q-1)(q-1) \\ q^2 & \mapsto 0000 \cdots 100 \\ q^2+1 & \mapsto 0000 \cdots 101 \\ & \vdots \\ q^n-1 & \mapsto (q-1)(q-1)(q-1)(q-1) \cdots (q-1)(q-1)(q-1) \end{array}$$

We now show that φ as in (5.3) preserves distances.

Theorem 5.2. *For any $n, q \in \mathbb{N}$ with $q \geq 2$ the map $\varphi : (\mathbb{Z}_q^n, d_{RT}) \rightarrow (\mathbb{Z}_{q^n}, d_q)$ as in (5.3) is an isometry.*

Proof. To see that φ is an isometry between metric groups we must show that φ preserves distances and that the involved metrics are translation invariant.

Let $x, y \in \mathbb{Z}_q^n$ and suppose that $d_{RT}(x, y) = k$. This means that $x_k \neq y_k$ and $x_i = y_i$ for $i = k+1, \dots, n$. On the other hand,

$$d_q(\varphi(x), \varphi(y)) = \min_{0 \leq i \leq n} \{i : q^{n-i} \mid \varphi(x) - \varphi(y)\}$$

where, by (5.3), we have that

$$\varphi(x) - \varphi(y) = (x_1 - y_1)q^{n-1} + (x_2 - y_2)q^{n-2} + \cdots + (x_k - y_k)q^{n-k} \pmod{q^n},$$

with $x_k - y_k \neq 0$. Thus

$$d_q(\varphi(x), \varphi(y)) = k = d_{RT}(x, y)$$

and hence φ preserves distances. Finally, we have previously observed that both d_{RT} and d_q are translation invariant and the result thus follows. \square

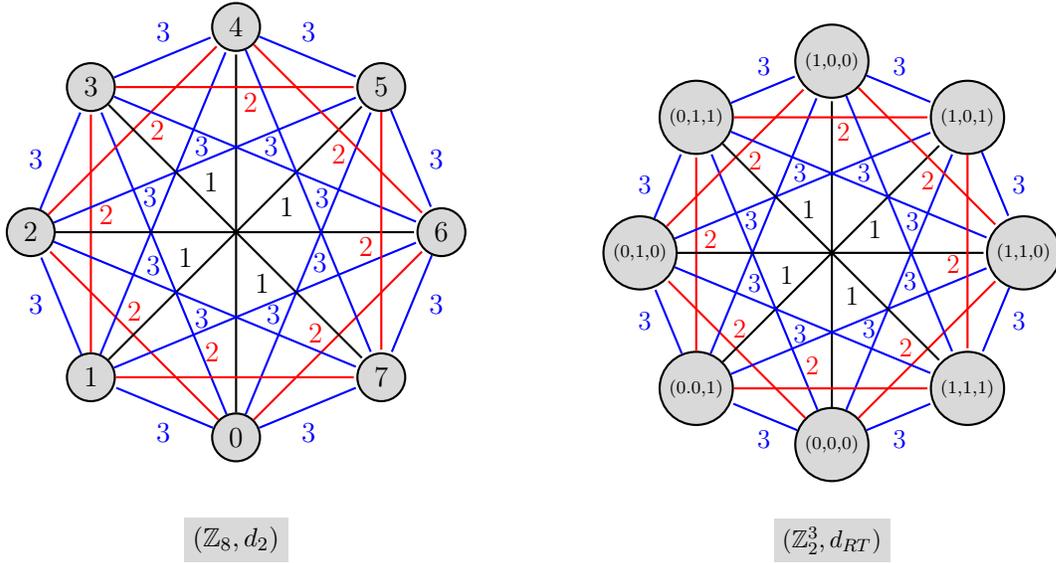
Notice that, by (5.1), (5.2) and Theorem 5.2, the weight enumerators are

$$(5.6) \quad \mathcal{W}_{(\mathbb{Z}_{q^n}, d_q)}(t) = \mathcal{W}_{(\mathbb{Z}_q^n, d_{RT})}(t) = \sum_{i=0}^n (q^i - q^{i-1}) t^i = (q-1) \sum_{i=0}^n q^{i-1} t^i.$$

Example 5.3. We now illustrate the previous theorem showing that the groups \mathbb{Z}_2^3 and \mathbb{Z}_8 are isometric. We take the d_{RT} metric on \mathbb{Z}_2^3 and the 2-adic metric d_2 on \mathbb{Z}_8 . In this case, the map $\varphi^{-1} : \mathbb{Z}_8 \rightarrow \mathbb{Z}_2^3$ in (5.4) is given by

$$\begin{array}{llll} 0 & \mapsto (0, 0, 0), & 2 & \mapsto (0, 1, 0), & 4 & \mapsto (1, 0, 0), & 6 & \mapsto (1, 1, 0), \\ 1 & \mapsto (0, 0, 1), & 3 & \mapsto (0, 1, 1), & 5 & \mapsto (1, 0, 1), & 7 & \mapsto (1, 1, 1). \end{array}$$

The graphs of distances of the groups are as follows:



One can easily check that the map φ preserves distances.

