# Minimum embedding of any Steiner triple system into a 3-sun system via matchings \*

Giovanni Lo Faro

Dipartimento di Scienze Matematiche e Informatiche, Scienze Fisiche e Scienze della Terra Università di Messina, Messina, Italia email: lofaro@unime.it, atripodi@unime.it

Antoinette Tripodi Dipartimento di Scienze Matematiche e Informatiche, Scienze Fisiche e Scienze della Terra Università di Messina, Messina, Italia email: lofaro@unime.it, atripodi@unime.it

#### Abstract

Let G be a simple finite graph and G' be a subgraph of G. A G'-design  $(X, \mathcal{B})$  of order n is said to be embedded into a G-design  $(X \cup U, \mathcal{C})$  of order n + u, if there is an injective function  $f : \mathcal{B} \to \mathcal{C}$  such that B is a subgraph of f(B) for every  $B \in \mathcal{B}$ . The function f is called an embedding of  $(X, \mathcal{B})$  into  $(X \cup U, \mathcal{C})$ . If u attains the minimum possible value, then f is a minimum embedding. Here, by means of König's Line Coloring Theorem and edge coloring properties a complete solution is given to the problem of determining a minimum embedding of any  $K_3$ -design (well-known as Steiner Triple System or, shortly, STS) into a 3-sun system or, shortly, a 3SS (i.e., a G-design where G is a graph on six vertices consisting of a triangle with three pendant edges which form a 1-factor).

**Keywords:** STS; 3-sun system; embedding; matching **MSC:** 05B05, 05B30.

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## 1 Introduction

If G is a graph, then let V(G) and  $\mathcal{E}(G)$  denote the vertex-set and edge-set of G, respectively. Given a set  $\Gamma$  of pairwise non-ismorphic simple graphs, a  $\Gamma$ -design of order n is a pair  $(X, \mathcal{B})$  where  $\mathcal{B}$  is a collection of graphs (called blocks) each isomorphic to some element of  $\Gamma$ , whose edges partition  $\mathcal{E}(K_n)$ , where  $K_n$  is the complete graph of order n on X; if the edges of the blocks of  $\mathcal{B}$  partition a proper spanning subgraph of  $K_n$ , then we speak of partial  $\Gamma$ -design of order n. If  $\Gamma = \{G\}$ , then we simply write G-design. Let  $\Sigma(G)$ denote the set of all integers n such that there exists a G-design of order n. A  $K_3$ -design of order n is known as Steiner triple system and denoted by STS(n); it is well-known that  $\Sigma(K_3) = \{n \in N : n \equiv 1, 3 \pmod{6}\}$ .

Let G be a simple finite graph and G' be a subgraph of G. A G'-design  $(X, \mathcal{B})$  of order n is said to be *embedded* into a G-design  $(X \cup U, \mathcal{C})$  of order n + u, if there is an injective function  $f : \mathcal{B} \to \mathcal{C}$  such that B is a subgraph of f(B) for every  $B \in \mathcal{B}$ . The function f is called an *embedding* of  $(X, \mathcal{B})$  into  $(X \cup U, \mathcal{C})$ . If u attains the minimum possible value, then f is a *minimum* embedding. Note that a special case occurs when G = G' and the related embedding problem is better known as Doyen-Wilson problem (see [6, 8, 11, 12, 13]).

The embedding problems have interesting applications to networks ([5]), that is why they have been investigated in several papers. In particular, the minimum embedding problem of STSs into G-designs have been studied in the case when  $G = K_4$ ,  $G = K_4 - e$  (the complete graph on four vertices with one deleted edge), or  $G = K_3 + e$  (a *kite*, i.e., a triangle with one pendant edge) have been solved in [3], [4], [9], [14].

