# Decomposability and co-modular indices of tournaments 

Houmem Belkhechine<br>University of Carthage, Bizerte Preparatory Engineering Institute, Bizerte, Tunisia<br>Cherifa Ben Salha<br>University of Carthage, Faculty of Sciences of Bizerte, Bizerte, Tunisia


#### Abstract

Given a tournament $T$, a module of $T$ is a subset $X$ of $V(T)$ such that for $x, y \in X$ and $v \in V(T) \backslash X,(x, v) \in A(T)$ if and only if $(y, v) \in A(T)$. The trivial modules of $T$ are $\varnothing,\{u\}(u \in V(T))$ and $V(T)$. The tournament $T$ is indecomposable if all its modules are trivial; otherwise it is decomposable. The decomposability index of $T$, denoted by $\delta(T)$, is the smallest number of arcs of $T$ that must be reversed to make $T$ indecomposable. The first author conjectured that for $n \geq 5$, we have $\delta(n)=\left\lceil\frac{n+1}{4}\right\rceil$, where $\delta(n)$ is the maximum of $\delta(T)$ over the tournaments $T$ with $n$ vertices. In this paper we prove this conjecture by introducing the co-modular index of a tournament $T$, denoted by $\Delta(T)$, as the largest number of disjoint co-modules of $T$, where a co-module of $T$ is a subset $M$ of $V(T)$ such that $M$ or $V(T) \backslash M$ is a nontrivial module of $T$. We prove that for $n \geq 3$, we have $\Delta(n)=\left\lceil\frac{n+1}{2}\right\rceil$, where $\Delta(n)$ is the maximum of $\Delta(T)$ over the tournaments $T$ with $n$ vertices. Our main result is the following close relationship between the above two indices: for every tournament $T$ with at least 5 vertices, we have $\delta(T)=\left\lceil\frac{\Delta(T)}{2}\right\rceil$. As a consequence, we obtain $\delta(n)=\left\lceil\frac{\Delta(n)}{2}\right\rceil=\left\lceil\frac{n+1}{4}\right\rceil$ for $n \geq 5$, and we answer some further related questions.


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## 1. Introduction and presentation of results

A tournament $T$ consists of a finite set $V(T)$ of vertices together with a set $A(T)$ of ordered pairs of distinct vertices, called arcs, such that for all

[^0]$x \neq y \in V(T),(x, y) \in A(T)$ if and only if $(y, x) \notin A(T)$. Such a tournament is denoted by $(\mathrm{V}(\mathrm{T}), \mathrm{A}(\mathrm{T}))$. The cardinality of $T$, denoted by $v(T)$, is that of $V(T)$. Given a tournament $T$, with each subset $X$ of $V(T)$ is associated the subtournament $T[X]=(X, A(T) \cap(X \times X))$ of $T$ induced by $X$. For $X \subseteq V(T)$ (resp. $\quad x \in V(T)$ ), the subtournament $T[V(T) \backslash X]$ (resp. $T[V(T) \backslash\{x\}])$ is simply denoted by $T-X$ (resp. $T-x$ ). Two tournaments $T$ and $T^{\prime}$ are isomorphic, which is denoted by $T \simeq T^{\prime}$, if there exists an isomorphism from $T$ onto $T^{\prime}$, that is, a bijection $f$ from $V(T)$ onto $V\left(T^{\prime}\right)$ such that for every $x, y \in V(T),(x, y) \in A(T)$ if and only if $(f(x), f(y)) \in A\left(T^{\prime}\right)$. With each tournament $T$ is associated its dual tournament $T^{\star}$ defined by $V\left(T^{\star}\right)=V(T)$ and $A\left(T^{\star}\right)=\{(x, y):(y, x) \in A(T)\}$.

A transitive tournament is a tournament $T$ such that for every $x, y, z \in$ $V(T)$, if $(x, y) \in A(T)$ and $(y, z) \in A(T)$, then $(x, z) \in A(T)$. Let $n$ be a positive integer. We denote by $\underline{n}$ the transitive tournament whose vertex set is $\{0, \ldots, n-1\}$ and whose arcs are the ordered pairs $(i, j)$ such that $0 \leq i<j \leq n-1$. Up to isomorphism, $\underline{n}$ is the unique transitive tournament with $n$ vertices.

The paper is based on the following notions. Given a tournament $T$, a subset $M$ of $V(T)$ is a module [6] (or a clan [2] or an interval [4]) of $T$ provided that for every $x, y \in M$ and for every $v \in V(T) \backslash M,(v, x) \in A(T)$ if and only if $(v, y) \in A(T)$. For example, $\varnothing,\{x\}$, where $x \in V(T)$, and $V(T)$ are modules of $T$, called trivial modules. A tournament is indecomposable [4, 5] (or prime [6] or primitive [2]) if all its modules are trivial; otherwise it is decomposable. Let us consider some examples. To begin, consider the case of small tournaments. The tournaments with at most two vertices are clearly indecomposable. The tournaments $\underline{3}$ and $C_{3}=$ $(\{0,1,2\},\{(0,1),(1,2),(2,0)\})$ are, up to isomorphism, the unique tournaments with three vertices. The tournament $C_{3}$ is indecomposable, whereas $\underline{3}$ is decomposable. Up to isomorphism, the tournaments with four vertices are the four tournaments $\underline{4}, T_{4}=(\{0,1,2,3\},\{(0,1),(1,2),(2,0),(3,0),(3,1),(3,2)\}), T_{4}^{\star}$, and $(\{0,1,2,3\},\{(0,1),(1,2),(2,0),(3,0),(3,1),(2,3)\})$, all of them are decomposable. We now consider the case of transitive tournaments. For every integer $n \geq 3$, the transitive tournament $\underline{n}$ is decomposable. More precisely,
the modules of $\underline{n}$ are the intervals of the usual total order on $V(\underline{n})$.
Lastly, consider the cases of isomorphic tournaments and dual tournaments. Let $T$ and $T^{\prime}$ be isomorphic tournaments. If $f$ is an isomorphism from $T$ onto $T^{\prime}$, then a subset $M$ of $V(T)$ is a module of $T$ if and only if $f(M)$ is a module of $T^{\prime}$. In particular, $T$ is indecomposable if and only if $T^{\prime}$ is. Similarly, a tournament $T$ and its dual share the same modules. In particular, $T$ is indecomposable if and only if $T^{\star}$ is. These remarks justify that in certain proofs, tournaments are considered up to isomorphism and/or duality.

Let $T$ be a tournament. An inversion of an arc $a=(x, y) \in A(T)$ consists of replacing the arc $a$ by $a^{\star}$ in $A(T)$, where $a^{\star}=(y, x)$. The tournament obtained from $T$ after reversing the arc $a$ is denoted by $\operatorname{Inv}(T, a)$ or $\operatorname{Inv}(T,\{x, y\})$. Thus $\operatorname{Inv}(T, a)=\operatorname{Inv}(T,\{x, y\})=\left(V(T),(A(T) \backslash\{a\}) \cup\left\{a^{\star}\right\}\right)$. More generally, for $B \subseteq$
$A(T)$, we denote by $\operatorname{Inv}(T, B)$ the tournament obtained from $T$ after reversing all the arcs of $B$, that is $\operatorname{Inv}(T, B)=\left(V(T),(A(T) \backslash B) \cup B^{\star}\right)$, where $B^{\star}=$ $\left\{b^{\star}: b \in B\right\}$. For example, $T^{\star}=\operatorname{Inv}(T, A(T))$.

Given a tournament $T$ with at least five vertices, the decomposability index of $T$, denoted by $\delta(T)$, was defined by the first author 1] as the least integer $m$ for which there exists $B \subseteq A(T)$ such that $|B|=m$ and $\operatorname{Inv}(T, B)$ is indecomposable. The index $\delta(T)$ is well-defined because, as observed in [1], for every integer $n \geq 5$, there exist indecomposable tournaments with $n$ vertices. Notice that $\delta(T)=$ $\delta\left(T^{\star}\right)$. Similarly, isomorphic tournaments have the same decomposability index. The exact value of the decomposability index of transitive tournaments was found in 1].

Proposition 1.1 ([1]). Given a transitive tournament $T_{n}$ with $n$ vertices, where $n \geq 5$, we have $\delta\left(T_{n}\right)=\left\lceil\frac{n+1}{4}\right\rceil$.

For $n \geq 5$, let $\delta(n)$ be the maximum of $\delta(T)$ over the tournaments $T$ with $n$ vertices. The first author [1] conjectured that $\delta(n)=\left\lceil\frac{n+1}{4}\right\rceil$ and asked some related questions. The original purpose of the paper is to prove or disprove this conjecture. We prove that this conjecture holds by establishing related results involving a new index, called co-modular index. As a consequence, we obtain the following theorem as well as answers for some further questions asked in [1].

Theorem 1.1. For every integer $n \geq 5$, we have $\delta(n)=\left\lceil\frac{n+1}{4}\right\rceil$.
Convention. Given a tournament $T$, for $X \subseteq V(T), V(T) \backslash X$ is denoted by $\bar{X}$.

Given a tournament $T$, a co-module of $T$ is a subset $M$ of $V(T)$ such that $M$ or $\bar{M}$ is a nontrivial module of $T$. For instance,
a tournament $T$ is decomposable if and only if $T$ admits a co-module.
Observe that

$$
\begin{equation*}
\text { neither } \varnothing \text { nor } V(T) \text { is a co-module of some tournament } T \text {. } \tag{1.3}
\end{equation*}
$$

Moreover, given a tournament $T$, contrary to the set of modules of $T$, the set of co-modules of $T$ is closed under complementation.

Given a tournament $T$, a co-modular decomposition of $T$ is a set of pairwise disjoint co-modules of $T$. For instance, a tournament $T$ is decomposable if and only if it admits a nonempty co-modular decomposition. The co-modular index of a tournament $T$, denoted by $\Delta(T)$, is the largest size of a co-modular decomposition of $T$, i.e., the maximum of $\{|D|: D$ is a co-modular decomposition of $T\}$.

Contrary to the co-modular index, which is defined for all tournaments, the decomposability index is not defined for the tournaments with four vertices because, as observed above, these tournaments are all decomposable. Therefore, since the few tournaments with at most three vertices can easily be checked
separately, we only consider tournaments with at least five vertices when the decomposability index is considered. For instance, given a tournament $T$ such that $v(T) \geq 5$,

$$
\begin{equation*}
T \text { is indecomposable if and only if } \Delta(T)=\delta(T)=0 \tag{1.5}
\end{equation*}
$$

Notice that $\Delta(T)$ is never equal to 1. Indeed, by (1.2) and (1.4),

$$
\begin{equation*}
\text { a tournament } T \text { is decomposable if and only if } \Delta(T) \geq 2 \text {. } \tag{1.6}
\end{equation*}
$$

Given a tournament $T$, a $\Delta$-decomposition of $T$ is a co-modular decomposition $D$ of $T$ which is of maximum size, i.e., such that $|D|=\Delta(T)$. As observed for the decomposability index, for every isomorphic tournaments $T$ and $T^{\prime}$, we have $\Delta(T)=\Delta\left(T^{\prime}\right)=\Delta\left(T^{\star}\right)$. The next result is the analogue of Proposition 1.1 for co-modular index.

Proposition 1.2. Given a transitive tournament $T_{n}$ with $n$ vertices, where $n \geq 3$, we have $\Delta\left(T_{n}\right)=\left\lceil\frac{n+1}{2}\right\rceil$.
Proof. Up to isomorphism, we may assume $T_{n}=\underline{n}$. Let us consider the comodular decomposition $D_{n}$ of $T_{n}$ defined as follows (see (1.1)).

$$
D_{n}=\{\{0\},\{n-1\}\} \cup\left\{\{2 i-1,2 i\}: 1 \leq i \leq\left\lfloor\frac{n-2}{2}\right\rfloor\right\} .
$$

Clearly $\left|D_{n}\right|=\left\lceil\frac{n+1}{2}\right\rceil$ and $D_{n}$ is a co-modular decomposition of $T_{n}$. Thus $\Delta\left(T_{n}\right) \geq$ $\left\lceil\frac{n+1}{2}\right\rceil$. Let $D_{n}^{\prime}$ be another co-modular decomposition of $T_{n}$. Since 0 and $n-1$ are the unique vertices $x$ of $T_{n}$ such that $\{x\}$ is a co-module of $T_{n}$, then $D_{n}^{\prime}$ contains at most two singletons. Therefore, for every $M \in D_{n}^{\prime} \backslash\{\{0\},\{n-1\}\}$, we have $|M| \geq 2$ (see (1.3)). It follows that $n \geq 2\left|D_{n}^{\prime}\right|-2$, i.e., $\left|D_{n}^{\prime}\right| \leq \frac{n+2}{2}$. Thus, $\left|D_{n}^{\prime}\right| \leq$ $\left\lfloor\frac{n+2}{2}\right\rfloor=\left\lceil\frac{n+1}{2}\right\rceil$ so that $\Delta\left(T_{n}\right) \leq\left\lceil\frac{n+1}{2}\right\rceil$. We conclude that $\Delta\left(T_{n}\right)=\left\lceil\frac{n+1}{2}\right\rceil$.

Now, for $n \geq 3$, let $\Delta(n)$ be the maximum of $\Delta(T)$ over the tournaments $T$ with $n$ vertices. The analogue of Theorem 1.1 for co-modular index (see Theorem (1.3) is a consequence of Proposition 1.2 and the following theorem due to Erdős et al. [3].

Notation 1.1. Given a tournament $T$, the set of the modules of $T$ is denoted by $\mathcal{M}(T)$.
Theorem 1.2 ([3]). Given a non-transitive tournament $T$, there exists a transitive tournament $T^{\prime}$ such that $V\left(T^{\prime}\right)=V(T)$ and $\mathcal{M}(T) \mp \mathcal{M}\left(T^{\prime}\right)$.
Theorem 1.3. For every integer $n \geq 3$, we have $\Delta(n)=\left\lceil\frac{n+1}{2}\right\rceil$.
Proof. Let $n \geq 3$. By Proposition 1.2, it suffices to prove that $\Delta(n) \leq\left\lceil\frac{n+1}{2}\right\rceil$. Let $T$ be a tournament with $n$ vertices. By Theorem 1.2, there exists a transitive tournament $T_{n}$ such that $V\left(T_{n}\right)=V(T)$ and $\mathcal{M}(T) \subseteq \mathcal{M}\left(T_{n}\right)$. Thus, every co-modular decomposition of $T$ is also a co-modular decomposition of $T_{n}$. It follows that $\Delta(T) \leq \Delta\left(T_{n}\right)$. Since $\Delta\left(T_{n}\right)=\left\lceil\frac{n+1}{2}\right\rceil$ by Proposition 1.2, we obtain $\Delta(n) \leq\left\lceil\frac{n+1}{2}\right\rceil$ as desired.

