

Proof of the Caccetta-Haggkvist conjecture for digraphs with small independence number

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Abstract

For a digraph G and $v \in V(G)$, let $\delta^+(v)$ be the number of out-neighbors of v in G . The Caccetta-Häggkvist conjecture states that for all $k \geq 1$, if G is a digraph with $n = |V(G)|$ such that $\delta^+(v) \geq n/k$ for all $v \in V(G)$, then G contains a directed cycle of length at most k . In [2], N. Lichiardopol proved that this conjecture is true for digraphs with independence number equal to two. In this paper, we generalize that result, proving that the conjecture is true for digraphs with independence number at most $(k + 1)/2$.

1 Introduction and definitions

For the rest of the paper, we use the words cycle and path to refer to a directed cycle and directed path, respectively, and every graph considered is a digraph. Furthermore, every digraph G is simple, meaning it has no loops or parallel edges. Let the girth $g(G)$ of a digraph G be the length of its shortest cycle, and for a vertex $v \in V(G)$, let $\delta^+(v)$ denote the number of out-neighbors of v in G . Let $\Delta^+(G) = \min_{v \in V(G)} \delta^+(v)$ be the minimum out-degree of a vertex in G . For vertices $u, v \in V(G)$,

let the distance $d(u, v)$ from u to v be the length of the shortest path from u to v (define this to be zero if $u = v$). For $v \in V(G)$ and $i \geq 1$, let $N_i^+(v)$ be the set of vertices u with $d(v, u) = i$, and let $N_i^-(v)$ be the set of vertices u with $d(u, v) = i$. For a digraph G , call a set of vertices $H \subset V(G)$ independent if there are no edges between any two vertices of H . Let the independence number $\alpha(G)$ of a digraph G be the size of the largest independent set $H \subset V(G)$. For disjoint sets $S_1, S_2 \subset V(G)$, say that S_1 is stable with S_2 if there are no edges between a vertex in S_1 and a vertex in S_2 .

We begin with the following simple observation.

Lemma 1.1 *Suppose that G is a digraph containing a cycle; then $g(G) \leq 2\alpha(G) + 1$.*

Proof. Let C be a cycle of G with minimum length, and suppose C has at least $2\alpha(G) + 2$ vertices. Then there exists a subset $S \subset V(C)$ of size $\alpha(G) + 1$ such that no pair of vertices of S are adjacent in C . Then there is an edge in G between some pair of vertices in S , which gives a shorter cycle in G , a contradiction. This proves Lemma 1.1. ■

The next lemma immediately follows from Lemma 1.1, and is used repeatedly throughout the paper.

Lemma 1.2 *Suppose G is a digraph with $g(G) \geq 2\alpha(G)$, and that $H \subset G$ is a subgraph of G with $\alpha(H) \leq \alpha(G) - 1$. Then H is acyclic.*

Proof. If H contains a cycle, then Lemma 1.1 shows that H contains a cycle of length at most $2\alpha(G) - 1$, which is a contradiction. This proves Lemma 1.2. ■

In this paper, we deal with the following formulation of the Caccetta-Haggkvist conjecture, which was introduced in [1]:

Conjecture 1.1 (Caccetta-Haggkvist) *For $d \geq 1$, $k \geq 1$, if G is a digraph with $n = |V(G)| \leq kd$ and $\Delta^+(G) \geq d$, then $g(G) \leq k$.*

For $k = 1$ and $k = 2$ it follows that the digraph is not simple, a contradiction. So, to prove Conjecture 1.1, we can assume $k \geq 3$.

Now, Lemma 1.1 gives that Conjecture 1.1 is true for $\alpha(G) \leq (k - 1)/2$. In this paper, we prove that Conjecture 1.1 is true for $\alpha(G) \leq (k + 1)/2$.

2 Main Results

We need the following two lemmas.

Lemma 2.1 *Suppose that G is an acyclic digraph; then for all $v \in V(G)$, there exists a path of length at most $2\alpha(G) - 1$ to a vertex of out-degree zero in G .*

Proof. Since G is acyclic, there exists a path from v to a vertex of out-degree zero in G . Let $P = (v, v_2, \dots, v_k)$ be a shortest such path. Then P is induced, so if $k \geq 2\alpha(G) + 1$ then $\{v, v_3, \dots, v_k\} \subset V(G)$ is an independent set of size at least $\alpha(G) + 1$, which is a contradiction. Thus P has length at most $2\alpha(G) - 1$, as desired. \blacksquare

