Upper bounds on the signed edge domination number of a graph

Fengming Dong[1]*, Jun Ge[2]†, and Yan Yang[3]‡

[1]National Institute of Education, Nanyang Technological University, Singapore
[2]School of Mathematical Sciences, Sichuan Normal University, China
[3]School of Mathematics, Tianjin University, China.

Abstract

A signed edge domination function (or SEDF) of a simple graph $G = (V, E)$ is a function $f : E \to \{1, -1\}$ such that $\sum_{e' \in N[e]} f(e') \geq 1$ holds for each edge $e \in E$, where $N[e]$ is the set of edges in G that share at least one endpoint with e. Let $\gamma'_s(G)$ denote the minimum value of $f(G)$ among all SEDFs f, where $f(G) = \sum_{e \in E} f(e)$. In 2005, Xu conjectured that $\gamma'_s(G) \leq n - 1$, where n is the order of G. This conjecture has been proved for the two cases $v_{odd}(G) = 0$ and $v_{even}(G) = 0$, where $v_{odd}(G)$ (resp. $v_{even}(G)$) is the number of odd (resp. even) vertices in G. This article proves Xu's conjecture for $v_{even}(G) \in \{1, 2\}$. We also show that for any simple graph G of order n, $\gamma'_s(G) \leq n + v_{odd}(G)/2$ and $\gamma'_s(G) \leq n - 2 + v_{even}(G)$ when $v_{even}(G) > 0$, and thus $\gamma'_s(G) \leq (4n - 2)/3$. Our result improves the best current upper bound of $\gamma'_s(G) \leq \lceil 3n/2 \rceil$.

Keywords: signed edge domination function, signed edge domination number, trail decomposition

1 Introduction

This article considers simple and undirected graphs only. For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. For any $v \in V(G)$, let $E_G(v)$ be the set of edges in G incident to v, let $N_G(v)$ be the set of vertices in G adjacent to v, and let $N_G[v] = N_G(v) \cup \{v\}$. $E_G(v)$, $N_G(v)$ and $N_G[v]$ are simply written as $E(v)$, $N(v)$ and $N[v]$, respectively, when there is no confusion. For any $v \in V(G)$, we use $d_G(v)$ (or simply $d(v)$ when there is no confusion) to denote the degree of v in G.

*Email: fengming.dong@nie.edu.sg and donggraph@163.com.
†Corresponding author. Email: mathsgejun@163.com.
‡Email: yanyang@tju.edu.cn.

For a graph $G = (V, E)$, a *signed domination function* of G is a function $f : V \to \{1, -1\}$ with the property that $f(N[v]) \geq 1$ holds for every $v \in V$, where $f(S) = \sum_{v \in S} f(v)$ for each $S \subseteq V$. The *signed domination number* of G, denoted by $\gamma_s(G)$, is defined to be the minimum value of $f(V)$ over all signed domination functions f of G. The parameter $\gamma_s(G)$ was introduced by Dunbar, Hedetniemi, Henning, and Slater [4] and has been studied by many authors, e.g., [3, 5–7, 10, 15].

In 2001, Xu [11] introduced signed edge domination functions. For a graph $G = (V, E)$, a function $f : E \to \{1, -1\}$ is called a *signed edge domination function* (SEDF) of G if $\sum_{e' \in N[e]} f(e') \geq 1$ holds for every $e \in E$, where $e = uv$ and $N[e] = E_G(u) \cup E_G(v)$. Let $\mathcal{F}_{sed}(G)$ denote the set of SEDFs of G. The *signed edge domination number* of G, denoted by $\gamma'_s(G)$, is defined to be the minimum value of $f(G)$ over all $f \in \mathcal{F}_{sed}(G)$, where $f(G) = \sum_{e \in E} f(e)$.

Observe that the parameter $\gamma'_s(G)$ is an extension of $\gamma_s(G)$, as each member f in $\mathcal{F}_{sed}(G)$ is actually a signed domination function of the line graph $L(G)$, thus implying that $\gamma'_s(G) = \gamma_s(L(G))$. The parameter $\gamma'_s(G)$ has been studied by many authors, e.g., [1, 2, 8, 11–14]. The following are some known results on $\gamma'_s(G)$ for a graph G of order n and size m:

(i). $\gamma'_s(G) \geq \frac{-n^2}{16}$ [1];

(ii). for any positive integer r, there exists an r-connected graph H such that $\gamma'_s(H) \leq -\frac{r}{6}|V(H)|$ [1];

(iii). $\gamma'_s(G) \geq \frac{2\alpha'(G) - m}{3}$, where $\alpha'(G)$ is the size of a largest matching of G [2];

(iv). $\gamma'_s(G) \geq n - m$ for $n \geq 4$ [12];

(v). $\gamma'_s(G) \leq \frac{11n}{6} - 1$ [13];

(vi). $\gamma'_s(G) \leq \lceil \frac{3n}{2} \rceil$ [8].

In this article, we will improve the upper bounds of $\gamma'_s(G)$ by establishing the following result. A vertex in a graph G is called an *odd vertex* (resp. *even vertex*) if it is of odd degree (resp. even degree) in G. Let $v_{odd}(G)$ (resp. $v_{even}(G)$) denote the number of odd (resp. even) vertices in G. Clearly, $v_{odd}(G)$ is even.

Theorem 1 *For any graph G of order n,*

(a) $\gamma'_s(G) \leq n + v_{odd}(G)/2$;

(b) $\gamma'_s(G) \leq n - 2 + v_{even}(G)$ when $v_{even}(G) > 0$;

and hence $\gamma'_s(G) \leq (4n - 2)/3$.

The most challenging and interesting problem on $\gamma'_s(G)$ may be the following conjecture proposed by Xu [12] in 2005.

Conjecture 1 ([12]) *For any simple graph G of order n, $\gamma'_s(G) \le n-1$ holds.*

As far as we know, Conjecture 1 has been only proved for a few cases. Karami, Khodkar, and Sheikholeslami [8] showed that Conjecture 1 holds when $v_{odd}(G) \in \{0, n\}$. In the case $v_{odd}(G) = n$, Akbari, Esfandiari, Barzegary, and Seddighin [2] strengthened the result to $\gamma'_s(G) \le n - \frac{2\alpha'(G)}{3}$, where $\alpha'(G)$ is the size of a maximum matching in G. In this paper, we prove that $\gamma'_s(G) \le n-1$ if $v_{even}(G) \in \{1, 2\}$.

Theorem 2 *Conjecture 1 holds for any simple graph G with $v_{even}(G) \in \{1, 2\}$.*

In Section 2, we introduce a subfamily $\mathcal{F}^0_{sed}(G)$ of $\mathcal{F}_{sed}(G)$ and establish some basic results for proving the main results in the following sections. Theorem 1 (a) and (b) are proved in Sections 3 and 4, respectively. By Theorem 1 (b), Conjecture 1 holds for $v_{even}(G) = 1$. In Section 5, we show that Conjecture 1 holds for $v_{even}(G) = 2$, and thus Theorem 2 follows. In Section 6, we propose a conjecture to replace Conjecture 1, as we think there exists a member f in $\mathcal{F}^0_{sed}(G)$ with $f(G) \le n-1$ for any graph G of order n. We also propose a conjecture for the lower bound of $\gamma'_s(G)$ when G is 2-connected.

