# On existence of perfect bitrades in Hamming graphs 

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#### Abstract

A pair $\left(T_{0}, T_{1}\right)$ of disjoint sets of vertices of a graph $G$ is called a perfect bitrade in $G$ if any ball of radius 1 in $G$ contains exactly one vertex in $T_{0}$ and $T_{1}$ or none simultaneously. The volume of a perfect bitrade $\left(T_{0}, T_{1}\right)$ is the size of $T_{0}$. In particular, if $C_{0}$ and $C_{1}$ are distinct perfect codes with minimum distance 3 in $G$ then $\left(C_{0} \backslash C_{1}, C_{1} \backslash C_{0}\right)$ is a perfect bitrade. For any $q \geq 3, r \geq 1$ we construct perfect bitrades in the Hamming graph $H(q r+1, q)$ of volume $(q!)^{r}$ and show that for $r=1$ their volume is minimum.


Keywords: perfect code, one-error-correcting code, trade, bitrade, spherical bitrade, perfect bitrade, MDS code, alternating group

## 1 Introduction

Bitrades are used for constructing large classes of codes and designs and investigating nontrivial structural properties of these combinatorial objects. It should be noted that in general bitrades are defined independently on including them into codes or designs. Bitrades could exist even regardless of existence of the parent objects with the corresponding parameters. These facts provide a serious additional motivation for constructing bitrades and studying their properties. Classical problems in this area are the existence of bitrades and bounds on their volumes.

In the paper we consider the problem of constructing perfect bitrades, which concerns to the classical problem of existence of perfect codes in the non prime power case. In 1973 Zinoviev and Leontiev [19] and independently Tietäväinen

[^0][15] proved that if $q$ is a power of a prime number then there are only perfect codes with the parameters of $q$-ary Hamming codes, binary and ternary Golay codes. The number of perfect one-error-correcting codes is double exponential [8, however full classification and enumeration are still open problems. By attempts of several authors it was proved that for minimum distances more than 5 there are no perfect codes over a non prime alphabets. In 1 Bassalygo at al. established the nonexistence of perfect codes for $q=2^{i} 3^{j}, i, j \geq 1$ with minimum distance at least 5. In 1964 Golomb and Posner proved the nonexistence of perfect codes of length 7 with minimum distance 3 over the alphabet of six elements. More information could be found in [20] and in the survey [4] with the lists of references there. A special case of bitrades arize from components ( $i$-components, $\alpha$-components) of codes. We refer to 14 for a survey on the switchings of $i$-components in perfect codes, see also [10, 11]. We note that a concept of bitrades was developed for MDS codes, see a work of Potapov [12].

A Steiner $(w-1, w, n)$-bitrade $\left(T_{0}, T_{1}\right)$ is defined as a pair of disjoint collections $T_{0}$ and $T_{1}$ of $w$-subsets of a $n$-element point set, such that any $(w-1)$ subset of the point set is a subset of exactly one set in $T_{0}$ and $T_{1}$ or none. A lower bound on the volume (i.e. the size of $\left|T_{0}\right|$ ) of such bitrade is obtained by Hwang in [6] along with a characterization for the bitrades of minimum volume. For a survey on Steiner and other related bitrades we refer to [5. Krotov et al. (9) suggested a generalization of this concept for $q$-ary Steiner bitrades and established an attainable lower bound for their volumes.

For perfect bitrades that are embedded into perfect codes in $H(n, 2)$ the lower bound $2^{(n-1) / 2}$ for their volumes is known, see Etzion and Vardy [3] or Solov'eva [13]. The argument of [3] holds for the perfect bitrades regardless of being embedded into a perfect code and for any odd $n$ not necessarily being a power of two but one. The bound is attained on so-called minimum $i$-components, that were used for constructing first nonlinear perfect binary codes by Vasil'ev. A classification of perfect bitrades in the binary case was obtained by Krotov for $n=9$ in 7 .

Given a perfect bitrade $\left(T_{0}, T_{1}\right)$ in the Hamming graph $H(n, q)$, it is not hard to see that the vector $\chi_{T_{0}}-\chi_{T_{1}}$ is an eigenvector of the adjacency matrix of $H(n, q)$ with the eigenvalue -1 . Here $\chi_{T_{0}}$ and $\chi_{T_{1}}$ denote the characteristic vectors of $T_{0}$ and $T_{1}$ respectively in the vertex set of $H(n, q)$. This fact relates the problem of determining the minimum volume of perfect bitrades to the problem of finding eigenvectors of $H(n, q)$ with minimum size of support. For eigenvalue -1 this problem was firstly considered in 18 and solved for the eigenvalue $(n-1) q-n$ in [16] by Valyuzhenich. The approach of work [16] was further extended in [17] for arbitrary eigenvalue of the Hamming graph with complete solution for all eigenvalues in case when $q$ is at least 4 . In particular, the result [17] implies a lower bound $2^{n-\frac{n-1}{q}}(q-1)^{\frac{n-1}{q}}$ for the volume of the perfect bitrades in $H(n, q), q \geq 4$. A construction for perfect bitrades from [18] gives the upper bound $2^{\frac{n-1}{q}+1} q^{\frac{(n-1)(q-2)}{q}}$ on the minimum volume of a bitrade in $H(n, q)$ for $q$ being powers of primes.