Also, note that the associated weight functions $w_{RT} : \mathbb{Z}_2^3 \rightarrow \llbracket 0, 3 \rrbracket$ and $w_2 : \mathbb{Z}_8 \rightarrow \llbracket 0, 3 \rrbracket$ are given by

$$w_{RT}(x) = \begin{cases} 0 & \text{if } x = (0, 0, 0), \\ 1 & \text{if } x = (1, 0, 0), \\ 2 & \text{if } x = (a, 1, 0), \\ 3 & \text{if } x = (a, b, 1), \end{cases} \quad \text{and} \quad w_2(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x = 4, \\ 2 & \text{if } x = 2, 6, \\ 3 & \text{if } x = 1, 3, 5, 7, \end{cases}$$

with $a, b \in \mathbb{Z}_2$. The weight enumerators are thus

$$\mathcal{W}_{(\mathbb{Z}_2^3, d_{RT})}(t) = \mathcal{W}_{(\mathbb{Z}_8, d_2)}(t) = 4t^3 + 2t^2 + t + 1.$$

6. EXTENDING ISOMETRIES OF SUBGROUPS

In this section we show that any isometry between metric subgroups can be extended to an isometry between the ambient groups with extended metrics. We recall from Remark 2.7 that all the metrics considered are G -invariant. More precisely, we have the following

Theorem 6.1. *Let G_1 and G_2 be two finite groups with $|G_1| = |G_2|$ and let $H_1 \subsetneq G_1$, $H_2 \subsetneq G_2$ be non-trivial proper subgroups with $|H_1| = |H_2|$. Then, any isometry between H_1 and H_2 can be extended to an isometry between G_1 and G_2 .*

Proof. Suppose that $(H_1, d_1) \simeq (H_2, d_2)$ and let $\tau : H_1 \rightarrow H_2$ be the isometry. Now, for $i = 1, 2$, we extend the metrics d_i of H_i to metrics \tilde{d}_i of G_i as follows:

$$(6.1) \quad \tilde{d}_i(x, y) := \begin{cases} d_i(x, y) & \text{if } x - y \in H_i, \\ \max_{u, v \in H_i} \{d_i(u, v)\} + 1 & \text{if } x - y \notin H_i. \end{cases}$$

Clearly $\tilde{d}_i(x, y) = 0$ if and only if $x = y$ and $\tilde{d}_i(x, y) = \tilde{d}_i(y, x)$. We must check that \tilde{d}_i satisfies the triangular inequality. Let $x, y, z \in G_i$. If $x - y, x - z, z - y \in H_i$ it follows from

the triangular inequality from d_i . Now, if one of $x - z$ or $z - y$ is not in H_i , say $x - z$, then

$$\tilde{d}_i(x, z) = \max_{u, v \in H_i} \{d_i(u, v)\} + 1$$

and we have $\tilde{d}_i(x, z) = \tilde{d}_i(x, y)$ if $x - y \notin H_i$ or $\tilde{d}_i(x, z) \geq \tilde{d}_i(x, y)$ if $x - y \in H_i$, and the claim follows.

Now, suppose that $m = |G_1| = |G_2|$ and $h = |H_1| = |H_2|$. Let T_i be a complete set of representatives of the right cosets of H_i in G_i for $i = 1, 2$. Consider any bijection $\rho : T_1 \rightarrow T_2$ and define the map

$$(6.2) \quad \begin{aligned} G_1 &\xrightarrow{\eta} G_2 \\ h + g_j &\longmapsto \tau(h) + \rho(g_j), \end{aligned}$$

where $g_1, g_2, \dots, g_{\frac{m}{h}}$ are the elements of T_1 . It is clear that η is bijective.

Note that x, y belong to the same coset of H_1 if and only if $\eta(x), \eta(y)$ belong to the same coset of H_2 . Therefore we conclude that

$$\tilde{d}_1(x, y) = d_1(x, y) = d_2(\eta(x), \eta(y)) = \tilde{d}_2(\eta(x), \eta(y))$$

and hence $(G_1, \tilde{d}_1) \simeq (G_2, \tilde{d}_2)$, as it was to be shown. \square

Remark 6.2. In the previous proof, the isometry η given by (6.2) is not unique, since $\eta = \eta_\rho$ depends on the bijection ρ between the complete set of representatives of right cosets T_1 on G_1 and T_2 on G_2 chosen. However, two such metrics differ by a distance preserving map. That is, if ρ and ρ' are two bijections from T_1 to T_2 then there is some $f \in \Gamma(G, \tilde{d}_2)$ such that $\eta_\rho = f \circ \eta_{\rho'}$. In fact, if $f = \eta_\rho \circ \eta_{\rho'}^{-1}$ then

$$\tilde{d}_2(f(x), f(y)) = \tilde{d}_2(\eta_\rho(\eta_{\rho'}^{-1}(x)), \eta_\rho(\eta_{\rho'}^{-1}(y))) = \tilde{d}_1(\eta_{\rho'}^{-1}(x), \eta_{\rho'}^{-1}(y)) = \tilde{d}_2(x, y)$$

and hence f preserves distances.

A direct consequence of this result is that every pair of groups of the same size are isometric. For groups of prime cardinality, the isometry is trivial in the sense that both metrics are Hamming metrics.

Corollary 6.3. *Let G_1 and G_2 be groups of the same cardinality. Then, there exists an isometry $\phi : (G_1, d_1) \rightarrow (G_2, d_2)$ where d_1 and d_2 are certain metrics in G_1 and G_2 . Moreover, if $|G_i|$ is not prime then d_i can be chosen in such a way that $d_i \neq d_{Ham}$ for $i = 1, 2$.*

Proof. Suppose $m = |G_1| = |G_2|$. If m is not prime, consider a prime p dividing m . Then, there are non-trivial proper subgroups $H_1 < G_1$ and $H_2 < G_1$ with $p = |H_1| = |H_2|$. Considering the Hamming metric in both H_1 and H_2 it is clear that these subgroups are isometric, that is $(H_1, d_{Ham}) \simeq (H_2, d_{Ham})$. From Theorem 6.1, this isometry lifts to an isometry

$$(G_1, d_1) \simeq (G_2, d_2),$$

where the metric $d_i = \tilde{d}_{Ham}$ for $i = 1, 2$ and \tilde{d} is as in (6.1). That is, $d_i(x, x) = 0$,

$$(6.3) \quad d_i(x, x+h) = 1 \quad \text{if } h \in H \setminus \{0\} \quad \text{and} \quad d_i(x, x+g) = 2 \quad \text{if } g \in G \setminus H,$$

for any $x \in G_i$ and $i = 1, 2$.