2 A subset $\mathcal{F}^0_{sed}(G)$ of $\mathcal{F}_{sed}(G)$

Let G be a simple graph. For any $f : E(G) \to \{1, -1\}$ and $v \in V(G)$, let $f(v) = \sum\limits_{e \in E_G(v)} f(e)$ and let $f(S) = \sum\limits_{e \in S} f(e)$, where $S \subseteq E(G)$. Let $\mathcal{F}^0_{sed}(G)$ denote the set of functions $f : E(G) \to \{1, -1\}$ satisfying the two conditions below:

(a) $f(v) \ge 0$ for all $v \in V(G)$; and

(b) $f(u) + f(v) \ge 2$ for each $e = uv \in E(G)$ with $f(e) = 1$.

Lemma 1 *$\mathcal{F}^0_{sed}(G) \subseteq \mathcal{F}_{sed}(G)$.*

Proof. Let f be any member in $\mathcal{F}^0_{sed}(G)$ and let $e = v_1v_2 \in E(G)$. It follows from the definition of $\mathcal{F}^0_{sed}(G)$ that $f(v_i) \ge 0$ for $i = 1, 2$ and $f(v_1) + f(v_2) \ge 2$ holds whenever $f(e) = 1$, thus implying $f(v_1) + f(v_2) \ge 1 + f(e)$. Consequently,

$$f(N[e]) = f(v_1) + f(v_2) - f(e) \ge 1 + f(e) - f(e) = 1.$$

Hence $f \in \mathcal{F}_{sed}(G)$ and the result holds. $\square$

For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S. If E_1 and E_2 form a partition of $E(G)$ and $f_i : E_i \to \{1, -1\}$, let $f_1 * f_2$ be the function $f : E(G) \to \{1, -1\}$ defined by $f(e) = f_i(e)$ whenever $e \in E_i$.

Lemma 2 *Let G be a separable graph with $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \{v_0\}$ and $E(G) = E(G[V_1]) \cup E(G[V_2])$. If $f_i \in \mathcal{F}^0_{sed}(G[V_i])$ for $i = 1, 2$, then $f = f_1 * f_2 \in \mathcal{F}^0_{sed}(G)$ with*

$$f(G) = f_1(G[V_1]) + f_2(G[V_2]).$$

Proof. Note that $E(G[V_1])$ and $E(G[V_2])$ form a partition of $E(G)$ and thus $f_1 * f_2$ is well defined. By the definition of f, it is obvious that $f(G) = f_1(G[V_1]) + f_2(G[V_2])$. Next, by the definition of f, for any $v \in V(G)$,

$$f(v) = \begin{cases} f_1(v_0) + f_2(v_0), & \text{if } v = v_0; \\ f_i(v), & \text{if } v \in V_i - \{v_0\}, i = 1, 2. \end{cases} \tag{1}$$

As $f_i \in \mathcal{F}^0_{sed}(G[V_i])$ for $i = 1, 2$, we have $f(v) \geq 0$ for each $v \in V(G)$ by (1). Now, let e be any edge in $E(G)$ with $f(e) = 1$. We may assume that $e = v_1v_2 \in E(G[V_1])$, and thus $f_1(e) = f(e) = 1$. As $f_1 \in \mathcal{F}^0_{sed}(G[V_1])$, $f_1(v_1) + f_1(v_2) \geq 2$. By (1) and the assumption that $f_2 \in \mathcal{F}^0_{sed}(G[V_2])$, we have

$$f(v_1) + f(v_2) \geq f_1(v_1) + f_1(v_2) \geq 2.$$

Hence $f \in \mathcal{F}^0_{sed}(G)$ as required. □

In the following, we assume that v_0 is a vertex in a 2-connected graph G with $d_G(v_0) = 2$.

Lemma 3 *Let G be a simple graph, and let $v_0 \in V(G)$ with $N_G(v_0) = \{u_1, u_2\}$.*

(i). *For $u_1u_2 \in E(G)$ and $g \in \mathcal{F}^0_{sed}(G')$, where $G' = G - u_1u_2 - v_0$, as shown in Figure 1(b), let $f : E(G) \to \{1, -1\}$ be defined below:*

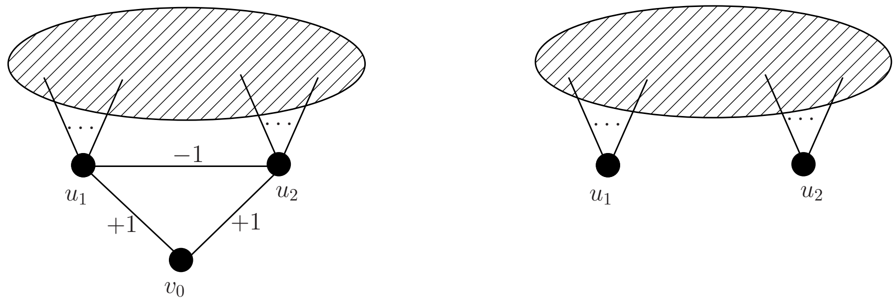
$$f(e) = \begin{cases} g(e), & \text{if } e \in E(G'); \\ 1, & \text{if } e = v_0u_i,\ i = 1, 2; \\ -1, & \text{if } e = u_1u_2. \end{cases}$$

Then, $f \in \mathcal{F}^0_{sed}(G)$ with $f(G) = g(G') + 1$.

(ii). *For $u_1u_2 \notin E(G)$ and $g \in \mathcal{F}^0_{sed}(G')$, where $G' = G + u_1u_2 - v_0$, as shown in Figure 2(b), let $f : E(G) \to \{1, -1\}$ be defined below:*

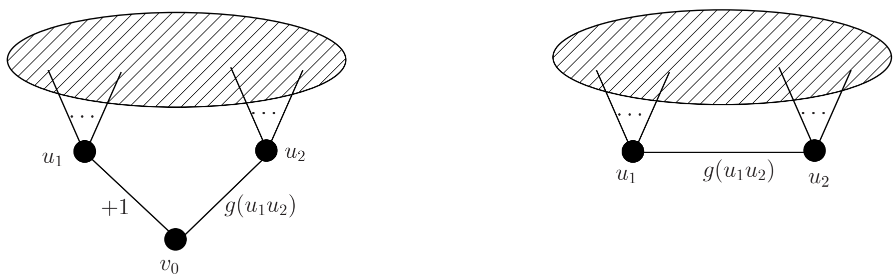
$$f(e) = \begin{cases} g(e), & \text{if } e \in (E(G') - \{u_1u_2\}); \\ 1, & \text{if } e = u_1v_0; \\ g(u_1u_2), & \text{if } e = u_2v_0. \end{cases}$$

Then, $f \in \mathcal{F}^0_{sed}(G)$ with $f(G) = g(G') + 1$.



(a) G (b) G' $(= G - u_1u_2 - v_0)$

Figure 1: Graphs G and G' $(= G - u_1u_2 - v_0)$.



(a) G (b) G' $(= G + u_1u_2 - v_0)$

Figure 2: Graphs G and G' $(= G + u_1u_2 - v_0)$.

Proof. (i). By the definition of f, for any $v \in V(G)$, we have

$$f(v) = \begin{cases} g(v), & \text{if } v \in V(G) - \{v_0\}; \\ 2, & \text{if } v = v_0. \end{cases}$$

For any $uv \in E(G) - \{v_0u_1, v_0u_2, u_1u_2\}$ such that $f(uv) = 1$, we have $f(u) + f(v) = g(u) + g(v) \geq 2$. For v_0u_i, $i \in \{1, 2\}$, we have $f(v_0) + f(u_i) = 2 + g(u_i) \geq 2$. Thus, $g \in \mathcal{F}^0_{sed}(G')$ implies $f \in \mathcal{F}^0_{sed}(G)$.