The current paper is organized as follows. Basic definitions and a revision of some previous results are given in Section 2. In particular, we introduce a concept of a spherical bitrade which is crucial for obtaining the main results of the paper. Basic theory regarding spherical and perfect bitrades is presented in Section 3. We reveal interrelations between perfect and spherical bitrades and eigenfunctions of Hamming graphs and obtain a natural recursive construction for spherical bitrades. In particular, the results of Section 3 allow to reduce the problem of constucting perfect bitrades in $H(q r+1, q)$ to constructing spherical bitrades in $H(q, q)$. The latter is solved in Section 4 as we split the opposite of the repetition code by the parity corresponding permutation in order to obtain bitrades. Thus we construct perfect bitrades in the Hamming graph $H(q r+1, q)$ of the volume $(q!)^{r}$ for any $q \geq 3, r \geq 1$. In Section 5 using a combinatorial argument we show that the actual value of the minimum volume of perfect bitrade in $H(q+1, q)$ is $q!$.

## 2 Definitions and preliminaries

The vertex set of the Hamming graph $H(n, q)$ consists of the tuples of length $n$ over the alphabet set $\{0, \ldots, q-1\}$ which we denote by $\mathcal{A}$ and tuples $x$ and $y$ are adjacent if they differ in exactly one coordinate position. A code in a graph $G$ is a subset of its vertices. The size of $C$ is $|C|$ and its minimum distance $d_{C}$ is $d_{C}=\min _{x, y \in C, x \neq y} d(x, y)$, where $d(x, y)$ is the length of a shortest path connecting $x$ and $y$. A code $C$ is one-error-correcting perfect (in throughout what follows perfect) if the balls of radius 1 centered at the vertices of $C$ part the vertex set of $G$. If $C$ and $D$ are codes in $G, d(C, D)$ denotes $\min \{d(x, y)$ : $x \in C, y \in D\}$.

When $C$ is a code in the Hamming graph $H(n, q)$ we use a traditional expression $q$-ary code of length $n$. If $q$ is a power of a prime, the alphabet set $\mathcal{A}$ is associated with the Galois field $F_{q}$ of order $q$. In this case we consider the linear space $F_{q}^{n}$ on the set vertices of $H(n, q)$ naturally inherited from $F_{q}$. With this regard, a code is called linear, if it is a linear subspace of $F_{q}^{n}$. The well-known Singleton bound states that the size of a q-ary code $C$ of length $n$ with minimum distance $d_{C}$ is not greater than $q^{n-d_{C}+1}$. If the size of the code $C$ attains the Singleton bound, $C$ is called a $M D S$ code.

Let $x$ be a vertex of a graph $G$. Denote by $\mathcal{O}(x)$ the sphere of radius one centered at $x$, i.e. the set of the neighbors of $x$ in $G$. Let $S_{0}$ and $S_{1}$ be two disjoint codes in $G$. The ordered pair $\left(S_{0}, S_{1}\right)$ is called a spherical bitrade in $G$ if for any vertex $x$ of $G$ we have

$$
\left|\mathcal{O}(x) \cap S_{0}\right|=\left|\mathcal{O}(x) \cap S_{1}\right| \in\{0,1\} .
$$

Denote by $\overline{\mathcal{O}}(x)$ the ball of radius one centered at a vertex $x$ in $G$, i.e. $\overline{\mathcal{O}}(x)=$ $\mathcal{O}(x) \cup\{x\}$. The ordered pair $\left(S_{0}, S_{1}\right)$ is called a perfect bitrade in $G$ if for any vertex $x$ of $G$ we have

$$
\left|\overline{\mathcal{O}}(x) \cap S_{0}\right|=\left|\overline{\mathcal{O}}(x) \cap S_{1}\right| \in\{0,1\}
$$

The volume of a spherical or perfect bitrade $\left(S_{0}, S_{1}\right)$ is $\left|S_{0}\right|$ (or $\left.\left|S_{1}\right|\right)$.
A real-valued function $f$ defined on the vertex set of a graph $G$ is called a $\lambda$-eigenfunction, if it is not the all-zero function and

$$
\lambda f(x)=\sum_{y \in \mathcal{O}(x)} f(y)
$$

In other words, the vector of the values of $f$ is an eigenvector of the adjacency matrix of the graph $G$ with the eigenvalue $\lambda$. It is well-known that the eigenvalues of the adjacency matrix of the Hamming graph $H(n, q)$ is the set $\{n(q-1)-q i: i \in\{0, \ldots, n\}\}$.

For a pair of tuples $x$ and $y$ over the same alphabet define their concatenation by $x \mid y$. If $C$ and $D$ are two codes then denote by $C \times D$ the following code $\{x \mid y: x \in C, y \in D\}$. Let $f$ and $g$ be two real-valued functions on the vertices of $H(n, q)$ and $H\left(n^{\prime}, q\right)$ respectively. The tensor product of $f$ and $g$ is the function $f \cdot g$ defined on the set of vertices of $H\left(n+n^{\prime}, q\right)$ as follows: $(f \cdot g)(x \mid y)=f(x) g(y)$.

Lemma 1. 17][Corollary 1] Let $f$ be a $\lambda$-eigenfunction of $H(n, q)$ and $g$ be a $\mu$-eigenfunction of $H\left(n^{\prime}, q\right)$. Then the function $f \cdot g$ is a $\left.\lambda+\mu\right)$-eigenfunction of $H\left(n+n^{\prime}, q\right)$.

