

# On the anti-Ramsey numbers of linear forests\*

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## Abstract

For a fixed graph  $F$ , the *anti-Ramsey number*,  $AR(n, F)$ , is the maximum number of colors in an edge-coloring of  $K_n$  which does not contain a rainbow copy of  $F$ . In this paper, we determine the exact value of anti-Ramsey numbers of linear forests for sufficiently large  $n$ , and show the extremal edge-colored graphs. This answers a question of Fang, Győri, Lu and Xiao.

**Key words:** Anti-Ramsey numbers, Linear forests.

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## 1 Introduction

In this paper, only finite graphs without loops and multiple edges will be considered. Let  $K_n$  and  $P_n$  be the clique and path on  $n$  vertices, respectively. An *even path* or *odd path* is a path on *even* or *odd* number of vertices. A *linear forest* is a forest whose components are paths. For a given graph  $G = (V(G), E(G))$ , if  $v \in V(G)$  is a vertex of  $G$ , let  $N_G(v)$ ,  $d_G(v)$  be the neighborhood and degree of  $v$  in graph  $G$  respectively, and if  $U \subseteq V$ , let  $N_U(v)$  be the neighborhood of  $v$  in  $U$ , and  $N_G(U)$  be the common neighborhood of  $U$  in  $G$ . An *universal vertex* is a vertex in  $V(G)$  which is adjacent to all other vertices in  $V(G)$ . Denote the minimum degree of  $G$  by  $\delta(G)$ . Let  $W$  and  $U$  be two subsets of  $V(G)$ , denote the induced subgraph on  $W$  of  $G$  by  $G[W]$ , denote the subgraph of  $G$  with vertex set  $U \cup W$  and edge set  $E(U, W) = \{uw \in E(G), u \in U, w \in W\}$  by  $G[U, W]$ . An *edge-colored graph* is a graph  $G = (V(G), E(G))$  with a map  $c : E(G) \rightarrow S$ . The members in  $S$  are called colors. A subgraph of an edge-colored graph is *rainbow* if its all edges have different colors. The *representing graph* of an edge-colored graph  $G$  is a spanning subgraph of  $G$  obtained by taking one edge of each color in  $c$ . Denoted by  $\mathcal{R}(c, G)$  the family of representing subgraphs of an edge-colored graph  $G$  with coloring  $c$ .

For a fixed graph  $F$  and an integer  $n$ , the *anti-Ramsey number* of  $F$  is the maximum number of colors in an edge-coloring of  $K_n$  which does not contain  $F$  as a rainbow subgraph, and denote it by  $AR(n, F)$ . The anti-Ramsey number was introduced by Erdős, Simonovits and Sós [3] in 1975. They determined the anti-Ramsey numbers of cliques when  $n$  is sufficiently large. Later, in 1984, Simonovits and Sós [7] determined the anti-Ramsey number of paths.

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**Theorem 1.1.** (Simonovits and Sós, [7]) Let  $P_k$  be a path on  $k$  vertices with  $k \geq 2$ . If  $n$  is sufficiently large, then

$$\text{AR}(n, P_k) = \binom{\lfloor \frac{k-1}{2} \rfloor - 1}{2} + \left( \left\lfloor \frac{k-1}{2} \right\rfloor - 1 \right) \left( n - \left\lfloor \frac{k-1}{2} \right\rfloor + 1 \right) + 1 + \varepsilon,$$

where  $\varepsilon = 1$  if  $k$  is even and  $\varepsilon = 0$  otherwise.

Moreover, they have given the unique extremal edge-colorings as following. Let  $U$  be a vertex subset of  $K_n$  with  $|U| = \lfloor \frac{k-1}{2} \rfloor - 1$ , all the edges which are incident with  $U$  have different colors, the all edges of  $K_n[V(K_n) - U]$  colored by another one color if  $k$  is odd or other two colors otherwise. Denoted by  $\mathcal{C}_{P_k}(n)$  the family of above extremal edge-colorings of  $K_n$  of  $P_k$ .