We will now see how the co-modular index is closely related to the decomposability one.

Notation 1.2. Let $T$ be a tournament. For an $\operatorname{arc} a=(x, y) \in A(T)$, the vertex set $\{x, y\}$ is denoted by $\mathcal{V}(a)$. Similarly, for an arc set $B \subseteq A(T)$, the vertex set $\bigcup_{b \in B} \mathcal{V}(b)$ is denoted by $\mathcal{V}(B)$.

Let $T$ be a (decomposable) tournament. Let $B \subseteq A(T)$ and let $D$ be a co-modular decomposition of $T$. If $\mathcal{V}(B) \cap M=\varnothing$ for some $M \in D$, then $M$ is still a co-module of $\operatorname{Inv}(T, B)$, and in particular $\operatorname{Inv}(T, B)$ is still decomposable. Therefore, if $\operatorname{Inv}(T, B)$ is indecomposable, then $\mathcal{V}(B) \cap M \neq \varnothing$ for every $M \in D$, so that $|\mathcal{V}(B)| \geq|D|$ and thus $|B| \geq \frac{|\mathcal{V}(B)|}{2} \geq \frac{|D|}{2}$. It follows that when $v(T) \geq 5$, we have $\delta(T) \geq \frac{|D|}{2}$ and thus $\delta(T) \geq\left\lceil\frac{|D|}{2}\right\rceil$. We have shown that

$$
\begin{equation*}
\delta(T) \geq\left\lceil\frac{\Delta(T)}{2}\right\rceil \text { for every tournament } T \text { with at least five vertices. } \tag{1.7}
\end{equation*}
$$

The most important result of the paper is certainly that equality holds in (1.7).
Theorem 1.4. For every tournament $T$ with at least five vertices, we have $\delta(T)=\left\lceil\frac{\Delta(T)}{2}\right\rceil$.

We will now see how the main problems posed in [1] follow from Theorem 1.4 We begin by Theorem 1.1, which is an immediate consequence of Theorems 1.4 and 1.3

Proof of Theorem 1.1. Let $n \geq 5$. By Theorem 1.4] $\delta(n)=\left\lceil\frac{\Delta(n)}{2}\right\rceil$. By Theo$\operatorname{rem}$ 1.3, $\Delta(n)=\left\lceil\frac{n+1}{2}\right\rceil$. It follows that $\delta(n)=\left\lceil\frac{\left\lceil\frac{n+1}{2}\right\rceil}{2}\right\rceil=\left\lceil\frac{n+1}{4}\right\rceil$.

The next result, also conjectured in [1], is another consequence of Theorem 1.4 .

Corollary 1.1. Given two tournaments $T$ and $T^{\prime}$ such that $V(T)=V\left(T^{\prime}\right)$ and $v(T) \geq 5$, if $\mathcal{M}(T) \subseteq \mathcal{M}\left(T^{\prime}\right)$, then $\delta(T) \leq \delta\left(T^{\prime}\right)$.

Proof. Suppose $\mathcal{M}(T) \subseteq \mathcal{M}\left(T^{\prime}\right)$. Since $V(T)=V\left(T^{\prime}\right)$ and $\mathcal{M}(T) \subseteq \mathcal{M}\left(T^{\prime}\right)$, every co-modular decomposition of $T$ is also a co-modular decomposition of $T^{\prime}$. It follows that $\Delta(T) \leq \Delta\left(T^{\prime}\right)$. Thus $\delta(T) \leq \delta\left(T^{\prime}\right)$ by Theorem 1.4 ,

Notice that Theorem 1.1 is also an immediate consequence of Corollary 1.1 Theorem 1.2, and Proposition 1.1.

Another application of Theorem 1.4 is about upward hereditary properties of the decomposability index based on the following question asked in [1]. For which values of the positive integer $k$ does the following property $\left(P_{k}\right)$ hold?
$\left(P_{k}\right)$ For every tournament $T$ such that $v(T) \geq 5+k$, there exists a subset $X$ of $V(T)$ such that $|X|=k$ and $\delta(T) \leq \delta(T-X)+1$.

As observed in 1], Property $\left(P_{k}\right)$ is false for $k \geq 5$. However, the trueness of Property $\left(P_{k}\right)$ has been proved for $k \in\{1,2,3\}$, while Property $\left(P_{4}\right)$ has been conjectured because it implies Theorem 1.1 (see [1]). Property $\left(P_{4}\right)$ is a consequence of Theorem 1.4 In fact, for each $k \in\{1,2,3,4\}$, Property $\left(P_{k}\right)$ is a consequence of Theorem 1.4. More precisely, let us consider the analogues $\left(Q_{k}\right)$ of Properties $\left(P_{k}\right)$ for the co-modular index: for which values of the positive integer $k$ does the following property $\left(Q_{k}\right)$ hold?
$\left(Q_{k}\right)$ For every tournament $T$ such that $v(T) \geq 3+k$, there exists a subset $X$ of $V(T)$ such that $|X|=k$ and $\Delta(T) \leq \Delta(T-X)+2$.

By Theorem 1.4 $\left(Q_{k}\right)$ implies $\left(P_{k}\right)$. Since $\left(P_{k}\right)$ is false for $k \geq 5,\left(Q_{k}\right)$ is also false for $k \geq 5$. By using Theorem 1.4, we prove that for every $k \in\{1,2,3,4\}$, Property $\left(Q_{k}\right)$, and thus Property $\left(P_{k}\right)$, holds. We obtain the following theorem.

Theorem 1.5. For every integer $k \in\{1,2,3,4\}$, the following two assertions are satisfied.

1. For every tournament $T$ such that $v(T) \geq 3+k$, there exists a subset $X$ of $V(T)$ such that $|X|=k$ and $\Delta(T) \leq \Delta(T-X)+2$.
2. For every tournament $T$ such that $v(T) \geq 5+k$, there exists a subset $X$ of $V(T)$ such that $|X|=k$ and $\delta(T) \leq \delta(T-X)+1$.

Proof. Let $k \in\{1,2,3,4\}$. As observed above, by Theorem 1.4, the first assertion implies the second one. Therefore, we only have to prove the first assertion. Assertion 1 clearly holds when $\Delta(T) \leq 2$. Let $T$ be a tournament such that $v(T) \geq 3+k$.

To begin, suppose $\Delta(T)=3$ or 4 . Since $T$ is decomposable, $T$ admits a nontrivial module $M$. Let $x, y$ be distinct elements of $M$, let $z$ be an element of $\bar{M}$, and let $X$ be a subset of $V(T) \backslash\{x, y, z\}$ such that $|X|=k$. Since $M \backslash X=M \cap \bar{X}$ is a nontrivial module of $T[\bar{X}]=T-X$ (see Assertion 1 of Proposition 2.1), the tournament $T-X$ is decomposable, i.e., $\Delta(T-X) \geq 2$ (see (1.6)). Thus $\Delta(T) \leq \Delta(T-X)+2$ as desired.

To finish, suppose $\Delta(T) \geq 5$. Let $D$ be a $\Delta$-decomposition of $T$. There exist two distinct elements $M$ and $N$ of $D$ such that $|M| \geq 2$ and $|N| \geq 2$ (see (1.3) and Assertion 1 of Lemma 2.2). Let $X$ be a subset of $M \cup N$ such that $|X|=k$. Since $|D|=\Delta(T) \geq 5$, the elements of $D \backslash\{M, N\}$ are co-modules of $T-X$ (see Assertion 1 of Proposition 2.1). Thus, $D \backslash\{M, N\}$ is a co-modular decomposition of $T-X$. Since $|D \backslash\{M, N\}|=\Delta(T)-2$ because $|D|=\Delta(T)$, it follows that $\Delta(T) \leq \Delta(T-X)+2$, which completes the proof.

We end this section by showing how Theorem 1.4 results from the following propositions.

Proposition 1.3. Given a tournament $T$ with at least five vertices, $\delta(T)=1$ if and only if $\Delta(T)=2$.

Proposition 1.4. Given a tournament $T$ with at least five vertices, if $\Delta(T)=3$, then $\delta(T)=2$.

Proposition 1.5. Given a tournament $T$ such that $\Delta(T) \geq 4$, there exists an arc $a \in A(T)$ such that $\Delta(\operatorname{Inv}(T, a))=\Delta(T)-2$.

Proof of Theorem 1.4. Let $T$ be a tournament such that $v(T) \geq 5$. We proceed by induction on $\Delta(T)$. By (1.5), (1.6), and Propositions 1.3 and 1.4 the theorem holds when $\Delta(T) \leq 3$. Suppose $\Delta(T) \geq 4$. Since $\delta(T) \geq\left\lceil\frac{\Delta(T)}{2}\right\rceil$ (see (1.7)), it suffices to show that $\delta(T) \leq\left\lceil\frac{\Delta(T)}{2}\right\rceil$. By Proposition 1.5, there exists $a \in A(T)$ such that $\Delta(\operatorname{Inv}(T, a))=\Delta(T)-2$. By the induction hypothesis, $\delta(\operatorname{Inv}(T, a))=\left\lceil\frac{\Delta(\operatorname{Inv}(T, a))}{2}\right\rceil=\left\lceil\frac{\Delta(T)-2}{2}\right\rceil$. Since $\delta(T) \leq \delta(\operatorname{Inv}(T, a))+1$, we obtain $\delta(T) \leq\left\lceil\frac{\Delta(T)-2}{2}\right\rceil+1=\left\lceil\frac{\Delta(T)}{2}\right\rceil$, as desired.

The rest of the paper aims to prove Propositions 1.3, 1.4 and 1.5. It is organized as follows. The next three sections contain the main preliminary results. Section 2 contains the basic properties of modules and co-modules. Section 3 contains a structural study of minimal co-modules. In Section 4 we review some useful results about $\delta$-decompositions, i.e., $\Delta$-decompositions in which every element is a minimal co-module. Section 5 is divided into two subsections. We prove Proposition 1.3 in Subsection 5.1. Propositions 1.4 and 1.5 in Subsection 5.2. The way of algorithmic considerations is left open.

## 2. Modules and co-modules

To manipulate modules of tournaments, it is convenient to introduce the following notations. Let $T$ be a tournament. For every distinct vertices $x, y \in$ $V(T)$, we set

$$
T(x, y)= \begin{cases}1 & \text { if } \quad(x, y) \notin A(T) \\ 0 & \text { if } \quad(x, y) \notin A(T)\end{cases}
$$

Let $X$ and $Y$ be two disjoint subsets of $V(T)$. The notation $X \equiv_{T} Y$ signifies that $T(x, y)=T\left(x^{\prime}, y^{\prime}\right)$ for every $x, x^{\prime} \in X$ and $y, y^{\prime} \in Y$. For more precision when $X \equiv_{T} Y$, we write $T(X, Y)=1$ (resp. $T(X, Y)=0$ ) to indicate that for every $x \in X$ and $y \in Y$, we have $T(x, y)=1$ (resp. $T(x, y)=0$ ). When $X$ is a singleton $\{x\}$, we write $x \equiv_{T} Y$ for $\{x\} \equiv_{T} Y, T(x, Y)$ for $T(\{x\}, Y)$, and $T(Y, x)$ for $T(Y,\{x\})$. For instance, given a subset $M$ of $V(T), M$ is a module of $T$ if and only if for every $x \in \bar{M}$, we have $x \equiv_{T} M$, or, equivalently, if and only if $T(x, u)=T(x, v)$ for every $x \in \bar{M}$ and $u, v \in M$.

We now review some useful properties of the modules of a tournament. We begin by the following properties which resemble those of the intervals in a total order.

Proposition 2.1. Let $T$ be a tournament.

1. Given a subset $W$ of $V(T)$, if $M$ is a module of $T$, then $M \cap W$ is a module of $T[W]$.
2. Given a module $M$ of $T$, if $N$ is a module of $T[M]$, then $N$ is also a module of $T$.
3. If $M$ and $N$ are modules of $T$, then $M \cap N$ is also a module of $T$.
4. If $M$ and $N$ are modules of $T$ such that $M \cap N \neq \varnothing$, then $M \cup N$ is also a module of $T$.
5. If $M$ and $N$ are modules of $T$ such that $M \backslash N \neq \varnothing$, then $N \backslash M$ is also $a$ module of $T$.
6. If $M$ and $N$ are disjoint modules of $T$, then $M \equiv_{T} N$.

Now we examine the modules of a tournament $T$ together with those of a tournament $T^{\prime}$ obtained from $T$ by reversing an arc. We say that two sets $E$ and $F$ overlap if $E \cap F \neq \varnothing, E \backslash F \neq \varnothing$ and $F \backslash E \neq \varnothing$.

Lemma 2.1. Given a tournament $T$, consider an arc $a \in A(T)$ and let $T^{\prime}=$ $\operatorname{Inv}(T, a)$.

1. Given a module $M$ of $T, M$ is a module of $T^{\prime}$ if and only if $M$ and $\mathcal{V}(a)$ do not overlap.
2. If $M$ is a module of $T$ and $M^{\prime}$ is a module of $T^{\prime}$, then $M \cap M^{\prime}$ is a module of $T$ or of $T^{\prime}$.
3. Given a module $M$ of $T$ and a module $M^{\prime}$ of $T^{\prime}$ such that $M \cap M^{\prime} \neq \varnothing$, if $M \cup M^{\prime}$ and $\mathcal{V}(a)$ do not overlap, then $M \cup M^{\prime}$ is a module of both $T$ and $T^{\prime}$.

Proof. The first assertion is obvious because for every distinct $x, y \in V(T)$, $T^{\prime}(x, y)=T(x, y)$ if and only if $\{x, y\} \neq \mathcal{V}(a)$. Let $M$ be a module of $T$, and let $M^{\prime}$ be a module of $T^{\prime}$.