Let $i_{1}, \ldots, i_{k}$ be pairwise distinct coordinate positions, $\left\{i_{1}, \ldots, i_{k}\right\} \subseteq$ $\{1, \ldots, n\}$ and $a_{1}, \ldots, a_{k}$ be symbols of the alphabet set $\mathcal{A}$. The set of all tuples $x$ of length $n$ over $\mathcal{A}$ such that $x_{i_{l}}=a_{l}$ for any $l \in\{1, \ldots, k\}$ is called a face in $H(n, q)$ and is denoted by $\Gamma_{i_{1} \ldots i_{k}}^{a_{1} \ldots a_{k}}$. Any position from $\{1, \ldots, n\} \backslash\left\{i_{1}, \ldots, i_{k}\right\}$ in the face $\Gamma_{i_{1} \ldots i_{k}}^{a_{1} \ldots a_{k}}$ is called free. We finish the preliminary part of the paper with a well-known result of Delsarte.

Theorem 1. [2] Let $f$ be a $(n(q-1)-m q)$-eigenfunction of $H(n, q)$ for $m \in$ $\{0, \ldots, n\}$. Then for any $k \leq m-1$, any pairwise distinct elements $i_{1}, \ldots, i_{k}$ of $\{1, \ldots, n\}$ and symbols $a_{1}, \ldots, a_{k}$ of $\mathcal{A}$ the sum of the values of $f$ on the face $\Gamma_{i_{1} \ldots i_{k}}^{a_{1} \ldots a_{k}}$ is zero.
Corollary 1. Let $f$ be a $(n(q-1)-m q)$-eigenfunction of $H(n, q)$ for $j \in$ $\{0, \ldots, n\}$. Then for any pairwise distinct elements $i_{1}, \ldots, i_{m-1}$ of $\{1, \ldots, n\}$ and symbols $a_{1}, \ldots, a_{k}$ of $\mathcal{A}$ the function $f$ has at least two nonzero values on the face $\Gamma_{i_{1} \ldots i_{m-1}}^{a_{1} \ldots a_{m-1}}$ or all values of $f$ on $\Gamma_{i_{1} \ldots i_{m-1}}^{a_{1} \ldots a_{m-1}}$ are zeroes.

## 3 Spherical and perfect bitrades

### 3.1 Bitrades and eigenfunctions

We now give a characterization for perfect bitrades in graphs and spherical bitrades in Hamming graphs in terms of eigenfunctions of these graphs.

Proposition 1. Let $T_{0}$ and $T_{1}$ be disjoint codes in a graph $G$ of diameter at least 3. The ordered pair $\left(T_{0}, T_{1}\right)$ is a perfect bitrade in $G$ if and only if $d_{T_{0}}=d_{T_{1}}=3$ and $\chi_{T_{0}}-\chi_{T_{1}}$ is a $(-1)$-eigenfunction of $G$.

Proof. Let for a pair of disjoint codes $T_{0}$ and $T_{1}$ the equality $\left|\overline{\mathcal{O}}(x) \cap T_{0}\right|=$ $\left|\overline{\mathcal{O}}(x) \cap T_{1}\right|$ be fulfilled for any vertex $x$ of $G$. We see that this property holds if and only if $-\left(\chi_{T_{0}}-\chi_{T_{1}}\right)(x)=\sum_{y \in \mathcal{O}(x)} \chi_{T_{0}}(y)-\chi_{T_{1}}(y)$ for any vertex $x$ of $G$, i.e. $\chi_{T_{0}}-\chi_{T_{1}}$ is a (-1)-eigenfunction of $G$.

Obviously, the balls of radius one centered at the vertices of a code with minimum distance $d$ are disjoint if and only if $d$ is at least 3 . Let $\left(T_{0}, T_{1}\right)$ be a perfect bitrade, $x$ and $y$ be neighbors from $T_{0}$ and $T_{1}$. Let $z$ be a neighbor of $x$ but not a neighbor of $y$. By the definition of a perfect bitrade, $z$ has a neighbor in $T_{1}$ at distance 3 from $y$.

A graph $G$ of diameter $d$ is called distance-regular, if there are constants $p_{i j}^{k}, i, j, k \in\{0, \ldots, d\}$ such that for any pair of vertices $x, y, d(x, y)=k p_{i j}^{k}=$ $|\{z: d(x, z)=i, d(z, y)=j\}|$. In throughout what follows $\oplus$ denotes the addition via modulo 2 .

Lemma 2. Let $G$ be a distance-regular graph with $p_{11}^{1}, p_{21}^{2} \neq 0$. We have the following:

1. Let $C$ be a code in $G$. Then $|\mathcal{O}(x) \cap C| \leq 1$ if and only if $d_{C} \geq 3$.
2. Let $S_{0}$ and $S_{1}$ be disjoint codes in $G$. Then $\chi_{S_{0}}-\chi_{S_{1}}$ is a 0 -eigenfunction of $G$ if and only if $\left|\mathcal{O}(x) \cap S_{0}\right|=\left|\mathcal{O}(x) \cap S_{1}\right|$ for any vertex $x$ of $G$.
3. Let $\left(S_{0}, S_{1}\right)$ be a spherical bitrade in $G$ or $d_{S_{0}}, d_{S_{1}} \geq 3$, $\chi_{S_{0}}-\chi_{S_{1}}$ be a 0 -eigenfunction of $G$. Then $d_{S_{0}}=d_{S_{1}}=3$.

