

Equitable partition of planar graphs

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Abstract

An *equitable k -partition* of a graph G is a collection of induced subgraphs $(G[V_1], G[V_2], \dots, G[V_k])$ of G such that (V_1, V_2, \dots, V_k) is a partition of $V(G)$ and $-1 \leq |V_i| - |V_j| \leq 1$ for all $1 \leq i < j \leq k$. We prove that every planar graph admits an equitable 2-partition into 3-degenerate graphs, an equitable 3-partition into 2-degenerate graphs, and an equitable 3-partition into two forests and one graph.

Keywords: induced forest; degenerate graph; equitable partition; planar graph.

1 Introduction

All graphs in this paper are simple and finite. A *k -partition* of a graph G is a collection of induced subgraphs $(G[V_1], G[V_2], \dots, G[V_k])$ such that

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(V_1, V_2, \dots, V_k) is a partition of $V(G)$. Such a k -partition is *equitable* if

$$||V_i| - |V_j|| \leq 1$$

for all $i, j \in \{1, 2, \dots, k\}$. If there is no confusion, then we use (V_1, V_2, \dots, V_k) to denote a k -partition $(G[V_1], G[V_2], \dots, G[V_k])$ of G . We write $\Delta(G)$ to denote the maximum degree of a graph G .

In 1970, Hajnal and Szemerédi [9] proved a conjecture of Erdős, stating that every graph admits an equitable k -partition into empty subgraphs, if $k > \Delta(G)$. In 2008, Kierstead and Kostochka [10] found a short proof. In 2010, Kierstead, Kostochka, Mydlarz, and Szemerédi [12] designed a fast algorithm to find such an equitable k -partition. The bound on k in the Hajnal-Szemerédi Theorem is sharp because of complete graphs for instance. Thus, there have been many results in this field trying to obtain better lower bounds on the number k of parts for special graph classes. Motivated by Brooks' theorem, Chen, Lih, and Wu [5] conjectured that a connected graph G admits an equitable $\Delta(G)$ -partition into empty graphs if and only if it is not $K_{\Delta(G)+1}$, an odd cycle, or $K_{\Delta(G), \Delta(G)}$ (for odd $\Delta(G)$). They proved this conjecture for $\Delta(G) \leq 3$ and Kierstead and Kostochka [11] proved the conjecture for $\Delta(G) = 4$. For planar graphs, Zhang and Yap [20] proved this conjecture for $\Delta(G) \geq 13$, and Nakprasit [15] proved it for $\Delta(G) \geq 9$; in other words, he proved that every planar graph G has an equitable k -partition into empty subgraphs if $k \geq \max(\Delta(G), 9)$.

If we relax the condition on each part, then it is possible to reduce the number of parts significantly. For instance, Williams, Vandenbussche, and Yu [17] proved that for all $k \geq 3$, every planar graph of minimum degree at least 2 and girth at least 10 has an equitable k -partition into graphs of maximum degree at most 1.

We will mostly focus on the degeneracy of graphs. A graph is *d-degenerate* if every non-null subgraph has a vertex of degree at most d . Note that a graph is 0-degenerate if it has no edges, and 1-degenerate if it is a forest. Kostochka, Nakprasit, and Pemmaraju [13] studied the existence of an equitable k -partition of a d -degenerate graph into $(d - 1)$ -degenerate graphs.

Theorem 1.1 (Kostochka, Nakprasit, and Pemmaraju [13]). *For $k \geq 3$ and $d \geq 2$, every d -degenerate graph has an equitable k -partition into $(d - 1)$ -degenerate subgraphs.*

This implies that every 5-degenerate graph admits an equitable 3-partition into 4-degenerate subgraphs, an equitable 9-partition into 3-degenerate subgraphs, an equitable 27-partition into 2-degenerate subgraphs, and an equitable 81-partition into forests.

Now we restrict our attention to planar graphs. As planar graphs are 5-degenerate, every planar graph admits an equitable 81-partition into forests. How far can we reduce 81? Esperet, Lemoine, and Maffray [7] proved that 81 can be improved to 4.