To prove the second assertion, we first suppose that $M$ and $\mathcal{V}(a)$ do not overlap. By the first assertion, $M$ is also a module of $T^{\prime}$. Therefore, $M \cap M^{\prime}$ is a module of $T^{\prime}$ by Assertion 3 of Proposition 2.1. Now suppose that $M$ and $\mathcal{V}(a)$ overlap. In this instance, $T^{\prime}[M]=T[M]$. Therefore, $M \cap M^{\prime}$ is a module of $T[M]$ because $M \cap M^{\prime}$ is a module of $T^{\prime}[M]$ by Assertion 1 of Proposition 2.1. By Assertion 2 of Proposition [2.1, $M \cap M^{\prime}$ is also a module of $T$.

For the third assertion, suppose that $M \cap M^{\prime} \neq \varnothing$ and that $\mathcal{V}(a)$ do not overlap $M \cup M^{\prime}$. Since $T^{\prime}=\operatorname{Inv}(T, \mathcal{V}(a))$ and $T=\operatorname{Inv}\left(T^{\prime}, \mathcal{V}(a)\right)$, we may interchange $T$ and $T^{\prime}$ so that it suffices to show that $M \cup M^{\prime}$ is a module of $T$. If $M^{\prime}$ and $\mathcal{V}(a)$ do not overlap, then since $M^{\prime}$ is also a module of $T$ by the first assertion of the lemma, $M \cup M^{\prime}$ is a module of $T$ by Assertion 4 of Proposition 2.1. Suppose that $M^{\prime}$ and $\mathcal{V}(a)$ overlap. Since $M \cup M^{\prime}$ and $\mathcal{V}(a)$ do not overlap, then $\mathcal{V}(a) \subseteq M \cup M^{\prime}$. Let $v \in \overline{M \cup M^{\prime}}$.

$$
\begin{equation*}
\text { For every } u \in M \cup M^{\prime}, \text { we have } T(v, u)=T^{\prime}(v, u) \tag{2.1}
\end{equation*}
$$

because $\{u, v\} \neq \mathcal{V}(a)$. Let $x, y \in M \cup M^{\prime}$. Fix $z \in M \cap M^{\prime}$. Since $M$ and $M^{\prime}$ are modules of $T$ and $T^{\prime}$ respectively, it follows from (2.1) that $T(v, x)=T(v, z)$ and $T(v, y)=T(v, z)$. Thus $T(v, x)=T(v, y)$. Therefore, $M \cup M^{\prime}$ is a module of $T$.

We now review some useful properties of co-modules and co-modular decompositions.

Lemma 2.2. Given a decomposable tournament $T$, consider a co-modular decomposition $D$ of $T$. The following assertions are satisfied.

1. The tournament $T$ admits at most two singletons which are co-modules of $T$. In particular, $D$ contains at most two singletons.
2. If $D$ contains an element $M$ which is not a module of $T$, then the elements of $D \backslash\{M\}$ are nontrivial modules of $T$.
3. If $D$ is a $\Delta$-decomposition of $T$, then $\cup D$ is never included in a co-module of $T$.
4. If $D$ is a $\Delta$-decomposition of $T$ and $v(T) \geq 4$, then $D$ contains a nontrivial module of $T$.

Proof. For the first assertion, suppose there are distinct $x, y \in V(T)$ such that $\{x\}$ and $\{y\}$ are co-modules of $T$, i.e., $\overline{\{x\}}$ and $\overline{\{y\}}$ are nontrivial modules of $T$. Let $z \in V(T) \backslash\{x, y\}$. We have $T(x, z)=T(x, y)$ and $T(y, x)=T(y, z)$. Thus $T(x, z) \neq T(y, z)$. Therefore, $\{z\}$ is a not a co-module of $T$. Since the elements of $D$ are co-modules of $T$, it follows that $D$ contains at most two singletons.

For the second assertion, suppose that $D$ contains an element $M$ which is not a module of $T$. Let $N$ be an element of $D \backslash\{M\}$. We have to prove that $N$ is a nontrivial module of $T$. Suppose not. Since $\bar{M}$ and $\bar{N}$ are modules of $T$ and $\bar{M} \backslash \bar{N}=N \neq \varnothing$, then $\bar{N} \backslash \bar{M}=M$ is a module of $T$ by Assertion 5 of Proposition 2.1, a contradiction. Thus, $N$ is a nontrivial module of $T$.

The third assertion holds because if $\cup D$ is included in a co-module $M$ of $T$, then $D \cup\{\bar{M}\}$ is a co-modular decomposition of $T$ so that $D$ is not a $\Delta$ decomposition of $T$.

To prove the fourth assertion, suppose that $D$ is a $\Delta$-decomposition of $T$, and that $v(T) \geq 4$. Recall that $|D|=\Delta(T) \geq 2$ because $T$ is decomposable (see (1.6)). Suppose for a contradiction that $D$ does not contain a nontrivial module of $T$. By the second assertion of the lemma, all the elements of $D$ are trivial modules of $T$. Since the elements of $D$ are co-modules of $T$, it follows that they are singletons (see (1.3)). By the first assertion of the lemma, $D=\{\{x\},\{y\}\}$ for some distinct $x, y \in V(T)$. In particular $\Delta(T)=2$. Since $\overline{\{x\}}$ and $\overline{\{y\}}$ are modules of $T$, then by Assertion 3 of Proposition 2.1, $\overline{\{x\}} \cap \overline{\{y\}}=\overline{\{x, y\}}$ is a module of $T$. Moreover the module $\overline{\{x, y\}}$ of $T$ is nontrivial because $v(T) \geq 4$. Therefore, $\{x, y\}$ is a co-module of $T$, which contradicts the third assertion of the lemma. Thus, $D$ contains a nontrivial module of $T$.

## 3. Minimal co-modules

Let $T$ be a tournament. A minimal co-module of $T$ is a co-module $M$ of $T$ which is minimal in the set of co-modules of $T$ ordered by inclusion, i.e., such that $M$ does not contain any other co-module of $T$.

Notation 3.1. Given a tournament $T$, the set of minimal co-modules of $T$ is denoted by $\mathrm{mc}(T)$.

For example, in the case of transitive tournaments, for every integer $n \geq 3$, we have (see (1.1))

$$
\begin{equation*}
\operatorname{mc}(\underline{n})=\{\{0\},\{n-1\}\} \cup\{\{i, i+1\}: 1 \leq i \leq n-3\} . \tag{3.1}
\end{equation*}
$$

The following remark is the analogue of Assertion 1 of Lemma 2.1 for minimal co-modules.

Remark 3.1. Let $T$ be a tournament, let $a \in A(T)$, and let $T^{\prime}=\operatorname{Inv}(T, a)$. Given $M \subseteq V(T) \backslash \mathcal{V}(a), M \in \operatorname{mc}(T)$ if and only if $M \in \operatorname{mc}\left(T^{\prime}\right)$.

Similarly, a minimal nontrivial module of a tournament $T$ is a nontrivial module of $T$ which is minimal in the set of nontrivial modules of $T$ ordered by inclusion.

Remark 3.2. Given a nontrivial module $M$ of a tournament $T$, if $M$ is a minimal co-module of $T$, then $M$ is a minimal nontrivial module of $T$.

The next remark is an immediate consequence of Assertion 2 of Proposition 2.1 .

Remark 3.3. If $M$ is a minimal nontrivial module of a tournament $T$, then $T[M]$ is indecomposable.

Given a tournament $T$, the elements of $\operatorname{mc}(T)$ are clearly pairwise incomparable with respect to inclusion. Therefore, given distinct $M, N \in \operatorname{mc}(T)$, either $M \cap N=\varnothing$ or $M$ and $N$ overlap. To study the overlapping case, we need the following notations.

Notation 3.2. Let $T$ be a tournament, and let $M \in \operatorname{mc}(T)$. The set of the elements $N \in \operatorname{mc}(T)$ that overlap $M$ is denoted by $O_{T}(M)$, i.e., $O_{T}(M)=\{N \in$ $\operatorname{mc}(T): N$ overlaps $M\}$. Moreover, we set $o_{T}(M)=\left|O_{T}(M)\right|$.

For example, in the case of transitive tournaments, we obtain the following fact.

Fact 3.1. Consider the transitive tournament $\underline{n}$, where $n \geq 3$. We have $\operatorname{mc}(\underline{n})=$ $\{\{0\},\{n-1\}\} \cup\{\{i, i+1\}: 1 \leq i \leq n-3\}$ (see (3.1)). Moreover, we have the following.

- $o_{\underline{n}}(\{0\})=o_{\underline{n}}(\{n-1\})=0$.
- Suppose $n \geq 5$. We have $o_{\underline{n}}(\{1,2\})=o_{\underline{n}}(\{n-2, n-3\})=1$. More precisely, $O_{\underline{n}}(\{1,2\})=\{2,3\}$ and $\overline{O_{\underline{n}}}(\{n-2, n-3\})=\{n-3, n-4\}$.
- Suppose $n \geq 6$, and let $i \in\{2, \ldots, n-4\}$. We have $o_{\underline{n}}(\{i, i+1\})=2$. More precisely, $O_{\underline{n}}(\{i, i+1\})=\{\{i-1, i\},\{i+1, i+2\}\}$.
The next result (see Lemma 3.1) leads us to distinguish the modules with two vertices as specific modules.

Definition 3.1. A twin of a tournament $T$ is a module of cardinality 2 of $T$.

Lemma 3.1. Let $T$ be a tournament and let $M \in \operatorname{mc}(T)$. We have $o_{T}(M) \leq 2$. Moreover, if $M$ is not a twin of $T$, then $o_{T}(M)=0$.

Proof. To begin, suppose $o_{T}(M) \neq 0$. We will prove that $M$ is a twin of $T$. Let $N \in O_{T}(M)$.

If $M \cup N=V(T)$, then $\bar{M}=N \backslash M$ is a co-module of $T$ (see (1.4)), which contradicts the minimality of the co-module $N$ of $T$ because $M \cap N \neq \varnothing$ (see Notation 3.2). Thus

$$
\begin{equation*}
M \cup N \neq V(T) \tag{3.2}
\end{equation*}
$$

Suppose for a contradiction that $M$ is not a module of $T$. Since $\bar{M}$ is a module of $T, N$ or $\bar{N}$ is a module of $T, \bar{M} \cap \bar{N} \neq \varnothing$ (see (3.2)), and $\bar{M} \cap N \neq \varnothing$ because $N \in O_{T}(M)$, then by Assertion 4 of Proposition 2.1. $\bar{M} \cup \bar{N}$ or $\bar{M} \cup N$ is a module of $T$. Moreover, since $M$ and $N$ overlap, then $2 \leq|\bar{M} \cup \bar{N}| \leq v(T)-1$ and $2 \leq|\bar{M} \cup N| \leq v(T)-1$. Thus, $\bar{M} \cup \bar{N}$ or $\bar{M} \cup N$ is a nontrivial module of $T$. In particular, $M \cap N$ or $M \cap \bar{N}$ is a co-module of $T$, which contradicts the minimality of the co-module $M$ of $T$. Thus, $M$ is a module of $T$. Similarly, $N$ is also a module of $T$. By Assertions 3 and 5 of Proposition 2.1. $M \cap N$ and $M \backslash N$ are modules of $T$. Moreover, each of these modules is trivial because $M \in \operatorname{mc}(T)$. Since $M$ and $N$ overlap, it follows that $|M \backslash N|=|M \cap N|=1$ and thus $|M|=2$. Since $M$ is a module of $T$, then it is a twin of $T$ as desired.

Now we prove that $o_{T}(M) \leq 2$. Suppose not and consider three pairwise distinct elements $I, J, K$ of $O_{T}(M)$. As shown above, $M, I, J$ and $K$ are twins of $T$. By interchanging $I, J$ and $K$, we may assume $|M \cap I \cap J|=1$. Thus, there are four pairwise distinct vertices $x, y, z, t \in V(T)$ such that $M=\{x, y\}$, $I=\{y, z\}$, and $J=\{y, t\}$. By Assertion 4 of Proposition 2.1. $\{x, y, z\},\{x, y, t\}$ and $\{y, z, t\}$ are modules of $T$. Since $\{y, z, t\}$ is a module of $T$, by interchanging $T$ and $T^{\star}$, we may assume $T(x,\{y, z, t\})=1$. Thus, $T(z,\{x, y, t\})=0$ because $T(z, x)=0$ and $\{x, y, t\}$ is a module of $T$. Hence $T(t, z)=1 \neq T(t, x)=0$, a contradiction because $\{x, y, z\}$ is a module of $T$. Thus $o_{T}(M) \leq 2$.

To continue the examination of minimal co-modules, we extend the notion of twin to that of transitive module (see Definition 3.2).

Definition 3.2. Let $T$ be a tournament. A transitive module of $T$ is a module $M$ of $T$ such that the subtournament $T[M]$ is transitive. A transitive module $M$ of $T$ is nontrivial if the module $M$ of $T$ is nontrivial. A transitive component of $T$ is a transitive module of $T$ which is maximal (under inclusion) among the transitive modules of $T$.

For example, the modules with at most two vertices are transitive.
Remark 3.4. Given a tournament $T$, consider two disjoint subsets $M$ and $N$ of $T$ such that the tournaments $T[M]$ and $T[N]$ are transitive. If $M \equiv_{T} N$, then the tournament $T[M \cup N]$ is transitive. In particular, if $M$ and $N$ are transitive modules of $T$, then the tournament $T[M \cup N]$ is transitive.

We need the next two results about transitive modules and transitive components. The following lemma is the analogue of Assertion 4 of Proposition 2.1 for transitive modules.

Lemma 3.2. Given a tournament $T$, if $M$ and $N$ are transitive modules of $T$ such that $M \cap N \neq \varnothing$, then $M \cup N$ is also a transitive module of $T$.

Proof. Let $M$ and $N$ be two transitive modules of $T$ such that $M \cap N \neq \varnothing$. By Assertion 4 of Proposition [2.1, $M \cup N$ is a module of $T$. We have to prove that the tournament $T[M \cup N]$ is transitive. If $M$ and $N$ do not overlap, then we are done because in this instance, $M \cup N=M$ or $N$. Hence suppose that $M$ and $N$ overlap. By Assertion 5 of Proposition 2.1. $N \backslash M$ is a module of $T$. By Assertion 6 of Proposition 2.1, we have $M \equiv_{T} N \backslash M$. Moreover, the module $N \backslash M$ of $T$ is transitive because the module $N$ is. It follows that the tournament $T[M \cup N]=T[M \cup(N \backslash M)]$ is transitive (see Remark (3.4).

Corollary 3.1. Given a tournament $T$, the transitive components of $T$ form a partition of $V(T)$.