In 2004, Schiermeyer [8] determined the anti-Ramsey number of matchings for  $n \geq 3t + 3$ .

**Theorem 1.2.** (Schiermeyer, [8])  $\text{AR}(n, tK_2) = \binom{t-2}{2} + (t-2)(n-t+2) + 1$  for all  $t \geq 2$  and  $n \geq 3t + 3$ .

And after that, Chen, Li and Tu [9] and Fujita, Magnant and Ozeki [10] independently showed that  $\text{AR}(n, tK_2) = \binom{t-2}{2} + (t-2)(n-t+2) + 1$  for all  $t \geq 2$  and  $n \geq 2t + 1$ .

In 2016, Gilboa and Roditty [5] determined that for large enough  $n$ , the anti-Ramsey number of  $L \cup tP_2$  and  $L \cup kP_3$  when  $t$  and  $k$  are large enough and  $L$  is a graph satisfying some conditions.

Very recently, Fang, Győri, Lu and Xiao [4] have given an approximate value of anti-Ramsey number of linear forests and determined the anti-Ramsey number of linear forests whose all components are odd paths.

**Theorem 1.3.** (Fang, Győri, Lu and Xiao, [4]) Let  $F = \cup_{i=1}^k P_{t_i}$  be a linear forest, where  $k \geq 2$ , and  $t_i \geq 2$  for all  $1 \leq i \leq k$ . Then

$$\text{AR}(n, F) = \left( \sum_{i=1}^k \left\lfloor \frac{t_i}{2} \right\rfloor - \epsilon \right) n + O(1),$$

where  $\epsilon = 1$  if all  $t_i$  are odd and  $\epsilon = 2$  otherwise.

For a given graph family  $\mathcal{F}$ , the *Turán number* of  $\mathcal{F}$  is the maximum number of edges of a graph on  $n$  vertices which does not contain a copy of any graph in  $\mathcal{F}$  as a subgraph, denote it by  $\text{ex}(n, \mathcal{F})$ .

The anti-Ramsey problem of linear forest is strongly connected with the Turán number of linear forest. Hence, we introduce some results of the Turán numbers of paths and linear forests.

In 1959, Erdős and Gallai showed the upper bound of the Turán number of  $P_k$  as the following theorem.

**Theorem 1.4.** (Erdős and Gallai, [2]) For any integers  $k, n \geq 1$ , we have  $\text{ex}(n, P_k) \leq \frac{k-2}{2}n$ .

The Turán number of linear forest have been determined by Bushaw and Kettle [1] and Lidicky, Liu and Palmer [6] for sufficiently large  $n$ .

**Theorem 1.5.** (Bushaw and Kettle, [1]) Let  $k \cdot P_3$  be the vertex disjoint union of  $k$  copies of  $P_3$ . Then for  $n \geq 7k$ , we have

$$\text{ex}(n, k \cdot P_3) = \binom{k-1}{2} + (k-1)(n-k+1) + \left\lfloor \frac{n-k+1}{2} \right\rfloor.$$

**Remark** Later, Yuan and Zhang [11] determined  $\text{ex}(n, k \cdot P_3)$  for all values of  $k$  and  $n$ .

**Theorem 1.6.** (Lidicky, Liu and Palmer, [6]) Let  $F = \cup_{i=1}^k P_{t_i}$  be a linear forest, where  $k \geq 2$  and  $t_i \geq 2$  for all  $1 \leq i \leq k$ . If at least one  $t_i$  is not 3, then for  $n$  sufficiently large,

$$\text{ex}(n, F) = \binom{\sum_{i=1}^k \lfloor t_i/2 \rfloor - 1}{2} + \left( \sum_{i=1}^k \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) \left( n - \sum_{i=1}^k \left\lfloor \frac{t_i}{2} \right\rfloor + 1 \right) + c,$$

where  $c = 1$  if all  $t_i$  are odd and  $c = 0$  otherwise. Moreover, the extremal graph is unique.