(ii). If $g(u_1u_2) = 1$, then by the definition of f, we have

$$f(v) = \begin{cases} g(v), & \text{if } v \in V(G) - \{v_0\}; \\ 2, & \text{if } v = v_0. \end{cases}$$

For any $uv \in E(G) - \{v_0u_1, v_0u_2\}$ such that $f(uv) = 1$, we have $f(u)+f(v) = g(u)+g(v) \geq 2$. For v_0u_i, $i \in \{1, 2\}$, we have $f(v_0) + f(u_i) = 2 + g(u_i) \geq 2$. Thus, $f \in \mathcal{F}^0_{sed}(G)$ in this case.

If $g(u_1u_2) = -1$, then, by the definition of f, we obtain

$$f(v) = \begin{cases} g(v), & \text{if } v \in V(G) - \{u_1, v_0\}; \\ g(u_1) + 2, & \text{if } v = u_1; \\ 0, & \text{if } v = v_0. \end{cases}$$

For any $uv \in E(G)-\{v_0u_1, v_0u_2\}$ such that $f(uv)=1$, we have $f(u)+f(v) \geq g(u)+g(v) \geq 2$. For the positive edge v_0u_1, we have $f(v_0)+f(u_1)=0+g(u_1)+2 \geq 2$. Thus, $f \in \mathcal{F}^0_{sed}(G)$ in this case. □

3 $\gamma'_s(G) \leq n + v_{odd}(G)/2$

For any graph G and $f : E(G) \to \{1, -1\}$, let $I_f(G) = \{v \in V(G) : f(v) = 0\}$. We will prove the main result in this section by applying the following result due to Karami, Khodkar, and Sheikholeslami [8].

Theorem 3 ([8]) *For any simple graph G of order n with $v_{odd}(G) = 0$, there exists $f \in \mathcal{F}^0_{sed}(G)$ with $I_f(G) \neq \emptyset$ and $f(u) \in \{0, 2\}$ for all $u \in V(G)$.*

Proposition 1 *For any simple graph G of order n, there is $f \in \mathcal{F}^0_{sed}(G)$ such that $f(G) \leq n + v_{odd}(G)/2$.*

Proof. It is sufficient to prove for the case when G is connected. It can be easily verified that the result holds whenever $n \leq 3$. Now assume that $n \geq 4$ and the result holds for any connected graph of order at most $n-1$.

If G is not 2-connected, by assumption, the result holds for each block of G. Assume that G has k blocks, then by using Lemma 2 $k-1$ times, we will see the result holds for G. If G is 2-connected and $\delta(G) = 2$, then the result also holds by assumption and Lemma 3.

If $v_{odd}(G) = 0$, then the result follows from Theorem 3. In the following, we assume that G is 2-connected with $\delta(G) \geq 3$ and $v_{odd}(G) > 0$.

For convenience, let $k = v_{odd}(G)/2$ in the proof, where $k \geq 1$. Let $U = \{u_1, u_2, \dots, u_{2k}\}$ be the set of odd vertices in G, and let G' be the graph obtained from G by adding a new vertex w and $2k$ new edges joining w to all vertices in U. Clearly, $v_{odd}(G') = 0$ and $E(G') = E(G) \cup \{wu_i : 1 \leq i \leq 2k\}$.

By Theorem 3, there exists $g \in \mathcal{F}^0_{sed}(G')$ with $I_g(G') \neq \emptyset$ and $g(u) \in \{0, 2\}$ for all $u \in V(G')$. Thus $g(w) \in \{0, 2\}$. As $E(G') = E(G) \cup E_{G'}(w)$, $g(G') = g(w) + g(E(G))$ holds. Thus, we have the following conclusion.

Claim 1: $g(E(G)) = g(G') - g(w)$.

Let U_1 be the set of vertices $u_i \in U$ with $g(wu_i) = +1$ and $U_2 = U - U_1$. As $g(w) = |U_1| - |U_2|$ and $|U_1| + |U_2| = d_{G'}(w) = 2k$, the following conclusion holds.

Claim 2: $|U_1| = k + g(w)/2$.

U_1 is then partitioned into A and B, where A is the set of $u_i \in U_1$ with $g(u_i) = 2$. Let C be the set of vertices $v \in V(G) - U$ with $g(v) = 0$ as shown in Figure 3. Then $B \cup C \subseteq I_g(G')$. Note that $w \in I_g(G')$ if and only if $g(w) = 0$. Thus the following claim holds.

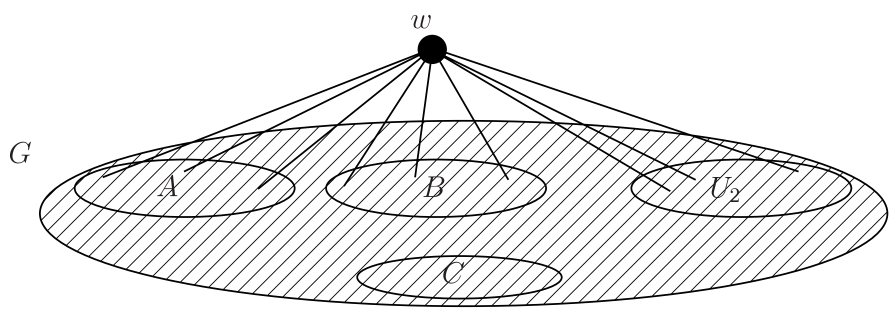


Figure 3: $G = G' - w$, $N_{G'}(w) = A \cup B \cup U_2$ and $B \cup C \subseteq I_g(G')$.

Claim 3: $|I_g(G')| \geq |B| + |C| + 1 - g(w)/2$.

By Theorem 3, we have $g(G') = \frac{1}{2}\sum_{u \in V(G')} g(u) = (n+1) - |I_g(G')|$. Thus, the following conclusions follows from Claims 1 and 3.

Claim 4: $g(E(G)) \leq n - (|B| + |C| + g(w)/2)$.

Let v be any vertex in $V(G)$. As $\delta(G) \geq 3$, $d_{G'}(v) \geq 4$ holds. Since $g(v) \in \{0, 2\}$, v is incident with some edge $e \in E(G)$ with $g(e) = -1$. Thus, there exists a subset E_1 of $E(G)$ with $g(e) = -1$ for all $e \in E_1$ such that each $v \in A \cup B$ is incident with some edge in E_1 (recall that $g(uw) = 1$ for $u \in A \cup B = U_1$). Let E_1 be a minimal one of such sets; note that $|E_1| \leq |A| + |B|$.

Let $f : E(G) \to \{+1, -1\}$ be the function defined by $f(e) = +1$ for all $e \in E_1$ and $f(e) = g(e)$ for all $e \in E(G) - E_1$. It can be easily verified that $f \in \mathcal{F}^0_{sed}(G)$ holds by the following facts:

(i). For each $u_i \in A \cup B$, we have $f(u_i) \geq g(u_i) - g(wu_i) + 2 = g(u_i) + 1 \geq 1$.

(ii). For each $u_i \in U_2$, we have $f(u_i) \geq g(u_i) - g(wu_i) = g(u_i) + 1 \geq 1$.

(iii). For each $v \in V(G) - U$, we have $f(v) \geq g(v) \geq 0$.