Proof. Let $\mathcal{C}(T)$ be the set of the transitive components of the tournament $T$. Let $v \in V(T)$. The singleton $\{v\}$ is obviously a transitive module of $T$. Let $C_{v}$ be the union of all the transitive modules of $T$ containing $v$. By Lemma 3.2, $C_{v}$ is a transitive module of $T$. Thus, clearly $C_{v} \in \mathcal{C}(T)$. It follows that $V(T) \subseteq \cup \mathcal{C}(T)$. Now let $C$ and $C^{\prime}$ be two elements of $\mathcal{C}(T)$. Suppose $C \cap C^{\prime} \neq \varnothing$. Again by Lemma 3.2. $C \cup C^{\prime}$ is a transitive module of $T$. It follows from the maximality of the transitive modules $C$ and $C^{\prime}$ of $T$ that $C=C^{\prime}$. Thus, $\mathcal{C}(T)$ is a partition of $V(T)$.

The following observation is a consequence of Lemmas 3.1 and 3.2 ,
Observation 3.1. Let $T$ be a decomposable tournament and let $M \in \operatorname{mc}(T)$. If $o_{T}(M)=2$, then $M$ is contained in a transitive component $C$ of $T$ such that $|C| \geq 4$.

Proof. Suppose $o_{T}(M)=2$. Let $N$ and $L$ be the two distinct elements of $O_{T}(M)$. By Lemma 3.1 $M, N$, and $L$ are twins of $T$. Let $C$ be the transitive component of $T$ such that $M \subseteq C$. By Lemma3.2, $C$ is the union of the transitive modules of $T$ containing $M$. Moreover, again by Lemma 3.2, $M \cup N \cup L$ is a transitive module of $T$. It follows that $M \cup N \cup L \subseteq C$. To complete the proof, it suffices to verify that $|M \cup N \cup L|=4$. Since $M, N$, and $L$ are pairwise distinct twins of $T[M \cup N \cup L]$ (see Assertion 1 of Proposition 2.1), and since the tournaments with three vertices do not admit three pairwise distinct twins, then $|M \cup N \cup L| \geq 4$, and thus $|M \cup N \cup L|=4$ because $O_{T}(M)=\{N, L\}$.

Now we will see how the minimal co-modules of a tournament $T$ are delimited by the transitive components of $T$, in the sense that an element of $\mathrm{mc}(T)$ never overlaps a transitive component of $T$ (see Lemma 3.3). To introduce a notation indicating the minimal co-modules that are contained in a transitive component (see Notation 3.3), we have to use the following fact.

Fact 3.2. Let $T$ be a tournament with at least three vertices. If $T$ admits a twin $W=\{x, y\}$, then $|\operatorname{mc}(T) \cap\{W,\{x\},\{y\}\}|=1$.

Proof. Suppose that $T$ admits a twin $W=\{x, y\}$. If $W \in \operatorname{mc}(T)$, then by minimality of $W, \operatorname{mc}(T) \cap\{W,\{x\},\{y\}\}=\{W\}$. Hence suppose $W \notin \operatorname{mc}(T)$. In this instance, since $W$ is a co-module of $T$, then $\{x\} \in \operatorname{mc}(T)$ or $\{y\} \in \operatorname{mc}(T)$. By interchanging $x$ and $y$, we may assume $\{x\} \in \operatorname{mc}(T)$. We have to prove that $\{y\} \notin \operatorname{mc}(T)$. Let $z \in V(T) \backslash\{x, y\}$. We have $T(x, y)=T(x, z)$ because $\overline{\{x\}}$ is a module of $T$. Moreover, $T(x, z)=T(y, z)$ because $\{x, y\}$ is a module of $T$. Thus $T(x, y)=T(y, z)$, i.e., $T(y, x) \neq T(y, z)$. Therefore, $\overline{\{y\}}$ is not a module of $T$ and thus $\{y\} \notin \operatorname{mc}(T)$.

Notation 3.3. Let $T$ be a tournament with at least three vertices. Suppose that $T$ admits a transitive component $C$ such that $|C|=n \geq 2$. Let us denote the elements of $C$ by $v_{0}, \ldots, v_{n-1}$, in such a way that $T[C]=\left(C,\left\{\left(v_{i}, v_{j}\right): 0 \leq i<\right.\right.$ $j \leq n-1\}$ ). For every $k \in\{0, \ldots, n-2\}$, the pair $\left\{v_{k}, v_{k+1}\right\}$ is a twin of $T[C]$ (see (1.1)) and thus of $T$ (see Assertion 2 of Proposition 2.1). The unique element of $\mathrm{mc}(T)$ that is contained in $\left\{v_{k}, v_{k+1}\right\}$ (see Fact 3.2) is denoted by $C(k)$.

Example 3.1. Consider the case where $T$ is the transitive tournament $\underline{n}$, where $n \geq 3$. The unique transitive component of $T$ is $C=V(T)=\{0, \ldots, n-1\}$. We have $C(0)=\{0\}, C(n-2)=\{n-1\}$, and for every integer $k$ such that $1 \leq k \leq n-3$, we have $C(k)=\{k, k+1\}$.

Observation 3.2 contains more details about $C(k)$ in the general case.
Observation 3.2. Let $T$ be a tournament with at least four vertices. Suppose that $T$ admits a transitive component $C$ such that $|C|=n \geq 3$. Let us denote the elements of $C$ by $v_{0}, \ldots, v_{n-1}$, in such a way that $T[C]=\left(C,\left\{\left(v_{i}, v_{j}\right): 0 \leq i<\right.\right.$ $j \leq n-1\}$ ). The following assertions are satisfied.

1. For every $k \in\{0, \ldots, n-2\}$, we have

$$
C(k)= \begin{cases}\left\{v_{0}\right\} \text { or }\left\{v_{0}, v_{1}\right\} & \text { if } k=0 \\ \left\{v_{n-1}\right\} \text { or }\left\{v_{n-2}, v_{n-1}\right\} & \text { if } k=n-2 \\ \left\{v_{k}, v_{k+1}\right\} & \text { otherwise }\end{cases}
$$

2. We have $C=\bigcup_{k=0}^{n-2} C(k)$.

Proof. To verify the first assertion, let $k \in\{0, \ldots, n-2\}$. Since $T\left(v_{1}, v_{2}\right)=$ $1 \neq T\left(\underline{\left.v_{1}, v_{0}\right)}=0\left(\right.\right.$ resp. $\left.T\left(v_{n-2}, v_{n-1}\right)=1 \neq T\left(v_{n-2}, v_{n-3}\right)=0\right)$, then $\overline{\left\{v_{1}\right\}}$ (resp. $\overline{\left\{v_{n-2}\right\}}$ ) is not a module of $T$. Therefore, neither $\left\{v_{1}\right\}$ nor $\left\{v_{n-2}\right\}$ is a co-module of $T$. It follows from the definitions of $C(0)$ and $C(n-2)$ that $C(0) \in\left\{\left\{v_{0}\right\},\left\{v_{0}, v_{1}\right\}\right\}$ and $C(n-2) \in\left\{\left\{v_{n-1}\right\},\left\{v_{n-2}, v_{n-1}\right\}\right\}$. Now suppose $1 \leq k \leq n-3$. Since $T\left(v_{k}, v_{k+1}\right)=1 \neq T\left(v_{k}, v_{k-1}\right)=0\left(\mathrm{resp} . T\left(v_{k+1}, v_{k+2}\right)=\right.$ $1 \neq T\left(v_{k+1}, v_{k}\right)=0$ ), then $\overline{\left\{v_{k}\right\}}$ (resp. $\overline{\left\{v_{k+1}\right\}}$ ) is not a module of $T$. Therefore, neither $\left\{v_{k}\right\}$ nor $\left\{v_{k+1}\right\}$ is a co-module of $T$. It follows from the definition of $C(k)$ that $C(k)=\left\{v_{k}, v_{k+1}\right\}$.

We now verify the second assertion. If $n \geq 4$, the second assertion is an immediate consequence of the first one. Hence suppose $n=3$. We
have $C=\left\{v_{0}, v_{1}, v_{2}\right\}$. By the first assertion $C(0) \in\left\{\left\{v_{0}\right\},\left\{v_{0}, v_{1}\right\}\right\}$ and $C(1) \in\left\{\left\{v_{2}\right\},\left\{v_{1}, v_{2}\right\}\right\}$. Suppose for a contradiction that $C(0)=\left\{v_{0}\right\}$ and $C(1)=\left\{v_{2}\right\}$. In this instance, $\overline{\left\{v_{0}\right\}}$ and $\overline{\left\{v_{2}\right\}}$ are modules of $T$. Thus $T\left(v_{0}, \bar{C}\right)=1 \neq T\left(v_{2}, \bar{C}\right)=0$, contradicting that $C$ is a nontrivial module of $T$. It follows that $C(0)=\left\{v_{0}, v_{1}\right\}$ or $C(1)=\left\{v_{1}, v_{2}\right\}$. Thus $C=C(0) \cup C(1)$, as desired.

Lemma 3.3. Let $T$ be a tournament with at least three vertices. Suppose that $T$ admits a transitive component $C$ such that $|C|=n \geq 2$. Given $M \subseteq V(T)$, the following assertions are equivalent.

1. $M \in \operatorname{mc}(T)$ and $M \cap C \neq \varnothing$.
2. $M \in\{C(0), \ldots, C(n-2)\}$.

Proof. If the tournament $T$ is transitive, then $C=V(T)$ and the lemma follows immediately from (3.1) and Example 3.1. Hence suppose that $T$ is nontransitive. In this instance, $C$ is a nontrivial transitive component of $T$. In particular, $|C| \neq v(T)-1$ (see Remark (3.4). Thus

$$
\begin{equation*}
2 \leq|C|=n \leq v(T)-2 . \tag{3.3}
\end{equation*}
$$

The second assertion implies the first one by the definition of $C(k)$ for $k \in$ $\{0, \ldots, n-2\}$ (see Notation 3.3). Conversely, suppose $M \in \operatorname{mc}(T)$ and $M \cap C \neq \varnothing$. Up to isomorphism, we may assume $T[C]=\underline{n}$.

First suppose $n \geq 3$. In this instance, we have $C=\bigcup_{k=0}^{n-2} C(k)$ (see Assertion 2 of Observation (3.2). Thus, there is $k \in\{0, \ldots, n-2\}$ such that $M \cap C(k) \neq \varnothing$. If $M=C(k)$, then we are done. Hence suppose $M \neq C(k)$. In this instance, since $M$ and $C(k)$ are distinct elements of $\operatorname{mc}(T)$ and $M \cap C(k) \neq \varnothing$, then $M$ and $C(k)$ overlap. It follows from Lemma 3.1 that $M$ is a twin of $T$. By Lemma3.2 $C \cup M$ is a transitive module of $T$. By maximality of the transitive module $C$ of $T$, we obtain $M \subseteq C$. Since $M$ is a twin of $T$ and $M \subseteq C$, then by Assertion 1 of Proposition 2.1, $M$ is a also a twin of $T[C]=\underline{n}$. Thus, $M=\{i, i+1\}$ for some $i \in\{0, \ldots, n-2\}$ (see (1.1)). Since $M \in \operatorname{mc}(T)$, it follows that $M=C(i)$ (see Notation 3.3).

Second suppose $n=2$. We have to prove that $M=C(0)$. Since $T[C]=\underline{2}$, we have $C(0)=\{0\},\{1\}$, or $\{0,1\}$. By interchanging the vertices 0 and 1 , as well as the tournaments $T$ and $T^{\star}$, we may assume that $C(0)=\{0\}$ or $C(0)=C=\{0,1\}$.

To begin, suppose $C(0)=C=\{0,1\}$. In this instance, $C \in \operatorname{mc}(T)$. For a contradiction, suppose that $M$ overlaps $C$. By Lemma 3.1, $M$ is a twin of $T$. It follows from Lemma 3.2 that $C \cup M$ is a transitive module of $T$. This contradicts the maximality of the transitive module $C$ of $T$. Thus, $M$ and $C$ do not overlap. Therefore, since $M$ and $C$ are minimal co-modules of $T$ and $M \cap C \neq \varnothing$, we obtain $M=C=C(0)$.

Finally, suppose $C(0)=\{0\}$. By minimality of the co-modules $M$ and $C(0)$ of $T$, we have $M=C(0)=\{0\}$ or $M \cap C=\{1\}$. Suppose for a contradiction
that $M \cap C=\{1\}$. Recall that since $\overline{\{0\}}$ and $C=\{0,1\}$ are modules of $T$, and $T[C]=\underline{2}$, then $T(0, \overline{\{0\}})=1$ and

$$
\begin{equation*}
T(C, \bar{C})=1 \tag{3.4}
\end{equation*}
$$

Therefore, $\{1\}$ is not a co-module of $T$ because $T(1,0)=0 \neq T(1, \bar{C})=1$. It follows that $M \neq\{1\}$ and since $M \cap C=\{1\}$, we have $M \backslash C \neq \varnothing$. Consider a vertex $x \in M \backslash C$. Since the tournament $T[C \cup\{x\}]$ is transitive (see Remark 3.4), it follows from the maximality of the transitive module $C$ of $T$ that $C \cup\{x\}$ is not a module of $T$. Moreover, since $T(C, \bar{C})=1$ (see (3.4)) and $C \cup\{x\}$ is not a module of $T$, there exists a vertex $y \in V(T) \backslash(C \cup\{x\})$ such that $T(y, x)=1$. It follows that if $M$ is a module of $T$, then $y \in M$, and by Assertion 5 of Proposition 2.1, $M \backslash C$ is a nontrivial module of $T$, which contradicts $M \in \operatorname{mc}(T)$. Thus, $M$ is not a module of $T$. Since $\bar{M}$ is a module of $T$, we obtain $T(1, \bar{M})=0$ because $T(1,0)=0,0 \in \bar{M}$, and $1 \notin \bar{M}$. But $T(1, \bar{C})=1$ (see (3.4)). It follows that $\bar{M}=\{0\}$. In particular $\bar{C} \varsubsetneqq M$, which contradicts $M \in \operatorname{mc}(T)$ because $\bar{C}$ is a nontrivial module of $T$ (see (3.4) and (3.3)). We conclude that $M=C(0)=\{0\}$, completing the proof.

