# Exponential Lower Bounds on the Generalized Erdős-Ginzburg-Ziv Constant 

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#### Abstract

For a finite abelian group $G$, the generalized Erdős-Ginzburg-Ziv constant $\mathbf{s}_{k}(G)$ is the smallest $m$ such that a sequence of $m$ elements in $G$ always contains a $k$-element subsequence which sums to zero. If $n=\exp (G)$ is the exponent of $G$, the previously best known bounds for $\mathrm{s}_{k n}\left(C_{n}^{r}\right)$ were linear in $n$ and $r$ when $k \geq 2$. Via a probabilistic argument, we produce the exponential lower bound $$
\mathrm{s}_{2 n}\left(C_{n}^{r}\right)>\frac{n}{2}[1.25+o(1)]^{r}
$$


for $n>0$. For the general case, we show

$$
\mathrm{s}_{k n}\left(C_{n}^{r}\right)>\frac{k n}{4}\left(1+\frac{1}{e k+1}+o(1)\right)^{r}
$$

## 1 Introduction

In 1961, Erdős, Ginzburg, and Ziv [3] proved that among any $2 n-1$ integers, some $n$ of them sum to a multiple of $n$. Equivalently, among every sequence (with repetition) of $2 n-1$ elements in $C_{n}:=\mathbb{Z} / n \mathbb{Z}$, some $n$ sum to zero (for brevity, given a sequence, we call a subsequence of length $n$ an $n$-subsequence).

This result led naturally to the study of restricted-length zero-sum subsequences in finite abelian groups. Recall that the exponent $\exp (G)$ of a finite group is the largest order among its elements.
Definition 1. If $G$ is a finite abelian group and $n=\exp (G)$, the $k$-th generalized $E G Z$ constant of $G$, denoted $\mathrm{s}_{k n}(G)$, is the smallest $m$ for which any sequence of $m$ elements of $G$ contains a zero-sum $k n$-subsequence.

The EGZ problem, especially in the "smallest" case of $k=1$, has proven to be surprisingly difficult. The only $r$ for which $s_{n}\left(C_{n}^{r}\right)$ has been exactly determined are $r=1$ and $r=2$. The fact that $\mathrm{s}_{n}\left(C_{n}\right)=2 n-1$ is the original Erdős-Ginzburg-Ziv Theorem [3], while $\mathbf{s}_{n}\left(C_{n}^{2}\right)=4 n-3$ is the famous Kemnitz conjecture and was only recently settled by Reiher [10. Reiher's theorem builds on the polynomial method of Rónyai [11, which has proved fruitful for providing upper bounds to $\mathrm{s}_{k n}\left(C_{n}^{r}\right)$ in higher ranks $r \geq 3$, especially when $k$ is somewhat large compared to $r$.

Major progress on the hardest case $k=1$ was recently made by Naslund [9, obtaining an exponential improvement on the upper bounds on $\mathrm{s}_{p}\left(C_{p}^{r}\right)$ for general $r$ by using the "multi-slice-rank" method. This technique generalizes that of Ellenberg and Gijswijt [2] and Croot, Lev, and Pach [1] on the closely related cap-set problem. Specifically, Naslund shows that

$$
\mathrm{s}_{p}\left(C_{p}^{r}\right)<3 p!(2 p-1)(J(p) p)^{r}
$$

where $J(p)$ is a constant, depending on $p$, which lies between .841 and .918 and decreases as $p$ grows.
We turn our attention to $\mathrm{s}_{k n}\left(C_{n}^{r}\right)$ for higher values of $k$. Gao and Thangadurai 5] proved that $5 p+\frac{p-1}{2}-3 \leq \mathrm{s}_{2 p}\left(C_{p}^{3}\right) \leq 6 p-3$, and that $\mathrm{s}_{k p}\left(C_{p}^{3}\right)=(k+3) p-3$ for $k \geq 4$. Kubertin [8] extended the latter result to the $k=3$ case, and also showed that $s_{4 p}\left(C_{p}^{4}\right)=8 p-4$. These results contribute to the following conjecture, made in a more general form by Gao and Thangadurai [5].