Lemma 2.2 *Let G be a simple digraph with minimum out-degree $d \geq 1$, $\alpha(G) \geq 3$, and $g(G) \geq 2\alpha(G)$. Set $p = 2\alpha(G) - 3$, and suppose $v \in V(G)$ is a vertex with $\delta^+(v) = d$. For odd $1 \leq i \leq p$, let S_i be the subgraph of G induced by the vertex set $V(G) \setminus (N_1^+(v) \cup \{v\} \cup (\bigcup_{j=1}^i N_j^-(v)))$. Then, for each odd $1 \leq i \leq p$, there exists a unique vertex $v_i \in S_i$ such that $N_1^+(v_i) \subset N_i^-(v)$. Furthermore, $|V(G) \setminus S_p| \geq (2\alpha(G) - 2)d + 1$.*

Proof. Every $w \in N_1^+(v)$ has $N_1^+(w) \subset N_1^+(v) \cup S_p$, since otherwise we obtain a cycle of length at most $2\alpha(G) - 1$, a contradiction. Since $|N_1^+(v)| = d$, it follows that $V(S_i) \neq \emptyset$ for odd $1 \leq i \leq p$.

Now, for odd $1 \leq i \leq p$, we iteratively choose $v_i \in S_i$ such that $N_1^+(v_i) \subset N_i^-(v)$. Let $\{v_1, v_3, \dots, v_{i-2}\}$ be vertices such that $N_1^+(v_j) \subset N_j^-(v)$ for all odd $1 \leq j \leq i - 2$ (if $i = 1$, this set of vertices is empty). The set $T = \{v, v_1, v_3, \dots, v_{i-2}\}$ (if $i = 1$, then $T = \{v\}$) is stable with S_i , so $\alpha(S_i) \leq \alpha(G) - (i + 1)/2$, and thus S_i is acyclic by Lemma 1.1. Thus there exists $v_i \in S_i$ with out-degree zero in S_i .

Now, we claim that $N_1^+(v_i) \subset N_i^-(v)$. If not, then v_i has an edge to a vertex $w_1 \in N_1^+(v)$, which has an edge to a vertex $w_2 \in S_i$. Let H be the subgraph of S_i induced by the set of vertices with no edge to v_i . We may assume $w_2 \in H$. We have that $\{v, v_1, \dots, v_i\}$ is stable with H , so $\alpha(H) \leq \alpha(G) - (i + 3)/2$. Then by Lemma 2.1 there exists a path $(w_2 \dots w_j)$ of length at most $2\alpha(G) - i - 4$ from w_2 to a vertex $w_j \in H$ with out-degree zero in H . If w_j has out-degree in S_i equal to zero, then since $|N_1^+(v)| = d$, it follows that w_j has an out-neighbor in $N_i^-(v)$ and we obtain a cycle of length at most $2\alpha(G) - 1$, a contradiction. If instead w_j has an out-neighbor to $w_{j+1} \in S_i \setminus H$, then we again obtain a cycle of length at most $2\alpha(G) - 1$, a contradiction. It follows that v_i has $N_1^+(v_i) \subset N_i^-(v)$ for odd $1 \leq i \leq p$, as claimed.

Now, for odd $1 \leq i \leq p$, let V_i be the set of vertices $u \in S_i$ such that u has out-degree zero in S_i . Let $H = \{v\} \cup V_1 \cup V_3 \cup \dots \cup V_p$. For $v_i \in V_i$, since $N_1^+(v_i) \subset N_i^-(v)$, it follows that H is an independent set, so $|V(H)| \leq \alpha(G)$. We also know that the V_i are nonempty, so $|V(H)| \geq \alpha(G)$. Thus $|V_i| = 1$ for all odd $1 \leq i \leq p$. This proves the first part of the lemma, namely that for each odd $1 \leq i \leq p$ there exists a unique vertex $v_i \in S_i$ with $N_1^+(v_i) \subset N_i^-(v)$. For the remainder of the proof, let $\{v_1, v_3, \dots, v_p\}$ be those unique vertices.

For odd $3 \leq i \leq p$, define $X_i = N_1^+(v_i) \subset N_i^-(v)$, and let $T_i = N_i^-(v) \cup N_{i-1}^-(v) \setminus X_i$. $\{v\}$ is stable with X_i , so by Lemma 1.2, X_i is acyclic and contains a vertex $u_i \in V(X_i)$ with out-degree zero in X_i . We claim that $N_1^+(u_i) \subset T_i$, and consequently $|T_i| \geq d$. If not, then there exists a path of length at most 2 from u_i to a vertex $w_2 \in S_i$. Since $\{v, v_1, \dots, v_{i-2}\}$ is stable with S_i , $\alpha(S_i) \leq \alpha(G) - (i + 1)/2$. Lemma 2.1 gives a path from w_2 to v_i of length at most $2\alpha(G) - 5$. These two paths together form a cycle in G of length at most $2\alpha(G) - 2$, which is a contradiction.