If $m = p$, then $G_1 \simeq G_2 \simeq \mathbb{Z}_p$ and they are trivially isometric with the Hamming metrics. This completes the proof. \square

Remark 6.4. By Corollary 6.3, for any $m, n \geq 2$ there exist, for instance, non-trivial isometries $\mathbb{Z}_{m^n} \simeq (\mathbb{Z}_m)^n$, $\mathbb{F}_{q^n} \simeq (\mathbb{F}_q)^n$, $\mathbb{D}_m \simeq \mathbb{Z}_{2m}$, etc.

Let $H \subsetneq G$ be a non-trivial proper subgroup and d a metric in H . In the sequel we will denote by

$$(6.4) \quad \tilde{d} = Ext_H^G(d)$$

(or simply $Ext_H(d)$ when G is understood) the metric in G induced by the extension given in Theorem 6.1 (see (6.1)). We will call this the *extended metric* of d from H to G .

We now illustrate the previous theorem for groups of small order $n = 4, 6, 8$.

Example 6.5 ($n = 4$). Let $G_1 = \mathbb{Z}_4$ and $G_2 = \mathbb{Z}_2^2$. It is known that (\mathbb{Z}_4, d_{Lee}) is isometric to $(\mathbb{Z}_2^2, d_{Ham}^2)$ via the Gray map. We now show that they are isometric by using Theorem 6.1.

Consider the subgroups $H_1 = \mathbb{Z}_2 \subsetneq \mathbb{Z}_4$ and $H_2 = \mathbb{Z}_2 \times \{0\} \subsetneq \mathbb{Z}_2 \times \mathbb{Z}_2$, both with the Hamming metric d_{Ham} . These subgroups are isometric via the inclusion map $\iota : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \times \{0\}$ given by $x \mapsto (x, 0)$. By Theorem 6.1 and (6.4) we have that

$$(\mathbb{Z}_4, \tilde{d}_1 = Ext_{\mathbb{Z}_2}^{\mathbb{Z}_4}(d_{Ham})) \simeq (\mathbb{Z}_2^2, \tilde{d}_2 = Ext_{\mathbb{Z}_2 \times \{0\}}^{\mathbb{Z}_2 \times \mathbb{Z}_2}(d_{Ham})).$$

Notice that $Ext_{\mathbb{Z}_2}^{\mathbb{Z}_4}(d_{Ham}) = d_2$, the 2-adic metric, and $Ext_{\mathbb{Z}_2 \times \{0\}}^{\mathbb{Z}_2 \times \mathbb{Z}_2}(d_{Ham}) = d_{RT}$, the *RT*-metric. In fact, by (5.1) and (5.2) we have

$$d_2(x, y) = \min_{0 \leq i < 2} \{i : 2^{2-i} \mid x - y\} \quad \text{and} \quad d_{RT}(x, y) = \max_{1 \leq i \leq 2} \{i : x_i - y_i \neq 0\}$$

respectively. Hence, by (6.1) or (6.3), for any $x, y \in \mathbb{Z}_4$ we have

$$\begin{aligned} \tilde{d}_1(x, y) &= 1 = d_2(x, y) & \text{if } x - y = 2, \\ \tilde{d}_1(x, y) &= 2 = d_2(x, y) & \text{if } x - y = 1, 3, \end{aligned}$$

while for any $u, v \in \mathbb{Z}_2^2$ we get

$$\begin{aligned} \tilde{d}_2(u, v) &= 1 = d_{RT}(u, v) & \text{if } u - v = (1, 0), \\ \tilde{d}_2(u, v) &= 2 = d_{RT}(u, v) & \text{if } u - v = (0, 1), (1, 1). \end{aligned}$$

We wish to point out that, although d_2 and \tilde{d}_{RT} are different from d_{Lee} and d_{Ham} , these metrics are correspondingly equivalent, i.e. $d_2 \simeq d_{Lee}$ and $d_{RT} \simeq d_{Ham}$, in a precise sense that we will not discuss here (this will be treated in another work).

Example 6.6 ($n = 6$). We now consider the groups \mathbb{Z}_6 and \mathbb{D}_3 . By Theorem 6.1, we have two different isometries between them, taking as H_1, H_2 subgroups of order 2 or 3, respectively. Namely,

$$(\mathbb{Z}_6, Ext_{\mathbb{Z}_2}(d)) \simeq (\mathbb{S}_3, Ext_{\langle \rho \rangle}(d)) \quad \text{and} \quad (\mathbb{Z}_6, Ext_{\mathbb{Z}_3}(d)) \simeq (\mathbb{S}_3, Ext_{\langle \tau \rangle}(d)),$$

where ρ is a 2-cycle, τ a 3-cycle and d is the Hamming metric in \mathbb{Z}_2 and \mathbb{Z}_3 , respectively.

Apart from the Hamming and Lee metrics, in addition we have the metrics obtained by the previous subgroup construction. The corresponding weight functions and enumerators are given by

\mathbb{Z}_6	0	1	2	3	4	5	enumerator
w_{Ham}	0	1	1	1	1	1	$5t + 1$
$w_{\mathbb{Z}_2}$	0	2	2	1	2	2	$4t^2 + t + 1$
$w_{\mathbb{Z}_3}$	0	2	1	2	1	2	$3t^2 + 2t + 1$
w_{Lee}	0	1	2	3	2	1	$t^3 + 2t^2 + 2t + 1$

and

\mathbb{S}_3	id	(12)	(13)	(23)	(123)	(132)	enumerator
w_{Ham}	0	1	1	1	1	1	$5t + 1$
$w_{\langle(12)\rangle}$	0	1	2	2	2	2	$4t^2 + t + 1$
$w_{\langle\tau\rangle}$	0	2	2	2	1	1	$3t^2 + 2t + 1$

where τ is any 3-cycle.