The extremal graph in Theorem 1.6, denote by  $G_F(n)$ , is a graph on  $n$  vertices with a vertex set  $U$  of order  $(\sum_{i=1}^k \lfloor t_i/2 \rfloor - 1)$  such that all the vertices in  $U$  are universal vertices and  $G_F(n) - U$  contains a single edge if each  $t_i$  is odd or  $V(G) - U$  is an independent subset otherwise.

The anti-Ramsey numbers of linear forests which consist of odd paths are determined by Gilboa and Roditty [5] for  $\text{AR}(n, k \cdot P_3)$  and Fang, Györi, Lu and Xiao [4] otherwise. In [4], they asked the following question: determining the exact value of anti-Ramsey number of a linear forest when it contains even paths. We will establish the following theorem.

From now on, let  $F = \cup_{i=1}^k P_{t_i}$  be a linear forest with at least one  $t_i$  is even, where  $k \geq 2$ ,  $t_i \geq 2$  for all  $1 \leq i \leq k$ . Define  $\mathcal{C}_F(n)$  to be a family of edge-colorings of  $K_n$  with a subset  $U$  of order  $\sum_{i=1}^k \lfloor t_i/2 \rfloor - 2$ , the all edges which are incident with  $U$  have different colors and the edges in  $V(K_n) - U$  are colored by another  $1 + \varepsilon$  colors, where  $\varepsilon = 1$  if exactly one  $t_i$  is even or  $\varepsilon = 0$  if at least two  $t_i$  are even. (see Figure 1).



Figure 1:  $\mathcal{C}_F(n)$ , (a) when  $F$  contains at least two components with even vertices; (b) when  $F$  contains exact one component with even vertices

**Theorem 1.7.** There is a function  $f(t_1, \dots, t_k)$  such that if  $n \geq f(t_1, \dots, t_k)$ , then

$$\text{AR}(n, F) = \binom{\sum_{i=1}^k \lfloor t_i/2 \rfloor - 2}{2} + \left( \sum_{i=1}^k \left\lfloor \frac{t_i}{2} \right\rfloor - 2 \right) \left( n - \sum_{i=1}^k \left\lfloor \frac{t_i}{2} \right\rfloor + 2 \right) + 1 + \varepsilon,$$

where  $\varepsilon = 1$  if exactly one  $t_i$  is even or  $\varepsilon = 0$  if at least two  $t_i$  are even. Moreover, the extremal edge-colorings must be in  $\mathcal{C}_F(n)$ .

## 2 Proof of Theorem 1.7

First, we prove a useful lemma for the extremal problems of linear forests.

**Lemma 2.1.** *Let  $G$  be an  $F$ -free graph on  $n$  vertices with  $|V(F)| = f$ . Let  $F_1 = \cup_{i \in L} P_{t_i}$  be a subgraph of  $F$ , where  $L \subseteq [k]$  and  $\sum_{i \in L} \lfloor t_i/2 \rfloor \geq 2$ , let  $F_2 = F - F_1$ . If  $G$  contains a copy of  $F_1$  as a subgraph and*

$$e(G) - \binom{|F_1|}{2} - \text{ex}(n - |F_1|, F_2) \geq \left( \sum_{i \in L} \left\lfloor \frac{t_i}{2} \right\rfloor - \frac{7}{4} \right) n,$$

*then any copy of  $F_1$  in  $G$  contains a subset  $U$  of order  $\sum_{i \in L} \lfloor t_i/2 \rfloor - 1$  with common neighborhood of size at least  $2f^2 + 8f$  in  $V(G) - U$ .*