The following fact is an immediate consequence of Lemma 3.3 ,
Fact 3.3. Given a tournament $T$ with at least three vertices, if $T$ admits a transitive component $C$ such that $|C| \geq 2$, then $o_{T}(C(0)) \leq 1$ and $o_{T}(C(|C|-2)) \leq$ 1.

## 4. Minimal $\Delta$-decompositions (or $\delta$-decompositions)

Minimal $\Delta$-decompositions form a basic tool in our proofs of Propositions 1.3, 1.4 and 1.5. A minimal $\Delta$-decomposition (or a $\delta$-decomposition) of a tournament $T$ is a $\Delta$-decomposition $D$ of $T$ in which every element is a minimal co-module of $T$, i.e., such that $D \subseteq \operatorname{mc}(T)$. To see that every tournament $T$ admits a $\delta$-decomposition, let $D$ be a $\Delta$-decomposition of $T$. For every element $M$ of $D$, since $M$ is a co-module of $T$, there exists a minimal co-module $M^{-}$of $T$ such that $M^{-} \subseteq M$. Clearly $\left\{M^{-}: M \in D\right\}$ is a $\delta$-decomposition of $T$. For example, consider the case of transitive tournaments. The $\delta$-decompositions of $\underline{n}$ are the sets of maximum size among the subsets of $\operatorname{mc}(\underline{n})$ whose elements are pairwise disjoint. Therefore, we obtain the following fact by using (3.1).

Fact 4.1. Consider the transitive tournament $\underline{n}$, where $n \geq 4$. The following two assertions hold.

1. If $n$ is even, then $\{\{0\},\{n-1\}\} \cup\left\{\{2 i-1,2 i\}: 1 \leq i \leq \frac{n-2}{2}\right\}$ is the unique $\delta$-decomposition of $\underline{n}$.
2. If $n$ is odd, then a subset $D$ of $2^{\{0, \ldots, n-1\}}$ is a $\delta$-decomposition of $\underline{n}$ if and only if $D$ is a $\delta$-decomposition of $\underline{n}-i$ for some odd integer $i \in\{1, \ldots, n-2\}$.
In particular, $\Delta(\underline{n})=\left\lceil\frac{n+1}{2}\right\rceil$ as found in Proposition 1.2.
The next remark is an immediate consequence of Assertion 4 of Lemma 2.2,

Remark 4.1. Given a decomposable tournament $T$ such that $v(T) \geq 4$, every $\delta$-decomposition of $T$ contains a nontrivial module of $T$. In particular, $T$ admits a minimal co-module which is a nontrivial module of $T$.

The starting points of our proofs of Propositions 1.3, 1.4 and 1.5 are based on the following result.

Proposition 4.1. Given a (decomposable) tournament $T$, the following three assertions are satisfied.

1. If $\Delta(T)=2$, then for every $M \in \operatorname{mc}(T)$, we have $o_{T}(M) \leq 1$.
2. If $\Delta(T)=3$, then $T$ admits a $\delta$-decomposition $D$ such that $o_{T}(M) \leq 1$ for every $M \in D$.
3. If $\Delta(T) \geq 4$, then $T$ admits a $\delta$-decomposition which contains four elements $M_{1}, M_{2}, M_{3}$ and $M_{4}$ satisfying the following conditions.
(a) For every $i \in\{1,3,4\}, o_{T}\left(M_{i}\right) \leq 1$.
(b) $T\left(M_{1}, M_{2}\right)=T\left(M_{2}, M_{3}\right)=1$.
(c) There exists $x \in M_{4}$ such that $T\left(x, M_{1}\right)=1$ or $T\left(M_{3}, x\right)=1$.

The aim of the rest of this section is to prove Proposition 4.1 For this purpose, we need some preliminary results. The following three ones are about $\delta$-decompositions. They are principally consequences of Lemmas 3.1 and 3.3.

Corollary 4.1. Given a decomposable tournament $T$, consider a $\delta$ decomposition $D$ of $T$ and let $M \in \operatorname{mc}(T)$. If $M \notin D$, then $o_{T}(M) \in\{1,2\}$ and $D \cap O_{T}(M) \neq \varnothing$.

Proof. Suppose that $o_{T}(M) \notin\{1,2\}$ or $D \cap O_{T}(M)=\varnothing$. By Lemma 3.1, we have $o_{T}(M)=0$ or $D \cap O_{T}(M)=\varnothing$. In both instances, if $M \notin D$, then $D \cup\{M\}$ would be a co-modular decomposition of $T$, which contradicts the hypothesis that $D$ is a $\delta$-decomposition of $T$. Thus $M \in D$.

Corollary 4.2. Let $T$ be a tournament. There exists a $\delta$-decomposition $D$ of $T$ such that for every transitive component $C$ of $T$ with $|C| \geq 4$, we have $\{C(0), C(|C|-2)\} \subseteq D$.

Proof. Consider a $\delta$-decomposition $D$ of $T$. Let $C$ be a transitive component of $T$ such that $|C| \geq 4$. By Assertion 1 of Observation 3.2 and by Lemma 3.3, we have $O_{T}(C(0))=\varnothing$ or $O_{T}(C(0))=\{C(1)\}$. In the first instance, $C(0) \in D$ by Corollary 4.1. In the second one, again by Corollary4.1 if $C(0) \notin D$, then $C(1) \in$ $D$. Similarly, we have $O_{T}(C(|C|-2))=\varnothing$ or $O_{T}(C(|C|-2))=\{C(|C|-3)\}$. In the first instance, $C(|C|-2) \in D$. In the second one, if $C(|C|-2) \notin D$, then $C(|C|-3) \in D$. To summarize, we have shown that for every transitive component $C$ of $T$ such that $|C| \geq 4$, the following two claims hold.

1. If $C(0) \notin D$, then $C(1) \in D$ and $O_{T}(C(0))=\{C(1)\}$.
2. If $C(|C|-2) \notin D$, then $C(|C|-3) \in D$ and $O_{T}(C(|C|-2))=\{C(|C|-3)\}$.

Now let $\mathcal{C}$ (resp. $\mathcal{C}^{\prime}$ ) be the set of the transitive components $C$ of $T$ such that $|C| \geq 4$ and $C(0) \notin D$ (resp. $|C| \geq 4$ and $C(|C|-2) \notin D)$. It follows from Claim 1 (resp. Claim 2) above that for every $C \in \mathcal{C}$ (resp. $C \in \mathcal{C}^{\prime}$ ), we have $C(1) \in D$ and $O_{T}(C(0))=\{C(1)\}\left(\right.$ resp. $C(|C|-3) \in D$ and $\left.O_{T}(C(|C|-2))=\{C(|C|-3)\}\right)$. Therefore, by taking $D^{\prime}=\left(D \cup\{C(0): C \in \mathcal{C}\} \cup\left\{C(|C|-2): C \in \mathcal{C}^{\prime}\right\}\right)$ ) ( $\{C(1): C \in \mathcal{C}\} \cup\left\{C(|C|-3): C \in \mathcal{C}^{\prime}\right\}$ ), we obtain that $D^{\prime}$ is a $\delta$-decomposition of $T$ such that for every transitive component $C$ of $T$ with $|C| \geq 4$, we have $\{C(0), C(|C|-2)\} \subseteq D^{\prime}$, as desired.

Corollary 4.3. Given a tournament $T$ admitting a nontrivial transitive component $C$, there exists $M \in \operatorname{mc}(T)$ such that $M \cap C=\varnothing$ and $o_{T}(M) \leq 1$.
Proof. Since $\bar{C}$ is a co-module of $T$, there exists $N \in \operatorname{mc}(T)$ such that $N \subseteq \bar{C}$. By Lemma 3.1 we can suppose $o_{T}(N)=2$. By Observation 3.1 $N$ is contained in a transitive component $C^{\prime}$ of $T$ such that $\left|C^{\prime}\right| \geq 4$. By Corollary 3.1, $C \cap C^{\prime}=\varnothing$ and thus $C \cap C^{\prime}(0)=\varnothing$. By Fact 3.3, $o_{T}\left(C^{\prime}(0)\right) \leq 1$. Thus, it suffices to take $M=C^{\prime}(0)$.

We also need the following observation for the proof of Assertion 3 of Proposition 4.1.

Observation 4.1. Given a tournament $T$ such that $\Delta(T) \geq 4$, consider a comodular decomposition $D$ of $T$ such that $|D|=4$. The elements of $D$ can be denoted by $M_{1}, M_{2}, M_{3}, M_{4}$, in such a way that $T\left(M_{1}, M_{2}\right)=T\left(M_{2}, M_{3}\right)=1$, and there exists $x \in M_{4}$ such that $T\left(x, M_{1}\right)=1$ or $T\left(M_{3}, x\right)=1$.

Proof. By Assertion 2 of Lemma 2.2, the elements of $D$ can be denoted by $M_{1}, M_{2}, M_{3}, M_{4}$, in such a way that $M_{1}, M_{2}$ and $M_{3}$ are modules of $T$. By interchanging $M_{1}, M_{2}$ and $M_{3}$, we may assume $T\left(M_{1}, M_{2}\right)=T\left(M_{2}, M_{3}\right)=1$ (see Assertion 6 of Proposition 2.1). If $T\left(x, M_{1}\right)=1$ or $T\left(M_{3}, x\right)=1$ for some $x \in M_{4}$, then we are done. Hence suppose $T\left(M_{1}, M_{4}\right)=T\left(M_{4}, M_{3}\right)=1$. In this instance $\overline{M_{4}}$ is not a module of $T$ because $T\left(M_{4}, M_{1}\right)=0 \neq T\left(M_{4}, M_{3}\right)=1$. Thus, $M_{4}$ is a module of $T$. Since $M_{2}$ and $M_{4}$ are modules of $T$, then by interchanging them, we may assume $T\left(M_{2}, M_{4}\right)=1$ (see Assertion 6 of Proposition 2.1). Thus $T\left(M_{1}, M_{2}\right)=T\left(M_{2}, M_{4}\right)=T\left(M_{4}, M_{3}\right)=1$. This completes the proof.

Proof of Proposition 4.1. It is straightforward to verify the proposition for the tournament with at most four vertices. In the rest of the proof, we suppose $v(T) \geq 5$.

First suppose that $o_{T}(M) \leq 1$ for every $M \in \operatorname{mc}(T)$. In this instance, the first two assertions are obviously satisfied. The third one follows from Observation 4.1

Second suppose that the tournament $T$ is transitive. Up to isomorphism, we may assume $T=\underline{n}$ for some integer $n \geq 5$. Suppose $n=5$. By Fact 4.1 $\Delta(T)=3$ and $\{\{0\},\{1,2\},\{4\}\}$ is a $\delta$-decomposition of $T$. Moreover, $o_{T}(\{0\})=o_{T}(\{4\})=$ 0 and $o_{T}(\{1,2\})=1$ (see Fact 3.1). Thus, the second assertion is satisfied. Hence suppose $n \geq 6$. By Fact 4.1 $\Delta(T) \geq 4$ and $T$ admits a $\delta$-decomposition which contains $M_{1}=\{0\}, M_{2}=\{1,2\}, M_{3}=\{n-2, n-3\}$, and $M_{4}=\{n-1\}$. We have
$T\left(M_{1}, M_{2}\right)=T\left(M_{2}, M_{3}\right)=T\left(M_{3}, M_{4}\right)=1$. Moreover, $o_{T}\left(M_{1}\right)=o_{T}\left(M_{4}\right)=0$ and $o_{T}\left(M_{2}\right)=o_{T}\left(M_{3}\right)=1$ (see Fact 3.1). Thus, the third assertion is satisfied. We conclude that the proposition holds for transitive tournaments.

Third, suppose that the tournament $T$ is non-transitive, and that there exists $Y \in \operatorname{mc}(T)$ such that $o_{T}(Y) \geq 2$. By Lemma 3.1, $o_{T}(Y)=2$. By Observation 3.1, $Y$ is contained in a transitive component $C$ of $T$ such that $|C|=n \geq 4$. Moreover, $C \neq V(T)$ because $T$ is non-transitive. Therefore, $\{C(0), C(n-2), \bar{C}\}$ is a comodular decomposition of $T$. In particular $\Delta(T) \geq 3$. By Fact 3.3, we have

$$
\begin{equation*}
o_{T}(C(0)) \leq 1 \text { and } o_{T}(C(n-2)) \leq 1 \tag{4.1}
\end{equation*}
$$

Moreover, since $C$ is a nontrivial transitive component of $T$, then by Corollary 4.3 there exists $N \in \operatorname{mc}(T)$ such that $N \cap C=\varnothing$ and $o_{T}(N) \leq 1$. It follows that if $\Delta(T)=3$, then $\{C(0), C(n-2), N\}$ is a $\delta$-decomposition of $T$ in which every element $X$ satisfies $o_{T}(X) \leq 1$, as desired. Hence suppose $\Delta(T) \geq 4$. First, suppose that
there exists $Z \in \operatorname{mc}(T)$ such that $Z \cap C=\varnothing$ and $o_{T}(Z)=2$.
By Observation 3.1, $Z$ is contained in a transitive component $C^{\prime}$ of $T$ such that $\left|C^{\prime}\right|=n^{\prime} \geq 4$. By Corollary 3.1] $C \cap C^{\prime}=\varnothing$. By Corollary 4.2, there exists a $\delta$-decomposition $D$ of $T$ such that $\left\{C(0), C(n-2), C^{\prime}(0), C^{\prime}\left(n^{\prime}-2\right)\right\} \subseteq D$. By interchanging the transitive components $C$ and $C^{\prime}$, we may assume $T\left(C, C^{\prime}\right)=1$ (see Assertion 6 of Proposition [2.1). Thus, the third assertion is satisfied by taking the $\delta$-decomposition $D$ with its four elements $M_{1}=C(0), M_{2}=C(n-2)$, $M_{3}=C^{\prime}(0)$, and $M_{4}=C^{\prime}\left(n^{\prime}-2\right)$. Indeed, $o_{T}\left(M_{i}\right) \leq 1$ for every $i \in\{1,2,3,4\}$ by Fact 3.3, and $T\left(M_{1}, M_{2}\right)=T\left(M_{2}, M_{3}\right)=T\left(M_{3}, M_{4}\right)=1$ by construction.

