Exponential Lower Bounds on the Generalized Erdős-Ginzburg-Ziv Constant

Jared Bitz, Sarah Griffith, and Xiaoyu He

September 1, 2017

Abstract

For a finite abelian group G, the generalized Erdős–Ginzburg–Ziv constant $\mathbf{s}_k(G)$ is the smallest m such that a sequence of m elements in G always contains a k-element subsequence which sums to zero. If $n = \exp(G)$ is the exponent of G, the previously best known bounds for $\mathbf{s}_{kn}(C_n^r)$ were linear in n and r when $k \geq 2$. Via a probabilistic argument, we produce the exponential lower bound

$$s_{2n}(C_n^r) > \frac{n}{2}[1.25 + o(1)]^{\frac{n}{2}}$$

for n > 0. For the general case, we show

$$\mathsf{s}_{kn}(C_n^r) > \frac{kn}{4} \Big(1 + \frac{1}{ek+1} + o(1) \Big)^r.$$

1 Introduction

In 1961, Erdős, Ginzburg, and Ziv [3] proved that among any 2n-1 integers, some n of them sum to a multiple of n. Equivalently, among every sequence (with repetition) of 2n-1 elements in $C_n := \mathbb{Z}/n\mathbb{Z}$, some n sum to zero (for brevity, given a sequence, we call a subsequence of length n an n-subsequence).

This result led naturally to the study of restricted-length zero-sum subsequences in finite abelian groups. Recall that the exponent $\exp(G)$ of a finite group is the largest order among its elements.

Definition 1. If G is a finite abelian group and $n = \exp(G)$, the k-th generalized EGZ constant of G, denoted $\mathbf{s}_{kn}(G)$, is the smallest m for which any sequence of m elements of G contains a zero-sum kn-subsequence.

The EGZ problem, especially in the "smallest" case of k = 1, has proven to be surprisingly difficult. The only r for which $s_n(C_n^r)$ has been exactly determined are r = 1 and r = 2. The fact that $s_n(C_n) = 2n - 1$ is the original Erdős-Ginzburg-Ziv Theorem [3], while $s_n(C_n^2) = 4n - 3$ is the famous Kemnitz conjecture and was only recently settled by Reiher [10]. Reiher's theorem builds on the polynomial method of Rónyai [11], which has proved fruitful for providing upper bounds to $s_{kn}(C_n^r)$ in higher ranks $r \geq 3$, especially when k is somewhat large compared to r.

Major progress on the hardest case k = 1 was recently made by Naslund [9], obtaining an exponential improvement on the upper bounds on $s_p(C_p^r)$ for general r by using the "multi-slice-rank" method. This technique generalizes that of Ellenberg and Gijswijt [2] and Croot, Lev, and Pach [1] on the closely related cap-set problem. Specifically, Naslund shows that

$$s_p(C_p^r) < 3p!(2p-1)(J(p)p)^r,$$

where J(p) is a constant, depending on p, which lies between .841 and .918 and decreases as p grows.

We turn our attention to $\mathbf{s}_{kn}(C_n^r)$ for higher values of k. Gao and Thangadurai [5] proved that $5p + \frac{p-1}{2} - 3 \leq \mathbf{s}_{2p}(C_p^3) \leq 6p - 3$, and that $\mathbf{s}_{kp}(C_p^3) = (k+3)p - 3$ for $k \geq 4$. Kubertin [8] extended the latter result to the k = 3 case, and also showed that $\mathbf{s}_{4p}(C_p^4) = 8p - 4$. These results contribute to the following conjecture, made in a more general form by Gao and Thangadurai [5].

Conjecture 1. For all $k \ge r$, and sufficiently large n, $s_{kn}(C_n^r) = (k+r)n - r$.

Extending the work of Rónyai[11] and Kubertin[8], it was shown by He [7] that the conjecture holds for prime n when $k \ge p + r$ and $2p \ge 7r - 3$. For a survey of related zero-sum problems and their generalizations, see [4].