Theorem 1.2 (Esperet, Lemoine, and Maffray [7]). *For all $k \geq 4$, every planar graph admits an equitable k -partition into forests.*

However it is not known whether 4 is tight. Indeed, Esperet, Lemoine, and Maffray [7] proposed the following problem:

Problem 1.3 (Esperet, Lemoine, and Maffray [7]). *Does every planar graph G admit an equitable 3-partition into forests?*

This problem still remains open and is known to have affirmative answers in the following cases:

- G is 2-degenerate, by Theorem 1.1 (even if G is non-planar),
- the girth of G is at least 5, due to Wu, Zhang, and Li [18],
- no two cycles of length at most 4 share vertices in G , due to Zhang [19],
- G has no triangles, and no two cycles of length 4 are adjacent, due to Zhang [19],
- G has an acyclic 4-coloring, due to Esperet, Lemoine, and Maffray [7].

By relaxing the condition further, we may ask the following question.

Problem 1.4. *For each i , what is the minimum integer k_i such that for all integers $k \geq k_i$, every planar graph admits an equitable k -partition into i -degenerate subgraphs?*

It is easy to see that $k_0 = \infty$ by considering $K_{1,n}$ for large n , see Meyer [14]. Since every planar graph is 5-degenerate, $k_i = 1$ for all $i \geq 5$. Theorem 1.2 implies that $k_1 \leq 4$. Not every planar graph admits a (not necessarily equitable) 2-partition into forests, shown by Chartrand and Kronk [4]. Thus, $k_1 \geq 3$.

Our first and second theorems prove that $k_3 = k_4 = 2$ and $k_2 \in \{2, 3\}$.

Theorem 2.1. *Every planar graph admits an equitable 2-partition into 3-degenerate graphs.*

Theorem 2.2. *Every planar graph admits an equitable 3-partition into 2-degenerate graphs.*

Our third theorem shows a weaker variant of Problem 1.3.

Theorem 3.1. *Every planar graph admits an equitable 3-partition into two forests and one graph.*

The rest of this paper is organized as follows. In Section 2, we prove Theorems 2.1 and 2.2, and moreover, show that every triangle-free planar graph admits an equitable 2-partition into 2-degenerate graphs. In Section 3, we prove Theorem 3.1 and illustrate some discussions towards Problem 1.3 and its relative problems.

2 Equitable partition into degenerate graphs

For a graph G and disjoint sets U, V of vertices of G , we denote by $e_G(U, V)$ the number of edges between U and V . If $U = \{u\}$ or $V = \{v\}$, then we simply write $e_G(u, V)$ or $e_G(U, v)$ for $e_G(U, V)$. For a vertex set $S \subseteq V(G)$ and vertices $v \in S$ and $u \notin V(G) - S$, let us write $S - v$ for the set $S - \{v\}$ and $S + u$ for the set $S \cup \{u\}$.

Our first theorem shows that $k_3 = k_4 = 2$.

Theorem 2.1. *Every planar graph admits an equitable 2-partition into 3-degenerate graphs.*

Proof. Let G be an n -vertex planar graph. We proceed by induction on $|E(G)|$. We may assume that G has at least one edge and $n \geq 4$.

As G is planar, it has a vertex v such that $0 < \deg(v) \leq 5$. Let v_1 be a neighbor of v . By the induction hypothesis, there is an equitable 2-partition (V_1, V_2) of $G - vv_1$ into 3-degenerate graphs. We may assume, without loss of generality, that $v \in V_1$. If $e_G(v, V_1 - v) \leq 3$, then (V_1, V_2) is an equitable 2-partition of G into 3-degenerate graphs. So we may assume that $e_G(v, V_1 - v) \geq 4$, and so $e_G(v, V_2) \leq 1$. Therefore, $V_2 + v$ induces a 3-degenerate subgraph of G .

If there is a vertex $w \in V_2$ so that $e_G(w, V_1 - v) \leq 3$, then $(V_1 - v + w, V_2 - w + v)$ is an equitable 2-partition of G into 3-degenerate graphs. Hence we assume that $e_G(w, V_1 - v) \geq 4$ for every $w \in V_2$, which implies that

$$e_G(V_2, V_1 - v) \geq 4|V_2| \geq 4\lfloor n/2 \rfloor \geq 2n - 2.$$

On the other hand, the graph induced by the edges between $V_1 - v$ and V_2 is a bipartite planar graph on $n - 1$ vertices, and therefore $e_G(V_2, V_1 - v) \leq 2(n - 1) - 4 = 2n - 6$, contradicting the other inequality. \square

Now we show that $2 \leq k_2 \leq 3$.