Proof. 1. The sufficiency is clear. If there are vertices from $C$ at distance 1 or 2 in $G$, then they obviously have a common neighbor $x$, so $|\mathcal{O}(x) \cap C| \geq 2$.
2. Follows from the definition of a 0 -eigenfunction of $G$.
3. Since $p_{21}^{2}$ is nonzero there is a path of length 4: $x, x^{1}, x^{2}, x^{3}$ in $G$ such that $x \in S_{i}, x^{2} \in S_{i \oplus 1}, d\left(x^{3}, x\right)=2, i \in\{0,1\}$. The vertex $x^{3}$ is a neighbor of $x^{2}$ from $S_{i \oplus 1}$. By the definition of a spherical bitrade and the second statement of the current lemma $x^{3}$ must have a neighbor from $S_{i}$. The latter is at distance 3 from $x$ because $d_{S_{i}} \geq 3$ (see the first statement of the current lemma), so $d_{S_{i}}=3$.

Proposition 2. Let $G$ be a distance-regular graph with $p_{11}^{1}, p_{21}^{2} \neq 0$. The following statements are equivalent for codes $S_{0}$ and $S_{1}$ in $G$ :
i. The pair $\left(S_{0}, S_{1}\right)$ is a spherical bitrade in $G$.
ii. The minimum distances of $S_{0}$ and $S_{1}$ are 3 and $\chi_{S_{0}}-\chi_{S_{1}}$ is a 0eigenfunction of $G$.
iii. The minimum distances of $S_{0}$ and $S_{1}$ are 3, $d\left(S_{0}, S_{1}\right)=2$ and for any $x \in S_{i}$ there are exactly $p_{11}^{0} / p_{11}^{2}$ vertices in $S_{i \oplus 1}$ at distance 2 from $x$.

Proof. (i)~(ii) Follows from the first and the second statements of Lemma 2
(i) $\sim$ (iii) Let $\left(S_{0}, S_{1}\right)$ be a spherical bitrade. Then by the third statement of Lemma 2 we have $d_{S_{0}}=d_{S_{1}}=3$. Suppose that $x \in S_{0}$ is a neighbor of $y \in S_{1}$. Then by the definition of a spherical bitrade, $x$ has a neighbor from $S_{0}$, which contradicts the first statement of Lemma 2. Therefore, $d\left(S_{0}, S_{1}\right)$ is 2.

Now let $S_{0}$ and $S_{1}$ be two codes with minimum distances three such that $d\left(S_{0}, S_{1}\right)=2$. A vertex $x$ from $S_{0} \cup S_{1}$ has no neighbors in $S_{0}$ and $S_{1}$ because $d_{S_{0}}=d_{S_{1}}=3, d\left(S_{0}, S_{1}\right)=2$.

$$
\begin{equation*}
\text { If } \mathrm{x} \text { is in } S_{0} \cup S_{1} \text { then }\left|\mathcal{O}(x) \cap S_{0}\right|=\left|\mathcal{O}(x) \cap S_{1}\right|=0 \tag{1}
\end{equation*}
$$

Given a vertex $x$ from $S_{i}$ let us consider the set $\left\{y \in S_{i \oplus 1}: d(x, y)=2\right\}$, $i \in\{0,1\}$. Each of the vertices from this set has exactly $p_{11}^{2}$ common neighbors with $x$. Moreover, distinct vertices from $\left\{y \in S_{i \oplus 1}: d(x, y)=2\right\}$ have disjoint sets of common neighbors with $x$ because $d_{S_{i \oplus 1}}=3$. Now each of the neighbors of $x$ is a neighbor of exactly one vertex from $S_{i \oplus 1}$ if and only if there are exactly $|\mathcal{O}(x)| / p_{11}^{2}=p_{11}^{0} / p_{11}^{2}$ vertices from $S_{i \oplus 1}$ at distance 2 from $x$. Taking into account the property (1), the proposition follows.

In nonbipartite case we have the following characterization for Hamming graphs.

Corollary 2. The following assertions are equivalent for $q$ greater or equal to 3 and for two disjoint $q$-ary codes $S_{0}$ and $S_{1}$ of length $n$ :
i. The pair $\left(S_{0}, S_{1}\right)$ is a spherical bitrade in $H(n, q)$.
ii. The minimum distances of $S_{0}$ and $S_{1}$ are 3 and $\chi_{S_{0}}-\chi_{S_{1}}$ is aeigenfunction of $H(n, q)$.
iii. The minimum distances of $S_{0}$ and $S_{1}$ are 3, d $\left(S_{0}, S_{1}\right)=2$ and for any $x \in S_{i}$ there are exactly $(q-1) n / 2$ tuples of $S_{i \oplus 1}$ at distance 2 from $x$.

The eigenvalues of the Hamming graph are $\{n(q-1)-q i: i \in\{0, \ldots, n\}\}$. Then taking into account the eigenfunction representations given in Proposition 1 and Corollary 2 we see that a spherical (perfect respectively) bitrade exists in $H(n, q)$ then necessarily $n$ is $q r(q r+1$ respectively), for some $r \geq 1$.

### 3.2 Perfect bitrades from spherical bitrades

Proposition 3. Let $\left(S_{0}, S_{1}\right)$ be a spherical bitrade in $H(q r, q)$. Then $\left(S_{0} \times\right.$ $\left.\{0\} \cup S_{1} \times\{1\}, S_{0} \times\{1\} \cup S_{1} \times\{0\}\right)$ is a perfect bitrade.