*Proof.* Let  $G = (V, E)$  be an  $F$ -free graph on  $n$  vertices. Assume that  $G$  contains a copy of  $F_1$  on subset  $P$  and  $p = |P|$ . Let  $t = \sum_{i \in L} \lfloor t_i/2 \rfloor$ . Since  $G$  is  $F$ -free,  $G[V - P]$  contains no copy of  $F_2$ . Hence  $e(G[V - P]) \leq \text{ex}(n - p, F_2)$  and the number of edges between  $P$  and  $V - P$  in  $G$  is at least  $e(G) - \binom{p}{2} - \text{ex}(n - p, F_2) \geq (t - \frac{7}{4})n$ . Let  $n_0$  be the number of vertices in  $V - P$  which have at least  $t - 1$  neighbors in  $P$ , this is,

$$n_0 = |\{v \in V - P : |N_P(v)| \geq t - 1\}|.$$

Then the number of edges between  $V - P$  and  $P$  is at most  $n_0 p + (n - p - n_0)(t - 2)$ . Hence

$$n_0 p + (n - p - n_0)(t - 2) \geq \left( t - \frac{7}{4} \right) n.$$

So,

$$n_0 \geq \frac{n/4 + p(t - 2)}{p - t + 2}.$$

Since there are  $\binom{p}{t-1}$  subsets with size  $t - 1$  in  $P$ , and  $n$  is large enough, there is a subset  $U$  of size  $t - 1$  in  $P$  which has at least  $n_0 / \binom{p}{t-1} \geq 2f^2 + 8f$  common neighbors in  $V - P$ . ■

**Lemma 2.2.** *Let  $K_n$  be a complete graph on  $n$  vertices with an edge-coloring  $c$ . Let  $U$  and  $W$  be vertex disjoint subsets of  $V(K_n)$  with  $|U| = u$ ,  $|W| = w$  and  $u, w > 0$ . If there are two representing graphs  $L_n^1$  and  $L_n^2$  in  $\mathcal{R}(c, K_n)$  such that  $U$  has at least  $s$  common neighbours in  $L_n^1$  and  $W$  has at least  $s + su$  common neighbours in  $L_n^2$ , then there is a representing graph  $L_n$  in  $\mathcal{R}(c, K_n)$  such that  $U$  and  $W$  have at least  $s$  common neighbours in  $L_n$  respectively.*

*Proof.* Let  $X$  with size  $s$  and  $Y$  with size  $s + su$  be the common neighbours of  $U$  in  $L_n^1$  and the common neighbours of  $W$  in  $L_n^2$  respectively. Since there are  $su$  colors between  $X$  and  $U$ , there is a subset  $Y'$  of  $Y$  with size at least  $s$  such that the colors between  $W$  and  $Y'$  are all different from the colors between  $X$  and  $U$ . The result follows. ■

The following lemma is trivial. We left its proof to the readers.

**Lemma 2.3.** *For large  $n$ ,  $\text{AR}(n, P_2 \cup P_2) = 1$  and  $\text{AR}(n, P_3 \cup P_2) = 2$ .*

Let  $t_1 \geq t_2 \geq \dots \geq t_k \geq 2$  and

$$f_F(n) = \binom{\sum_{i=1}^k \lfloor t_i/2 \rfloor - 2}{2} + \left( \sum_{i=1}^k \left\lfloor \frac{t_i}{2} \right\rfloor - 2 \right) \left( n - \sum_{i=1}^k \left\lfloor \frac{t_i}{2} \right\rfloor + 2 \right) + 1 + \varepsilon,$$

where  $\varepsilon = 1$  if exactly one  $t_i$  is even and  $\varepsilon = 0$  if at least two  $t_i$  are even. Let  $F_0 = F - P_{t_k}$ .

We begin with a minimal degree version of the anti-Ramsey problem of linear forests.