Theorem 2.2. *Every planar graph admits an equitable 3-partition into 2-degenerate graphs.*

Proof. Let G be an n -vertex planar graph. We proceed by induction on $|E(G)|$. We may assume that G has at least one edge and at least 4 vertices.

Since G is planar, there is a vertex v such that $1 \leq \deg(v) \leq 5$. Let v_1 be a neighbor of v . By applying the induction hypothesis to the graph $G - vv_1$, we obtain an equitable 3-partition (V_1, V_2, V_3) of $G - vv_1$ into 2-degenerate graphs. We may assume, without loss of generality, that $v \in V_1$. If $e_G(v, V_1 - v) \leq 2$, then (V_1, V_2, V_3) is also an equitable 3-partition of G into 2-degenerate graphs. So we may assume that $e_G(v, V_1 - v) \geq 3$, which implies that $e_G(v, V_2) \leq 2$ and $e_G(v, V_3) \leq 2$. Therefore, both $V_2 + v$ and $V_3 + v$ induce 2-degenerate subgraphs of G .

If there is a vertex $w \in V_2$ so that $e_G(w, V_1 - v) \leq 2$, then $V_1 - v + w$ induces a 2-degenerate subgraph of G . Hence, $(V_1 - v + w, V_2 - w + v, V_3)$ is an equitable 3-partition of G into 2-degenerate graphs. Now we assume that $e_G(w, V_1 - v) \geq 3$ for every $w \in V_2$, and by symmetry, we assume further that $e_G(w, V_1 - v) \geq 3$ for every $w \in V_3$. This implies that

$$e_G(V_2 \cup V_3, V_1 - v) \geq 3|V_2 \cup V_3| \geq 3 \cdot 2\lfloor n/3 \rfloor \geq 2(n - 2).$$

On the other hand, the graph induced by the edges between $V_2 \cup V_3$ and $V_1 - v$ is a bipartite planar graph on $n - 1$ vertices, and therefore

$$e_G(V_2 \cup V_3, V_1 - v) \leq 2(n - 1) - 4,$$

contradicting the previous inequality. □

We do not know whether $k_2 = 2$.

Problem 2.3. *Does every planar graph admit an equitable 2-partition into 2-degenerate graphs?*

We remark that Thomassen [16] proved that every planar graph admits a 2-partition into a 2-degenerate graph and a forest.

The following theorem shows that a possible counterexample to Problem 2.3 shall contain triangles.

Theorem 2.4. *Every triangle-free planar graph admits an equitable 2-partition into 2-degenerate graphs.*

Proof. We first prove by using the discharging method that every triangle-free planar graph contains a vertex of degree at most 2 or an edge with one end having degree 3 and the other having degree at most 6.

Suppose there exists a triangle-free planar graph G not satisfying the condition above. We assign initial charge $\mu(v) = \deg(v)$ to each vertex v of G , and let each vertex of degree at least 7 send $1/3$ to each of its neighbors of degree 3. After this discharging, the final charge $\mu'(v)$ of each vertex v of degree 3 is exactly $3 + 3 \times 1/3 = 4$ since vertices of degree 3 are only adjacent to vertices of degree at least 7. If v is a vertex of degree in $\{4, 5, 6\}$, then $\mu'(v) = \mu(v) \geq 4$. Lastly, each vertex v of degree at least 7 has final charge $\mu'(v) \geq \deg(v) - \deg(v)/3 \geq 14/3 > 4$. Since G is triangle-free and has minimum degree at least 3,

$$4n > 2e(G) = \sum_{v \in V(G)} \deg(v) = \sum_{v \in V(G)} \mu(v) = \sum_{v \in V(G)} \mu'(v) \geq 4n,$$

a contradiction. Therefore, the claim holds.

Now we prove the theorem.

Let G be an n -vertex triangle-free planar graph. We proceed by induction on n . If there is a vertex v of degree at most two, then by applying the induction hypothesis to the graph $G - v$, we obtain an equitable 2-partition (V_1, V_2) of $G - v$ into 2-degenerate graphs with $|V_1| \leq |V_2|$. Then, $(V_1 + v, V_2)$ is an equitable 2-partition of G into 2-degenerate graphs.