Second, suppose that (4.2) does not hold. By Lemma 3.1,

$$
\begin{equation*}
\text { for every } Z \in \operatorname{mc}(T) \text { such that } Z \cap C=\varnothing, \text { we have } o_{T}(Z) \leq 1 \text {. } \tag{4.3}
\end{equation*}
$$

By Corollary 4.2 there exists a $\delta$-decomposition $D$ of $T$ such that $\{C(0), C(n-$ $2)\} \subseteq D$. To begin, suppose that $D \cap\{C(i): 0 \leq i \leq n-2\}=\{C(0), C(n-2)\}$. By Lemma 3.3, since $\Delta(T) \geq 4$, there exist distinct $M, L \in D$ such that $(M \cup L) \cap C=$ $\varnothing$. By (4.3), we have $o_{T}(M) \leq 1$ and $o_{T}(L) \leq 1$. Thus, $\{C(0), C(n-2), M, L\}$ is a co-modular decomposition of $T$ that is contained in the $\delta$-decomposition $D$ of $T$, and in which every element $X$ satisfies $o_{T}(X) \leq 1$ (see (4.1)). Therefore, the third assertion is satisfied by applying Observation4.1to $\{C(0), C(n-2), M, L\}$. To finish, suppose that there is an integer $i$ such that $1 \leq i \leq n-3$ and $C(i) \in D$. We have $\{C(0), C(i), C(n-2)\} \subseteq D$. Moreover, by maximality of $D$, it follows from Lemma 3.3 and Corollary 4.3 that $D$ contains an element $K$ such that $K \cap C=\varnothing$. We have $o_{T}(K) \leq 1$ (see (4.3)). Thus, the third assertion is satisfied by taking the $\delta$-decomposition $D$ with its four elements $M_{1}=C(0), M_{2}=C(i)$, $M_{3}=C(n-2)$, and $M_{4}=K$. Indeed, $o_{T}\left(M_{i}\right) \leq 1$ for every $i \in\{1,3,4\}$ (see (4.1)), $T\left(M_{1}, M_{2}\right)=T\left(M_{2}, M_{3}\right)=1$, and since $C$ is a module of $T$, then for $x \in M_{4}$, we have $T(x, C)=1$ or $T(C, x)=1$.

## 5. Proofs of Propositions 1.3, 1.4 and 1.5

The proofs use the following notation.
Notation 5.1. Given a decomposable tournament $T$, consider a minimal comodule of $T$ such that $o_{T}(M) \leq 1$. When $o_{T}(M)=1$, we denote by $M^{\prime}$ the element of $O_{T}(M)$. We set

$$
\widetilde{M}=\left\{\begin{array}{ccc}
M & \text { if } & o_{T}(M)=0 \\
M \cap M^{\prime} & \text { if } & o_{T}(M)=1
\end{array}\right.
$$

Notice that $\widetilde{M} \neq \varnothing$. More precisely, if $o_{T}(M)=1$, then $M$ and $M^{\prime}$ are twins of $T$ and $|\widetilde{M}|=1$ (see Lemma 3.1).

For a better understanding of Notation 5.1, notice the following remark which is a consequence of Lemma 3.1

Remark 5.1. Given a decomposable tournament $T$ and a minimal co-module $M$ of $T$ such that $o_{T}(M) \leq 1$, one of the following holds

1. $\widetilde{M}=M$,
2. $T$ admits a module $H$ such that $T[H] \simeq \underline{3}, M$ is a twin of $T[H]$ and thus of $T$, and $\widetilde{M}=\{f(1)\}$ where $f$ is the isomorphism from $\underline{3}$ onto $T[H]$.

### 5.1. Proof of Proposition 1.3

We need the next three lemmas.
Lemma 5.1. Given a tournament $T$ such that $\Delta(T)=2$, consider a $\delta$ decomposition $\{M, N\}$ of $T$. The following assertions are satisfied.

1. We have $o_{T}(M) \leq 1$ and $o_{T}(N) \leq 1$ so that $\widetilde{M}$ and $\widetilde{N}$ are well-defined.
2. Let $x \in \widetilde{M}$ and $y \in \widetilde{N}$. Suppose that the tournament $T^{\prime}=\operatorname{Inv}(T,\{x, y\})$ is decomposable. Let $L$ be a minimal co-module of $T^{\prime}$ which is a nontrivial module of $T^{\prime}$. The following two assertions hold.
2.1. $L$ and $\{x, y\}$ overlap.
2.2. We have $L \cap M \neq \varnothing$ and $L \cap N \neq \varnothing$.

Proof. By Assertion 1 of Proposition 4.1, we have $o_{T}(M) \leq 1$ and $o_{T}(N) \leq 1$ because $M, N \in \operatorname{mc}(T)$. Therefore $\widetilde{M}$ and $\widetilde{N}$ are well-defined (see Notation 5.1). Thus, the first assertion is satisfied.

We now prove Assertion 2.1. Suppose toward a contradiction that $L$ and $\{x, y\}$ do not overlap. By Assertion 1 of Lemma 2.1, $L$ is also a nontrivial module of $T$. By Assertion 2 of Lemma [2.2, and by interchanging $M$ and $N$, we may assume that $M$ is a module of $T$.

First suppose $L \cap\{x, y\}=\varnothing$. In this instance, $L$ is also a minimal co-module of $T$ (see Remark [3.1). Therefore, if $M \cap L \neq \varnothing$, then $\widetilde{M}=M \cap L$ so that $x \in M \cap L$, contradicting $L \cap\{x, y\}=\varnothing$. Thus $M \cap L=\varnothing$. Similarly, $N \cap L=\varnothing$. It follows that $\{L, M, N\}$ is a co-modular decomposition of $T$, which contradicts $\Delta(T)=2$.

Second suppose $\{x, y\} \subseteq L$. By Assertion 4 of Proposition 2.1, $L \cup M$ is a module of $T$. Recall that $N$ or $\bar{N}$ is a module of $T$. To begin, suppose that $\bar{N}$ is a module of $T$. By Assertion 4 of Proposition [2.1] $\overline{N \backslash L}=L \cup \bar{N}$ is a module of $T$. By minimality of the co-module $N$ of $T$, the module $\overline{N \backslash L}$ of $T$ is trivial. Thus $N \subseteq L$. It follows that the module $L \cup M$ of $T$ contains $M \cup N$. Therefore, $V(T)=L \cup M$ by Assertion 3 of Lemma [2.2. Thus $\overline{M \backslash L}=L$. It follows that $\overline{M \backslash L}$ is a nontrivial module of $T$, which contradicts the minimality of the comodule $M$ of $T$. Now suppose that $N$ is a module of $T$. In this instance, since $L, M$ and $N$ are modules of $T$, then $L \cup M, L \cup N$ and $L \cup M \cup N$ are also modules of $T$ by Assertion 4 of Proposition 2.1. It follows from Assertion 3 of Lemma 2.2 that the module $L \cup M \cup N$ of $T$ is trivial. Therefore $V(T)=L \cup M \cup N$. Thus $\overline{M \backslash L}=L \cup N$. Since $\overline{M \backslash L}=L \cup N$ is a module of $T$, it follows from the minimality of the co-module $M$ of $T$ that the module $L \cup N$ of $T$ is trivial. Thus $V(T)=L \cup N$. Similarly, we have $V(T)=L \cup M$. Since $V(T)=L \cup M=L \cup N$ and $M \cap N=\varnothing$, then $V(T)=L$, a contradiction because $L$ is a nontrivial module of $T$. This completes the proof of Assertion 2.1.

For the proof of Assertion 2.2, since $L$ and $\{x, y\}$ overlap by Assertion 2.1, then by interchanging $M$ and $N$, we may assume $L \cap\{x, y\}=\{y\}$. Thus, we only have to prove that $L \cap M \neq \varnothing$. Suppose not. We have $x \equiv_{T^{\prime}} L$ because $L$ is a module of $T^{\prime}$ and $x \notin L$. Since $L \cap\{x, y\}=\{y\}$ and $|L| \geq 2$, we obtain $x \neq T_{T} L$. In particular, $x \neq{ }_{T} \bar{M}$ because $L \subseteq \bar{M}$. Therefore, $\bar{M}$ is not a module of $T$ so that $M$ is a nontrivial module of $T$. Since $L$ and $M$ are disjoint nontrivial modules, there are distinct $z, t \in V(T) \backslash\{x, y\}$ such that $\{x, z\} \subseteq M$ and $\{y, t\} \subseteq L$. Since $T^{\prime}(x, t)=T^{\prime}(x, y)$ because $L$ is a module of $T^{\prime}$, then $T(x, t) \neq T(x, y)$. Moreover, $T(x, t)=T(z, t)$ and $T(x, y)=T(z, y)$ because $M$ is a module of $T$. It follows that $T(z, y) \neq T(z, t)$ and thus $T^{\prime}(z, y) \neq T^{\prime}(z, t)$, a contradiction because $L$ is a module of $T^{\prime},\{y, t\} \subseteq L$ and $z \notin L$. Thus $L \cap M \neq \varnothing$ as desired.

Lemma 5.2. Let $T$ be a tournament such that $\Delta(T)=2$ and satisfying the following hypothesis (H).

$$
\begin{equation*}
\text { For every } x \in V(T) \text {, the tournament } T-x \text { is decomposable. } \tag{H}
\end{equation*}
$$

Consider a $\delta$-decomposition $\{M, N\}$ of $T$ such that $M$ is a nontrivial module of T. Recall that $\widetilde{M}$ and $\widetilde{N}$ are well-defined (see Assertion 1 of Lemma 5.1). Let $x \in \widetilde{M}$ and $y \in \widetilde{N}$. If the tournament $T^{\prime}=\operatorname{Inv}(T,\{x, y\})$ is decomposable, then the following three assertions are satisfied.

1. $N$ is not a nontrivial module of $T$. In particular $\widetilde{N}=N$.
2. Given a minimal co-module $L$ of $T^{\prime}$, if $L$ is a nontrivial module of $T^{\prime}$, then $L=\{x, z\}$ for some $z \in N \backslash\{y\}$.
3. There exists a unique vertex $z \in N \backslash\{y\}$ such that $\{x, z\}$ is a twin of $T^{\prime}$.

Proof. We easily verify that no tournament $U$ on at most four vertices satisfies both hypotheses $(\underline{H})$ and $\Delta(U)=2$. Thus $v(T) \geq 5$. Suppose that the tournament $T^{\prime}=\operatorname{Inv}(T,\{x, y\})$ is decomposable. It follows from Remark 4.1 that $T^{\prime}$ admits a minimal co-module $L$ which is a nontrivial module of $T^{\prime}$.

For the first assertion，suppose for a contradiction that $N$ is a nontrivial module of $T$ ．In this instance，we may interchange $M$ and $N$ ．Thus，by As－ sertion 2.1 of Lemma［5．1，we can suppose $L \cap\{x, y\}=\{x\}$ ．Since $L \cap N \neq \varnothing$ by Assertion 2.2 of Lemma 5．1］then $L \cup N$ is a module of $T$ by Assertion 3 of Lemma 2．1．By Assertion 4 of Proposition［2．1，$L \cup N \cup M$ is a module of $T$ ．It follows from Assertion 3 of Lemma 2．2 that $L \cup N \cup M=V(T)$ ．Thus $\overline{M \backslash L}=L \cup N$ ．Since $M \backslash L$ is not a co－module of $T$ by minimality of the co－module $M$ of $T$ ，it follows that the module $L \cup N$ of $T$ is trivial．Therefore， $L \cup N=V(T)$ and thus $M \subseteq L$ ．On the other hand，since $y \equiv_{T} M$ and $|M| \geq 2$ because $M$ is a nontrivial module of $T$ ，then $y{⿻ 三 丨 T^{\prime}}^{M}$ ．Since $M \subseteq L$ ，it follows that $y \not \equiv T^{\prime} L$ ，which contradicts that $L$ is a module of $T^{\prime}$ ．Thus，$N$ is not a nontrivial module of $T$ ，as desired．Therefore，$\widetilde{N}=N$ because $o_{T}(N)=0$ by Lemma 3．1．

We now prove the second assertion．To begin，we prove that $L \cap\{x, y\}=\{x\}$ ． Suppose not．By Assertion 2.1 of Lemma 5．1，we have $L \cap\{x, y\}=\{y\}$ ．Recall that $N$ is not a nontrivial module of $T$ by the first assertion of the lemma．Thus， $\bar{N}$ is a module of $T$ ．It follows from Assertions 2 and 1 of Lemma2．1that $L \cap \bar{N}$ is a module of $T^{\prime}$ ．More precisely，$L \cap \bar{N}$ is a trivial module of $T^{\prime}$ because $L \cap \bar{N} \varsubsetneqq L$ and $L$ is a minimal co－module of $T^{\prime}$ ．Moreover，$L \cap M \neq \varnothing$ by Assertion 2.2 of Lemma5．1．Thus $|L \cap \bar{N}|=|L \cap M|=1$ ．It follows that $L \subseteq M \cup N$ ．On the other hand， $\bar{N} \cup L$ is a module of $T$ by Assertion 3 of Lemma 2．1．But $\bar{N} \cup L=\overline{N \backslash L}$ ． It follows from the minimality of the co－module $N$ of $T$ that the module $\overline{N \backslash L}$ of $T$ is trivial．Therefore $N \backslash L=\varnothing$ ，i．e．，$N \subseteq L$ ．Thus $L \cup M \cup N=L \cup M$ ． Since $L \cup M$ is a module of $T$ by Assertion 3 of Lemma 2．1 it follows that $V(T)=L \cup M$ by Assertion 3 of Lemma 2．2，Recall that $L \subseteq M \cup N$ ．Thus

$$
\begin{equation*}
V(T)=L \cup M=M \cup N . \tag{5.1}
\end{equation*}
$$

Notice that since $M$ is a minimal nontrivial module of $T$（see Remark 3．2），the tournament $T[M]$ is indecomposable（see Remark 3．3）．It follows that $N \neq\{y\}$ ， otherwise $T[M]=T-y$ because $V(T)=M \cup N$（see（5．1）），so that $T-y$ is indecomposable contrary to the hypothesis（H）．Suppose for a contradiction that $L \neq \overline{\{x\}}$ ．Since $V(T) \backslash(L \cup\{x\})=M \backslash(L \cup\{x\})$ because $V(T)=M \cup L$ （see（5．1）），then $M \backslash(L \cup\{x\}) \neq \varnothing$ because $L \neq \overline{\{x\}}$ ．Pick $u \in N \backslash\{y\}$ and $w \in M \backslash(L \cup\{x\})$ ．We have $T(w, y)=T(x, y)$ and $T(w, u)=T(x, u)$ because $M$ is a module of $T$ ．Moreover，since $L$ is a module of $T^{\prime},\{y, u\} \subseteq N \subseteq L$（see （5．1））and $x \notin L$ ，then $T^{\prime}(x, y)=T^{\prime}(x, u)$ and thus $T(x, y) \neq T(x, u)$ ．It follows that $T(w, y) \neq T(w, u)$ and thus $T^{\prime}(w, y) \neq T^{\prime}(w, u)$ ，a contradiction because $L$ is a module of $T^{\prime}$ ．Therefore $L=\overline{\{x\}}$ ．It follows from Remarks 3.2 and 3.3 that $T^{\prime}[L]=T^{\prime}-x$ is indecomposable．Since $T^{\prime}-x=T-x$ ，this again contradicts the hypothesis（H）．We conclude that $L \cap\{x, y\}=\{x\}$ as claimed．