Proof. By Corollary 2 the function $\chi_{S_{0}}-\chi_{S_{1}}$ is a 0-eigenfunction of $H(q r, q)$, $d\left(S_{0}, S_{1}\right)=2$ and $d_{S_{0}}$ and $d_{S_{1}}$ are three. Consider the difference of characteristic functions $\chi_{0}$ and $\chi_{1}$ of vertices (symbols) 0 and 1 in the complete graph $H(1, q)$. It is clear that $\chi_{0}-\chi_{1}$ is a $(-1)$-eigenfunction of $H(1, q)$. By Lemma 1 we conclude that

$$
\left(\chi_{S_{0}}-\chi_{S_{1}}\right) \cdot\left(\chi_{0}-\chi_{1}\right)=\chi_{S_{0} \times\{0\} \cup S_{1} \times\{1\}}-\chi_{S_{0} \times\{1\} \cup S_{1} \times\{0\}}
$$

is a $(-1)$-eigenfunction of $H(q r+1, q)$. The minimum distances of $S_{0}$ and $S_{1}$ are three and the distances between vertices of $S_{0}$ and $S_{1}$ are at least two. This implies that the minimum distances of $S_{0} \times\{0\} \cup S_{1} \times\{1\}$ and $S_{0} \times\{1\} \cup S_{1} \times\{0\}$ are three. By Proposition 1 we conclude that $\left(S_{0} \times\{0\} \cup S_{1} \times\{1\}, S_{0} \times\{1\} \cup\right.$ $\left.S_{1} \times\{0\}\right)$ is a perfect bitrade .

### 3.3 A recursive construction for spherical bitrades

Theorem 2. Let $\left(S_{0}, S_{1}\right)$ and $\left(S_{0}^{\prime}, S_{1}^{\prime}\right)$ be spherical bitrades in $H(q r, q)$ and $H\left(q r^{\prime}, q\right)$. Then $\left(S_{0} \times S_{0}^{\prime} \cup S_{1} \times S_{1}^{\prime}, S_{0} \times S_{1}^{\prime} \cup S_{1} \times S_{0}^{\prime}\right)$ is a spherical bitrade in $H\left(q\left(r+r^{\prime}\right), q\right)$.

Proof. By Corollary 2 the functions $\chi_{S_{0}}-\chi_{S_{1}}$ and $\chi_{S_{0}^{\prime}}-\chi_{S_{1}^{\prime}}$ are 0-eigenfunctions of $H(q r, q)$ and $H\left(q r^{\prime}, q\right)$ and the minimum distances of $S_{0}, S_{1}, S_{0}^{\prime}, S_{1}^{\prime}$ are three. Consider the tensor product $\left(\chi_{S_{0}}-\chi_{S_{1}}\right) \cdot\left(\chi_{S_{0}^{\prime}}-\chi_{S_{1}^{\prime}}\right)$. By Lemma 1 this function is a 0 -eigenfunction of $H\left(q\left(r+r^{\prime}\right), q\right)$. We have the following equalities:

$$
\begin{aligned}
& \left(\chi_{S_{0}}-\chi_{S_{1}}\right) \cdot\left(\chi_{S_{0}^{\prime}}-\chi_{S_{1}^{\prime}}\right)=\left(\chi_{S_{0}} \cdot \chi_{S_{0}^{\prime}}\right)-\left(\chi_{S_{1}} \cdot \chi_{S_{0}^{\prime}}\right)-\left(\chi_{S_{0}} \cdot \chi_{S_{1}^{\prime}}\right)+\left(\chi_{S_{1}} \cdot \chi_{S_{1}^{\prime}}\right)= \\
& \quad=\chi_{S_{0} \times S_{0}^{\prime}}+\chi_{S_{1} \times S_{1}^{\prime}}-\chi_{S_{0} \times S_{1}^{\prime}}-\chi_{S_{1} \times S_{0}^{\prime}}=\chi_{S_{0} \times S_{0}^{\prime} \cup S_{1} \times S_{1}^{\prime}}-\chi_{S_{0} \times S_{1}^{\prime} \cup S_{1} \times S_{0}^{\prime}} .
\end{aligned}
$$

We see that the function $\chi_{S_{0} \times S_{0}^{\prime} \cup S_{1} \times S_{1}^{\prime}}-\chi_{S_{0} \times S_{1}^{\prime} \cup S_{1} \times S_{0}^{\prime}}$ is a 0 -eigenfunction of $H\left(q\left(r+r^{\prime}\right), q\right)$. Moreover it is easy to see that the minimum distances of $S_{0} \times S_{0}^{\prime} \cup S_{1} \times S_{1}^{\prime}$ and $S_{0} \times S_{1}^{\prime} \cup S_{1} \times S_{0}^{\prime}$ are also three which follows from the minimum distances of $S_{0}, S_{1}, S_{0}^{\prime}, S_{1}^{\prime}$. By Corollary 2 the pair $\left(S_{0} \times S_{0}^{\prime} \cup S_{1} \times\right.$ $\left.S_{1}^{\prime}, S_{0} \times S_{1}^{\prime} \cup S_{1} \times S_{0}^{\prime}\right)$ is a spherical bitrade.