(iv). For each $e = v_1v_2 \in E(G)$ with $f(e) = +1$, if $e \in E_1$, then $f(v_1) + f(v_2) \geq g(v_1) + g(v_2) + 2 \geq 2$; if $e \in E(G) - E_1$, then $f(v_1) + f(v_2) \geq g(v_1) + g(v_2) \geq 2$.

By the definition of f and Claim 4, we have

$$\begin{aligned} f(G) &= g(E(G)) + 2|E_1| \\ &\leq n - (|B| + |C| + g(w)/2) + 2(|A| + |B|) \\ &= n + 2|A| + |B| - |C| - g(w)/2. \end{aligned}$$

Thus, the following conclusion holds.

Claim 5: $\gamma_s'(G) \leq n + 2|A| + |B| - |C| - g(w)/2$.

Similarly, there exists a subset E_2 of $E(G)$ with $g(e) = -1$ for all $e \in E_2$ such that each $v \in B \cup C$ is incident with some edge in E_2. Let E_2 be a minimal one of such sets; note that $|E_2| \leq |B| + |C|$.

Let $f' : E(G) \to \{+1, -1\}$ be the function defined by $f'(e) = +1$ for all $e \in E_2$ and $f'(e) = g(e)$ for all $e \in E(G) - E_2$. Again, it can be verified easily that $f' \in \mathcal{F}^0_{sed}(G)$ holds by the following facts:

(i). For each $u \in A$, we have $f'(u) \geq g(u) - 1 = 2 - 1 = 1$.

(ii). For each $u \in B$, we have $f'(u) \geq g(u) - 1 + 2 = 0 - 1 + 2 = 1$.

(iii). For each $u \in U_2$, we have $f'(u) \geq g(u) + 1 \geq 0 + 1 = 1$.

(iv). For each $u \in C$, we have $f'(u) \geq g(u) + 2 = 0 + 2 \geq 2$.

(v). For each $u \in V(G) - U - C$, we have $f'(u) \geq g(u) = 2$.

(vi). For each $e = v_1v_2 \in E(G)$ with $f'(e) = +1$, we have $f(v_1) + f(v_2) \geq 1 + 1 = 2$.

By the definition of f' and Claim 4, we have

$$\begin{aligned} f'(G) &= g(E(G)) + 2|E_2| \\ &\leq n - (|B| + |C| + g(w)/2) + 2(|B| + |C|) \\ &= n + |B| + |C| - g(w)/2. \end{aligned}$$

Thus, the following conclusion holds.

Claim 6: $\gamma_s'(G) \leq n + |B| + |C| - g(w)/2$.

By Claims 5 and 6, $\gamma_s'(G) \leq n + |A| + |B| - g(w)/2 = n + |U_1| - g(w)/2$ holds. By Claim 2, we have $\gamma_s'(G) \leq n + k = n + v_{odd}(G)/2$. □

4 $\gamma_s'(G) \leq n - 2 + v_{even}(G)$ when $v_{even}(G) > 0$

In this section, the following exact values of $\gamma_s'(K_{m,n})$ will be used.

Theorem 4 ([1]) *Let m and n be two positive integers, where $m \leq n$. Then:*

(i). *If m and n are even, then $\gamma_s'(K_{m,n}) = \min\{2m, n\}$.*

(ii). *If m and n are odd, then* $\gamma'_s(K_{m,n}) = \min\{2m-1, n\}$.

(iii). *If m is even and n is odd, then* $\gamma'_s(K_{m,n}) = \min\{3m, \max\{2m, n+1\}\}$.

(iv). *If m is odd and n is even, then* $\gamma'_s(K_{m,n}) = \min\{3m-1, \max\{2m, n\}\}$.

In the proof of the part (b) of Theorem 1, we shall need the parts (i) and (iii) of Theorem 4. In these two cases, actually Akbari et al. proved that there exists $f \in \mathcal{F}^0_{sed}(G)$ such that $f(K_{m,n}) = \gamma'_s(K_{m,n})$.

Proposition 2 *For any simple graph G of order n, if $v_{even}(G) > 0$, then there is an $f \in \mathcal{F}^0_{sed}(G)$ such that $f(G) \leq n - 2 + v_{even}(G)$, and thus $\gamma'_s(G) \leq n - 2 + v_{even}(G)$.*

Proof. When $v_{even}(G) = 0$, by Theorem 7 in [8], there exists $f \in \mathcal{F}^0_{sed}(G)$ with $f(G) \leq |V(G)| - 1$. So it is sufficient to prove the case when G is connected. It can be easily verified that the result holds whenever $n \leq 3$. Now assume that $n \geq 4$ and the result holds for any graph of order at most $n-1$. By Lemma 2, we only need to prove the result for 2-connected graphs. Let $v_{even}(G) = t \geq 1$, and let $W = \{w_1, w_2, \dots w_t\}$ be the set of all even vertices.

Claim 1: If W is an independent set, then there exists $f \in \mathcal{F}^0_{sed}(G)$ such that $f(G) \leq n - 2 + v_{even}(G)$.

Assume that w_1 has the minimum degree among all elements in W. Let $d_G(w_1) = 2s$, $s \geq 1$, and assume that $N_G(w_1) = \{u_1, u_2, \dots, u_{2s}\}$. Consider $G' = G - w_1$. Since G has no cut vertex, G' is connected. Clearly, $|V(G')| = n - 1$ and $v_{odd}(G') = n - t - 2s$.

Case 1.1. $n - t - 2s \geq 2$, i.e., G' is not an Eulerian graph.

In this case, G' can be decomposed into $(n-t-2s)/2$ trails $T_1, \dots, T_{(n-t-2s)/2}$, and the endpoints of these $(n-t-2s)/2$ trails correspond to all odd vertices of G'. Now, we define the function $f_1 : E(G) \to \{1, -1\}$ as follows:

(i). for each T_i, $1 \leq i \leq (n-t-2s)/2$, starting with $+1$, we assign $+1$ and -1 to the edges of T_i alternatively. When the trail has even number of edges, we change the value of the last edge to $+1$;

(ii). for each edge w_1u_i, $1 \leq i \leq 2s$, we set $f_1(w_1u_i) = +1$; and

(iii). for any w_i, $2 \leq w_i \leq t$, if the weight of w_i till now is 0, then we choose any negative edge incident to w_i and change it to a positive one.

It follows from the construction that

$$f_1(G) \leq 2 \cdot \frac{n-t-2s}{2} + 2s + 2(t-1) = n - 2 + t = n - 2 + v_{even}(G).$$

Next, after Step (i), the weight of any vertex in $V(G) - W - N_G(w_1)$ is at least 1, and the weight of vertices in $(W - \{w_1\}) \cup N_G(w_1)$ is 0 or at least 2. After Step (ii), $f(w_1)$ is $2s$; the weight of vertices in $N_G(w_1)$ has increased by 1, and others remain unchanged. Finally, after Step (iii), all the vertices in $W - \{w_1\}$ are of the weight at least 2.

Hence $f_1 \in \mathcal{F}^0_{sed}(G)$, and $\gamma'_s(G) \le f_1(G) \le n - 2 + v_{even}(G)$.

Case 1.2. $n - t - 2s = 0$, i.e., G' is an Eulerian graph, and $2s \ge 4$.

