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Abstract:	A vertex v of the underlying graph of a symmetric matrix A is called ‘Parter’ if the nullity of the matrix obtained from A by removing the row and column indexed by v is more than the nullity of A. Let A be a singular symmetric matrix with rank r whose underlying graph is a tree. It is known that the number of Parter vertices of A is at most r − 1. We prove that when r is odd this number is at most r − 2. We characterize the trees where these bounds are achieved.

Dear Editor and Reviewers,

We read carefully the reports of both reviewers on our manuscript: “The maximum number of Parter vertices of acyclic matrices” (DM 26746).

We thank the reviewers for their careful reading of our manuscript and for their very helpful comments and their very valuable suggestions. Hence, considering the reports we proceeded with the corresponding adjustments and corrections in our manuscript.

Yours sincerely,
Amélia Fonseca, Ângela Mestre, Ali Mohammadian, Cecília Perdigão, Maria Manuel Torres

The maximum number of Parter vertices of acyclic matrices

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Abstract

A vertex v of the underlying graph of a symmetric matrix A is called 'Parter' if the nullity of the matrix obtained from A by removing the row and column indexed by v is more than the nullity of A. Let A be a singular symmetric matrix with rank r whose underlying graph is a tree. It is known that the number of Parter vertices of A is at most $r-1$. We prove that when r is odd this number is at most $r-2$. We characterize the trees where these bounds are achieved.

Keywords: Acyclic Matrix, Nullity, Parter Vertex, Tree.

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1. Introduction

In this article, all graphs are assumed to be finite, undirected, and without loops or multiple edges. Let $\mathbb{F}$ denote a field and let G be a graph with vertex set $\mathcal{V}(G)$ and edge set $\mathcal{E}(G)$. Denote by $\mathcal{S}_{\mathbb{F}}(G)$ the set of all the symmetric matrices A with entries in $\mathbb{F}$, whose rows and columns are indexed by $\mathcal{V}(G)$, such that for every two distinct vertices $u, v \in \mathcal{V}(G)$, the (u, v)-entry of A is nonzero if and only if $\{u, v\} \in \mathcal{E}(G)$.

Indeed, since G is loopless, this definition imposes no restriction on the diagonal entries of matrices in $\mathcal{S}_{\mathbb{F}}(G)$. The *adjacency matrix* of G, denoted by $\mathcal{A}(G)$, is a $(0,1)$-matrix in $\mathcal{S}_{\mathbb{F}}(G)$ all of whose diagonal entries are equal to 0. In fact, the matrices in $\mathcal{S}_{\mathbb{F}}(G)$ can be seen as weighted adjacency matrices of G. For any tree T, we refer to the elements of $\mathcal{S}_{\mathbb{F}}(T)$ as *acyclic* matrices.

Before proceeding further, let us first set some notation and terminology. For an $n \times n$ matrix A with entries in a field $\mathbb{F}$, the *kernel* of A is defined as $\ker(A) = \{\boldsymbol{x} \in \mathbb{F}^n \,|\, A\boldsymbol{x} = \boldsymbol{0}\}$. The dimension of $\ker(A)$ is called the *nullity* of A and is denoted by $\eta(A)$. Moreover, the dimension of row (column) space of A is called the *rank* of A and is denoted by $\operatorname{rank}(A)$. For every matrix A in $\mathcal{S}_{\mathbb{F}}(G)$ and subset X of $\mathcal{V}(G)$, the principal submatrix of A obtained by deleting the rows and the columns indexed by X (respectively, $\mathcal{V}(G) \setminus X$) is denoted by $A(X)$ (respectively, $A[X]$). For simplicity, we write $A(v)$ instead of $A(\{v\})$. For a subset X of $\mathcal{V}(G)$, we use the notation $\langle X \rangle$ for the subgraph of G induced by X.

Let G be a graph with $n = |\mathcal{V}(G)|$ and let $A \in \mathcal{S}_{\mathbb{F}}(G)$. For each vertex $v \in \mathcal{V}(G)$, since $A(v)$ is an $(n-1) \times (n-1)$ submatrix of A and adding a row or a column can increase the rank by at most 1, $\operatorname{rank}(A) - \operatorname{rank}(A(v)) \in \{0, 1, 2\}$, which implies that $\eta(A) - \eta(A(v)) \in \{-1, 0, 1\}$. Following [7], we refer to a vertex $v \in \mathcal{V}(G)$ as a *Parter vertex of A* if $\eta(A(v)) = \eta(A) + 1$. Equivalently, a vertex $v \in \mathcal{V}(G)$ is a Parter vertex of A if and only if $\operatorname{rank}(A(v)) = \operatorname{rank}(A) - 2$. We denote the number of Parter vertices of A by $p(A)$. For a scalar $\sigma \in \mathbb{F}$, the geometric multiplicity of σ as an eigenvalue of A is denoted by $\eta_\sigma(A)$. Note that $\eta_\sigma(A) = \eta(A - \sigma I)$ and so, as there is no restriction on the diagonal entries of matrices in $\mathcal{S}_{\mathbb{F}}(G)$, the definitions and results in case $\sigma = 0$ can be generalized for any eigenvalue σ.