## 4 Constructions of spherical and perfect bitrades

It is a well-known fact that the action of any automorphism of $H(n, q)$ can be represented as the action of a permutation $\pi$ on the coordinate positions $\{1, \ldots, n\}$ followed by the action of $n$ permutations $\sigma_{1}, \ldots, \sigma_{n}$ of the alphabet set $\mathcal{A}$ :

$$
\begin{gathered}
\pi\left(x_{1}, \ldots, x_{n}\right)=\left(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}\right), \\
\left(\sigma_{1}, \ldots, \sigma_{n}\right)\left(x_{1}, \ldots, x_{n}\right)=\left(\sigma_{1}\left(x_{1}\right), \ldots, \sigma_{n}\left(x_{n}\right)\right)
\end{gathered}
$$

Let $S y m_{\mathcal{A}}$ and $A l t_{\mathcal{A}}$ denote the symmetric and alternating groups on the elements of the set $\mathcal{A}$. Define the codes $S_{0}$ and $S_{1}$ as follows:

$$
\begin{array}{r}
S_{0}=\left\{(\pi(0), \pi(1), \ldots, \pi(q-1)): \pi \in \text { Alt }_{\mathcal{A}}\right\} \\
S_{1}=\left\{(\pi(0), \pi(1), \ldots, \pi(q-1)): \pi \in \text { Sym }_{\mathcal{A}} \backslash \text { Alt }_{\mathcal{A}}\right\} \tag{2}
\end{array}
$$

Note that

$$
S_{1}=\left\{(\pi(1), \pi(0), \pi(2), \ldots, \pi(q-1)): \pi \in A l t_{\mathcal{A}}\right\}
$$

So the group $\left\{(\pi, \pi, \ldots, \pi): \pi \in A l t_{\mathcal{A}}\right\}$ acts regularly on the tuples of the codes $S_{0}$ and $S_{1}$ and the group $\left\{(\pi, \pi, \ldots, \pi): \pi \in S y m_{\mathcal{A}}\right\}$ acts regularly on the tuples of $S_{0} \cup S_{1}$. Moreover, the code $S_{1}$ is obtained from $S_{0}$ by an authomorphism of $H(q, q)$, i.e. a transposition of coordinate positions.

Theorem 3. Let $S_{0}$ and $S_{1}$ be the codes defined by (2). Then $\left(S_{0}, S_{1}\right)$ is a spherical bitrade in $\mathcal{H}(q, q)$ of volume $q!/ 2$.

Proof. We prove that $\left(S_{0}, S_{1}\right)$ fulfills the statement (iii) of Corollary 2 i.e. $d_{S_{i}}=3$, where $i \in\{0,1\} ; d\left(S_{0}, S_{1}\right)=2$ and for any $x \in S_{i}$ there exist $\binom{q}{2}$ tuples in $S_{i \oplus 1}$ at distance 2 from $x$ for any $i \in\{0,1\}$.

Let $x$ be the tuple $(0,1, \ldots, q-1), x \in S_{0}$. If $\pi$ is in $S y m_{\mathcal{A}}$ then the distance between the tuples $(0,1, \ldots, q-1)$ and $(\pi(0), \pi(1), \ldots, \pi(q-1))$ is always at least 2 and equals 2 if and only if $\pi$ is a transposition. So by (22) the only tuples from $S_{0} \cup S_{1}$ at distance 2 from $x$ are $\binom{q}{2}$ tuples of $S_{1}$. Since $S_{0}$ and $S_{1}$ are orbits of the same group and the code $S_{1}$ is obtained from $S_{0}$ by an authomorphism of $H(q, q)$ we have the desired properties.

Theorem 4. For any integer $r \geq 1$ and $q \geq 3$ there is a spherical bitrade in $H(q r, q)$ of volume $(q!)^{r} / 2$ and a perfect bitrade in $H(q r+1, q)$ of volume $(q!)^{r}$.

Proof. In order to obtain a spherical bitrade in $H(q r, q)$ we apply $(r-1)$ times the construction of Theorem 2 with the initial bitrade being the spherical bitrade in $H(q, q)$ from Theorem 3. This spherical bitrade in $H(q r, q)$ implies the existence of a perfect bitrade according to Proposition 3.

In the linear case we also have bitrades from MDS codes of significantly larger volumes than those of constructed in Theorem 3,

Theorem 5. Let $q$ be $p^{r}$, p be a prime number. Let $M$ be a $q$-ary linear $M D S$ code with minimum distance 2, $C_{0}$ and $C_{1}$ be q-ary linear MDS codes of length $n$ with minimum distances 3, $C_{0}, C_{1} \subset M, C_{0} \neq C_{1}$. Then $\left(C_{0} \backslash C_{1}, C_{1} \backslash C_{0}\right)$ is a spherical bitrade in $\mathcal{H}(q, q)$ of volume $q^{q-2}-q^{q-3}$.

Proof. We show that $\left|\mathcal{O}(x) \cap C_{0}\right|=\left|\mathcal{O}(x) \cap C_{1}\right|$ is zero if $x \in M$ and one otherwise. Since $M$ has minimum distance $2, C_{0}$ and $C_{1}$ are subcodes of $M$ we see that any tuple of $M$ has no neighbors in $C_{0}$ and $C_{1}$. It remains to show that if $x$ is not in $M$ then $\left|\mathcal{O}(x) \cap C_{0}\right|=\left|\mathcal{O}(x) \cap C_{1}\right|=1$. Since $d_{C_{0}}=d_{C_{1}}=3$ we have that $\left|\mathcal{O}(x) \cap C_{0}\right|,\left|\mathcal{O}(x) \cap C_{1}\right| \leq 1$. Then the number of the neighbors of $C_{i}$ could be counted as

$$
\left|C_{i}\right| \cdot q(q-1)=q^{q-2} \cdot q(q-1)=q^{q}-|M| .
$$

The neighbors of $C_{i}$ cannot be in $M$, so the above implies that each of the tuples outside of $M$ is a neighbor of only one tuple in $C_{0}$ and in $C_{1}$. Since linear codes $C_{0}$ and $C_{1}$ are of dimension $q-2$, they meet in a subspace of dimension $q-3$ and the expression for the volume of the bitrade ( $C_{0} \backslash C_{1}, C_{1} \backslash C_{0}$ ) follows.