**Lemma 2.4.** *Let  $c$  be an edge-coloring of  $K_n$  which does not contain a rainbow copy of  $F$  with at least  $f_F(n)$  colors and  $n \geq g(t_1, \dots, t_k)$ . If the minimum degree of each representing graph is at least  $\sum_{i=1}^k \lfloor t_i/2 \rfloor - 2$ , then the number of colors of  $c$  is exactly  $f_F(n)$ , and the extremal edge-coloring must be in  $\mathcal{C}_F(n)$ .*

*Proof.* Let  $c$  be an edge-coloring of  $K_n$  on vertex set  $V$  with at least  $f_F(n)$  colors. Since  $f_F(n) > \text{ex}(n, F_0)$  when  $n$  is sufficiently large, each representing graph in  $\mathcal{R}(c, K_n)$  contains a copy of  $F_0$ . Let  $L_n^1 \in \mathcal{R}(c, K_n)$  be a representing graph. Let  $|V(F)| = f$ . By Lemma 2.3, we may assume that  $F$  is not  $P_2 \cup P_2$  nor  $P_3 \cup P_2$ , so  $\sum_{i=1}^k \lfloor t_i/2 \rfloor \geq 3$ .

**Claim 1.** *There is a representing graph  $L_n \in \mathcal{R}(c, K_n)$  such that it contains two disjoint vertex subset  $U$  and  $W$ <sup>1</sup> of order  $u = |U| = \sum_{i=1}^{k-1} \lfloor t_i/2 \rfloor - 1$  and  $w = |W| = \lfloor t_k/2 \rfloor - 1$  such that  $|N_{L_n}(U)| \geq 2f + 8$  and  $|N_{L_n}(W)| \geq 2f + 8$  if  $W \neq \emptyset$ .*

*Proof.* Since  $F$  is not  $P_2 \cup P_2$  nor  $P_3 \cup P_2$ , we have  $\sum_{i=1}^{k-1} \lfloor t_i/2 \rfloor \geq 2$  and

$$\begin{aligned} & e(L_n^1) - \binom{|V(F_0)|}{2} - \text{ex}(n - |V(F_0)|, P_{t_k}) \\ & \geq \binom{\sum_{i=1}^k \lfloor t_i/2 \rfloor - 2}{2} + \left( \sum_{i=1}^k \left\lfloor \frac{t_i}{2} \right\rfloor - 2 \right) \left( n - \sum_{i=1}^k \left\lfloor \frac{t_i}{2} \right\rfloor + 2 \right) + 1 + \varepsilon \\ & \quad - \left( \frac{\sum_{i=1}^{k-1} t_i}{2} \right) - \frac{t_k - 2}{2} \left( n - \sum_{i=1}^{k-1} t_i \right) \\ & > \left( \sum_{i=1}^{k-1} \left\lfloor \frac{t_i}{2} \right\rfloor - \frac{7}{4} \right) n. \end{aligned}$$

By Lemma 2.1, one can find a subset  $U$  of  $P$  with  $|U| = \sum_{i=1}^{k-1} \lfloor t_i/2 \rfloor - 1$  which has common neighborhoods of size at least  $2f^2 + 8f$  in  $L_n^1$ .

Now we consider the subgraph  $L_n^1[V - U]$ . Then we have

$$\begin{aligned} e(L_n^1[V - U]) & \geq e(L_n^1) - \binom{\sum_{i=1}^{k-1} \lfloor \frac{t_i}{2} \rfloor - 1}{2} - \left( \sum_{i=1}^{k-1} \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) \left( n - \sum_{i=1}^{k-1} \left\lfloor \frac{t_i}{2} \right\rfloor + 1 \right) \\ & \geq \binom{\sum_{i=1}^k \lfloor t_i/2 \rfloor - 2}{2} + \left( \sum_{i=1}^k \left\lfloor \frac{t_i}{2} \right\rfloor - 2 \right) \left( n - \sum_{i=1}^k \left\lfloor \frac{t_i}{2} \right\rfloor + 2 \right) + 1 + \varepsilon \\ & \quad - \left( \frac{\sum_{i=1}^{k-1} \lfloor t_i/2 \rfloor - 1}{2} \right) - \left( \sum_{i=1}^{k-1} \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) \left( n - \sum_{i=1}^{k-1} \left\lfloor \frac{t_i}{2} \right\rfloor + 1 \right) \\ & \geq \binom{\lfloor t_k/2 \rfloor - 1}{2} + \left( \left\lfloor \frac{t_k}{2} \right\rfloor - 1 \right) \left( n - \sum_{i=1}^k \left\lfloor \frac{t_i}{2} \right\rfloor + 2 \right) + 1 + \varepsilon \\ & \geq \binom{\lfloor (t_k - 1)/2 \rfloor - 1}{2} + \left( \left\lfloor \frac{t_k - 1}{2} \right\rfloor - 1 \right) \left( n - \sum_{i=1}^{k-1} \left\lfloor \frac{t_i}{2} \right\rfloor - \left\lfloor \frac{t_k - 1}{2} \right\rfloor + 2 \right) + 1 + \varepsilon' \end{aligned}$$