Conjecture 1. For all $k \geq r$, and sufficiently large $n$, $\mathrm{s}_{k n}\left(C_{n}^{r}\right)=(k+r) n-r$.
Extending the work of Rónyai [11] and Kubertin [8, it was shown by He 7 that the conjecture holds for prime $n$ when $k \geq p+r$ and $2 p \geq 7 r-3$. For a survey of related zero-sum problems and their generalizations, see 4].

When $n$ is an odd prime and $k=1$, Harborth [6] gives the elementary lower bound $\mathrm{s}_{p}\left(C_{p}^{r}\right)>$ $2^{r}(p-1)$, but no similar bound exponential in $r$ was known for $k \geq 2$. In this paper, we provide lower bound constructions for $\mathrm{s}_{k n}\left(C_{n}^{r}\right)$ when $k \geq 2$ which are effective when $k$ is much smaller than $r$.

Theorem 1. The generalized $E G Z$ constant $\mathrm{s}_{2 n}\left(C_{n}^{r}\right)$ satisfies the bound

$$
\mathrm{s}_{2 n}\left(C_{n}^{r}\right)>\frac{n}{2}\left[\frac{5}{4}+o(1)\right]^{r}
$$

That is, there is a sequence of this length in $C_{n}^{r}$ that contains no zero-sum $2 n$-subsequence. This result is the first exponential lower bound on $\mathrm{s}_{2 n}\left(C_{n}^{r}\right)$ for an arbitrary $r$.

Generalizing this to arbitrary values of $k$, we also prove
Theorem 2. Let $k>2$. The generalized $E G Z$ constant $\mathrm{s}_{k n}\left(C_{n}^{r}\right)$ satisfies the bound

$$
\mathrm{s}_{k n}\left(C_{n}^{r}\right)>\frac{k n}{4}\left[1+\frac{1}{e k+1}+o(1)\right]^{r}
$$

In both results, the $o(1)$ term goes to zero as $n$ grows.

## 2 Lower Bounds on $\mathrm{s}_{k n}\left(C_{n}^{r}\right)$

We first give the proof of Theorem 1.
Proof of Theorem 1. Let

$$
N=\frac{n}{2} A^{r}
$$

for a value of $A$ which will be specified later. Choose a sequence $X$ of $N$ random vectors in $\{0,1\}^{r}$ as follows. For each $v=\left(v_{1}, \ldots, v_{r}\right)$ that is a term of $X$, let each $v_{i}=1$ with probability $q$ and $v_{i}=0$ with probability $1-q$, with each $v_{i}$ chosen independently. Let $Z$ be the number of zero-sum length $2 n$-subsequences in $X$. We will produce an $A$ such that $\mathbb{E}[Z]<1$, so that there must be some possible $X$ with no such subsequences.

Consider some arbitrary $2 n$-subsequence $Y$ of $X$. For $Y$ to be zero-sum, each of the $r$ coordinates must sum to $0 \bmod n$, and so contain exactly $0, n$, or $2 n$ ones. For any coordinate $i \leq r$, let $P_{0}$ be the
probability that coordinate $i$ contains 0 ones, $P_{n}$ the probability that it contains $n$ ones, and $P_{2 n}$ the probability that it contains $2 n$ ones (clearly, this is not dependent on $i$ ). We have

$$
\begin{aligned}
P_{0} & =(1-q)^{2 n} \\
P_{n} & =\binom{2 n}{n} q^{n}(1-q)^{n} \\
P_{2 n} & =q^{2 n},
\end{aligned}
$$

the values from a standard binomial distribution with probability of success $q$.
Now, if $Q$ is the probability that coordinate $i$ sums to zero, we have