In this article, we deal with the maximum number of Parter vertices of singular acyclic matrices. We know by Proposition 4.4 of [7] that the number of Parter vertices of a singular matrix with rank r whose underlying graph has no isolated vertices is at most $r - 1$. This upper bound is tight. Further, we know from [3] and [6] that the maximum number of Parter vertices of $n \times n$ singular acyclic matrices is $2\lfloor \frac{n-1}{2} \rfloor - 1$. As a generalization, we prove in this paper that the number of Parter vertices of singular acyclic matrices with rank r is at most $2\lfloor \frac{r}{2} \rfloor - 1$. We also characterize the structure of trees which achieve this upper bound. It is noteworthy that, by [1], the maximum number of Parter vertices of $n \times n$ nonsingular acyclic matrices is $2\lfloor \frac{n}{2} \rfloor$.

Some other type results on Parter vertices of acyclic matrices are considered in the literature. For instance, we refer to [2], [4], and [9] among others.

2. Results

We begin this section with the following definition. Recall that an edge of a graph is called a *cut-edge* if it is not contained in any cycle of the graph.

Definition 1. *Let G be a graph and $A \in \mathcal{S}_{\mathbb{F}}(G)$. Use $\mathcal{V}(G)$ to index the components of each vector in* $\ker(A)$*. Define $G_A^{\downarrow}$ to be the set of vertices $v \in \mathcal{V}(G)$ such that there exists a vector $\boldsymbol{x} \in \ker(A)$ with $\boldsymbol{x}_v \neq 0$. Also, define $G_A^{\uparrow}$ to be the set of vertices $v \in \mathcal{V}(G)$ such that there exist a cut-edge $\{v, w\} \in \mathcal{E}(G)$ and a vector $\boldsymbol{x} \in \ker(A)$ with $\boldsymbol{x}_v = 0$ and $\boldsymbol{x}_w \neq 0$. Put $G_A^{\rightharpoondown} = \mathcal{V}(G) \setminus (G_A^{\downarrow} \cup G_A^{\uparrow})$.*

All the assertions of the following theorem are proved in [8].

Theorem 2. *For any tree T with $|\mathcal{V}(T)| \geqslant 2$ and any singular matrix $A \in \mathcal{S}_{\mathbb{F}}(T)$, the following hold.*

(i) *For any $v \in \mathcal{V}(T)$, $\eta(A(v)) = \eta(A) - 1$ if and only if $v \in T_A^{\downarrow}$.*

(ii) *For any $v \in T_A^{\uparrow}$, $\eta(A(v)) = \eta(A) + 1$.*

(iii) $T_A^{\downarrow} = \mathcal{V}(T)$ *if and only if* $T_A^{\uparrow} = \emptyset$*. If one of these occurs, then* $\eta(A) = 1$.

(iv) *The number of connected components of* $\langle T_A^{\downarrow} \rangle$ *is* $\eta(A) + |T_A^{\uparrow}|$.

(v) *If* $T_A^{\rightharpoonup} \neq \emptyset$*, then* $A[T_A^{\rightharpoonup}]$ *is nonsingular.*

(vi) *For any connected component* C *of* $\langle T_A^{\downarrow} \rangle$, $\ker(A[C])$ *is spanned by a nowhere-zero vector.*

Let T be a tree with $|\mathcal{V}(T)| \geqslant 2$ and let $A \in \mathcal{S}_{\mathbb{F}}(T)$ be a singular matrix. Parts (i) and (ii) of Theorem 2 imply that the subsets $T_A^{\downarrow}, T_A^{\uparrow}, T_A^{\rightharpoonup}$ are mutually disjoint, and moreover, the set of Parter vertices of A contains the vertices in $T_A^{\uparrow}$ and is contained in $T_A^{\uparrow} \cup T_A^{\rightharpoonup}$. The following example shows that, in general, the set of Parter vertices of A does not coincide with $T_A^{\uparrow}$ or $T_A^{\uparrow} \cup T_A^{\rightharpoonup}$.

Example 3. *Let T be the tree depicted in Figure 1 and consider the singular matrix*

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \in \mathcal{S}_{\mathbb{F}}(T).$$

Easy computations show that $T_A^{\downarrow} = \{v_1, v_2\}$, $T_A^{\uparrow} = \{v_3\}$, $T_A^{\rightharpoonup} = \{v_4, v_5, v_6\}$*, and the set of Parter vertices of* A *is* $\{v_3, v_4, v_6\}$.

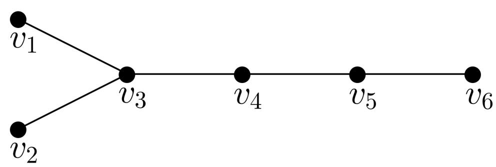

Figure 1. The tree in Example 3.

The following corollary is a partial consequence of Theorem 2 (vi).

Corollary 4. *Let T be a tree and let $A \in \mathcal{S}_{\mathbb{F}}(T)$ be a singular matrix. Assume that $v \in T_A^{\downarrow}$ is of degree* 1 *and its neighbor is contained in $T_A^{\uparrow}$. Then the (v, v)-entry of A is* 0.