Because G' is an Eulerian graph, so it has an Eulerian circuit. Now we define the function $f_2 : E(G) \to \{1, -1\}$ as follows:

(i). for a fixed Eulerian circuit of G', starting from the vertex u_1, walking along the Eulerian circuit, we assign $+1$ and -1 alternatively starting with $+1$;

(ii). we set $f_2(w_1u_i) = 1$, $1 \le i \le 2s$ if $|E(G')|$ is even; otherwise, if $|E(G')|$ is odd, we set $f_2(w_1u_1) = -1$ and $f_2(w_1u_i) = 1$, $2 \le i \le 2s$; and

(iii). for any w_i, $2 \le w_i \le t$, choose any negative edge incident to w_i and change it to a positive one.

After Step (i), if G' has an even number of edges, then each vertex in G' has weight 0; if G' has an odd number of edges, then $f_2(u_1) = 2$, and all other vertices have weight 0. Next, after Step (ii), all vertices in $N_G(w_1)$ have weight 1, and $f_2(w_1) \ge 2s - 2 \ge 2$, and all vertices in $W - \{w_1\}$ have weight 0. Finally, after Step (iii), all vertices in $W - \{w_1\}$ have weight 2, and others do not decrease.

Hence $f_2 \in \mathcal{F}^0_{sed}(G)$. Recall that in this case $n - t - 2s = 0$, we have

$$\gamma'_s(G) \le f_2(G) = 0 + 2s + 2(t-1) = 2s + 2t - 2 = n + t - 2 = n - 2 + v_{even}(G)$$

when $|E(G')|$ is even, and

$$\gamma'_s(G) \le f_2(G) = 1 + (2s-2) + 2(t-1) = 2s + 2t - 3 = n + t - 3 = n - 3 + v_{even}(G)$$

when $|E(G')|$ is odd.

Case 1.3. $n - t - 2s = 0$ and $2s = 2$.

In this case, if n is odd, then $G = K_{2,n-2}$. Then, if $n \ge 5$, there exists $f_3 \in \mathcal{F}^0_{sed}(G)$ such that $f_3(G) = \gamma'_s(G) = \gamma'_s(K_{2,n-2}) = \min\{6, \max\{4, n-1\}\}$. Hence there exists $f_3 \in \mathcal{F}^0_{sed}(G)$ such that

$$\gamma'_s(G) = f_3(G) = \begin{cases} 2, & \text{if } n = 3, \\ 4, & \text{if } n = 5, \\ 6, & \text{if } n \ge 7. \end{cases}$$

Therefore $\gamma'_s(G) = f_3(G) \le n - 2 + v_{even}(G)$ holds.

If n is even, then $G = K_{2,n-2} + u_1u_2$. There exists $f \in \mathcal{F}^0_{sed}(K_{2,n-2})$ such that $f(K_{2,n-2}) = \gamma'_s(K_{2,n-2}) = \min\{4, n-2\}$, $n \geq 4$. Now we extend f by assigning $+1$ to u_1u_2, thus obtain $f_4 \in \mathcal{F}^0_{sed}(G)$ such that $f_4(G) = \gamma'_s(K_{2,n-2}) + 1 = \min\{4, n-2\} + 1$. That is, there exists $f_4 \in \mathcal{F}^0_{sed}(G)$ such that

$$\gamma'_s(G) \leq f_4(G) = \begin{cases} 3, & \text{if } n = 4, \\ 5, & \text{if } n \geq 6. \end{cases}$$

Therefore $\gamma'_s(G) \leq f_4(G) \leq n - 2 + v_{even}(G)$ holds.

Claim 2: If W is not an independent set, then there is an $f \in \mathcal{F}^0_{sed}(G)$ such that $f(G) \leq n - 2 + v_{even}(G)$.

In this case, we find a maximal matching M in $G[W]$. Assume that $|M| = p$ and consider the graph $G'' = G - M$, with n vertices and $v_{even}(G'') = t - 2p$.

Case 2.1. $t - 2p \geq 1$.

Since M is maximal in $G[W]$, the $t - 2p$ even vertices in G'' form an independent set. By Claim 1, there is an $f_5 \in \mathcal{F}^0_{sed}(G'')$ such that $f_5(G'') \leq n - 2 + v_{even}(G'')$. We now extend f_5 by adding M to G'' and letting each edge in M be a positive edge. Thus we obtain $f'_5 \in \mathcal{F}^0_{sed}(G)$ such that $\gamma'_s(G) \leq f'_5(G) = f_5(G'') + p \leq n - 2 + t - 2p + p = n - 2 + t - p < n - 2 + v_{even}(G)$.

Case 2.2. $t - 2p = 0$, i.e., M is a perfect matching of $G[W]$.

In this subcase, $v_{odd}(G'') = n$. Karami et al. [8] proved that for a graph G with n vertices in which each vertex is of odd degree, there exists $f \in \mathcal{F}^0_{sed}(G)$ such that $\gamma'_s(G) \leq n - 1$. So there is $f_6 \in \mathcal{F}^0_{sed}(G'')$ such that $f_6(G'') \leq n - 1$. We now extend f_6 by adding M to G'' and letting each edge in M be a positive edge. Thus we obtain $f'_6 \in \mathcal{F}^0_{sed}(G)$ such that $\gamma'_s(G) \leq f'_6(G) = f_6(G'') + p \leq n - 1 + p = n - 1 + \frac{v_{even}(G)}{2} \leq n - 2 + v_{even}(G)$.

Thus, Claim 2 holds and the proof is complete. $\square$

Corollary 1 *Conjecture* 1 *holds for the case* $v_{even}(G) = 1$.

Now we prove Theorem 1.

Proof of Theorem 1. From Propositions 1 and 2, we can see that $\gamma'_s(G) \leq n + v_{odd}(G)/2$, and $\gamma'_s(G) \leq n - 2 + v_{even}(G)$ when $v_{even}(G) > 0$.

So when $v_{even}(G) > 0$, we have

$$3\gamma'_s(G) \leq 2(n + v_{odd}(G)/2) + (n - 2 + v_{even}(G)) = 3n + v_{odd}(G) + v_{even}(G) - 2 = 4n - 2,$$

and hence $\gamma'_s(G) \leq (4n - 2)/3$.

When $v_{even}(G) = 0$, i.e., $v_{odd}(G) = n$, it was proved in [8] that $\gamma'_s(G) \leq n - 1$. Hence $\gamma'_s(G) \leq (4n - 2)/3$ also holds. $\square$

5 Conjecture 1 for $v_{even}(G) = 2$

Proposition 3 *For any simple graph G of order n, if $v_{even}(G) = 2$, then there is an $f \in \mathcal{F}^0_{sed}(G)$ such that $f(G) \le n-1$, and thus $\gamma'_s(G) \le n-1$.*

Proof. It can be easily verified that the result holds whenever $n \le 3$. So assume that $n \ge 4$ and the result holds for any graph of order at most $n-1$.

Let G be a simple graph of order n with $v_{even}(G) = 2$, and let w_1, w_2 be the two vertices of even degree.

Claim 1: Proposition 3 holds for G when it is disconnected.

Assume that $G_1, G_2, \ldots, G_k$ are the components of G, where $k \ge 2$. Then $v_{even}(G_i) \le 2$ and $|V(G_i)| \le n-1$ for all $i = 1, 2, \ldots, k$. For any G_i, where $1 \le i \le k$, if $v_{even}(G_i) = 0$, by the proof in [8], there exists $f_i \in \mathcal{F}^0_{sed}(G_i)$ with $f_i(G_i) \le |V(G_i)| - 1$; otherwise, by the assumption above and Proposition 2, there exists $f_i \in \mathcal{F}^0_{sed}(G_i)$ such that $f_i(G_i) \le |V(G_i)| - 1$.
Let f be the mapping $E(G) \to \{1, -1\}$ defined by $f|_{E(G_i)} = f_i$ for all $1 \le i \le k$. It is easy to see that $f \in \mathcal{F}^0_{sed}(G)$ with

$$f(G) = \sum_{i=1}^{k} f(G_i) \le \sum_{i=1}^{k} (|V(G_i)| - 1) = n - k \le n - 2.$$

Thus, Claim 1 holds.