$$
Q=P_{0}+P_{n}+P_{2 n}
$$

We wish to minimize $Q$. We proceed by choosing $q$ so that $P_{0}$ grows slowly while still dominating. Calculations indicate that we ought to allow $q$ to equal $4 / 5$. Using the bound $\binom{2 n}{n} \leq 4^{n} / \sqrt{3 n+1}$, we then have

$$
Q \leq\left(\frac{4}{5}\right)^{2 n}+\frac{1}{\sqrt{3 n+1}}\left(\frac{4}{5}\right)^{2 n}+\left(\frac{1}{5}\right)^{2 n}<\left(1+\frac{1}{\sqrt{n}}\right)\left(\frac{4}{5}\right)^{2 n}
$$

Since $Q$ is the probability that any one coordinate in $Y$ sums to zero, the probability that $Y$ as a whole is zero-sum is $Q^{r}$. Since each $2 n$-subsequence of $X$ is zero-sum with equal probability, we have

$$
\begin{aligned}
\mathbb{E}[Z] & =\binom{N}{2 n} Q^{r} \\
& <\left(\frac{2 N}{n}\right)^{2 n}\left(1+\frac{1}{\sqrt{n}}\right)^{r}\left(\frac{4}{5}\right)^{2 n r} \\
& <A^{2 n r}\left(1+\frac{1}{\sqrt{n}}\right)^{r}\left(\frac{4}{5}\right)^{2 n r} .
\end{aligned}
$$

To obtain $\mathbb{E}[Z]<1$, it is sufficient to have

$$
\begin{aligned}
& A^{2 n r}\left(1+\frac{1}{\sqrt{n}}\right)^{r}\left(\frac{4}{5}\right)^{2 n r}<1 \\
& A<\frac{5}{4}\left(1+\frac{1}{\sqrt{n}}\right)^{-\frac{1}{2 n}},
\end{aligned}
$$

So, we can take $A=\frac{5}{4}+o(1)$.
It is possible to improve Theorem 1 by a subexponential factor using the Lovasz Local Lemma, but the exponential constant $5 / 4$ appears to be the natural limit of the method.

The proof of Theorem 2 is similar.
Proof of Theorem 2. If we repeat the above construction in the general case, we first must calculate the probability that a given coordinate $i \leq r$ sums to zero. We now have $k+1$ component probabilities, $P_{0}$ through $P_{k n}$. By the same logic as above, we see that

$$
P_{i n}=\binom{k n}{i n} q^{i n}(1-q)^{(k-i) n}
$$

We again wish to have $P_{0}$ dominate while growing slowly. This time calculations suggest allowing $q$ to equal $1 /(e k+1)$. Note that when $P_{0}$ dominates we have the bound $Q<(k+1)(1-q)^{k n}$. Then, if $N=\frac{k n}{4} A^{r}$, we have

$$
\mathbb{E}[Z]=\binom{N}{k} Q^{r}<\left(\frac{4 N}{k n}\right)^{k n}(k+1)^{r}(1-q)^{k n r}
$$

If we want $\mathbb{E}[Z]<1$, we must have

$$
\left(\frac{4 N}{k n}\right)^{k n}(k+1)^{r}(1-q)^{k n r}<1
$$

and thus

$$
\begin{aligned}
A & <\frac{1}{(k+1)^{1 / k n}(1-q)} \\
& <\frac{1}{(k+1)^{1 / k n}\left(1-\frac{1}{e k+1}\right)} .
\end{aligned}
$$

So, we have $A=1+\frac{1}{e k+1}+o(1)$, as desired.
When $q$ is determined as above, the permissible values of $A$ approach one as $k$ and $n$ grow. Therefore, the result gives us little if we seek a bound on $\mathrm{s}_{k n}\left(C_{n}^{r}\right)$ for completely general $k$ and $n$. However, if we fix a specific value of $k$ (for example, if we are interested in $\mathrm{s}_{3 n}$ ), we can still attempt to optimize $q$ in order to achieve an exponential lower bound - it just will not be as effective for large $k$.

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