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<sup>1</sup> $W$  can be empty set

$$= AR \left( n - \sum_{i=1}^{k-1} \left\lfloor \frac{t_i}{2} \right\rfloor + 1, P_{t_k} \right).$$

where  $\varepsilon' = 1$  if  $t_k$  is even or  $\varepsilon' = 0$  if  $t_k$  is odd. Moreover, the equality holds if and only if  $t_k$  is odd and at least two  $t_i$ 's are even. Thus, by Theorem 1.1, it is easy to see that  $c \in \mathcal{C}_F(n)$  and we are done, or  $K_n$  contains a rainbow copy of  $P_{t_k}$  on subset  $V - U$ . So there is a representing graph  $L_n^2 \in \mathcal{R}(c, K_n)$  such that  $L_n^2[V - U]$  contains a copy of  $P_{t_k}$ . If  $t_k \leq 3$ , let  $W = \emptyset$ ; if  $t_k \geq 4$ , by Lemma 2.1, there is a subset  $W$  of  $V - U$  of size  $\lfloor t_k/2 \rfloor - 1$  has common neighborhoods of size at least  $2f^2 + 8f$  in  $L_n^2$ . Now, by Lemma 2.2, there is a representing graph  $L_n$  satisfying the claim, the proof is completed.  $\blacksquare$

Since  $\sum_{i=1}^k \lfloor t_i/2 \rfloor \geq 3$ , we have  $|U \cup W| \geq 1$ . Let  $T = V - U - W$ . We may choose  $X \subseteq T$  and  $Y \subseteq T$  be the set of common neighbours of  $U$  and  $W$  in  $L_n$  respectively. By Claim 1, we have  $|X| \geq f + 8$  when  $|U| \geq 1$  and  $|Y| \geq f + 8$  when  $|W| \geq 1$ .