In [7] the authors embed a cyclic STS of order  $n \equiv 1 \pmod{6}$  into a 3sun system of order 2n-1, i.e., a *G*-design where *G* is a graph on six vertices consisting of a triangle with three pendant edges which form a 1-factor, and as an open problem they ask whether it is possible to embed any STS into a 3-sun system. Here we give an answer to this open problem by determining the minimum embedding for any Steiner triple system. More precisely, for every integer  $n \in \Sigma(K_3)$  detoted by  $u_{min}(n)$  the minimum integer u such that any STS(n) can be embedded into a 3-sun system of order n + u, as main result we prove the following theorem.

#### Main Theorem

(i) If  $n \equiv 1, 3, 9, 19 \pmod{24}$ , then  $u_{min}(n) = \frac{n-1}{2}$  for every  $n \neq 3, 9$ ,  $u_{min}(3) = 6$ , and  $u_{min}(9) = 7$ .

(ii) If  $n \equiv 7, 13, 15, 21 \pmod{24}$ , then  $u_{min}(n) = \frac{n-1}{2} + 2$  for every  $n \neq 7, 13, u_{min}(7) = 6$ , and  $u_{min}(13) = 11$ .

To obtain our result we make use of some results on edge colorings and, in particular, of König's Line Coloring Theorem, which here, for convenience, is formulated in terms of matchings (for definitions and results on edge colorings or matchings, the reader is referred to [1]).

**Theorem 1.1** (König's Line Coloring Theorem) Let G be a bipartite multigraph with maximum degree  $\Delta$ . Then  $\mathcal{E}(G)$  can be partitioned into  $M_1, M_2, \ldots, M_\Delta$  such that each  $M_i, 1 \leq i \leq \Delta$ , is a matching in G.

#### 2 Notation and basic results

In what follows, we will denote:

- the triangle on the vertices a, b and c by (a, b, c);
- the kite consisting of the triangle (a, b, c) and the pendant edge  $\{c, d\}$  by (a, b, c; d);
- the *bull* graph consisting of the triangle (a, b, c) and the pendant edges  $\{b, d\}$  and  $\{c, e\}$  by (a, b, c; d, e);
- the 3-sun consisting of the triangle (a, b, c) and the pendant edges  $\{a, d\}, \{b, e\}$  and  $\{c, f\}$  by (a, b, c; d, e, f).

If G is a kite, a bull, or a 3-sun, then its triangle will be denoted by t(G).

In this section we will give the necessary condition for embedding a Steiner triple system into a 3-sun system and prove some useful results to get our main result. From now on, if f is an embedding of  $(X, \mathcal{T})$  into  $(X \cup U, S)$ , then  $f(\mathcal{T})$  will be denoted by  $S_{\mathcal{T}}$ . Finally, we recall that a 3-sun system of order n, or shortly a 3SS(n), exists if and only if  $n \equiv 0, 1, 4, 9 \pmod{12}$  (see [7]).

**Lemma 2.1** If there exists a 3SS(n+u) embedding an STS(n), then  $u \ge \frac{n-1}{2}$ .

*Proof.* Since an STS(n) has  $\frac{n(n-1)}{6}$  triples, then in order to complete every triple so to abtain a 3-sun, necessarily  $n \cdot u \ge 3\frac{n(n-1)}{6}$  and so  $u \ge \frac{n-1}{2}$ .  $\Box$ 

In general, to construct a 3SS(m)  $(X \cup U, S)$  embedding a STS(n) (X, T), we need to complete each triangle of T to a 3-sun by using some edges of the complete bipartite graph  $K_{n,m-n}$  on  $X \cup U$  and partition into 3suns the remaining edges of  $K_{n,m-n}$  along with those of the complete graph  $K_{n,m-n}$  on U. In the following lemma a partial 3SS embedding an STS(n)is constructed by using all the edges of the above complete bipartite graph.