Definition 5. *Denote the tree on* 2 *vertices by P_2. Attaching a P_2 to a vertex of a tree T by one edge is called the* adding pendant P_2 operation *on T.*

We recall the next theorem whose proof can be found in [5].

Theorem 6 (Du–da Fonseca [5])**.** *Let T be a tree on $n \geqslant 2$ vertices.*

(i) *There is a nonsingular matrix A in $\mathcal{S}_{\mathbb{F}}(T)$ with $p(A) = n$ if and only if T is obtained from P_2 by a sequence of adding pendant P_2 operations.*

(ii) *There is a nonsingular matrix A in $\mathcal{S}_{\mathbb{F}}(T)$ with $p(A) = n - 1$ if and only if T is obtained from a star by a sequence of adding pendant P_2 operations.*

We need the following lemma to prove our main theorem.

Lemma 7. *Let T be a tree and let $A \in \mathcal{S}_{\mathbb{F}}(T)$ be a singular matrix such that $|T_A^{\uparrow}| = 1$, $|T_A^{\Rsh}| \geqslant 2$, and $\langle T_A^{\downarrow} \rangle$ has no edge. Then a vertex in $T_A^{\Rsh}$ is a Parter vertex of A if and only if it is a Parter vertex of $A[T_A^{\Rsh}]$.*

Proof. Fix a vertex $v \in T_A^{\Rsh}$ and put $B = A[T_A^{\Rsh}]$. By Definition 1 and Corollary 4, we may assume that

$$
A = \begin{array}{c c}
 & \begin{array}{cccc} v & T_A^{\Rsh}\backslash\{v\} & T_A^{\uparrow} & T_A^{\downarrow} \end{array} \\
\begin{array}{c} v \\ T_A^{\Rsh}\backslash\{v\} \\ T_A^{\uparrow} \\ T_A^{\downarrow} \end{array} &
\left[\begin{array}{c|c|c|c}
\alpha & \boldsymbol{x}^{\top} & \beta & \mathbf{0} \\ \hline
\boldsymbol{x} & C & \boldsymbol{y} & \mathbf{0} \\ \hline
\beta & \boldsymbol{y}^{\top} & \gamma & \boldsymbol{z}^{\top} \\ \hline
\mathbf{0} & \mathbf{0} & \boldsymbol{z} & \mathbf{0}
\end{array}\right]
\end{array}
$$

for some scalars $\alpha, \beta, \gamma \in \mathbb{F}$ and column matrices $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$. Since $\boldsymbol{z}$ is nonzero,

$$
\operatorname{rank}(A) = \operatorname{rank} \left[\begin{array}{c|c|c|c}
\alpha & \boldsymbol{x}^{\top} & \beta & \mathbf{0} \\ \hline
\boldsymbol{x} & C & \boldsymbol{y} & \mathbf{0} \\ \hline
0 & \mathbf{0} & \gamma & \boldsymbol{z}^{\top} \\ \hline
\mathbf{0} & \mathbf{0} & \boldsymbol{z} & \mathbf{0}
\end{array}\right] = \operatorname{rank} \left[\begin{array}{c|c}
\alpha & \boldsymbol{x}^{\top} \\ \hline
\boldsymbol{x} & C
\end{array}\right] + 2 = \operatorname{rank}(B) + 2 \tag{1}
$$

and

$$
\operatorname{rank}\big(A(v)\big) = \operatorname{rank} \left[\begin{array}{c|c|c}
C & \boldsymbol{y} & \mathbf{0} \\ \hline
\boldsymbol{y}^{\top} & \gamma & \boldsymbol{z}^{\top} \\ \hline
\mathbf{0} & \boldsymbol{z} & \mathbf{0}
\end{array}\right] = \operatorname{rank} \left[\begin{array}{c|c|c}
C & \boldsymbol{y} & \mathbf{0} \\ \hline
\mathbf{0} & \gamma & \boldsymbol{z}^{\top} \\ \hline
\mathbf{0} & \boldsymbol{z} & \mathbf{0}
\end{array}\right] = \operatorname{rank}(C) + 2 = \operatorname{rank}\big(B(v)\big) + 2. \tag{2}
$$

Now, it follows from (1) and (2) that v is a Parter vertex of A if and only if v is a Parter vertex of B. $\square$

We are now in the position to state and prove our main result. We establish below that, for every tree T and singular matrix $A \in \mathcal{S}_{\mathbb{F}}(T)$, the number of Parter vertices of A is at most $2\lfloor \frac{\operatorname{rank}(A)}{2} \rfloor - 1$. We also characterize the trees which achieve the upper bound.