**Claim 2.** *Let  $L$  be the set of common neighbours of  $U \cup W$  in  $L_n$ . Then  $|L| \geq f + 2$ .*

*Proof.* If  $U = \emptyset$  or  $W = \emptyset$ , then the claim is obviously true. Let  $U \neq \emptyset$  and  $W \neq \emptyset$ , then  $t_1 \geq t_2 \geq \dots \geq t_k \geq 4$ . We consider the following two cases: (a)  $t_k$  is even. Let  $L_n^3$  be the graph obtained from  $L_n$  by adding an edge  $ab$  in  $K_n[Y]$  and deleting the edge  $cd$  in  $L_n$  colored by  $c(ab)$ . Suppose that there are at least three vertices in  $X' = X \setminus \{a, b, c, d\}$  which are not adjacent to all vertices of  $W$ . Since  $\delta(L_n^3) \geq |U \cup W|$ , there are three vertices of  $X'$ , say  $x_1, x_2$  and  $x_3$ , such that  $x_i$  is adjacent to  $y_i \in T$  for  $i = 1, 2, 3$ . If two of  $\{y_1, y_2, y_3\}$  are not belong to  $\{a, b\}$ , then we can find a copy of  $F$  in  $L_n^3$  easily. Moreover, the edges in  $Y' = Y \setminus \{a, b, c, d, y_1, y_2, y_3, x_1, x_2, x_3\}$  can not be colored by  $c(ab)$ . Otherwise, the graph obtained from  $L_n$  by adding an edge colored by  $c(ab)$  in  $Y'$  and deleting the edge  $cd$  contains a copy of  $F$ . Now let  $L_n^4$  be the graph obtained from  $L_n$  by adding an edge  $x_4x_5$  inside  $Y'$  and deleting the edge colored by  $c(x_4x_5)$ . Note that at least two of  $x_iy_i$  for  $1 \leq i \leq 3$  belong to  $L_n^4$  and we delete at most one edge between  $\{x_1, x_2, x_3\}$  and  $U$  or between  $x_4x_5$  and  $W$ , we can easily find a copy of  $F$  in  $L_n^4$ , a contradiction. Thus there are at most two vertices in  $X'$  which are not adjacent to all vertices of  $W$ . Hence, we have  $|L| \geq f + 8 - 6 = f + 2$ . (b)  $t_k$  is odd. The proof of this case is similar as case (a) and be omitted. The proof of the claim is completed.  $\blacksquare$

**Claim 3.** *There are at most  $1 + \varepsilon$  colors in  $c(K_n[T])$ .*

*Proof.* We only prove the claim for  $\varepsilon = 1$ , since the case  $\varepsilon = 0$  is much easier. Then there is exactly one  $t_i$  is even. Take the representing graph  $L_n$  with  $e(L_n[U \cup W, T])$  maximum and  $|N_{L_n}(U \cup W)| \geq f$ . That is if  $z_1z_2$  with  $z_1 \in T$  and  $z_2 \in U \cup W$  is not an edge of  $L_n$ , then  $z_1z_2$  is colored by  $c(z'_1z'_2)$ , where  $z'_1 \in T$  and  $z'_2 \in U \cup W$ .

If  $L_n$  is connected, suppose that there are at least three colors in  $c(K_n[T])$ . Then by Lemma 2.3, we can assume that  $L_n[T]$  contains a copy of  $P_3 \cup P_2$ . Let  $P_2 = ab$  and  $P_3 = xyz$ . We take  $e(L_n[x, U \cup W])$  as large as possible. Thus  $x$  is adjacent to  $U \cup W$ . And  $ab$  is connected to  $U \cup W \cup \{x, y, z\}$  by a path, let  $P = wPw'$  be the shortest path starting from  $ab$  ending at  $U \cup W \cup \{x, y, z\}$  with  $w \in U \cup W \cup \{x, y, z\}$ . If  $w \in U \cup W$ , then  $V(P) \cap \{x, y, z\} = \emptyset$ . If  $|U \cup W| = 1$ , then  $\{u\} = U \cup W$ . Thus we can easily find a copy of  $F$  (Note that  $F = P_5 \cup P_2$ ,  $F = P_4 \cup P_3$  or  $F = P_3 \cup P_3 \cup P_2$ ). Let  $|U \cup W| \geq 2$ . If  $e(L_n[\{x\}, U \cup W]) = 1$ , the edges between  $x$  and  $U \cup W$  are colored by the same color. We can take any edge of  $K_n[\{x\}, U \cup W]$  for  $L_n$ . If  $e(L_n[\{x\}, U \cup W]) \geq 2$ , then there are at least two edges between  $L_n[\{x\}, U \cup W]$ . Thus in both