Since $L$ is a module of $T^{\prime}$ ，we have $y \equiv_{T^{\prime}} L$ ．Moreover，since $\bar{N}$ is a nontrivial module of $T$ by the first assertion of the lemma，then $y \equiv_{T} \bar{N}$ so that $y \not ⿻_{T^{\prime}} \bar{N}$ and $y \equiv_{T^{\prime}} \bar{N} \backslash\{x\}$ ．Therefore，if $\{x\} \mp L \cap \bar{N}$ ，then $y \not{⿻ 三 丨 T^{\prime}}^{L \cap} \bar{N}$ ，which contradicts $y \equiv_{T^{\prime}} L$ ．Thus $L \cap \bar{N}=\{x\}$ ．On the other hand，since $y \notin L$ ，and $L \cap N \neq \varnothing$ by Assertion 2.2 of Lemma 5．1．then $L \cap(N \backslash\{y\}) \neq \varnothing$ ．Let $z \in L \cap(N \backslash\{y\})$ ．Since
$L \cap \bar{N}=\{x\}$, to show that $L=\{x, z\}$, which finishes the proof of the second assertion, it suffices to show that $L \cap N=\{z\}$. By Assertion 1 of Proposition 2.1] $M$ and $L$ are modules of $T-y=T^{\prime}-y$. Since $M \backslash L \neq \varnothing$ because $|M| \geq 2$ and $L \cap \bar{N}=\{x\}$, then $L \backslash M$ is a module of $T-y$ by Assertion 5 of Proposition 2.1. Moreover, since $y \equiv_{T^{\prime}} L$ and thus $y \equiv_{T} L \backslash\{x\}$, then $y \equiv_{T} L \backslash M$. It follows that $L \backslash M$ is a module of $T$. But $L \backslash M=L \cap N$ because $L \cap \bar{N}=\{x\} \subseteq M$. Since $L \cap N \varsubsetneqq N$, it follows from the minimality of the co-module $N$ of $T$ that the module $L \cap N$ of $T$ is trivial. Since $z \in N \cap L$ and $N \cap L \mp V(T)$, we obtain $N \cap L=\{z\}$ as claimed.

Lastly, we prove the third assertion. It follows from the second assertion of the lemma and from Remark 4.1 that there exists $z \in N \backslash\{y\}$ such that $\{x, z\}$ is a twin of $T^{\prime}$. Let $z^{\prime} \in N \backslash\{y\}$ such that $\left\{x, z^{\prime}\right\}$ is a twin of $T^{\prime}$. We will prove that $\left\{z, z^{\prime}\right\}$ is a module of $T$, which implies that $z=z^{\prime}$ because $N$ is a minimal co-module of $T$. By Assertion 4 of Proposition 2.1. $\left\{x, z, z^{\prime}\right\}$ is a module of $T^{\prime}$ and thus of $T^{\prime}-y=T-y$. Since $M$ and $\left\{x, z, z^{\prime}\right\}$ are modules of $T-y$, and $M \backslash\left\{x, z, z^{\prime}\right\}=M \backslash\{x\} \neq \varnothing$ because the module $M$ is nontrivial, then $\left\{x, z, z^{\prime}\right\} \backslash M=\left\{z, z^{\prime}\right\}$ is a module of $T-y$ by Assertion 5 of Proposition 2.1. Moreover, since $T^{\prime}(y, z)=T^{\prime}\left(y, z^{\prime}\right)$ because $\left\{x, z, z^{\prime}\right\}$ is a module of $T^{\prime}$, then $T(y, z)=T\left(y, z^{\prime}\right)$. It follows that $\left\{z, z^{\prime}\right\}$ is a module of $T$, completing the proof.

The following lemma complements Lemma 5.2 when the hypothesis (H) is not satisfied.

Lemma 5.3 ([1]). Given a decomposable tournament $T$ such that $v(T) \geq 5$, if there exists $x \in V(T)$ such that $T-x$ is indecomposable, then $\delta(T)=1$.

Proof of Proposition 1.3. Let $T$ be a tournament with at least five vertices. If $\delta(T)=1$, then $\Delta(T)=2$ by (1.6) and (1.7). Conversely, suppose $\Delta(T)=2$. By Lemma 5.3, to prove that $\delta(T)=1$, we can suppose that for every $x \in V(T)$, the tournament $T-x$ is decomposable. Consider a $\delta$-decomposition $\{M, N\}$ of $T$. By Remark 4.1 and by interchanging $M$ and $N$, we may assume that $M$ is a nontrivial module of $T$. By Assertion 1 of Lemma $5.1, \widetilde{M}$ and $\widetilde{N}$ are well-defined. Fix $x \in \widetilde{M}$. Suppose toward a contradiction that for every $y \in \widetilde{N}, \operatorname{Inv}(T,\{x, y\})$ is decomposable. By Assertion 1 of Lemma 5.2, we have $N=\widetilde{N}$. It follows from Assertion 3 of Lemma 5.2 that there exists a sequence $\left(z_{n}\right)_{n \in \mathbb{N}}$ of vertices of $N$ such that for every positive integer $n,\left\{x, z_{n}\right\}$ is a twin of $\operatorname{Inv}\left(T,\left\{x, z_{n-1}\right\}\right)$, and $z_{n} \neq z_{n-1}$. Since $N$ is finite and $\left\{z_{n}: n \in \mathbb{N}\right\} \subseteq N$, the vertices $z_{n}$ are not pairwise distinct. Consider the smallest positive integer $k$ such that $z_{k} \in\left\{z_{0}, \ldots, z_{k-1}\right\}$. Since $z_{k} \neq z_{k-1}$, we have $k \geq 2$ and $z_{k} \in\left\{z_{0}, \ldots, z_{k-2}\right\}$. For every integer $n$, let $T_{n}=\operatorname{Inv}\left(T,\left\{x, z_{n}\right\}\right)$. If $z_{k} \neq z_{0}$, i.e., $k \geq 3$ and $z_{k}=z_{i}$ for some $i \in\{1, \ldots k-2\}$, then $\left\{x, z_{k}\right\}$ is a module of both $T_{k-1}$ and $T_{i-1}$, which is not possible because $z_{k-1} \neq z_{i-1}$. Thus $z_{k}=z_{0}$. To show that $\left\{z_{0}, \ldots, z_{k-1}\right\}$ is a module of $T$, we first consider a vertex $v \in V(T) \backslash\left(\left\{z_{0}, \ldots, z_{k-1}\right\} \cup\{x\}\right)$. For every $j \in\{0, \ldots, k-1\}$, we have $T_{j}\left(v, z_{j+1}\right)=T_{j}(v, x)$ and thus $T\left(v, z_{j+1}\right)=T(v, x)$. It follows that $v \equiv_{T}\left\{z_{1}, \ldots, z_{k}\right\}$. Since $\left\{z_{1}, \ldots, z_{k}\right\}=\left\{z_{0}, \ldots, z_{k-1}\right\}$ because $z_{k}=z_{0}$, we obtain

$$
\begin{equation*}
v \equiv_{T}\left\{z_{0}, \ldots, z_{k-1}\right\} \text { for every } v \in V(T) \backslash\left(\left\{z_{0}, \ldots, z_{k-1}\right\} \cup\{x\}\right) \tag{5.2}
\end{equation*}
$$

Thus, $\left\{z_{0}, \ldots, z_{k-1}\right\}$ is a module of $T-x$. Moreover, we have $x \equiv_{T}\left\{z_{0}, \ldots, z_{k-1}\right\}$ because $\bar{N}$ is a nontrivial module of $T$ (see Assertion 1 of Lemma 5.2), $\{x\} \mp \bar{N}$, and for every $w \in \bar{N} \backslash\{x\}$, we have $w \equiv_{T}\left\{z_{0}, \ldots, z_{k-1}\right\}$ by (5.2). It follows that $\left\{z_{0}, \ldots, z_{k-1}\right\}$ is a module of $T$. Moreover, the module $\left\{z_{0}, \ldots, z_{k-1}\right\}$ of $T$ is nontrivial because $k \geq 2$. Therefore, since $\left\{z_{0}, \ldots, z_{k-1}\right\} \subseteq N$ and $N$ is not a nontrivial module of $T$ (see Assertion 1 of Lemma5.2), we obtain $\left\{z_{0}, \ldots, z_{k-1}\right\} \mp$ $N$. This contradicts the minimality of the co-module $N$ of $T$. We conclude that there exists a vertex $y \in \widetilde{N}$ such that $\operatorname{Inv}(T,\{x, y\})$ is indecomposable. Thus $\delta(T)=1$.

Discussion. Let $T$ be a tournament with at least five vertices such that $\Delta(T)=$ 2, i.e., such that $\delta(T)=1$ (see Proposition 1.3). Suppose a $\delta$-decomposition $\{M, N\}$ of $T$ is given. We would like to discuss the following question. For which arcs $a$ of $T$, is the tournament $\operatorname{Inv}(T, a)$ indecomposable?

Suppose that $\operatorname{Inv}(T, a)$ is indecomposable. Recall that $o_{T}(M) \leq 1$ and $o_{T}(N) \leq 1$ (see Assertion 1 of Lemma 5.1). Clearly, $\mathcal{V}(a) \cap M \neq \varnothing$ and $\mathcal{V}(a) \cap N \neq \varnothing$ (see Remark 3.1). We will prove that $\mathcal{V}(a) \cap \widetilde{M} \neq \varnothing$ and $\mathcal{V}(a) \cap \widetilde{N} \neq \varnothing$. Suppose for a contradiction that $\mathcal{V}(a) \cap \widetilde{M}=\varnothing$. In this instance, $\widetilde{M} \neq M$. By Remark 5.1 and up to isomorphism, we can suppose that $\{0,1,2\}$ is a module of $T, T[\{0,1,2\}]=\underline{3}$, and $\widetilde{M}=\{1\}$. By Assertion 1 of Lemma 2.1, since $1 \notin \mathcal{V}(a)$ and $\{0,1\}$ (resp. $\{1,2\}$ ) is a module of $T$, then $0 \in \mathcal{V}(a)$ (resp. $2 \in \mathcal{V}(a))$. Thus $\mathcal{V}(a)=\{0,2\}$. Again by Assertion 1 of Lemma 2.1, $\{0,1,2\}$ is a module of the indecomposable tournament $\operatorname{Inv}(T, a)$, a contradiction. Thus $\mathcal{V}(a) \cap \widetilde{M} \neq \varnothing$. Similarly, we have $\mathcal{V}(a) \cap \widetilde{N} \neq \varnothing$.

Conversely, let $x \in \widetilde{M}$ and $y \in \widetilde{N}$. Suppose that $M$ and $N$ are nontrivial modules of $T$. In this instance the hypothesis (H) of Lemma 5.2 is satisfied. It follows from Assertion 1 of Lemma 5.2 that $\operatorname{Inv}(T,\{x, y\})$ is indecomposable. Here we would like to note that we often have $\widetilde{M}=M$ and $\widetilde{N}=N$ (see Remark 5.1). For example, this is the case if each of the modules M and N contains at least three vertices. In this instance, $\operatorname{Inv}(T,\{x, y\})$ is indecomposable for every $x \in M$ and $y \in N$.

Lastly, recall that we can suppose that $M$ is a nontrivial module of $T$ (see Remark 4.1). Suppose that the hypothesis (H) is satisfied. By the proof of Proposition 1.3, for every $x \in \widetilde{M}$, there exists $y \in \widetilde{N}$ such that $\operatorname{Inv}(T,\{x, y\})$ is indecomposable. For some details about the case where the hypothesis (H) is not satisfied, see the proof of [1, Lemma 5.1].

### 5.2. Proofs of Propositions 1.4 and 1.5

We need the following lemma.
Lemma 5.4. Given a tournament $T$ such that $\Delta(T) \geq 3$, consider a co-modular decomposition $\{M, N, L\}$ of $T$ such that $M \in \operatorname{mc}(T)$. Suppose there are $x \in M$ and $y \in N$ such that $x \equiv_{T} L, y \equiv_{T} L$, and $T(x, L) \neq T(y, L)$. Consider the tournament $T^{\prime}=\operatorname{Inv}(T,\{x, y\})$, and let $I \in \operatorname{mc}\left(T^{\prime}\right)$. If $I \cap M \neq \varnothing$ and $I \cap N=\varnothing$, then $x \notin I$ and $I \in O_{T}(M)$.

Proof. Suppose $I \cap M \neq \varnothing$ and $I \cap N=\varnothing$. If $x \notin I$, then $I \in \operatorname{mc}(T)$ (see Remark (3.1), and since $M \cap I \neq \varnothing$, we obtain $I \in O_{T}(M)$. Therefore, it suffices to prove that $x \notin I$. Suppose for a contradiction that $x \in I$. We distinguish the following two cases.

First suppose that $I$ is a nontrivial module of $T^{\prime}$. In this instance, $y \equiv_{T^{\prime}} I$ and thus $y \not ⿻_{T} I$. Since $I \subseteq \bar{N}$, it follows that $\bar{N}$ is not a module of $T$. Thus, $N$ is a nontrivial module of $T$. Let $y^{\prime} \in N \backslash\{y\}$ and $x^{\prime} \in I \backslash\{x\}$. Since $I$ is a module of $T^{\prime}$, we have $T^{\prime}(y, x)=T^{\prime}\left(y, x^{\prime}\right)$ and $T^{\prime}\left(y^{\prime}, x\right)=T^{\prime}\left(y^{\prime}, x^{\prime}\right)$. Moreover, since $N$ is a module of $T$, we have $T\left(y, x^{\prime}\right)=T\left(y^{\prime}, x^{\prime}\right)$ and thus $T^{\prime}\left(y, x^{\prime}\right)=T^{\prime}\left(y^{\prime}, x^{\prime}\right)$. It follows that $T^{\prime}(y, x)=T^{\prime}\left(y^{\prime}, x\right)$. Therefore $T(y, x) \neq T\left(y^{\prime}, x\right)$, contradicting that $N$ is a module of $T$.