Claim 2: Proposition 3 holds for G when $d(w_i) = 2$ for some $i \in \{1, 2\}$.

Assume that $d(w_1) = 2$. Let $N(w_1) = \{u_1, u_2\}$.

If $u_1u_2 \in E(G)$, then consider the graph $G - u_1u_2 - w_1$. $G - u_1u_2 - w_1$ is a simple graph of order $n-1$ and $v_{even}(G - u_1u_2 - w_1) = 1$. By Proposition 2, there exists $g \in \mathcal{F}^0_{sed}(G - u_1u_2 - w_1)$ such that $g(G - u_1u_2 - w_1) \le (n-1) - 2 + 1 = n - 2$. Then, by Lemma 3 (i), there exists $f \in \mathcal{F}^0_{sed}(G)$ such that $f(G) = g(G - u_1u_2 - w_1) + 1 \le n - 1$. If $u_1u_2 \notin E(G)$, then similarly, by applying Proposition 2 and Lemma 3 (ii), we can show that there exists $f \in \mathcal{F}^0_{sed}(G)$ with $f(G) \le n-1$.

Thus, Claim 2 holds.

According to Claims 1 and 2, in the following, we may assume that G is connected and $d(w_i) \ge 4$ for $i = 1, 2$.

Let $N_0 = N_G(w_1) \cap N_G(w_2)$, $N_1 = N_G(w_1) - N_G(w_2) - \{w_2\}$, $N_2 = N_G(w_2) - N_G(w_1) - \{w_1\}$, and $N_3 = V(G) - (N_0 \cup N_1 \cup N_2 \cup \{w_1, w_2\})$. Set $n_i = |N_i|$ for $0 \le i \le 3$. Then $n_0 + n_1 + n_2 + n_3 = v_{odd}(G) = n - 2$.

Case 1. $w_1w_2 \notin E(G)$.

We divide this case into two subcases, depending on whether w_1 and w_2 share a neighbour or not, i.e., $n_0 \geq 1$ or $n_0 = 0$.

Case 1.1. $w_1w_2 \notin E(G)$ and $n_0 \geq 1$.

Consider the graph $G' = G - w_1$. Note that $v_{odd}(G') = n_2 + n_3$, and so G' can be decomposed into $(n_2 + n_3)/2$ trails $\{T_1, T_2, \ldots, T_{(n_2+n_3)/2}\}$, whose endpoints corresponds to all odd vertices of G'.

If T_i has odd length, we assign $+1$ and -1 alternatively to the edges of T_i, starting and ending with $+1$; this weight assignment in an odd trail is called a *proper assignment*. When T_i has even length t, there are exactly $\frac{t}{2}$ vertices on T_i, each of which can naturally divide T_i into two subtrails with odd length. We call these $\frac{t}{2}$ vertices *good*. For each T_i with even length, choose a good vertex u_i of T_i. We can assign $+1$ and -1 alternatively to edges in the two subtrails of T_i divided by u_i such that both starting and ending edges in each subtrail are assigned $+1$. This weight assignment of edges in an even trail T_i is called a *proper assignment with respect to* u_i. Let $\mathbb{T}_1 = \{T_1, T_2, \ldots, T_{(n_2+n_3)/2}\}$.

Claim 3: In Case 1.1, Proposition 3 holds for G when there is at least one trail of odd length in $\mathbb{T}_1$.

Assume that there is at least one trail of odd length in $\mathbb{T}_1$. For each $T_i \in \mathbb{T}_1$ with even length, let u_i be a good vertex of T_i. We define a function $f_1 : E(G) \to \{1, -1\}$ as follows: each odd trial $T_i \in \mathbb{T}_1$ is equipped with a proper assignment, and each even trail $T_i \in \mathbb{T}_1$ is equipped with a proper assignment with respect to u_i. Then we assign $+1$ to each edge incident to w_1. If the weight of w_2 till now is 0, we choose any negative edge incident to w_2 and change it to a positive one.

Now we have $f_1(w_1) = d(w_1) \geq 4$, $f_1(w_2) \geq 2$, and $f_1(u) \geq 1$ for each $u \in V(G) - \{w_1, w_2\}$. So $f_1 \in \mathcal{F}^0_{sed}(G)$ and hence $\gamma'_s(G) \leq f_1(G) \leq 1 + 2(\frac{n_2+n_3}{2} - 1) + n_1 + n_0 + 2 = n - 1$. Thus Claim 3 holds.

Claim 4: In Case 1.1, if all trails in $\mathbb{T}_1$ have even length, then either w_2 or some vertex $x \in N_0$ is a good vertex of some trail $T_j \in \mathbb{T}_1$.

Assume that all trails in $\mathbb{T}_1$ have even length. Then, some edge w_2x, where $x \in N_0$, must be in some $T_j \in \mathbb{T}_1$. Obviously, either w_2 or x is a good vertex in T_j. Thus Claim 4 holds.

Claim 5: In Case 1.1, Proposition 3 holds for G when all trails in $\mathbb{T}_1$ have even length.

Assume that all trails in $\mathbb{T}_1$ have even length. By Claim 4, either w_2 or some vertex $x \in N_0$ is a good vertex of some trail $T_j \in \mathbb{T}_1$.

If w_2 is good, we define a function $f_2 : E(G) \to \{1, -1\}$ as follows. We equip T_j with the proper assignment with respect to w_2, and for each $T_i \in \mathbb{T}_1 - \{T_j\}$, we equip T_i with a proper assignment (with respect to any good vertex). Then we assign $+1$ to each edge incident to w_1.

Now we have $f_2(w_1) = d(w_1) \geq 4$, $f_2(w_2) \geq 2$, and $f_2(u) \geq 1$ for each $u \in V(G) - \{w_1, w_2\}$. So $f_2 \in \mathcal{F}^0_{sed}(G)$ and hence $\gamma'_s(G) \leq f_2(G) = 2 \cdot \frac{n_2+n_3}{2} + n_1 + n_0 = n - 2$.

If x is good, we define a function $f_3 : E(G) \to \{1, -1\}$ as follows. We equip T_j with the proper assignment with respect to x, and for each $T_i \in \mathbb{T}_1 - \{T_j\}$, we equip T_i with a proper assignment (with respect to any good vertex). Then we assign -1 to w_1x and $+1$ to any other edge incident to w_1. If the weight of w_2 till now is 0, we choose any negative edge incident to w_2 and change it to a positive one.

Now we have $f_3(w_1) = d(w_1) - 2 \geq 2$, $f_3(w_2) \geq 2$, $f_3(x) \geq 2 - 1 = 1$, and $f_3(u) \geq 1$ for each $u \in V(G) - \{w_1, w_2, x\}$. So $f_3 \in \mathcal{F}^0_{sed}(G)$ and hence $\gamma'_s(G) \leq f_3(G) \leq 2 \cdot \frac{n_2+n_3}{2} + (n_1 + n_0 - 2) + 2 = n - 2$. Thus Claim 5 holds.