## **Lemma 2.2** Any STS(n), $n \ge 7$ , can be embedded into a partial $3SS(\frac{3n-1}{2})$ .

Proof. Let  $(X, \mathcal{T})$  be an STS(n) and consider its incidence graph  $\mathcal{I}$ , i.e., the bipartite graph whose vertex set is  $X \cup \mathcal{T}$  and whose edges are determined by joining  $x \in X$  to  $t \in \mathcal{T}$  if and only if  $x \in t$ . In the graph  $\mathcal{I}$  every vertex of X has degree  $\frac{n-1}{2}$  and every vertex of  $\mathcal{T}$  has degree 3. Since the maximum degree of  $\mathcal{I}$  is  $\Delta = \frac{n-1}{2}$ , by König's Line Coloring Theorem the edges of  $\mathcal{I}$  can be partitioned into  $\Delta$  matchings  $M_1, M_2, \ldots, M_\Delta$ , each of which satures the vertices of X, i.e., every vertex of X is incident to an edge of each matching. Let  $\mathcal{S}$  be the set of 3-suns on  $X \cup \{M_1, M_2, \ldots, M_\Delta\}$ obtained by completing each triple of  $\mathcal{T}$  to a 3-sun as follows: for every  $t = (x_1, x_2, x_3) \in \mathcal{T}$ , consider the 3-sun  $(x_1, x_2, x_3; M_{i_1}, M_{i_2}, M_{i_3})$ , where  $\{x_j, t\} \in M_{i_j}$  for every j = 1, 2, 3.  $(X \cup \{M_1, M_2, \ldots, M_\Delta\}, \mathcal{S})$  is a partial  $3SS(\frac{3n-1}{2})$  embedding  $(X, \mathcal{T})$ .

The lower bound given by Lemma 2.1 is attained if  $n \equiv 1, 3, 9, 19 \pmod{24}$ ,  $n \neq 3, 9$ , as it is established by the following proposition.

**Proposition 2.1** For every  $n \equiv 1, 3, 9, 19 \pmod{24}$ ,  $n \geq 19$ ,  $u_{min}(n) = \frac{n-1}{2}$ .

Proof. Let  $(X, \mathcal{T})$  be any  $\operatorname{STS}(n)$  with  $n \equiv 1, 3, 9, 19 \pmod{24}$ ,  $n \geq 19$ . By Lemma 2.2, it can be embedded into a partial  $\operatorname{3SS}(\frac{3n-1}{2}) (X \cup \{M_1, M_2, \dots, M_{\frac{n-1}{2}}\}, \mathcal{S})$ . Since  $\frac{n-1}{2} \equiv 0, 1, 4, 9 \pmod{12} \geq 9$ , there exists a  $\operatorname{3SS}(\frac{n-1}{2}) (\{M_1, M_2, \dots, M_{\frac{n-1}{2}}\}, \mathcal{S}')$ . Then  $(X \cup \{M_1, M_2, \dots, M_{\frac{n-1}{2}}\}, \mathcal{S} \cup \mathcal{S}')$  is a  $\operatorname{3SS}(\frac{3n-1}{2})$  which embeds  $(X, \mathcal{T})$ .  $\Box$ 

**Lemma 2.3** If n = 3, 9, then  $u_{min}(n) = 6, 7$ , respectively.

*Proof.* Any STS(3) can be trivially embedded into a 3SS of any admissible order  $v \ge 9$  and so  $u_{min}(3) = 6$ .

Let  $(X \cup U, S)$  be a 3SS(9+u) embedding an STS(9)  $(X, \mathcal{T})$ . By Lemma 2.1  $u \geq 4$ . If u = 4, then  $S \setminus S_{\mathcal{T}}$  contains only one 3-sun S such that  $V(S) \subseteq$ 

U, which is impossible and so  $u_{min}(9) \ge 7$ . To prove that  $u_{min}(9) = 7$ , on  $Z_{16}$  we give the blocks of a 3SS embedding the unique STS(9) (whose triangles are in bold):