Example 6.7 ($n = 8$). By Corollary 6.3, all the groups of the same size are isometric to each other. Thus, for instance, we have

$$\mathbb{Z}_2^3 \simeq \mathbb{Z}_2 \times \mathbb{Z}_4 \simeq \mathbb{Z}_8 \simeq \mathbb{D}_4 \simeq \mathbb{Q}_8.$$

In fact, note that all these groups have at least one isomorphic copy of \mathbb{Z}_2 as a subgroup. Thus, if we take any pair $G_1, G_2 \in \{\mathbb{Z}_2^3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_8, \mathbb{D}_4, \mathbb{Q}_8\}$, with the trivial identifications, we then have

$$(G_1, Ext_{\mathbb{Z}_2}^{G_1}(d_{Ham})) \simeq (G_2, Ext_{\mathbb{Z}_2}^{G_2}(d_{Ham})).$$

The associated weights $w_{\mathbb{Z}_2}$ are given by

\mathbb{Z}_8	0	1	2	3	4	5	6	7
w	0	2	2	2	1	2	2	2
\mathbb{Z}_2^3	(0, 0, 0)	(1, 0, 0)	(0, 1, 0)	(0, 0, 1)	(1, 1, 0)	(0, 1, 1)	(1, 0, 1)	(1, 1, 1)
w	0	1	2	2	2	2	2	2
$\mathbb{Z}_2 \times \mathbb{Z}_4$	(0, 0)	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(0, 1)	(0, 2)	(0, 3)
w	0	1	2	2	2	2	2	2
\mathbb{D}_4	e	ρ	ρ^2	ρ^3	τ	$\rho\tau$	$\rho^2\tau$	$\rho^3\tau$
w	0	2	2	2	1	2	2	2
\mathbb{Q}_8	1	-1	i	$-i$	j	$-j$	k	$-k$
w	0	1	2	2	2	2	2	2

The weight enumerator is $\mathcal{W}_{(G, w_{\mathbb{Z}_2})}(t) = 6t^2 + t + 1$, where G is any group of order 8.

We now compute the weight enumerators for all the subgroups of all the groups of order 8. Since isomorphic subgroups give the same metric, we consider subgroups up to isomorphism. It is clear that \mathbb{Z}_4 is a subgroup of $G_1 \in \{\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{D}_4, \mathbb{Q}_8\}$, that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a subgroup of $G_2 \in \{\mathbb{Z}_2^3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{D}_4\}$ and that \mathbb{Z}_2 is a subgroup of G_3 , any group of order 8.

Thus, the weight enumerators for the corresponding extended metrics are as follows

$$\begin{aligned}\mathcal{W}_{(G_1, \text{Ext}_{\mathbb{Z}_4}(d_{Ham}))}(t) &= 4t^2 + 3t + 1, \\ \mathcal{W}_{(G_1, \text{Ext}_{\mathbb{Z}_4}(d_{Lee}))}(t) &= 4t^3 + t^2 + 2t + 1, \\ \mathcal{W}_{(G_2, \text{Ext}_{\mathbb{Z}_2^2}(d_{Ham}))}(t) &= 4t^2 + 3t + 1, \\ \mathcal{W}_{(G_2, \text{Ext}_{\mathbb{Z}_2^2}(d_{Ham}^2))}(t) &= 4t^3 + t^2 + 2t + 1, \\ \mathcal{W}_{(G_3, \text{Ext}_{\mathbb{Z}_2}(d_{Ham}))}(t) &= 6t^2 + t + 1.\end{aligned}$$

7. CHAIN METRICS AND CHAIN ISOMETRIES

In this section we will consider chain metrics and chain isometries on groups with chains of subgroups, generalizing the construction and results of the previous section.

Definition 7.1. Let G be a group and \mathcal{C} a chain of subgroups of G ,

$$(7.1) \quad \langle 0 \rangle = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_n = G.$$

The *chain metric* on G associated to \mathcal{C} is defined by

$$(7.2) \quad d_{\mathcal{C}}(x, y) = i \quad \text{if } x - y \in H_i \setminus H_{i-1}$$

for $i = 0, \dots, n$. Here we use the convention $H_{-1} = \emptyset$.

We now check that $d_{\mathcal{C}}$ is indeed a metric. We only have to show that the triangular inequality holds. Let $x, y, z \in G$ and suppose that $d(x, y) = i$, $d(x, z) = j$ and $d(z, y) = k$. Thus, $x - y \in H_i \setminus H_{i-1}$, $x - z \in H_j \setminus H_{j-1}$ and $z - y \in H_k \setminus H_{k-1}$. We can assume that $k \geq j$, therefore

$$x - y = (x - z) - (y - z) \in H_k \setminus H_{k-1}.$$

This implies that $d(x, y) \leq k \leq d(x, z) + d(z, y)$, as we wanted to see.

As a direct consequence of Definition 7.1, the weight enumerator of G with the chain metric is given by

$$(7.3) \quad \mathcal{W}_{(G, d_{\mathcal{C}})}(x) = \sum_{i=0}^n (|H_i| - |H_{i-1}|) x^i.$$

Remark 7.2. It is worth noting that the q -adic metric in \mathbb{Z}_{q^n} and the RT -metric in \mathbb{Z}_q^n are chain metrics.