Theorem 8. *The following statements hold for any tree T.*

(i) *For any singular matrix $A \in \mathcal{S}_{\mathbb{F}}(T)$ with* $\operatorname{rank}(A) \geqslant 2$,

$$
p(A) \leqslant 2 \left\lfloor \frac{\operatorname{rank}(A)}{2} \right\rfloor - 1. \tag{3}
$$

(ii) *There exists a singular matrix $A \in \mathcal{S}_{\mathbb{F}}(T)$ with $p(A) = \operatorname{rank}(A) - 1$ if and only if T is of the form depicted in Figure 2 (a) for some trees $T_1, \ldots, T_k$ obtained from P_2 by a sequence of adding pendant P_2 operations.*

(iii) *There exists a singular matrix $A \in \mathcal{S}_{\mathbb{F}}(T)$ with odd rank and $p(A) = \operatorname{rank}(A) - 2$ if and only if either*

> *T is of the form shown in Figure 2 (a) for some trees $T_1, \ldots, T_k$ where one of them is obtained from a star with an odd number of vertices by a sequence of adding pendant P_2 operations, and the rest are obtained from P_2 by a sequence of adding pendant P_2 operations,*

or

T is of the form indicated in Figure 2 (b) for some trees $T_1, \ldots, T_k$ obtained from P_2 by a sequence of adding pendant P_2 operations.

Furthermore, the number of vertices in S_a and S_b are respectively equal to $\operatorname{rank}(A) - 1$ *and* $\operatorname{rank}(A) - 2$ *for every tree T of the form depicted in Figure 2 and singular matrix $A \in \mathcal{S}_{\mathbb{F}}(T)$ achieving the equality in* (3). *Note that in each of (a) and (b), all of $T_1, \ldots, T_k$ together may be absent.*

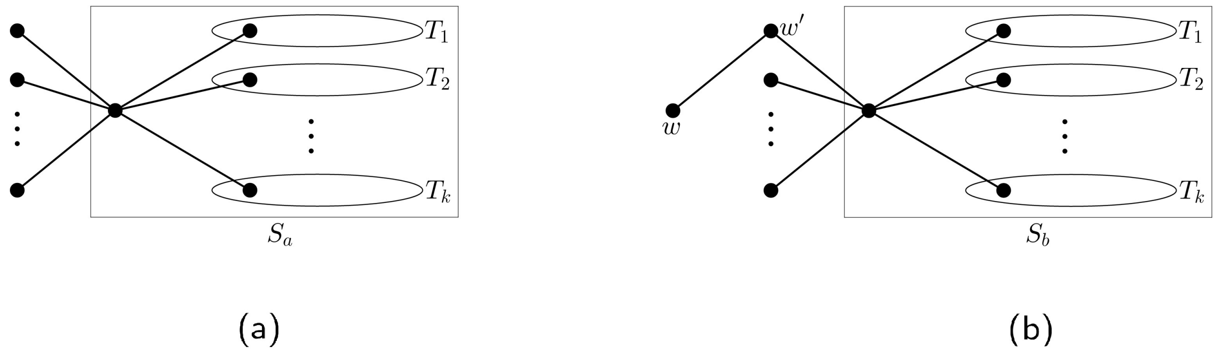

Figure 2. The extremal trees in Theorem 8.

Proof. Let $n = |\mathcal{V}(T)|$. Consider a singular matrix $A \in \mathcal{S}_{\mathbb{F}}(T)$ with rank $r \geqslant 2$. Note that $n \geqslant 3$. If $T_A^{\uparrow} = \emptyset$, then we find from Parts (i) and (iii) of Theorem 2 that $p(A) = 0$ and $r = n - 1$, so (3) holds and there is nothing more to prove. Hence, it what follows, we assume that $|T_A^{\uparrow}| \geqslant 1$. We first consider the case $r = 2$. It follows from Theorem 2 (iv) that the number of connected components of $\langle T_A^{\downarrow} \rangle$ is at least $n - 1$. As $|T_A^{\uparrow}| \geqslant 1$ and $T_A^{\uparrow} \cap T_A^{\downarrow} = \emptyset$, we conclude that $|T_A^{\downarrow}| = n - 1$ and therefore $p(A) = |T_A^{\uparrow}| = 1$ by Theorem 2 (ii). This proves (3) for $r = 2$. Furthermore, since $\langle T_A^{\downarrow} \rangle$ has $n - 1$ vertices and $n - 1$ connected components, $\langle T_A^{\downarrow} \rangle$ is an edgeless graph on $n-1$ vertices. As T is a tree, the unique vertex in $T_A^{\uparrow}$ is adjacent to all the vertices in $T_A^{\downarrow}$ and so T is of the form depicted in Figure 2 (a), where all of the trees $T_1, \ldots, T_k$ are absent. Thus, there is nothing more to prove in the case $r = 2$. From now on, we assume that $|T_A^{\uparrow}| \geqslant 1$ and $r \geqslant 3$.