By Claims 3 and 5, Proposition 3 holds for G in Case 1.1.

Case 1.2. $w_1w_2 \notin E(G)$ and $n_0 = 0$.

Claim 6: Proposition 3 holds for G in Case 1.2.

Choose edges e_1, e_2 incident to w_1 and edges e_3, e_4 incident to w_2.

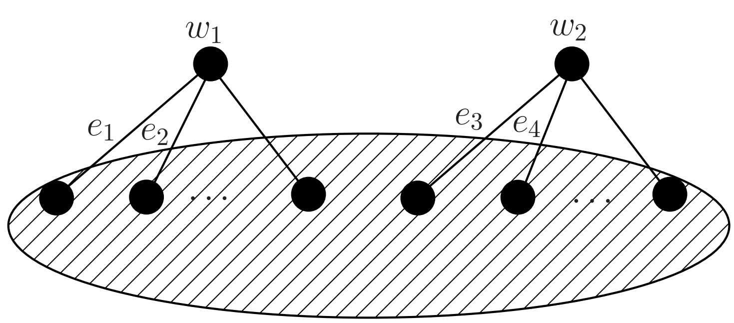


Figure 4: Case 1.2.

As $n_0 = 0$, $N_0 = N(w_1) \cap N(w_2) = \emptyset$. Thus, by the condition that $d(w_i) \geq 4$ for both $i = 1, 2$, we have $n \geq 2 + 4 \cdot 2 = 10$. Let G'' denote the graph $G - \{e_1, e_2, e_3, e_4\}$. Observe that $v_{odd}(G'') = v_{odd}(G) - 4 = n - 6 > 0$. Thus $E(G'')$ can be decomposed into $t = (n-6)/2$ trails, say $T_1, T_2, \ldots, T_t$. We now define the function $f_4 : E(G) \to \{1, -1\}$ as follows:

- $f_4(e_i) = 1$ for $i = 1, 2, 3, 4$; and
- each odd trial T_i is equipped with a proper assignment, and each even trail T_i is

equipped with a proper assignment with respect to some good vertex.

Observe that $f_4(w_i) \geq 2$ for $i = 1, 2$ and $f_4(u) \geq 1$ for all $u \in V(G) - \{w_1, w_2\}$. Thus $f_4 \in \mathcal{F}^0_{sed}(G)$. Also note that $f_4(G) \leq 2t + 4 = 2(n - 6)/2 + 4 = n - 2$. So Claim 6 holds.

Case 2. $w_1w_2 \in E(G)$.

Similarly as in Case 1, we divide this case into two subcases, depending on whether w_1 and w_2 share a neighbour or not.

Case 2.1. $w_1w_2 \in E(G)$ and $n_0 \geq 1$.

Claim 7: Proposition 3 holds for G in Case 2.1.

Consider the graph $G' = G - w_1$. Note that $v_{odd}(G') = n_2 + n_3 + 1$, and so G' can be decomposed into $(n_2+n_3+1)/2$ trails $\{T_1, T_2, \ldots, T_{(n_2+n_3+1)/2}\}$ whose endpoints correspond to all odd vertices of G'. Let $\mathbb{T}_2 = \{T_1, T_2, \ldots, T_{(n_2+n_3+1)/2}\}$.

Case 2.1 is now divided into two subcases.

Case 2.1.1. Some trail in $\mathbb{T}_2$ has an odd length.

We define a function $g_1 : E(G) \to \{1, -1\}$ as follows. Each trail $T_i \in \mathbb{T}_2$ of odd length is equipped with a proper assignment, and each trail $T_i \in \mathbb{T}_2$ of even length is equipped with a proper assignment with respect to some good vertex of T_i. Then we assign $+1$ to each edge incident to w_1.

Now we have $g_1(w_1) = d(w_1) \geq 4$, $g_1(w_2) \geq 2$, and $g_1(u) \geq 1$ for each $u \in V(G) - \{w_1, w_2\}$. So $g_1 \in \mathcal{F}^0_{sed}(G)$ and hence $\gamma'_s(G) \leq g_1(G) \leq 1 + 2(\frac{n_2+n_3+1}{2} - 1) + n_1 + n_0 + 1 = n - 1$.

Case 2.1.2. All trails in $\mathbb{T}_2$ have even length.

Choose any $x \in N_0$ and assume that w_2x is an edge in T_1. Then, either w_2 or x is good in T_1. Let $u_1 = w_2$ if w_2 is good in T_1, and $u_1 = x$ otherwise. For any $i = 2, 3, \ldots, (n_2 + n_3 + 1)/2$, let u_i be any good vertex of T_i.

We define a function $g_2 : E(G) \to \{1, -1\}$ as follows. We first equip each T_i with a proper assignment with respect to u_i. Then, we assign -1 to w_1u_1, and finally, we assign $+1$ to any other edge incident with w_1.

If $u_1 = w_1$, then $g_2(w_1) = d(w_1) - 2 \geq 2$, $g_2(w_2) \geq 2 - 1 = 1$, and $g_2(u) \geq 1$ for each $u \in V(G) - \{w_1, w_2\}$. So $g_2 \in \mathcal{F}^0_{sed}(G)$ and hence $\gamma'_s(G) \leq g_2(G) \leq 2 \cdot \frac{n_2+n_3+1}{2} + n_1 + n_0 - 1 = n - 2$.

If $u_1 = x$, then $g_2(w_1) = d(w_1) - 2 \geq 2$, $g_2(w_2) \geq 1$, $g_2(x) \geq 2 - 1 = 1$, and $g_2(u) \geq 1$ for each $u \in V(G) - \{w_1, w_2, x\}$. So $g_2 \in \mathcal{F}^0_{sed}(G)$ and hence $\gamma'_s(G) \leq g_2(G) \leq 2 \cdot \frac{n_2+n_3+1}{2} + (n_1 + n_0 - 2 + 1) = n - 2$.

Hence Claim 7 holds.

Case 2.2. $w_1w_2 \in E(G)$ and $n_0 = 0$.

Claim 8: Proposition 3 holds for G in Case 2.2.

Choose $x_1 \in N_1$ and $x_2 \in N_2$ and consider the graph $G''' = G - \{e_0, e_1, e_2\}$, where $e_0 = w_1w_2$, $e_1 = w_1x_1$, and $e_2 = w_2x_2$.

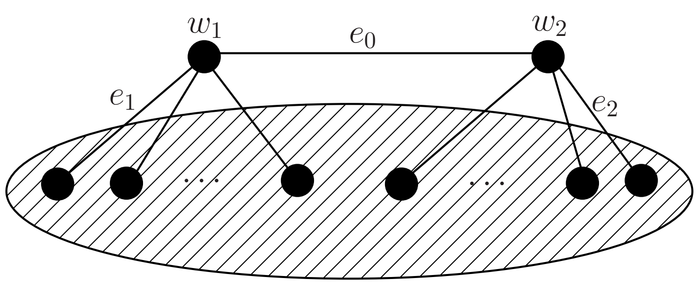


Figure 5: Case 2.2.

As $n_0 = 0$, $N_0 = N(w_1) \cap N(w_2) = \emptyset$. Thus, by the condition that $d(w_i) \geq 4$ for both $i = 1, 2$, we have $n \geq 2+3 \cdot 2 = 8$. Observe that $v_{odd}(G''') = v_{odd}(G) - 2 = n - 4 > 0$. Thus $E(G''')$ can be decomposed into $t = (n-4)/2$ trails, say $T_1, T_2, \ldots, T_t$. We now define a function $g_3 : E(G) \to \{1, -1\}$ as follows:

- $g_3(e_i) = 1$ for $i = 0, 1, 2$; and
- each odd trial T_i is equipped with a proper assignment and each even trail T_i is equipped with a proper assignment with respect to some good vertex.