(i) Let $G = \mathbb{Z}_{q^n}$ and consider the chain of subgroups \mathcal{C} given by

$$\langle 0 \rangle \subsetneq \mathbb{Z}_q \subsetneq \mathbb{Z}_{q^2} \subsetneq \cdots \subsetneq \mathbb{Z}_{q^n},$$

where we are identifying \mathbb{Z}_{q^i} with $\langle q^{n-i} \rangle = q^{n-i}\mathbb{Z}_{q^n}$ for $i = 0, \dots, n$. In fact, since for $x, y \in G$ we have

$$x - y \in \langle q^{n-i} \rangle \setminus \langle q^{n-(i-1)} \rangle \quad \Leftrightarrow \quad q^{n-i} \mid x - y \quad \text{and} \quad q^{n-(i-1)} \nmid x - y$$

then

$$d_{\mathcal{C}}(x, y) = \min_{0 \leq i \leq n} \{i : q^{n-i} \mid x - y\} = d_q(x, y)$$

holds for any $x, y \in G$, by (5.1) and (7.2).

(ii) Let $G = \mathbb{Z}_q^n$ and consider the following chain of subgroups \mathcal{C} ,

$$\langle 0 \rangle \subsetneq \mathbb{Z}_q \subsetneq \mathbb{Z}_q^2 \subsetneq \cdots \subsetneq \mathbb{Z}_q^n,$$

where by abuse of notation \mathbb{Z}_q^i denotes $\mathbb{Z}_q^i \times \{0\}^{n-i}$ for $i = 1, \dots, n$. In fact, since for $x, y \in G$ we have

$$\begin{aligned} x - y \in \mathbb{Z}_q^i \setminus \mathbb{Z}_q^{i-1} &\Leftrightarrow x_i - y_i \neq 0 \quad \text{and} \quad x_j - y_j \in \mathbb{Z}_q \\ &\Leftrightarrow x_i - y_i \neq 0 \quad \text{and} \quad x_j - y_j = 0 \quad \text{for } j > i. \end{aligned}$$

Then,

$$d_{\mathcal{C}}(x, y) = \max_{1 \leq i \leq n} \{i : x_i - y_i \neq 0\} = d_{RT}(x, y)$$

holds for any $x, y \in G$, by (5.2) and (7.2).

Notice that the weight enumerators given in (5.6) are of the form (7.3).

We now exhibit another chain metric. Let G be a finite group and r, n positive integers. Consider the following chain of groups

$$(7.4) \quad \mathcal{C} : \quad G \subset G^r \subset G^{r^2} \subset G^{r^3} \subset \cdots \subset G^{r^n},$$

where the inclusions are given by the diagonal maps δ_i . For instance, $\delta_0 : G \rightarrow G^r$ is given by $x \mapsto (x, x, \dots, x)$ with x repeated r -times, $\delta_1 : G^r \rightarrow G^{r^2}$ is given by

$$(x, x, \dots, x) \mapsto ((x, x, \dots, x), (x, x, \dots, x), \dots, (x, x, \dots, x)),$$

and so on. The chain metric $d_{\mathcal{C}}$ associated to \mathcal{C} is given as follows. If $x = (x_1, \dots, x_{r^n}) \in G^{r^n}$ the weight function associated to \mathcal{C} is given by

$$(7.5) \quad w_{\mathcal{C}}(x) = \min_{0 \leq i \leq n} \{i : x_j = x_j \text{ if } j \equiv k \pmod{r^{i-1}}\}.$$

Let $m = r^n$. The group \mathbb{S}_m acts on G^m by permutation of coordinates. If $\sigma = (12 \cdots m) \in \mathbb{S}_m$, one can check that this is equivalent to

$$(7.6) \quad w_{\mathcal{C}}(x) = \min_{1 \leq i \leq n} \{i : \sigma^{r^{i-1}}(x) = x\}$$

for $x \neq 0$ and $w_{\mathcal{C}}(x) = 0$ if $x = (0, 0, \dots, 0)$. The chain metric is given by $d_{\mathcal{C}}(x, y) = w_{\mathcal{C}}(x - y)$. We call this the *diagonal chain metric* of G , and we denote it by d_{Δ} .

Example 7.3. Take $G = \mathbb{Z}_2$, $r = 2$ and $n = 3$ in (7.4). Then we have

$$\langle 0 \rangle \subset \mathbb{Z}_2 \subset \mathbb{Z}_2^2 \subset \mathbb{Z}_2^4 \subset \mathbb{Z}_2^8.$$

The possible weights in \mathbb{Z}_2^8 are 0, 1, 2, 3, 4 given by

$$w_{\mathcal{C}}(x) = \begin{cases} 0 & \text{if } x = (0, 0, 0, 0, 0, 0, 0, 0), \\ 1 & \text{if } x = (1, 1, 1, 1, 1, 1, 1, 1), \\ 2 & \text{if } x = (x_1, x_2, x_1, x_2, x_1, x_2, x_1, x_2) \text{ with } x_1 \neq x_2, \\ 3 & \text{if } x = (x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4) \text{ with } x_1 \neq x_3 \text{ or } x_2 \neq x_4, \\ 4 & \text{otherwise.} \end{cases}$$

This is in coincidence with expressions (7.5) and (7.6). It is clear that the corresponding weight enumerator is given by

$$\mathcal{W}_{(\mathbb{Z}_2^8, d_{\Delta})}(t) = 240t^4 + 12t^3 + 2t^2 + t + 1.$$

Compare with the weight enumerator

$$\mathcal{W}_{(\mathbb{Z}_2^8, d_{RT})}(x) = 128t^8 + 64t^7 + 32t^6 + 16t^5 + 8t^4 + 4t^3 + 2t^2 + t + 1$$

of \mathbb{Z}_2^8 with the RT -metric. ◇

Let \mathcal{C} denote a chain of subgroups as in (7.1) and let d be a metric in H_1 . The metric in G obtained by repeated extensions is

$$(7.7) \quad \tilde{d} = Ext_{\mathcal{C}}(d) = Ext_{H_{n-1}}^{H_n} \circ \cdots \circ Ext_{H_1}^{H_2}(d).$$

Remark 7.4. In the above situation, if in (7.7) we take the Hamming metric in H_1 , the extended metric turns out to be the chain metric of \mathcal{C} , i.e.

$$\tilde{d}_{Ham} = d_{\mathcal{C}}.$$

Chain isometries. We now consider isometries between whole chains of groups. Let \mathcal{C} and \mathcal{C}' be two chains of subgroups of the same length of groups G and G' respectively, say $H_1 \subsetneq H_2 \subsetneq \cdots \subsetneq H_n = G$ and $H'_1 \subsetneq H'_2 \subsetneq \cdots \subsetneq H'_n = G'$.