We are going to establish (3). By Theorem 2 (i), each Parter vertex of A is contained in $T_A^{\uparrow} \cup T_A^{\nearrow}$, implying that

$$p(A) \leqslant \left|T_A^{\uparrow}\right| + \left|T_A^{\nearrow}\right|. \tag{4}$$

As $|T_A^{\uparrow}| \geqslant 1$, we have

$$\left|T_A^{\uparrow}\right| + \left|T_A^{\nearrow}\right| \leqslant \Big(\left|T_A^{\uparrow}\right| + \left|T_A^{\nearrow}\right|\Big) + \Big(\eta(A) + \left|T_A^{\uparrow}\right|\Big) - \big(\eta(A) + 1\big). \tag{5}$$

Also, we obtain from Theorem 2 (iv) that $\eta(A) + |T_A^{\uparrow}| \leqslant |T_A^{\downarrow}|$. Hence, it follows from $|T_A^{\uparrow}| + |T_A^{\nearrow}| = n - |T_A^{\downarrow}|$ that

$$\Big(\left|T_A^{\uparrow}\right| + \left|T_A^{\nearrow}\right|\Big) + \Big(\eta(A) + \left|T_A^{\uparrow}\right|\Big) - \big(\eta(A) + 1\big) \leqslant \Big(n - \left|T_A^{\downarrow}\right|\Big) + \left|T_A^{\downarrow}\right| - \big(\eta(A) + 1\big) = r - 1. \tag{6}$$

From (4), (5), and (6) we conclude that $p(A) \leqslant r - 1$. So, in order to prove $p(A) \leqslant 2\lfloor \frac{r}{2} \rfloor - 1$, it suffices to show that if $p(A) = r - 1$, then r is even. Suppose that $p(A) = r - 1$. Then, the equalities occur in (4)–(6).

It follows from (5) that $|T_A^{\uparrow}| = 1$ and hence $|T_A^{\rightharpoonup}| = r-2$ by (4). Hence, it follows from (6) that $\langle T_A^{\downarrow}\rangle$ is an edgeless graph on $n-r+1$ vertices. If $r=3$, then T is a star whose center is a vertex in $T_A^{\uparrow}$ and, using Corollary 4, it is straightforwardly checked that $p(A)=1$, which contradicts $p(A)=r-1$. Therefore, $r \geqslant 4$. It follows from Theorem 2 (v) that $A[T_A^{\rightharpoonup}]$ is nonsingular. Using Lemma 7 and Theorem 6 (i), r must be even, as required.

Now, we are going to determine the structure of T when the equality occurs in (3). First, suppose that $p(A)=r-1$. Then, r is an even number at least 4. Moreover, in this case, the equalities occur in (4)–(6). By (5), we conclude that $|T_A^{\uparrow}|=1$ and hence $|T_A^{\rightharpoonup}|=r-2$ by (4). Therefore, it follows from (6) that $\langle T_A^{\downarrow}\rangle$ is an edgeless graph on $n-r+1$ vertices. In addition, all the vertices of $T_A^{\rightharpoonup}$ are Parter vertices of A and $A[T_A^{\rightharpoonup}]$ is nonsingular by Theorem 2 (v). Using Lemma 7, we conclude that all the vertices of $T_A^{\rightharpoonup}$ are Parter vertices of $A[T_A^{\rightharpoonup}]$. Let $T_1,\ldots,T_k$ be the connected components of $\langle T_A^{\rightharpoonup}\rangle$. By Theorem 6 (i), we find that each T_i is obtained from P_2 by a sequence of adding pendant P_2 operations. Thus, T is of the form depicted in Figure 2 (a), as required. Next, suppose that r is odd and $p(A)=r-2$. We distinguish the three following cases:

Case 1. Assume that the inequality (4) is strict. In this case, the equalities occur in (5) and (6). By (5), we conclude that $|T_A^{\uparrow}|=1$ and hence $|T_A^{\rightharpoonup}|=r-2$ by (4). Therefore, it follows from (6) that $\langle T_A^{\downarrow}\rangle$ is an edgeless graph on $n-r+1$ vertices. If $r=3$, then T is a star whose center is a vertex in $T_A^{\uparrow}$, we are done. So, we assume that $r\geqslant 5$. It follows from Theorem 2 (v) that $A[T_A^{\rightharpoonup}]$ is nonsingular. Now, using Lemma 7 and Theorem 6 (ii), we conclude that T is of the form depicted in Figure 2 (a), as required.

Case 2. Assume that the inequality (5) is strict. So, the equalities occur in (4) and (6). By (5), we conclude that $|T_A^{\uparrow}|=2$ and thus $|T_A^{\rightharpoonup}|=r-4$ by (4). Since r is odd, $r\geqslant 5$. As the equality occurs in (6), we find that $\langle T_A^{\downarrow}\rangle$ is an edgeless graph on $n-r+2$ vertices. Moreover, Theorem 2 (v) yields that $B=A[T_A^{\rightharpoonup}]$ is nonsingular. We claim that $p(B)=r-4$. Assume that v is an arbitrary vertex in $T_A^{\rightharpoonup}$ and, for simplicity, suppose that v is corresponding to the first row of B. By Corollary 4, we may write