Observe that $g_3(w_i) \geq 2$ for $i = 1, 2$, and $g_3(u) \geq 1$ for all $u \in V(G) - \{w_1, w_2\}$. Thus $g_3 \in \mathcal{F}^0_{sed}(G)$. Also note that $g_3(G) \leq 2t + 3 = 2(n-4)/2 + 3 = n - 1$, and so Claim 8 holds, which eventually finishes the proof. □

Note that Theorem 2 follows directly from Proposition 2 for the case $v_{even}(G) = 1$ and from Proposition 3 for the case $v_{even}(G) = 2$.

6 Concluding remarks

Karami et al. [8] proved Conjecture 1 for the two cases $v_{odd}(G) = 0$ or n by showing the existence of $f \in \mathcal{F}^0_{sed}(G)$ with $f(G) \leq n - 1$. In the proof of Propositions 1, 2 and 3, all defined members in $\mathcal{F}_{sed}(G)$ also belong to $\mathcal{F}^0_{sed}(G)$. Therefore, we believe Conjecture 1 can be strengthened to the following one.

Conjecture 2 *For any simple graph G of order n, there exists $f \in \mathcal{F}^0_{sed}(G)$ with $f(G) \leq n-1$.*

In 2005, Xu [12] proved the following sharp lower bound of $\gamma'_s(G)$.

Theorem 5 ([12]) *Let G be a graph with n vertices, m edges and $\delta(G) \geq 1$. Then $\gamma'_s(G) \geq n - m$.*

Then Karami et al. [9] characterized all simple connected graphs G for which $\gamma'_s(G) = n - m$. These graphs all have many vertices of degree 1. If we restrict graphs to have higher connectivity or larger minimum degree, a better lower bound can be expected. So we raise the following conjecture.

Conjecture 3 *Let G be a 2-connected graph with n vertices and m edges, and without two adjacent degree 2 vertices. Then $\gamma'_s(G) \geq 2n - m$.*

If the conjecture above is correct, then the lower bound is also sharp. For example, $\gamma'_s(K_4 - e) = 3 = 2n - m$.

Now we show more examples that the bound in Conjecture 3 is reachable. Let G be a 2-connected Hamiltonian graph with $\delta(G) \geq 3$, $V(G) = \{v_1, v_2, \ldots, v_n\}$ and size m. Suppose C is one of its Hamiltonian cycles.

The *triangulation* of a graph H, denoted by $T(H)$, is the graph obtained from H by changing each edge uv of H into a triangle uwv, where w is a new vertex associated with uv. Let $G' = T(G - E(C)) + E(C)$, that is, the graph obtained from $T(G - E(C))$ by adding all the edges in the Hamiltonian cycle C. Then the order of G' is m and the size of G' is $3m - 2n$.

Observe that G' is 2-connected and does not have two adjacent degree 2 vertices. Consider a function $f : E(G') \to \{1, -1\}$, where $f(e) = 1$ if $e \in E(G)$, and $f(e) = -1$ otherwise. Then

$$f(G') = |E(G)| - |E(G') - E(G)| = m - 2(m - n) = 2n - m = 2|V(G')| - |E(G')|.$$

By the definition of f, $f_{G'}(v_i) = 2$ for each $i = 1, 2, \ldots, n$ whereas $f(u) = -2$ for each $u \in V(G') - V(G)$. Thus, for each $e = uv \in E(G)$, we have $f(e) = 1$ and $f(u) = f(v) = 2$, whereas for each $e = uv \in E(G') - E(G)$, we have $f(e) = -1$ and $f(u) + f(v) = 0$. Thus $f \in \mathcal{F}_{sed}(G)$. The graph shown in Figure 6 is an example of G' when $G = K_5$ (edges without a sign in the figure receive sign $+1$).

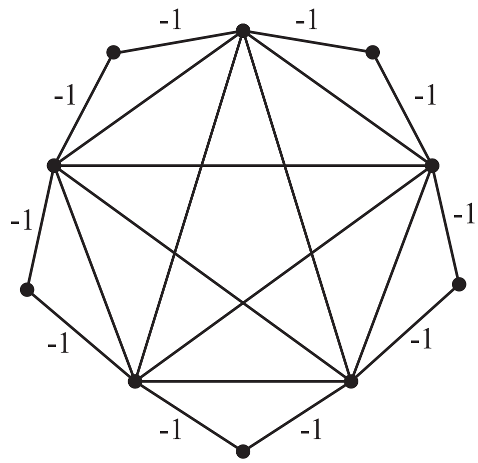


Figure 6: $G' = T(G - E(C)) + E(C)$, where $G = K_5$.

Acknowledgements

This paper is supported by NSFC (Nos. 11701401 and 11971346). The authors would like to thank two anonymous referees for their very helpful comments. This research was conducted during the second and third authors' visit at National Institute of Education, Nanyang Technological University, Singapore. They are grateful to staff in NIE/NTU for their help during their visits.

References

[1] S. Akbari, S. Bolouki, P. Hatami, and M. Siami, On the signed edge domination number of graphs, Discrete Math. 309(3) (2009) 587–594.

[2] S. Akbari, H. Esfandiari, E. Barzegary, and S. Seddighin, Some bounds for the signed edge domination number of a graph, Australas. J. Combin. 58 (2014) 60–66.

[3] W. Chen and E. Song, Lower bounds on several versions of signed domination number, Discrete Math. 308(10) (2008) 1837–1846.

[4] J. Dunbar, S. Hedetniemi, M.A. Henning, and P.J. Slater, Signed domination in graphs, in: Y. Alavi, A. Schwenk (Eds.), Graph Theory, Combinatorics, and Applications, Vol. 1, John Wiley & Sons, Inc., New York, 1995, pp. 311–322.

[5] O. Favaron, Signed domination in regular graphs, Discrete Math. 158(1-3) (1996) 287–293.

[6] Z. Füredi and D. Mubayi, Signed domination in regular graphs and set systems, J. Combin. Theory Ser. B 76(2) (1999) 223–239.

[7] M.A. Henning and P.J. Slater, Open packing in graphs, J. Combin. Math. Combin. Comput. 29 (1999) 3–16.

[8] H. Karami, A. Khodkar, and S.M. Sheikholeslami, An improved upper bound for signed edge domination numbers in graphs, Util. Math. 78 (2009) 121–128.

[9] H. Karami, S. M. Sheikholeslami, and A. Khodkar, Some notes on signed edge domination in graphs, Graphs Combin. 24(1) (2008) 29–35.

[10] J. Matoušek, On the signed domination in graphs, Combinatorica 20(1) (2000) 103–108.

[11] B. Xu, On signed edge domination numbers of graphs, Discrete Math. 239(1-3) (2001) 179–189.

[12] B. Xu, On edge domination numbers of graphs, Discrete Math. 294(3) (2005) 311–316.

[13] B. Xu, Two classes of edge domination in graphs, Discrete Appl. Math. 154(10) (2006) 1541–1546.

[14] B. Xu, On signed cycle domination in graphs, Discrete Math. 309(4) (2009) 1007–1012.

[15] B. Zelinka, Some remarks on domination in cubic graphs, Discrete Math. 158(1-3) (1996) 249–255.