So we assume that every vertex of G has degree at least 3. Then, by the claim, there is an edge uv with $\deg(u) = 3$ and $\deg(v) \leq 6$. Applying the induction hypothesis to the graph $G - \{u, v\}$, we obtain an equitable 2-partition (V_1, V_2) of $G - \{u, v\}$ into two 2-degenerate graphs. Since v has at most five neighbors in $G - \{u, v\}$, we may assume that V_1 contains at most two neighbors of v . Then $(V_1 + v, V_2 + u)$ is an equitable 2-partition of G into 2-degenerate graphs. This completes the proof. \square

3 Equitable partition into 2 forests and 1 graph

In this section, we aim to prove the following theorem.

Theorem 3.1. *Every planar graph admits an equitable 3-partition into two forests and one graph.*

An *acyclic k -coloring* of a graph is a proper k -coloring such that there is no cycle consisting of two colors. In other words, if a graph has an acyclic k -coloring, then its vertex set can be partitioned into k independent sets

A_1, A_2, \dots, A_k such that $A_i \cup A_j$ induces a forest for all $i, j \in \{1, 2, \dots, k\}$. Borodin proved the following theorem, initially conjectured by Grünbaum [8].

Theorem 3.2 (Borodin [1]). *Every planar graph has an acyclic 5-coloring.*

To prove Theorem 1.2, Esperet, Lemoine, and Maffray used Theorem 3.2 and try to combine two color classes in an acyclic 5-coloring of planar graphs to produce large induced forests. We extend their idea.

3.1 Key proposition

To prove Theorem 3.1, we prove the following stronger statement.

Proposition 3.3. *Let $k > \ell \geq 1$ be integers. Let A_1, A_2, \dots, A_k be sets such that $\left| \bigcup_{i=1}^k A_i \right| = n$. Let*

$$q = \left\lfloor \frac{2}{k + \ell - 1} \left(n + \frac{k - \ell}{2} \right) \right\rfloor.$$

Then there exists a partition $(B_0, B_1, \dots, B_\ell)$ of $\bigcup_{i=1}^k A_i$ into $\ell + 1$ sets, possibly empty, such that

- (i) *for each $1 \leq i \leq \ell$, B_i is a subset of the union of two members of $\{A_1, A_2, \dots, A_k\}$,*
- (ii) *$|B_i| \geq q + 1$ if $1 \leq i \leq \lceil n - \frac{k+\ell-1}{2}q \rceil$,*
- (iii) *$|B_i| \geq q$ if $\lceil n - \frac{k+\ell-1}{2}q \rceil < i \leq \ell$,*
- (iv) *there exists $I \subseteq \{1, 2, \dots, k\}$ with $|I| = k - \ell - 1$ such that $B_0 \subseteq \bigcup_{i \in I} A_i$.*

Let us first see why Proposition 3.3 together with Theorem 3.2 implies Theorem 3.1

Proof of Theorem 3.1. Let G be a planar graph with n vertices. Then G has an acyclic 5-coloring by Theorem 3.2 and so there exist sets A_1, A_2, \dots, A_5 such that $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 = V(G)$ and $A_i \cup A_j$ induces a forest for all $1 \leq i < j \leq 5$. By applying Proposition 3.3 with $k := 5$ and $\ell := 2$, we have a partition (B_0, B_1, B_2) of $V(G)$ such that $|B_1|, |B_2| \geq q$ and both $G[B_1]$ and $G[B_2]$ are forests, where $q = \lfloor \frac{2}{6}(n + \frac{3}{2}) \rfloor = \lfloor \frac{n+1}{3} \rfloor$. If $n \not\equiv 2 \pmod{3}$, then we take $B'_1 \subseteq B_1$ and $B'_2 \subseteq B_2$ such that $|B'_1| = |B'_2| = \lfloor n/3 \rfloor$. If $n \equiv 2 \pmod{3}$, then we take $B'_1 \subseteq B_1$ and $B'_2 \subseteq B_2$ such that $|B'_1| = |B'_2| = \lfloor n/3 \rfloor + 1$. Let $B'_0 = V(G) \setminus (B'_1 \cup B'_2)$. Then (B'_0, B'_1, B'_2) is a desired equitable 3-partition. \square

Here is a key lemma to prove Proposition 3.3 inductively.