When n is an odd prime and k = 1, Harborth [6] gives the elementary lower bound $s_p(C_p^r) > 2^r(p-1)$, but no similar bound exponential in r was known for $k \ge 2$. In this paper, we provide lower bound constructions for $s_{kn}(C_n^r)$ when $k \ge 2$ which are effective when k is much smaller than r.

Theorem 1. The generalized EGZ constant $s_{2n}(C_n^r)$ satisfies the bound

$$s_{2n}(C_n^r) > \frac{n}{2} \left[\frac{5}{4} + o(1)\right]^r.$$

That is, there is a sequence of this length in C_n^r that contains no zero-sum 2*n*-subsequence. This result is the first exponential lower bound on $s_{2n}(C_n^r)$ for an arbitrary r.

Generalizing this to arbitrary values of k, we also prove

Theorem 2. Let k > 2. The generalized EGZ constant $s_{kn}(C_n^r)$ satisfies the bound

$$s_{kn}(C_n^r) > \frac{kn}{4} \left[1 + \frac{1}{ek+1} + o(1) \right]^r.$$

In both results, the o(1) term goes to zero as n grows.

2 Lower Bounds on $s_{kn}(C_n^r)$

We first give the proof of Theorem 1.

Proof of Theorem 1. Let

$$N = \frac{n}{2}A^r$$

for a value of A which will be specified later. Choose a sequence X of N random vectors in $\{0, 1\}^r$ as follows. For each $v = (v_1, \ldots, v_r)$ that is a term of X, let each $v_i = 1$ with probability q and $v_i = 0$ with probability 1 - q, with each v_i chosen independently. Let Z be the number of zero-sum length 2n-subsequences in X. We will produce an A such that $\mathbb{E}[Z] < 1$, so that there must be some possible X with no such subsequences.

Consider some arbitrary 2*n*-subsequence Y of X. For Y to be zero-sum, each of the r coordinates must sum to 0 mod n, and so contain exactly 0, n, or 2n ones. For any coordinate $i \leq r$, let P_0 be the probability that coordinate *i* contains 0 ones, P_n the probability that it contains *n* ones, and P_{2n} the probability that it contains 2n ones (clearly, this is not dependent on *i*). We have

$$P_0 = (1-q)^{2n}$$
$$P_n = {\binom{2n}{n}} q^n (1-q)^n$$
$$P_{2n} = q^{2n},$$

the values from a standard binomial distribution with probability of success q.

Now, if Q is the probability that coordinate i sums to zero, we have

$$Q = P_0 + P_n + P_{2n}.$$

We wish to minimize Q. We proceed by choosing q so that P_0 grows slowly while still dominating. Calculations indicate that we ought to allow q to equal 4/5. Using the bound $\binom{2n}{n} \leq 4^n/\sqrt{3n+1}$, we then have

$$Q \le \left(\frac{4}{5}\right)^{2n} + \frac{1}{\sqrt{3n+1}} \left(\frac{4}{5}\right)^{2n} + \left(\frac{1}{5}\right)^{2n} < \left(1 + \frac{1}{\sqrt{n}}\right) \left(\frac{4}{5}\right)^{2n}$$

Since Q is the probability that any one coordinate in Y sums to zero, the probability that Y as a whole is zero-sum is Q^r . Since each 2n-subsequence of X is zero-sum with equal probability, we have

$$\mathbb{E}[Z] = \binom{N}{2n} Q^r$$

$$< \left(\frac{2N}{n}\right)^{2n} \left(1 + \frac{1}{\sqrt{n}}\right)^r \left(\frac{4}{5}\right)^{2nr}$$

$$< A^{2nr} \left(1 + \frac{1}{\sqrt{n}}\right)^r \left(\frac{4}{5}\right)^{2nr}.$$

To obtain $\mathbb{E}[Z] < 1$, it is sufficient to have

$$A^{2nr} \left(1 + \frac{1}{\sqrt{n}}\right)^r \left(\frac{4}{5}\right)^{2nr} < 1$$
$$A < \frac{5}{4} \left(1 + \frac{1}{\sqrt{n}}\right)^{-\frac{1}{2n}},$$

So, we can take $A = \frac{5}{4} + o(1)$.