Second suppose that $I$ is not a nontrivial module of $T^{\prime}$. In this instance, $\bar{I}$ is a module of $T^{\prime}$. On the other hand, since $L$ is a co-module of $T$, then by Assertion 1 of Lemma 2.1, $L$ is also a co-module of $T^{\prime}$. Moreover, $I \backslash L \neq \varnothing$ because $I \cap M \neq \varnothing$ and $M \cap L=\varnothing$. It follows from the minimality of the comodule $I$ of $T^{\prime}$ that $L \backslash I \neq \varnothing$. Let $z \in L \backslash I$. We have $T^{\prime}(x, y)=T^{\prime}(x, z)$ because $\bar{I}$ is a module of $T^{\prime}$. Thus $T(y, x)=T(x, z)$. Moreover, $T(x, z) \neq T(y, z)$ because $T(x, L) \neq T(y, L)$. It follows that $T(y, x) \neq T(y, z)$. Therefore, $\bar{N}$ is not a module of $T$. So $N$ is a nontrivial module of $T$. Let $y^{\prime} \in N \backslash\{y\}$. Because $\bar{I}$ is a module of $T^{\prime}$ and $N \cap I=\varnothing$, we have $T^{\prime}(x, y)=T^{\prime}\left(x, y^{\prime}\right)$ and thus $T(x, y) \neq T\left(x, y^{\prime}\right)$, which contradicts that $N$ is a module of $T$.

Proof of Proposition 1.4. Let $T$ be a tournament with at least five vertices such that $\Delta(T)=3$. There exists a $\delta$-decomposition $\{M, N, L\}$ of $T$ such that $o_{T}(M) \leq 1, o_{T}(N) \leq 1, o_{T}(L) \leq 1$ (see Assertion 2 of Proposition 4.1), and

$$
\begin{equation*}
\text { there are } x \in \widetilde{M}, y \in \widetilde{N}, \text { and } z \in \widetilde{L} \text { satisfying } T(x, z)=T(z, y)=1 \tag{5.3}
\end{equation*}
$$

Thereby, $\bar{L}$ is not a module of $T$. Thus, $L$ is a module of $T$. Therefore, it follows from (5.3) that

$$
\begin{equation*}
T(x, L)=T(L, y)=1 \tag{5.4}
\end{equation*}
$$

We consider the tournament $T^{\prime}=\operatorname{Inv}(T,\{x, y\})$. If $\operatorname{mc}\left(T^{\prime}\right)$ admits two disjoint elements $X$ and $Y$ such that $X \cap(M \cup N)=\varnothing$ and $Y \cap(M \cup N)=\varnothing$, then $\{X, Y, M, N\}$ is a co-modular decomposition of $T$ (see Remark 3.1), which contradicts $\Delta(T)=3$. It follows that

$$
\begin{equation*}
\text { for every } \delta \text {-decomposition } D \text { of } T^{\prime},|\{X \in D: X \cap(M \cup N)=\varnothing\}| \leq 1 \tag{5.5}
\end{equation*}
$$

We will prove that $\Delta\left(T^{\prime}\right) \leq 2$. Suppose not. By (5.5), there are disjoint $I, J \in$ $\operatorname{mc}\left(T^{\prime}\right)$ such that $I \cap(M \cup N) \neq \varnothing$ and $J \cap(M \cup N) \neq \varnothing$. Since $L \in \operatorname{mc}(T)$ and $\{x, y\} \cap L=\varnothing$, then $L \in \operatorname{mc}\left(T^{\prime}\right)$ (see Remark 3.1). Since $I, J$ and $L$ are pairwise distinct elements of $\mathrm{mc}\left(T^{\prime}\right)$, then by minimality of $I$ and $J$,

$$
\begin{equation*}
L \backslash I \neq \varnothing \text { and } L \backslash J \neq \varnothing . \tag{5.6}
\end{equation*}
$$

Suppose for a contradiction that

$$
\begin{equation*}
I \cap M \neq \varnothing, I \cap N \neq \varnothing, J \cap M \neq \varnothing, \text { and } J \cap N \neq \varnothing . \tag{5.7}
\end{equation*}
$$

Suppose to the contrary that $M$ and $N$ are modules of $T$. By (5.4), $T(L, M)=$ $0 \neq T(L, N)=1$. Since $T^{\prime}(L, M)=T(L, M)$ and $T^{\prime}(L, N)=T(L, N)$, we obtain $T^{\prime}(L, M)=0 \neq T^{\prime}(L, N)=1$. Therefore, it follows from (5.6) and (5.7) that neither $I$ nor $J$ is a module of $T^{\prime}$, which contradicts Assertion 2 of Lemma 2.2 Thus, $M$ or $N$ is not a module of $T$. By interchanging $T$ and $T^{\star}$, as well as $M$ and $N$, we may assume that $N$ is not a module of $T$. By Lemma3.1, $o_{T}(N)=0$. It follows that $I \notin \operatorname{mc}(T)$ and $J \notin \mathrm{mc}(T)$, otherwise $I$ or $J$ belongs to $O_{T}(N)$, which contradicts $o_{T}(N)=0$. Thus, $I \cap\{x, y\} \neq \varnothing$ and $J \cap\{x, y\} \neq \varnothing$ (see Remark (3.1). Therefore, since $I \cap J=\varnothing$, then by interchanging $I$ and $J$, we may assume $I \cap\{x, y\}=\{x\}$ and $J \cap\{x, y\}=\{y\}$. Since $N$ is not a module of $T$, then by Assertion 2 of Lemma [2.2, $M$ is a module of $T$. Thus by (5.4) we get $T(L, M)=0 \neq T(L, y)=1$ and hence $T^{\prime}(L, M)=0 \neq T^{\prime}(L, y)=1$. Since $y \in J, M \cap J \neq \varnothing$ (see (5.7)), and $L \backslash J \neq \varnothing$ (see (5.6)), it follows that $J$ is not a module of $T^{\prime}$ and hence $\bar{J}$ is a module of $T^{\prime}$. Thus $T^{\prime}(y, x)=T^{\prime}(y, L \backslash J)$ and hence $T(y, x) \neq T(y, L \backslash J)$, contradicting that $\bar{N}$ is a module of $T$ because $N$ is not a module of $T$. We conclude that (5.7) does not hold.

By interchanging $I$ and $J$, we may assume that $I \cap M=\varnothing$ or $I \cap N=\varnothing$. By interchanging $T$ and $T^{\star}$, as well as $M$ and $N$, we may assume $I \cap N=\varnothing$. Moreover, $I \cap M \neq \varnothing$ because $I \cap(M \cup N) \neq \varnothing$. Thus, Lemma 5.4 applies to the $\delta$ decomposition $\{M, N, L\}$ of $T$, and yields $x \notin I$ and $I \in O_{T}(M)$. More precisely, $O_{T}(M)=\{I\}$ because $o_{T}(M) \leq 1$ and $I \in O_{T}(M)$. Therefore $\widetilde{M}=M \cap I$, and since $x \in \widetilde{M}$, we get $\widetilde{M}=M \cap I=\{x\}$ (see Notation [5.1), contradicting $x \notin I$. We conclude that $\Delta\left(T^{\prime}\right) \leq 2$. Since $T^{\prime}$ is decomposable because $\delta(T) \geq\left\lceil\frac{\Delta(T)}{2}\right\rceil=2$ (see (1.7)), then $\Delta\left(T^{\prime}\right) \geq 2$ (see (1.6)). Thus $\Delta\left(T^{\prime}\right)=2$. By Proposition 1.3 $\delta\left(T^{\prime}\right)=1$. Since $\delta(T) \leq 1+\delta\left(T^{\prime}\right)$, we obtain $\delta(T) \leq 2$. But since $\Delta(T)=3$, we have $\delta(T) \geq 2$ (see (1.7)). Thus $\delta(T)=2$, completing the proof.

Proof of Proposition 1.5. Let $T$ be a tournament such that $\Delta(T) \geq 4$. By Assertion 3 of Proposition 4.1, $T$ admits a $\delta$-decomposition $D$ which contains four elements $M_{1}, M_{2}, M_{3}$ and $M_{4}$ satisfying the following conditions.
(C1) For every $i \in\{1,3,4\}, o_{T}\left(M_{i}\right) \leq 1$.
(C2) $T\left(M_{1}, M_{2}\right)=T\left(M_{2}, M_{3}\right)=1$.
(C3) There exists $u \in M_{4}$ such that $T\left(u, M_{1}\right)=1$ or $T\left(M_{3}, u\right)=1$.
By (C1), $\widetilde{M_{1}}$ and $\widetilde{M_{3}}$ are well-defined (see Notation5.1). Recall that $\widetilde{M_{1}} \neq \varnothing$ and $\widetilde{M_{3}} \neq \varnothing$. Let $x \in \widetilde{M_{1}}$ and $y \in \widetilde{M_{3}}$. Consider the tournament $T^{\prime}=\operatorname{Inv}(T,\{x, y\})$. We claim that

$$
\begin{equation*}
\text { for every } I \in \operatorname{mc}\left(T^{\prime}\right) \text {, we have } I \cap\left(M_{1} \cup M_{3}\right)=\varnothing \text {. } \tag{5.8}
\end{equation*}
$$

Before proving (5.8), we first use it to show that $\Delta\left(T^{\prime}\right)=\Delta(T)-2$, which completes the proof. By (5.8) and by Remark 3.1, we have

$$
\begin{equation*}
\operatorname{mc}\left(T^{\prime}\right) \subseteq \operatorname{mc}(T) \backslash\left\{M_{1}, M_{3}\right\} \tag{5.9}
\end{equation*}
$$

Let $D^{\prime}$ be a $\delta$-decomposition of $T^{\prime}$. It follows from (5.9) and (5.8) that $D^{\prime} \cup$ $\left\{M_{1}, M_{3}\right\}$ is a co-modular decomposition of $T$. Thus $\left|D^{\prime} \cup\left\{M_{1}, M_{3}\right\}\right|=\left|D^{\prime}\right|+2 \leq$
$\Delta(T)$. Since $\left|D^{\prime}\right|=\Delta\left(T^{\prime}\right)$, we obtain $\Delta\left(T^{\prime}\right) \leq \Delta(T)-2$. On the other hand, $D \backslash\left\{M_{1}, M_{3}\right\}$ is a co-modular decomposition of $T^{\prime}$ (see Remark 3.1). Since $\left|D \backslash\left\{M_{1}, M_{3}\right\}\right|=|D|-2=\Delta(T)-2$, it follows that $\Delta\left(T^{\prime}\right) \geq \Delta(T)-2$. We conclude that $\Delta\left(T^{\prime}\right)=\Delta(T)-2$ as desired.

We now prove (5.8). Suppose toward a contradiction that there exists $I \in$ $\operatorname{mc}\left(T^{\prime}\right)$ such that $I \cap\left(M_{1} \cup M_{3}\right) \neq \varnothing$. We first show that $I \cap M_{1} \neq \varnothing$ and $I \cap M_{3} \neq \varnothing$. Suppose not. By interchanging $T$ and $T^{\star}$, as well as $M_{1}$ and $M_{3}$, we may assume $I \cap M_{3}=\varnothing$. Moreover, $I \cap M_{1} \neq \varnothing$ because $I \cap\left(M_{1} \cup M_{3}\right) \neq \varnothing$. Thus, Lemma 5.4 applies to the co-modular decomposition $\{M, N, L\}$ of $T$, where $(M, N, L)=\left(M_{1}, M_{3}, M_{2}\right)$, and yields $x \notin I$ and $I \in O_{T}\left(M_{1}\right)$. Since $o_{T}\left(M_{1}\right) \leq 1$ (see (C1)), we obtain $O_{T}\left(M_{1}\right)=\{I\}$. Therefore, $\widetilde{M_{1}}=M_{1} \cap I$ (see Notation5.1), which is a contradiction because $x \in \widetilde{M_{1}}$ and $x \notin I$. Thus $I \cap M_{1} \neq \varnothing$ and $I \cap M_{3} \neq \varnothing$. We now show that $I$ is not a module of $T^{\prime}$. Since $I$ and $M_{2}$ are distinct elements of $\operatorname{mc}\left(T^{\prime}\right)$ (see Remark 3.1), then by minimality of $I$ we have $M_{2} \backslash I \neq \varnothing$. Moreover, we have $T\left(M_{2} \backslash I, M_{1} \cap I\right)=0 \neq T\left(M_{2} \backslash I, M_{3} \cap I\right)=1$ (see (C2)) and thus $T^{\prime}\left(M_{2} \backslash I, M_{1} \cap I\right)=0 \neq T^{\prime}\left(M_{2} \backslash I, M_{3} \cap I\right)=1$. Therefore, $I$ is not a module of $T^{\prime}$. So $\bar{I}$ is a module of $T^{\prime}$. Moreover, since $o_{T^{\prime}}(I)=0$ by Lemma3.1, and $M_{2}, M_{4} \in \operatorname{mc}\left(T^{\prime}\right)$ by Remark 3.1 then $I \cap M_{\underline{2}}=I \cap M_{4}=\varnothing$. Let $v \in I \cap M_{1}$ and $w \in I \cap M_{3}$. We have $T^{\prime}(v, \bar{I})=1$ because $\bar{I}$ is a module of $T^{\prime}, M_{2} \subseteq \bar{I}$, and $T^{\prime}\left(v, M_{2}\right)=T\left(v, M_{2}\right)=1$ (see (C2)). Similarly, we have $T^{\prime}(w, \bar{I})=0$. Thus $T^{\prime}(v, \bar{I})=T^{\prime}(\bar{I}, w)=1$. Since $M_{4} \subseteq \bar{I}$, it follows that $T^{\prime}\left(v, M_{4}\right)=T^{\prime}\left(M_{4}, w\right)=1$ and thus $T\left(v, M_{4}\right)=T\left(M_{4}, w\right)=1$, which contradicts (C3). This completes the proof.

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[^0]:    Email addresses: houmem.belkhechine@ipeib.rnu.tn (Houmem Belkhechine), cherifa.bensalha@fsb.u-carthage.tn (Cherifa Ben Salha)