Lemma 3.4. *Let $2 \leq \ell < k$ and n be positive integers and let $q = \lfloor \frac{2n+k-\ell}{k+\ell-1} \rfloor$ and $n' = n - q$. Then*

$$\left\lfloor \frac{2n' + k - \ell}{k + \ell - 3} \right\rfloor = \begin{cases} q + 1 & \text{if } \lceil n - \frac{k+\ell-1}{2}q \rceil \geq \ell - 1, \\ q & \text{otherwise,} \end{cases}$$

and

$$n - \frac{k + \ell - 1}{2}q = n' - \frac{k + \ell - 3}{2}q.$$

Proof. Let $m := k + \ell - 1$, and let r be the integer such that $2n + k - \ell = mq + r$ and $0 \leq r < m$. Then

$$\left\lfloor \frac{2n' + k - \ell}{k + \ell - 3} \right\rfloor = \left\lfloor \frac{2(n - q) + k - \ell}{m - 2} \right\rfloor = \left\lfloor \frac{(m - 2)q + r}{m - 2} \right\rfloor = q + \left\lfloor \frac{r}{m - 2} \right\rfloor.$$

Note that $\lceil n - mq/2 \rceil \geq \ell - 1$ if and only if $r \geq k + \ell - 3 = m - 2$. Thus the first equation holds. The second equation is trivial. This completes the proof. \square

Proof of Proposition 3.3. We first observe that $\lceil n - \frac{k+\ell-1}{2}q \rceil \leq \ell - 1$, since

$$\frac{2n - 2\ell + 2}{k + \ell - 1} \leq \left\lfloor \frac{(2n - 2\ell + 2) + (k + \ell - 2)}{k + \ell - 1} \right\rfloor = \left\lfloor \frac{2n + k - \ell}{k + \ell - 1} \right\rfloor = q.$$

We may assume that $A_i \cap A_j = \emptyset$ for all $i \neq j$. We proceed by induction on ℓ . Let $a_i = |A_i|$ for all $1 \leq i \leq k$.

If $\ell = 1$, then by the pigeonhole principle, there exist $1 \leq i < j \leq k$ such that $a_i + a_j \geq \lceil \frac{2n}{k} \rceil$. Observe that

$$q = \left\lfloor \frac{2}{k} \left(n + \frac{k-1}{2} \right) \right\rfloor = \left\lfloor \frac{2n}{k} + \frac{k-1}{k} \right\rfloor = \left\lceil \frac{2n}{k} \right\rceil$$

and therefore we take a set $B_1 = A_i \cup A_j$. Then $|B_1| \geq q$ and B_0 is the union of the $k - 2$ members of $\{A_1, A_2, \dots, A_k\} \setminus \{A_i, A_j\}$. Now we may assume that $\ell > 1$.

Suppose that there exist $i \neq j$ such that $a_i \leq q \leq a_i + a_j$. Without loss of generality, let us assume $i = 1$ and $j = 2$. Then there exists B_ℓ such that $A_1 \subseteq B_\ell \subseteq A_1 \cup A_2$ and $|B_\ell| = q$. Let

$$n' = n - q \quad \text{and} \quad q' = \left\lfloor \frac{2}{k + \ell - 3} \left(n' + \frac{k - \ell}{2} \right) \right\rfloor.$$

By applying the induction hypothesis to the $k-1$ sets $A_2-B_\ell, A_3, A_4, \dots, A_k$, we obtain a partition $(B_0, B_1, B_2, \dots, B_{\ell-1})$ of $\bigcup_{i=2}^k A_i - B_\ell$, such that for $1 \leq i \leq \ell-1$, the set B_i is a subset of the union of two members of $\{A_2, A_3, \dots, A_k\}$ and B_0 is a subset of the union of $k-\ell-1$ members of $\{A_2, A_3, \dots, A_k\}$. If $\lceil n - \frac{k+\ell-1}{2}q \rceil = \ell-1$, then by Lemma 3.4, $q' = q+1$ and therefore the induction hypothesis provides that $|B_i| \geq q+1$ for all $1 \leq i \leq \ell-1$. If $\lceil n - \frac{k+\ell-1}{2}q \rceil < \ell-1$, then again by Lemma 3.4, $q' = q$ and $n - \frac{k+\ell-1}{2}q = n' - \frac{k+\ell-3}{2}q'$ and therefore we deduce (ii) and (iii) by the induction hypothesis.

Thus, we may assume that for all $i \neq j$, $a_i > q$ or $a_i + a_j < q$. Note that if $a_i > q$, then $a_i + a_j > q$ and therefore $a_j > q$. Thus we deduce that either

(I) $a_i > q$ for all $1 \leq i \leq k$, or

(II) $a_i + a_j < q$ for all $i \neq j$.