$$A = \begin{array}{c} \\ v \\ T_A^{\rightharpoonup}\setminus\{v\} \\ T_A^{\uparrow} \\ T_A^{\downarrow} \end{array} \begin{array}{c} \begin{array}{cccc} v & T_A^{\rightharpoonup}\setminus\{v\} & T_A^{\uparrow} & T_A^{\downarrow} \end{array} \\ \left[\begin{array}{c|c|c|c} \alpha & \boldsymbol{x}^{\top} & \boldsymbol{t}^{\top} & \mathbf{0} \\ \hline \boldsymbol{x} & C & Y & \mathbf{0} \\ \hline \boldsymbol{t} & Y^{\top} & D & Z^{\top} \\ \hline \mathbf{0} & \mathbf{0} & Z & \mathbf{0} \end{array}\right] \end{array},$$

where $\boldsymbol{x},\boldsymbol{t}$ are some column matrices and D is a 2×2 matrix. Since $|T_A^{\downarrow}|\geqslant 3$ and in view of Definition 1, Z has two linearly independent rows and then it is straightforwardly seen that

$$\operatorname{rank}\big(A(v)\big) = \operatorname{rank}\left[\begin{array}{c|c|c} C & Y & \mathbf{0} \\ \hline Y^{\top} & D & Z^{\top} \\ \hline \mathbf{0} & Z & \mathbf{0} \end{array}\right] = \operatorname{rank}\left[\begin{array}{c|c|c} C & Y & \mathbf{0} \\ \hline \mathbf{0} & D & Z^{\top} \\ \hline \mathbf{0} & Z & \mathbf{0} \end{array}\right] = \operatorname{rank}(C)+4.$$

As $p(A)=r-2$, $|T_A^{\uparrow}|=2$, and $|T_A^{\rightharpoonup}|=r-4$, all the vertices of $T_A^{\rightharpoonup}$ are Parter vertices of A. In particular, v is a Parter vertex of A. Therefore, we conclude that $\operatorname{rank}(B(v))=\operatorname{rank}(C)=r-6=|T_A^{\rightharpoonup}|-2=\operatorname{rank}(B)-2$. This means that v is a Parter vertex of B which implies that all the vertices of $T_A^{\rightharpoonup}$ are Parter vertices of B, proving the claim. By Theorem 6 (i), r is even, a contradiction.

Case 3. Assume that the inequality (6) is strict. So, the equalities occur in (4) and (5). By (5), we conclude that $|T_A^{\uparrow}|=1$ and hence $|T_A^{\rightharpoonup}|=r-3$ by (4). Thus, it follows from (6) that $\langle T_A^{\downarrow}\rangle$ has $n-r+2$ vertices and 1 edge. If $r=3$, then T is of the form shown in Figure 2 (b) and we are done. So, we assume that $r \geqslant 5$. It follows from Theorem 2 (v) that $B = A[T_A^{\rightharpoonup}]$ is nonsingular. Denote by w the unique vertex in $T_A^{\downarrow}$ with no neighbor in $T_A^{\uparrow}$. We claim that $p(B) = r - 3$. Assume that v is an arbitrary vertex in $T_A^{\rightharpoonup}$ and, for simplicity, suppose that v is corresponding to the first row of A and w corresponding to the last row of A. By Corollary 4, we may assume that

$$A = \begin{array}{c} \\ v \\ T_A^{\rightharpoonup}\backslash\{v\} \\ T_A^{\uparrow} \\ T_A^{\downarrow}\backslash\{w\} \\ w \end{array} \begin{array}{c} \begin{array}{ccccc} v & T_A^{\rightharpoonup}\backslash\{v\} & T_A^{\uparrow} & T_A^{\downarrow}\backslash\{w\} & w \end{array} \\ \left[\begin{array}{c|c|c|c|c} \alpha & \boldsymbol{x}^\top & \beta & \mathbf{0} & 0 \\ \hline \boldsymbol{x} & C & \boldsymbol{y} & \mathbf{0} & \mathbf{0} \\ \hline \beta & \boldsymbol{y}^\top & \gamma & \boldsymbol{z}^\top & 0 \\ \hline \mathbf{0} & \mathbf{0} & \boldsymbol{z} & D & \boldsymbol{t} \\ \hline 0 & \mathbf{0} & 0 & \boldsymbol{t}^\top & \delta \end{array}\right] \end{array},$$

where $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are some column matrices, $D = \operatorname{diag}(\sigma, 0, \ldots, 0)$, $\boldsymbol{t}^\top = [\tau \ 0 \ \cdots \ 0]$, and $\alpha, \beta, \gamma, \delta, \sigma, \tau \in \mathbb{F}$. We know that $\operatorname{rank}(B) = r - 3$ and

$$\operatorname{rank} \left[\begin{array}{c|c|c|c} \alpha & \boldsymbol{x}^\top & \mathbf{0} & 0 \\ \hline \boldsymbol{x} & C & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & D & \boldsymbol{t} \\ \hline 0 & \mathbf{0} & \boldsymbol{t}^\top & \delta \end{array}\right] = \operatorname{rank}\Big(A\Big(T_A^{\uparrow}\Big)\Big) = r - 2.$$

This yields that $\delta\sigma = \tau^2$. Now, it is straightforward to check that

$$\operatorname{rank} \left[\begin{array}{c|c|c} \gamma & \boldsymbol{z}^\top & 0 \\ \hline \boldsymbol{z} & D & \boldsymbol{t} \\ \hline 0 & \boldsymbol{t}^\top & \delta \end{array}\right] = 3.$$