Theorem 6. For any integer $r \geq 1$ and $q \geq 3, q=p^{r}$, where $p$ is a prime there is a spherical bitrade in $H(q r, q)$ of volume $2^{r-1}\left(q^{q-2}-q^{q-3}\right)^{r}$ and a perfect bitrade in $H(q r+1, q)$ of volume $2^{r}\left(q^{q-2}-q^{q-3}\right)^{r}$.

Proof. A spherical bitrade in $H(q r, q)$ of volume $2^{r-1}\left(q^{q-2}-q^{q-3}\right)^{r}$ is obtained by Theorem 2 from the spherical bitrade in $H(q, q)$ described in Theorem 5 , We then apply the construction from Proposition 3 to the spherical bitrade in $H(q r, q)$ in order to obtain a perfect bitrade in $H(q r+1, q)$.

Remark. There are other spherical bitrades that have rather less symmetric structure than the described above. In $H(5,5)$ using a computer we have found spherical bitrades of the following volumes: $60,95,100,125$. A bitrade of volume 60 can be obtained by Theorem 3, a bitrade of volume 100 exists by Theorem 5. A spherical bitrade of volume 125 could be obtained by taking $C_{1}$ to be a coset of a linear MDS code $C_{0}$ in Theorem [5, the proof for this fact is the same. This bitrade coincides with the bitrade described in the work [18].

## 5 Lower bound for the volumes of perfect bitrades in $\mathcal{H}(q+1, q)$

Theorem 7. The volume of a perfect bitrade in $\mathcal{H}(q+1, q)$ is not less than $q$ !.
Proof. Let $\left(T_{0}, T_{1}\right)$ be a perfect bitrade in $\mathcal{H}(q+1, q)$, $\Gamma$ be a face in $\mathcal{H}(q+1, q)$ with $(k+1), q \geq k \geq 1$ free positions. We show that $\left|\Gamma \bigcap T_{0}\right|+\left|\Gamma \bigcap T_{1}\right| \geq 2(k!)$ or $\left|\Gamma \bigcap T_{0}\right|+\left|\Gamma \bigcap T_{1}\right|=0$. The proof is by induction on the number of free positions of $\Gamma$.

The base case is when $k$ is 1 . Let $\Gamma$ be a face in $\mathcal{H}(q+1, q)$ with exactly 2 free positions. By Proposition 1 the function $\chi_{T_{0}}-\chi_{T_{1}}$ is a ( -1 )-eigenfunction of $H(q+1, q)$. Taking into account Corollary 1 the function $\chi_{T_{0}}-\chi_{T_{1}}$ has 0 or at least 2 nonzero values on $\Gamma$, i.e. $\left|\Gamma \bigcap T_{0}\right|+\left|\Gamma \bigcap T_{1}\right|$ is 0 or at least 2 .

By induction hypothesis we have that any face with exactly $k$ free positions has at least $(k-1)!\cdot 2$ tuples of $T_{0} \cup T_{1}$ or none. Without restriction of generality, suppose that the face $\Gamma=\Gamma_{1 \ldots q-k}^{0 \ldots 0}$ in $\mathcal{H}(q+1, q)$ contains the all-zero tuple $\mathbf{0}$ from $T_{0}$. We show that there are at least $2(k!)$ tuples from $T_{0} \bigcup T_{1}$ in $\Gamma_{1 \ldots q-k}^{0 \ldots 0}$.

By the definition of a perfect bitrade, the ball centered at $\mathbf{0} \in T_{0}$ must contain a unique tuple $y$ of $T_{1}$. Let $y$ be different from $\mathbf{0}$ in the position $j$. All neighbors of $\mathbf{0}$ different from $y$ are neighbors of tuples from $T_{1}$. The neighbors that are different from $\mathbf{0}$ in the $j$ th coordinate position are common neighbors with $y \in T_{1}$. The remaining $(q-1) q$ neighbors are exactly covered by $(q-1) q / 2$ tuples from $T_{1}$ at distance 2 from $\mathbf{0}$. Let $l$ be any free position for the face $\Gamma_{1 \ldots q-k}^{0 \ldots 0}$, i.e. any position from $\{q-k+1, \ldots, q+1\}$. Since $k \geq 2$, we choose $l$ to be distinct from the position $j$.