By the pigeonhole principle there exist $1 \leq i < j \leq k$ such that $a_i + a_j \geq \lceil \frac{2n}{k} \rceil$. Since

$$\left\lceil \frac{2n}{k} \right\rceil = \left\lfloor \frac{2}{k} \left(n + \frac{k-1}{2} \right) \right\rfloor \geq \left\lfloor \frac{2}{k+\ell-1} \left(n + \frac{k-\ell}{2} \right) \right\rfloor = q,$$

(II) does not hold and so (I) holds. Then we take $B_i = A_i$ for $i = 1, 2, \dots, \ell-1$, $B_\ell = A_\ell \cup A_{\ell+1}$, and $B_0 = \bigcup_{i=\ell+2}^k A_i$. \square

Proposition 3.3 is best possible in the sense that we cannot increase q ; consider the case that A_1, A_2, \dots, A_k are disjoint sets with $|A_1| = \ell a$, $|A_2| = |A_3| = \dots = |A_k| = a$ for some positive integer a . There are no ℓ disjoint sets of size at least $2a+1$, each contained in the union of two members of $\{A_1, \dots, A_k\}$, so we cannot increase q from $2a$ to $2a+1$.

When $\ell = k-1$, then we obtain the following from Proposition 3.3, which is due to Esperet, Lemoine, and Maffray [7].

Corollary 3.5 (Esperet, Lemoine, and Maffray [7]). *Let $k > 1$ be an integer. Let A_1, A_2, \dots, A_k be sets such that $\left| \bigcup_{i=1}^k A_i \right| = n$. Then there exists a partition (B_1, \dots, B_{k-1}) of $\bigcup_{i=1}^k A_i$ into $k-1$ sets such that for each $1 \leq i \leq k-1$, B_i is a subset of the union of two members of $\{A_1, A_2, \dots, A_k\}$, and $|B_i| = \lceil \frac{n}{k-1} \rceil$ or $|B_i| = \lfloor \frac{n}{k-1} \rfloor$*

Proof. We apply Proposition 3.3 with $\ell = k-1$ to obtain $(B_0, B_1, \dots, B_{k-1})$. Then $q = \lfloor \frac{1}{k-1}(n + \frac{1}{2}) \rfloor = \lfloor \frac{n}{k-1} \rfloor$. As B_0 is a subset of the union of 0 sets, $B_0 = \emptyset$. And, $\lceil n - \frac{k+\ell-1}{2}q \rceil = n - (k-1)q$ is exactly the remainder when

dividing n by $k - 1$ and therefore $|B_i| = q + 1$ for all $1 \leq i \leq \lceil n - \frac{k+\ell-1}{2}q \rceil$ and $|B_i| = q$ for all $\lceil n - \frac{k+\ell-1}{2}q \rceil < i \leq k - 1$. Thus $(B_1, B_2, \dots, B_{k-1})$ is a desired partition. \square

3.2 Discussions

Borodin and Ivanova [2] and Chen and Raspaud [6] independently showed that every planar graph without cycles of length 4 or 5 has an acyclic 4-coloring. By Corollary 3.5, if a planar graph has no cycle of length 4 or 5, then it admits an equitable 3-partition into forests. By Proposition 3.3, we have the following variation of Problem 1.3.

Corollary 3.6. *If a planar graph has no cycle of length 4 or 5, then it admits a partition of its vertex set into three sets A_1, A_2, A_3 such that*

- (i) *both A_1 and A_2 induce forests and $|A_1|, |A_2| \geq \lfloor \frac{2}{5}(n+1) \rfloor$,*
- (ii) *A_3 is independent.*

By the four color theorem, Corollary 3.5 implies that every planar graph admits an equitable 3-partition into bipartite graphs. We also deduce the following variant.

Corollary 3.7. *Every n -vertex planar graph admits a partition of its vertex set into three sets A_1, A_2, A_3 such that*

- (i) *both A_1 and A_2 induce bipartite subgraphs and $|A_1|, |A_2| \geq \lfloor \frac{2}{5}(n+1) \rfloor$,*
- (ii) *A_3 is independent.*

A *linear forest* is a forest of maximum degree at most 2. Cai, Xie, and Yang [3] showed that for every planar graph G , its vertices can be colored by $\Delta(G) + 7$ colors such that the union of any two color classes induces a linear forest. Combined with Proposition 3.3 and Corollary 3.5, we deduce that every planar graph G admits an equitable $(\Delta(G) + 6)$ -partition into linear forests.

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