Definition 7.5. We say that \mathcal{C} is *isometric* to \mathcal{C}' , denoted $\mathcal{C} \simeq \mathcal{C}'$, if for every $i = 1, \dots, n$ there are metrics d_i of H_i and d'_i of H'_i such that $(H_i, d_i) \simeq (H'_i, d'_i)$. The groups G and G' are said to be *chain isometric* if they admit isometric chains.

That is, if two chains \mathcal{C} and \mathcal{C}' are isometric we have

$$(7.8) \quad \begin{array}{ccccccc} H_1 & \subsetneq & H_2 & \subsetneq & \cdots & \subsetneq & H_n = G \\ | \simeq & & | \simeq & & & & | \simeq \\ H'_1 & \subsetneq & H'_2 & \subsetneq & \cdots & \subsetneq & H'_n = G' \end{array}$$

We now show that chains of the same length and corresponding sizes are isometric.

Lemma 7.6. *Let G and G' be groups with chains of subgroups \mathcal{C} and \mathcal{C}' , respectively given by $\langle 0 \rangle \neq H = H_1 \subsetneq H_2 \subsetneq \cdots \subsetneq H_n = G$ and $\langle 0 \rangle \neq H' = H'_1 \subsetneq H'_2 \subsetneq \cdots \subsetneq H'_n = G'$. If $|H_i| = |H'_i|$ for $1 \leq i \leq n$ then we have the chain isometry*

$$(G, d_{\mathcal{C}}) \simeq (G', d_{\mathcal{C}'}).$$

Proof. Since $|H_1| = |H'_1|$ there is a bijection $\eta : H_1 \rightarrow H'_1$ inducing the trivial isometry $(H_1, d_{Ham}) \simeq (H'_1, d_{Ham})$. By applying part (b) of Theorem 6.1 we can lift this isometry to get $(H_2, Ext_{H_1}^{H_2}(d_{Ham})) \simeq (H'_2, Ext_{H'_1}^{H'_2}(d_{Ham}))$. Repeating this lifting procedure we obtain that \mathcal{C} and \mathcal{C}' are isometric chains with the extended metrics. □

Example 7.7. (i) The isometry $\mathbb{Z}_{q^n} \simeq (\mathbb{Z}_q)^n$ given explicitly in Section 5, can be seen as a chain isometry. In fact, the chains

$$\begin{array}{ccccccc} \mathbb{Z}_q & \subset & \mathbb{Z}_{q^2} & \subset & \cdots & \subset & \mathbb{Z}_{q^{n-1}} & \subset & \mathbb{Z}_{q^n} \\ | & & | & & & & | & & | \\ \mathbb{Z}_q & \subset & \mathbb{Z}_q^2 & \subset & \cdots & \subset & \mathbb{Z}_q^{n-1} & \subset & \mathbb{Z}_q^n \end{array}$$

are isometric by the previous lemma.

(ii) There is a chain isometry $\mathbb{Z}_{q^n} \simeq \mathbb{F}_q^n$ given by the chains $\mathbb{Z}_q \subset \mathbb{Z}_{q^2} \subset \cdots \subset \mathbb{Z}_{q^n}$ and $\mathbb{F}_q \subset \mathbb{F}_q^2 \subset \cdots \subset \mathbb{F}_q^n$. In fact, any bijection between \mathbb{F}_q and \mathbb{Z}_q with the Hamming metrics

induces a chain isometry between \mathbb{F}_q^n and \mathbb{Z}_{q^n} . One can replace \mathbb{F}_q and \mathbb{Z}_q by any group C_q of order q .

Example 7.8 (*Galois fields and rings*). Let p be a prime and r_1, r_2, \dots, r_n be positive integers such that $r_1 \mid r_2 \mid \dots \mid r_n$. Consider the Galois rings $R_i = GR(p^k, r_i)$ for $i = 1, \dots, n$. Then we have the isometric chains of rings

$$\begin{array}{ccccccc} GR(p^k, r_1) & \subset & GR(p^k, r_1)^{\frac{r_2}{r_1}} & \subset & \dots & \subset & GR(p^k, r_1)^{\frac{r_n}{r_1}} \\ | & & | & & & & | \\ GR(p^k, r_1) & \subset & GR(p^k, r_2) & \subset & \dots & \subset & GR(p^k, r_n) \end{array}$$

and, in particular taking $k = 1$, $GR(p, r_i) \simeq \mathbb{F}_{p^{r_i}}$, so this becomes

$$\begin{array}{ccccccc} \mathbb{F}_{p^{r_1}} & \subset & (\mathbb{F}_{p^{r_1}})^{\frac{r_2}{r_1}} & \subset & \dots & \subset & (\mathbb{F}_{p^{r_1}})^{\frac{r_n}{r_1}} \\ | & & | & & & & | \\ \mathbb{F}_{p^{r_1}} & \subset & \mathbb{F}_{p^{r_2}} & \subset & \dots & \subset & \mathbb{F}_{p^{r_n}} \end{array}$$

Geometric chains. Let \mathcal{C} be a chain $\langle 0 \rangle = H_0 \subset H_1 \subset H_2 \subset \dots \subset H_n = G$ with the sizes of the terms in geometric progression, that is

$$(7.9) \quad [H_i : H_{i-1}] = m \quad \text{for } i = 1, \dots, n.$$

We will call this a *geometric chain*.