It is possible to improve Theorem 1 by a subexponential factor using the Lovasz Local Lemma, but the exponential constant 5/4 appears to be the natural limit of the method.

The proof of Theorem 2 is similar.

Proof of Theorem 2. If we repeat the above construction in the general case, we first must calculate the probability that a given coordinate $i \leq r$ sums to zero. We now have k+1 component probabilities, P_0 through P_{kn} . By the same logic as above, we see that

$$P_{in} = \binom{kn}{in} q^{in} (1-q)^{(k-i)n}.$$

We again wish to have P_0 dominate while growing slowly. This time calculations suggest allowing q to equal 1/(ek+1). Note that when P_0 dominates we have the bound $Q < (k+1)(1-q)^{kn}$. Then, if $N = \frac{kn}{4}A^r$, we have

$$\mathbb{E}[Z] = \binom{N}{k} Q^r < \left(\frac{4N}{kn}\right)^{kn} (k+1)^r (1-q)^{knr}.$$

If we want $\mathbb{E}[Z] < 1$, we must have

$$\left(\frac{4N}{kn}\right)^{kn} (k+1)^r (1-q)^{knr} < 1$$

and thus

$$A < \frac{1}{(k+1)^{1/kn}(1-q)} < \frac{1}{(k+1)^{1/kn}\left(1-\frac{1}{ek+1}\right)}$$

So, we have $A = 1 + \frac{1}{ek+1} + o(1)$, as desired.

When q is determined as above, the permissible values of A approach one as k and n grow. Therefore, the result gives us little if we seek a bound on $s_{kn}(C_n^r)$ for completely general k and n. However, if we fix a specific value of k (for example, if we are interested in s_{3n}), we can still attempt to optimize q in order to achieve an exponential lower bound - it just will not be as effective for large k.

3 Acknowledgements

The authors would like to thank George Schaeffer and the Stanford Undergraduate Research in Mathematics program for support during the development of these results. We indebted to Jacob Fox for valuable guidance on the probabilistic method, and to Christian Reiher for general insights regarding the Erdős-Ginzburg-Ziv problem. We would also like to thank Jesse Geneson for the helpful comments he provided us with.

References

- E. Croot, V. Lev, and P. Pach, Progression-free sets in Zⁿ₄ are exponentially small, Annals of Mathematics 185 (2017), 331–337.
- [2] J. Ellenberg and D. Gijswijt, On large subsets of Fⁿ_q with no three-term arithmetic progressions, Annals of Mathematics 185 (2016), 1–4.
- [3] P. Erdős, A. Ginzburg, and A. Ziv, Theorem in the additive number theory, Bull. Research Council Israel 10 (1961), 41–43.
- [4] W. Gao and A. Geroldinger, Zero-sum problems in finite abelian groups: A survey, Expositiones Mathematicae 24 (2006), 337–369.
- [5] W. Gao and R. Thangadurai, On zero-sum sequences of prescribed length, Aequationes Mathematicae 72 (2006), 201–212.

- [6] H. Harborth, Ein Extremalproblem fur Gitterpunkte, J. Reine Angew. Math. 262 (1973), 356–360.
- [7] X. He, Zero-sum subsequences of length kq over finite abelian p-groups, Discrete Mathematics 339 (2015), 399–407.
- [8] S. Kubertin, Zero-sums of length kq in \mathbb{Z}_q^d , Acta Arithmetica **116** (2005), 145–152.
- [9] E. Naslund, Exponential bounds for the Erdős-Ginzburg-Ziv constant, preprint (2017), available at arXiv:1701.04942v1.
- $[10]\ {\rm C.}\ {\rm Reiher},\ {\it On}\ {\it Kemnitz'}\ {\it conjecture}\ {\it concerning}\ {\it lattice-points}\ {\it in}\ {\it the}\ {\it plane},\ {\rm Ramanujan}\ {\it Journal}\ {\bf 13}\ (2007),\ {\it 333-337}.$
- [11] L. Rónyai, On a Conjecture of Kemnitz, Combinatorica 20 (2000), 569–573.