Consider the tuples that are different from $\mathbf{0}$ only in the $l$ th position. Since $l$ is not $j$ for any $\alpha \in \mathcal{A} \backslash\{0\}$ the tuple $(0, \ldots, 0, \underset{l}{\alpha}, 0, \ldots, 0)$ is a neighbor of a unique tuple from $T_{1}$ at distance 2 from $\mathbf{0}$. We denote this tuple by $y^{\alpha}$. By Proposition 1 we have $d_{T_{1}}=3$. Then for any distinct $\alpha$ and $\beta, \alpha, \beta \in \mathcal{A} \backslash\{0\}$ we see that

$$
y^{\alpha}=(0, \ldots, 0, *, 0,0, \ldots, 0, \underset{l}{\alpha}, 0, \ldots \ldots, 0)
$$

and

$$
y^{\beta}=\left(0, \ldots, 0,0, *, 0, \ldots, 0, \beta_{l}, 0, \ldots \ldots, 0\right)
$$

have only one common nonzero coordinate position which is $l$. We have $(q-1)$ such tuples $y^{\alpha}, \alpha \in \mathcal{A} \backslash\{0\}$ and at most $q-k$ of them have nonzero positions among nonfree positions $\{1, \ldots, q-k\}$ of $\Gamma_{1 \ldots q-k}^{0 \ldots 0}$. Then there are at least $k-1$ values for $\alpha \in \mathcal{A} \backslash\{0\}$ such that the face $\Gamma_{1 \ldots q-k l}^{0 \ldots 0 \alpha}$ contains a tuple $y^{\alpha} \in T_{1}$. Moreover the face $\Gamma_{1 \ldots q-k l}^{0 \ldots 00}$ contains a tuple $\mathbf{0} \in T_{0}$. We see that there are $k$ disjoint faces with $k$ free positions that are subsets of $\Gamma_{1 \ldots q-k}^{0 \ldots 0}$ and each of them contains at least $(k-1)!\cdot 2$ tuples from $T_{0} \bigcup T_{1}$ by induction hypothesis. The theorem follows when $k$ is $q$.

Corollary 3. The volume of a spherical bitrade in $H(q, q)$ is greater or equal to $q!/ 2$.

Proof. The existence of a spherical bitrade in $H(q, q)$ of volume $v$ implies the existence of a perfect bitrade in $H(q+1, q)$ of volume $2 v$ by the construction of Proposition 3. The result follows from the lower bound given by Theorem 7

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## References

[1] L. A. Bassalygo, V. A. Zinoviev, V. K. Leontiev, N. I. Feldman, Nonexistence of perfect codes over some non prime alphabets. Probl. Inf. Transm. 11(3):181-189, 1975.
[2] P. Delsarte, An Algebraic Approach to the Association Schemes of Coding Theory. N.V. Philips Gloeilampenfabrieken, Eindhoven, 1973, p. 97,.
[3] T. Etzion, A. Vardy, On perfect codes and tilings: problems and solutions. SIAM J. Discr. Math., (11)2:205-223, 1998.
[4] O. Heden, A survey of perfect codes. Advances in Math. of Communications, 2(2):223-247, 2008.
[5] A. S. Hedayat and G. B. Khosrovshahi, Trades. In C. J. Colbourn and J. H. Dinitz, editors. Handbook of Combinatorial Designs. Discrete Mathematics and Its Applications. Chapman Hall CRC, Boca Raton, London, New York, second edition, 2006., 2007, p. 984.
[6] H. L. Hwang, On the structure of (v, k, t) trades. J. Stat. Plann. Inference. 13:179-191, 1986.
[7] D. S. Krotov, The extended 1-perfect trades in small hypercubes. Discr. Math. 340(10):2559-2572, 2017.
[8] D. S. Krotov, S. V. Avgustinovich, On the number of 1-perfect binary codes: A lower bound. IEEE Trans. Inf. Theory. 54(4):1760-1765, 2008.
[9] D. Krotov, I. Mogilnykh, V. Potapov, To the theory of q-ary Steiner and othertype trades. Discr. Math. 339(3):1150-1157, 2016.
[10] P. R. J. Östergård, Switching codes and designs, Discrete Math., (312)3:621-632, 2012.
[11] K. T. Phelps, M. LeVan, Switching equivalence classes of perfect codes Des., Codes and Cryptogr. 16(2):179-184, 1999.
[12] V. N. Potapov, Multidimensional latin bitrades Siberian Mathematical Journal. 54(2):317-324, 2013.
[13] F. I. Solov'eva, Exact Bounds on the Connectivity of Code-Generated Disjunctive Normal Forms. Inst. Math. of the Siberian Branch of Acad. of Sciences, USSR, Preprint 1990, p. 10, (in Russian).
[14] F. I. Solov'eva, Switching Methods for Error-Correcting Codes. Aspects of Network and Information Security, IOS Press, NATO Science for Peace and Security, Series D: Information and Communication Security., 17:333-342, 2008.
[15] A. Tietäväinen, On the nonexistence of perfect codes over finite fields. SIAM J.Appl.Math. 24:88-96, 1973.
[16] A. Valyuzhenich, Minimum supports of eigenfunctions of Hamming graphs. Discr. Math. 340(5):1064-1068, 2017.
[17] A. Valyuzhenich, K. Vorob'ev, Minimum supports of functions on the Hamming graphs with spectral constraints. Discr. Math. 342(5):1351-1360, 2019.
[18] K. V. Vorobev, D. S. Krotov, Bounds for the Size of a Minimal 1-Perfect Bitrade in a Hamming Graph. J. Appl. and Industrial Math. 9(1):141-146, 2015.
[19] V. A. Zinoviev, V. K. Leontiev, Nonexistence of perfect codes over Galois fields. Probl. Control and Inform. Theory. 2(2):123-132, 1973.
[20] V. A. Zinoviev, Combinatorial methods to of constructing and analyzing nonlinear correcting codes. Doc. D. Thesis, Moscow, 1988 (in Russian).


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