Proposition 7.9. *If G admits a geometric chain \mathcal{C} of subgroups $H = H_1 \subsetneq \dots \subsetneq H_n = G$ with $H \neq \langle 0 \rangle$ then we have the isometry*

$$(G, d_{\mathcal{C}}) \simeq (H^n, d_{RT}).$$

Proof. Let \mathcal{C} be the chain $H = H_1 \subsetneq H_2 \subsetneq \dots \subsetneq H_n = G$ and consider the geometric chain \mathcal{C}' given by $H \subset H^2 \subset H^3 \subset \dots \subset H^n$. Starting from the trivial isometry $id : H \rightarrow H$ with the Hamming metrics and applying Theorem 6.1, we get that \mathcal{C} and \mathcal{C}' are isometric chains. In particular, $G \simeq H^n$ and

$$(\tilde{d}_{Ham})^n = d_{\mathcal{C}'} = d_{RT}$$

as we wanted to see. \square

Remark 7.10. The isometries $(\mathbb{Z}_{q^n}, d_q) \simeq (\mathbb{Z}_q^n, d_{RT})$ and $(\mathbb{Z}_{q^n}, d_q) \simeq (\mathbb{F}_q^n, d_{RT})$ given in Example 7.7 are instances of geometric chains and of chain isometries given by the previous proposition.

We now show that the result in Theorem 5.2, i.e. that $(\mathbb{Z}_{q^n}, d_q) \simeq (\mathbb{Z}_q^n, d_{RT})$, can be generalized to any pair of groups G and H^n of order q^n , with G and H not necessarily cyclic.

Theorem 7.11. *Let q be a prime power and G, H groups with $|G| = q^n$ and $|H| = q$. Then,*

$$(7.10) \quad (G, d_{\mathcal{C}}) \simeq (H^n, d_{RT}),$$

where $d_{\mathcal{C}}$ is the chain metric associated to some geometric chain of length n .

Proof. Since $|G| = q^n$, by Sylow's theorems we get that G has a geometric chain \mathcal{C} of length n , say $\langle 0 \rangle \subset H_1 \subset \dots \subset H_n = G$. By Proposition 7.9 we have that

$$(G, d_{\mathcal{C}}) \simeq (H_1^n, d_{RT}).$$

On the other hand, since $|H_1| = |H|$ there is a bijection $\tau : H_1 \rightarrow H$ which extends to $\tau : H_1^n \rightarrow H^n$ and induces the isometry

$$(H_1^n, d_{RT}) \simeq (H^n, d_{RT}).$$

In fact,

$$d_{RT}(\tau(x), \tau(y)) = \max_{1 \leq i \leq n} \{i : \tau(x_i) \neq \tau(y_i)\} = \max_{1 \leq i \leq n} \{i : x_i \neq y_i\} = d_{RT}(x, y).$$

This implies the result. \square

The theorem implies, for instance, that there exists a metric d in the generalized quaternion group \mathbb{Q}_{2^n} of order 2^n and a metric d' in the dihedral group $\mathbb{D}_{2^{n-1}}$ of order 2^n such that

$$(\mathbb{Z}_{2^n}, d_2) \simeq (\mathbb{Q}_{2^n}, d) \simeq (\mathbb{D}_{2^{n-1}}, d') \simeq (\mathbb{Z}_2^n, d_{RT}).$$

Also, in the above list one can add all the groups $\mathbb{Z}_{2^i} \times \mathbb{Z}_{2^{n-i}}$ with some metrics $d_{(i)}$ for $i = 1, \dots, n-1$.

8. BLOCK ROSENBLOOM–TSFASMAN METRIC

We will next extend the result for geometric chains given in the previous section for groups with arbitrary chains. For this, we must first consider a generalization of the RT -metric.

Definition 8.1. Let X be a group and $n \in \mathbb{N}$. Given a partition $n = m_1 + \dots + m_r$ consider $X^n = X^{m_1} \times \dots \times X^{m_r}$. We write $x = (\tilde{x}_1, \dots, \tilde{x}_r)$ for an element in X^n , where $\tilde{x}_i \in X^{m_i}$ for any i . We define the *block Rosenbloom–Tsfasman metric* (or *BRT-metric*) on X^n as

$$(8.1) \quad d_{BRT}(x, y) = \max_{1 \leq i \leq r} \{i : \tilde{x}_i \neq \tilde{y}_i\}.$$

Note that for $r = n$, then $m_1 = \dots = m_n = 1$ and hence the BRT -metric is just the RT -metric. Also, notice that this metric can be seen as the *block poset metric* (see [9]) associated to the chain poset $1 \preceq 2 \preceq \dots \preceq r$.

Theorem 8.2. *Let H be a proper subgroup of a group G and \mathcal{C} a chain of subgroups with initial term H . Then we have*

$$(8.2) \quad (G, d_{\mathcal{C}}) \simeq (H^{[G:H]}, d_{BRT}).$$

Proof. Suppose \mathcal{C} is the chain $H_1 = H \subset H_2 \subset \dots \subset H_n = G$. Consider the group $G' = H^{[G:H]}$. We will construct a chain \mathcal{C}' in G' of length n , say $H'_1 = H \subset H'_2 \subset \dots \subset H'_n = G'$, such that $|H'_i| = |H_i|$ for all $i = 1, \dots, n$. Consider $H'_2 = H^{[H_2:H_1]}$,

$$H'_3 = (H'_2)^{[H_2:H_1]} = (H^{[H_2:H_1]})^{[H_3:H_2]} = H^{[H_3:H_1]}$$

and in general for every $1 \leq i \leq n$ take

$$H'_i = H^{[H_i:H_1]}.$$

It is clear that $|H'_i| = |H_i|$ for $i = 1, \dots, n$.