Thus, for odd $3 \leq i \leq p$, we have $|X_i| + |T_i| \geq 2d$. Also, $N_1^+(v_i) \subset N_1^-(v)$ gives $|N_1^-(v)| \geq d$, and by the definition of v we have $|N_1^+(v)| = d$. Together with the vertex v , these give:

$$|V(G) \setminus S_p| \geq (p - 1)d + 2d + 1 = (2\alpha(G) - 2)d + 1$$

as desired. This proves Lemma 2.2. ■

Lemma 2.2 is used to prove the following two theorems.

Theorem 2.1 *Suppose that G is a digraph with minimum out-degree $d \geq 1$ and $n = |V(G)| \leq 2\alpha(G)d$; then $g(G) \leq 2\alpha(G)$.*

Proof. As mentioned above, it suffices to consider simple digraphs G with $\alpha(G) \geq 2$. The case $\alpha(G) = 2$ is proved in [2], so we may further assume that $\alpha(G) \geq 3$. Now, for the sake of contradiction, suppose that $g(G) \geq 2\alpha(G) + 1$. Then Lemma 2.2 implies that $|V(G) \setminus S_p| \geq (2\alpha(G) - 2)d + 1$, which together with $|V(G)| \leq 2\alpha(G)d$ gives $|S_p| \leq 2d - 1$. $H = \{v, v_1, v_3, \dots, v_{p-2}\}$ is stable with S_p , so $\alpha(S_p) = 1$ and S_p is a transitive tournament. Let $(w_1 \cdots w_r)$ be the unique Hamiltonian path of the transitive tournament S_p .

Now, $J = \{v_1, v_3, \dots, v_p\}$ is stable with $N_1^+(v)$, so $N_1^+(v)$ is a transitive tournament. Let its unique Hamiltonian path be $(u_1 \cdots u_d)$. $N_1^+(u_d) \subset S_p$, so there is an out-neighbor w_k of u_d with $k \geq d$. It follows that w_k has an edge to a vertex not in S_p . An edge from w_k to v or to $w' \in N_i^-(v)$ for some $1 \leq i \leq p$ yields a cycle of length at most $2\alpha(G)$, a contradiction. If, instead, w_k has an edge to $u' \in N_1^+(v)$, then u' has an edge to u_d , and we obtain a cycle of length at most three, a contradiction. This proves Theorem 2.1. ■

Theorem 2.2 *Suppose G is a digraph with minimum out-degree $d \geq 1$ and $n = |V(G)| \leq (2\alpha(G) - 1)d$; then $g(G) \leq 2\alpha(G) - 1$.*

Proof. As mentioned above, it suffices to consider simple digraphs G with $\alpha(G) \geq 2$. The case $\alpha(G) = 2$ is proved in [2], so we further assume that $\alpha(G) \geq 3$. For the sake of contradiction, suppose $g(G) \geq 2\alpha(G)$. Lemma 2.2 gives a set of vertices $\{v_i\}$ indexed by odd $1 \leq i \leq p$ such that $N_1^+(v_i) \subset N_i^-(v)$. v_1 is stable with $N_1^+(v)$ (otherwise we obtain a cycle of length four), so Lemma 1.2 gives that $N_1^+(v)$ is acyclic. So, there exists $u \in N_1^+(v)$ with out-degree zero in $N_1^+(v)$. If u has an edge to any vertex not in S_p , we obtain a cycle of length at most $2\alpha(G) - 1$, a contradiction. Thus, we must have $N_1^+(u) \subset S_p$ and it follows that $|S_p| \geq d$.

But Lemma 2.2 also gives that $|V(G) \setminus S_p| \geq (2\alpha(G) - 2)d + 1$, which together with $|S_p| \geq d$ implies that $|V(G)| \geq (2\alpha(G) - 1)d + 1$, contradicting the assumption that $|V(G)| \leq (2\alpha(G) - 1)d$. This proves Theorem 2.2. ■

Theorem 2.3 *Conjecture 1.1 is true for digraphs G with $\alpha(G) \leq (k + 1)/2$.*

Proof. Theorem 2.1 and Theorem 2.2 together with Lemma 1.1 give the desired result. This proves Theorem 2.3. ■

References

- [1] L. Caccetta, R. Häggkvist, “On minimal digraphs with given girth”, *Congr. Numer.*, 21:181-187, 1978.
- [2] N. Lichiardopol, “Proof of the Caccetta-Häggkvist conjecture for oriented graphs with positive minimum out-degree and of independence number two”, *Discrete Math*, 313(14):1540-1542, 2013.