As we have assumed the equality in (4), all the vertices in $T_A^{\rightharpoonup}$ are Parter vertices of A. So, v is a Parter vertex of A and, as $\boldsymbol{z}$ has at least two nonzero entries,

$$r - 2 = \operatorname{rank}\big(A(v)\big) = \operatorname{rank} \left[\begin{array}{c|c|c|c} C & \boldsymbol{y} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \gamma & \boldsymbol{z}^\top & 0 \\ \hline \mathbf{0} & \boldsymbol{z} & D & \boldsymbol{t} \\ \hline \mathbf{0} & 0 & \boldsymbol{t}^\top & \delta \end{array}\right] = \operatorname{rank}(C) + 3$$

and thus $\operatorname{rank}(B(v)) = \operatorname{rank}(C) = r - 5$. This means that v is a Parter vertex of B, proving that $p(B) = r - 3$. Using Theorem 6 (i), we conclude that T is of the form indicated in Figure 2 (b), as required.

In order to end the proof, let T be a tree of one of the forms illustrated in Figure 2 for some trees $T_1, \ldots, T_k$. Let $n_i = |\mathcal{V}(T_i)|$ for any $i \in \{1, \ldots, k\}$ and let $m = n_1 + \cdots + n_k$. In what follows, we will construct a singular matrix $A \in \mathcal{S}_{\mathbb{F}}(T)$ achieving the equality in (3).

First, assume that T is of the form depicted in Figure 2 (a) for some trees $T_1, \ldots, T_k$ obtained from P_2 by a sequence of adding pendant P_2 operations. By Theorem 6 (i), for any $i \in \{1, \ldots, k\}$, there is a nonsingular matrix A_i in $\mathcal{S}_{\mathbb{F}}(T_i)$ such that $p(A_i) = n_i$. Let $A \in \mathcal{S}_{\mathbb{F}}(T)$ be the matrix obtained from $\mathcal{A}(T)$ by replacing the submatrices $\mathcal{A}(T_1), \ldots, \mathcal{A}(T_k)$ with $A_1, \ldots, A_k$, respectively. It is easy to check that A is a singular matrix with $p(A) = \operatorname{rank}(A) - 1 = m + 1$. Actually, the set of Parter vertices of A is equal to S_a.

Next, assume that T is of the form depicted in Figure 2 (a) for some trees $T_1, \ldots, T_k$, where one of them is obtained from a star with an odd number of vertices by a sequence of adding pendant P_2 operations and the rest are obtained from P_2 by a sequence of adding pendant P_2 operations. Without loss of generality, suppose that T_1 is obtained from a star with an odd number of vertices by a sequence of adding pendant P_2 operations. By Theorem 6, there is a nonsingular matrix A_1 in $\mathcal{S}_{\mathbb{F}}(T_1)$ such that $p(A_1) = n_1 - 1$ and, for any $i \in \{2, \ldots, k\}$, there is a nonsingular matrix A_i in $\mathcal{S}_{\mathbb{F}}(T_i)$ such that $p(A_i) = n_i$. Let $A \in \mathcal{S}_{\mathbb{F}}(T)$ be the matrix obtained from $\mathcal{A}(T)$ by replacing the submatrices $\mathcal{A}(T_1), \ldots, \mathcal{A}(T_k)$ with $A_1, \ldots, A_k$, respectively. It is easy to check that A is a singular matrix with $p(A) = \text{rank}(A) - 2 = m$. Actually, the set of Parter vertices of A is equal to S_a. Note that in this case m is odd which implies that $\text{rank}(A)$ is odd.

Finally, assume that T is of the form depicted in Figure 2 (b) for some trees $T_1, \ldots, T_k$ obtained from P_2 by a sequence of adding pendant P_2 operations. By Theorem 6 (i), for any $i \in \{1, \ldots, k\}$, there is a nonsingular matrix A_i in $\mathcal{S}_{\mathbb{F}}(T_i)$ such that $p(A_i) = n_i$. Let $A \in \mathcal{S}_{\mathbb{F}}(T)$ be the matrix obtained from $\mathcal{A}(T)$ by replacing the submatrices $\mathcal{A}(T_1), \ldots, \mathcal{A}(T_k)$ with $A_1, \ldots, A_k$, respectively, and by replacing 0 with 1 on the positions (w, w) and (w', w'), where w and w' are introduced in Figure 2 (b). It is straightforward to check that A is a singular matrix with $p(A) = \text{rank}(A) - 2 = m + 1$. Actually, the set of Parter vertices of A is equal to S_b. Note that in this case m is even which implies that $\text{rank}(A)$ is odd. □

3. Concluding remarks

In this paper, we showed for every tree T and singular matrix $A \in \mathcal{S}_{\mathbb{F}}(T)$ that $p(A) \leqslant 2\lfloor \frac{\text{rank}(A)}{2} \rfloor - 1$ provided $\text{rank}(A) \geqslant 2$ and we determined all trees for which there exists a singular matrix attaining the upper bound. More precisely, we characterized the trees T for which there is a singular matrix $A \in \mathcal{S}_{\mathbb{F}}(T)$ with $p(A) = \text{rank}(A) - 1$, and moreover, we characterized the trees T for which there is a singular matrix $A \in \mathcal{S}_{\mathbb{F}}(T)$ having odd rank and satisfying $p(A) = \text{rank}(A) - 2$. It is worth to mention that our results do not depend on the ground field $\mathbb{F}$. Naturally, one may consider a more general problem: For a given integer $\ell \geqslant 2$, find all trees T for which there exists a singular matrix $A \in \mathcal{S}_{\mathbb{F}}(T)$ with $p(A) = \text{rank}(A) - \ell$.