cases, we can take an edge  $xu$  between  $x$  and  $U \cup W$  with  $u \neq w$ . Thus we can easily find a copy of  $F$ , a contradiction. Now we may suppose  $w \in \{x, y, z\}$ . If  $w \in \{y, z\}$ . Then  $xyPw'$  or  $xyzPw'$  contains a copy of  $P_4$  ending at  $x$ , so one can find a copy of  $F$  in  $L_n$ . If  $w = x$ . If  $xPw'$  contains at least four vertices, one can find a copy of  $F$  in  $L_n$ ; otherwise, we may assume that  $xa$  is an edge in  $L_n$ . If  $z$  is adjacent to  $U \cup W$  in  $L_n$ , then  $zyxa$  is a copy of  $P_4$  which is connected to  $U \cup W$ , then  $L_n$  contains a copy of  $F$ ; if for there is no edge between  $U \cup W$  and  $z$  in  $L_n$ , then we may add an edge  $zz'$  with  $z' \in U \cup W$  delete the edge in  $L_n$  colored by  $c(zz')$ . Thus the new representing graph contains a copy of  $F$ , a contradiction.

Assume that  $L_n$  is disconnected. Let  $C_1$  be the component of  $L_n$  containing  $U \cup W$ ,  $Z = V - V(C_1)$  and  $Q = V(C_1) - U \cup W$ . By the similar argument above,  $c(K_n[Q])$  contains at most two colors. Let  $L_n^5$  be the graph obtained from  $L_n$  by adding an edge  $vv'$  inside  $L$  and deleting the edge colored by  $c(vv')$ . Since  $L_n^5[V(C_1)]$  contains a copy of  $F_0$ ,  $L_n^5[Z]$  is  $P_{t_k}$ -free. So, we have  $e(L_n^5) \leq \binom{|U \cup W|}{2} + |U \cup W||Q| + 2 + \frac{t_k-2}{2}|Z| < f_F(n)$ , a contradiction. The claim is proved. ■

Since  $e(L_n) \geq f_F(n)$ , by Claim 3, we have  $c \in \mathcal{C}_F(n)$ . The proof is completed. ■

*Proof of Theorem 1.7.* Let  $c$  be an edge-coloring of  $K_n$  contains no rainbow copy of  $F$  with at least  $f_F(n)$  colors and  $n \geq f(t_1, \dots, t_k)$ , where  $f(t_1, \dots, t_k) \gg g(t_1, \dots, t_k)$ . Suppose that each representing graph in  $\mathcal{R}(c, K_n)$  has minimum degree at least  $\sum_{i=1}^k \lfloor t_i/2 \rfloor - 2$ . Hence, by Lemma 2.4, we have the number of edge-coloring in  $c$  is  $f_F(n)$  and  $c \in \mathcal{C}_F(n)$ .

Now, we may assume that there is a representing graph  $L_n \in \mathcal{R}(c, K_n)$  with  $\delta(L_n) \leq \sum_{i=1}^k \lfloor t_i/2 \rfloor - 3$ . So there is a vertex  $u_n$  in  $V$  with degree at most  $\sum_{i=1}^k \lfloor t_i/2 \rfloor - 3$  in  $L_n$ . Let  $G^n = K_n$  and  $G^{n-1} = K_n - u_n$ . Then  $G^{n-1}$  is an edge-colored completed graph on  $n-1$  vertices with at least  $f_F(n-1) + 1$  colors. If each representing graph in  $\mathcal{R}(c, G^{n-1})$  has minimum degree at least  $\sum_{i=1}^k \lfloor t_i/2 \rfloor - 2$ , then similar as the argument above, we have  $G^{n-1}$  contains a rainbow copy of  $F$ . Hence, there is a vertex  $u_{n-1}$  in  $G^{n-1}$  with degree at most  $\sum_{i=1}^k \lfloor t_i/2 \rfloor - 2$ . Thus we may construct a sequence of graphs  $G^n, G^{n-1}, \dots, G^{n-\ell}$  such that the number of coloring of  $G^{n-\ell}$  is at least  $f_F(n-\ell) + \ell$  (note that  $f(t_1, \dots, t_k) \gg g(t_1, \dots, t_k)$ ). Note that there are at most  $\binom{n-\ell}{2}$  colors in  $G^{n-\ell}$ , we get a contradiction when  $\ell$  is large. ■

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