By Theorem 6.1, the trivial isometry $\varphi_1 = id : (H_1, d_{Ham}) \rightarrow (H'_1, d_{Ham})$ can be lifted to an isometry

$$\varphi_2 : (H_2, Ext_{H_1}^{H_2}(d_{Ham})) \rightarrow (H'_2, Ext_{H'_1}^{H'_2}(d_{Ham})).$$

By iterating this process we arrive at an isometry

$$\varphi_n : (H_n, \text{Ext}_{H_{n-1}}^{H_n} \circ \dots \circ \text{Ext}_{H_1}^{H_2}(d_{Ham})) \rightarrow (H'_n, \text{Ext}_{H'_{n-1}}^{H'_n} \circ \dots \circ \text{Ext}_{H'_1}^{H'_2}(d_{Ham})).$$

That is, we have

$$(G, d_C) \simeq (H^{[G:H]}, d_{C'}).$$

It only remains to show that the chain metric $d_{C'}$ is the *BRT*-metric. Put $r = [G : H]$ and $r_i = [H_i : H_{i-1}]$ for $i = 1, \dots, n$ (where $H_{-1} = \langle 0 \rangle$). Consider the natural decomposition $H^r = H^{r_1} \times \dots \times H^{r_n}$. If $x \in H^r$ then $x = (\tilde{x}_1, \dots, \tilde{x}_n)$ with $\tilde{x}_i \in H^{r_i}$ for any i . Since

$$d_{BRT}(x, y) = \max_{1 \leq i \leq n} \{i : \tilde{x}_i \neq \tilde{y}_i\}$$

one can check that for $i = 1, \dots, n$ we have

$$d_{BRT}(x, y) = i \quad \Leftrightarrow \quad d_{C'}(x, y) = i.$$

Hence the metrics coincide and the result thus follows. \square

Example 8.3. Let $G = \mathbb{Z}_{q^n}$ and $H = \mathbb{F}_q$, $n \geq 2$. By the previous theorem, if we take the chains

$$\mathcal{C} : \quad \langle 0 \rangle \subset \mathbb{Z}_q \subset \mathbb{Z}_{q^n} \quad \text{and} \quad \mathcal{C}' : \quad \langle 0 \rangle \subset \mathbb{F}_q \subset \mathbb{F}_q^n$$

and we consider the decomposition $\mathbb{F}_q^n = \mathbb{F}_q \times \mathbb{F}_q^{n-1}$ we get

$$(\mathbb{Z}_{q^n}, d_C) \simeq (\mathbb{F}_q^n, d_{BRT}).$$

Note that the weight function associated to \mathcal{C} is

$$w_C = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in q^{n-1}\mathbb{Z}_{q^n} \setminus \{0\}, \\ 2 & \text{if } x \in \mathbb{Z}_{q^n} \setminus q^{n-1}\mathbb{Z}_{q^n}. \end{cases}$$

Now, by properly rescaling this weight, we get the following,

$$\tilde{w}_C = \begin{cases} 0 & \text{if } x = 0, \\ q^{n-1} & \text{if } x \in q^{n-1}\mathbb{Z}_{q^n} \setminus \{0\}, \\ q^{n-2}(q-1) & \text{if } x \in \mathbb{Z}_{q^n} \setminus q^{n-1}\mathbb{Z}_{q^n}. \end{cases}$$

Thus, we obtain

$$(8.3) \quad (\mathbb{Z}_{q^n}, \tilde{d}_C) \simeq (\mathbb{F}_q^n, \tilde{d}_{BRT}),$$

where \tilde{d}_{BRT} is a rescaled metric obtained from d_{BRT} . It is easy to check that the rescaled metrics are also metrics. In the case when $q = p$ is prime, the metric \tilde{d}_C coincides with the *homogeneous metric* (see [3]) defined over the ring \mathbb{Z}_p^n ,

$$(8.4) \quad (\mathbb{Z}_p^n, d_{Hom}) \simeq (\mathbb{F}_p^n, \tilde{d}_{BRT}).$$

Remark 8.4. As in the previous example, we have the isometry $(\mathbb{Z}_{q^n}, d_{Hom}) \simeq (\mathbb{F}_q^n, \tilde{d}_{BRT})$ for $q = p^r$. Consider the q -ary first order Reed-Muller code $RM(1, q^{n-1})$ and let G be any generating matrix of the code whose first row is the all ones vector $(1, 1, \dots, 1)$. The code $RM(1, q^{n-1})$ lies in $\mathbb{F}_q^{q^{n-1}}$ with the Hamming metric, and right multiplication by G encodes

the space \mathbb{F}_q^n into $RM(1, q^{n-1})$, that is $RM(1, q^{n-1}) = \{xG : x \in \mathbb{F}_q^n\}$. Putting these things together we get

$$(8.5) \quad (\mathbb{F}_q^n, \tilde{d}_{BRT}) \rightarrow (RM(1, q^{n-1}), d_{Ham}^{q^{n-1}}) \hookrightarrow (\mathbb{F}_q^{q^{n-1}}, d_{Ham}^{q^{n-1}}).$$

Combining the isometry (8.3) with the embedding (8.5) we get the isometric embedding

$$(\mathbb{Z}_{q^n}, \tilde{d}_C) \hookrightarrow (\mathbb{F}_q^{q^{n-1}}, d_{Ham}^{q^{n-1}}).$$

Taking $q = p$ prime, we obtain the following result of Greferath ([7])

$$(8.6) \quad (\mathbb{Z}_{p^n}, d_{Hom}) \hookrightarrow (\mathbb{Z}_p^{p^{n-1}}, d_{Ham}^{p^{n-1}}).$$

In this way, similarly as in Section 4, we get isometric embeddings of the form

$$\mathbb{Z}_{p^n} \hookrightarrow (\mathbb{F}_p^{p^{n-i}}, d_{Ham}^{p^{n-i}})$$

for $i = 1, \dots, n$.

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