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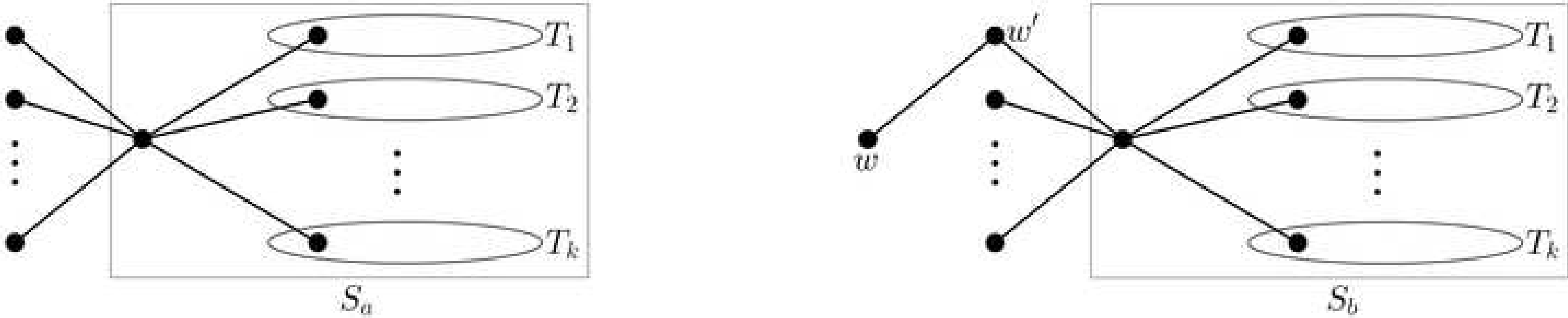
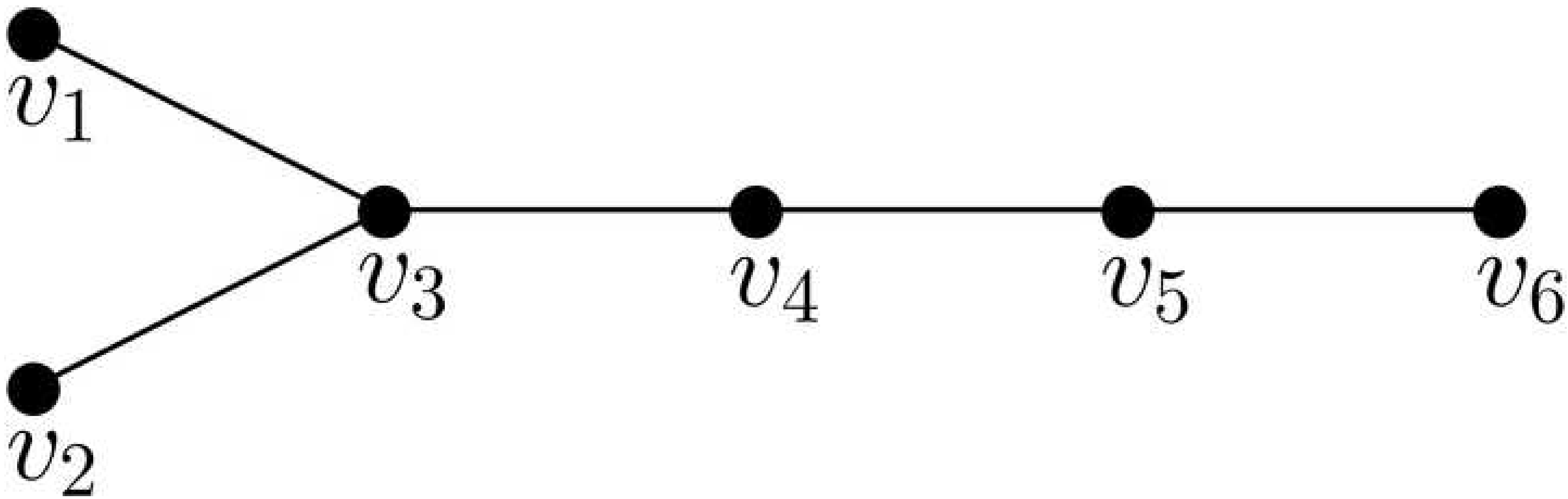
Figure



(a) (b)


v_1
v_2
v_3
v_4
v_5
v_6

Supplementary File

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