 $\begin{array}{l} (\mathbf{0},\mathbf{1},\mathbf{2};9,10,11), \ ((\mathbf{0},\mathbf{3},\mathbf{6};10,15,9), \ (\mathbf{0},\mathbf{4},\mathbf{8};11,9,13), \\ (\mathbf{0},\mathbf{5},\mathbf{7};12,9,15), \ (\mathbf{1},\mathbf{3},\mathbf{8};9,10,11), \ (\mathbf{1},\mathbf{4},\mathbf{7};11,10,9), \\ (\mathbf{1},\mathbf{5},\mathbf{6};12,10,13), \ (\mathbf{2},\mathbf{3},\mathbf{7};9,11,12), \ (\mathbf{2},\mathbf{4},\mathbf{6};10,11,12), \\ (\mathbf{2},\mathbf{5},\mathbf{8};12,13,14), \ (\mathbf{3},\mathbf{4},\mathbf{5};9,12,14), \ (\mathbf{6},\mathbf{7},\mathbf{8};11,10,9), \\ (0,13,15;14,7,8), \ (1,14,15;13,4,9), \ (3,12,14;13,15,11) \\ (2,13,14;15,4,7), \ (5,11,12;15,7,10), \ (6,10,14;15,8,9), \\ (9,12,13;10,8,11), \ (10,11,15;13,9,4). \end{array}$ 

**Lemma 2.4** Let  $n \equiv 7, 13, 15, 21 \pmod{24}$ . If there exists a 3SS(n+u) embedding an STS(n), then  $u \ge \frac{n-1}{2} + 2$ .

*Proof.* Let n = 24k + r,  $r \in \{7, 13, 15, 21\}$ . If (X, S) is a 3SS(n + u) embedding an STS(n), then by Lemma 2.1  $n + u \ge \frac{3n-1}{2} = 36k + \frac{3r-1}{2}$ , where  $\frac{3r-1}{2} \in \{10, 19, 22, 31\}$ . Since  $n + u \equiv 0, 1, 4, 9 \pmod{12}$ , this implies  $u \ge \frac{n-1}{2} + 2$ 

**Remark 2.1** For every  $n \equiv 7, 13, 15, 21 \pmod{24}$ , if  $(X \cup U, S)$  is a  $3SS(n + \frac{n+3}{2})$  embedding an STS(n) (X, T), then each vertex  $x \in X$  appears in exactly two block of  $S \setminus S_T$  as a pendant vertex (therefore, for every  $S \in S \setminus S_T$  the vertices of t(S) are in U).

The lower bound established by Lemma 2.4 is not attained when n = 7, 13, as it is showed by the following lemma.

**Lemma 2.5** If n = 7, 13, then  $u_{min}(n) = 6, 11$ , respectively.

*Proof.* Let  $(X \cup U, S)$  be a 3SS(n+u) embedding an STS(n)  $(X, \mathcal{T})$ , where n = 7, 13. By Lemma 2.4,  $u \ge \frac{n+3}{2}$ .

If n = 7 and u = 5, then  $|S \setminus S_{\mathcal{T}}| = 4$ , whereas by Remark 2.1  $|S \setminus S_{\mathcal{T}}| > 4$ , and so  $u_{min}(7) \ge 6$ . To prove that  $u_{min}(7) = 6$ , on  $Z_{13}$  we give the blocks of a 3SS embedding the unique STS(7):

(0, 1, 2; 7, 8, 9), ((0, 3, 4; 8, 7, 9), (0, 5, 6; 9, 8, 10), (1, 3, 5; 9, 8, 7), (1, 4, 6; 10, 7, 12), (2, 3, 6; 7, 9, 8), (2, 4, 5; 8, 11, 9), (0, 10, 11; 12, 5, 3), (1, 7, 12; 11, 8, 5), (2, 10, 12; 11, 9, 8), (4, 8, 10; 12, 9, 3),

(9, 11, 12; 7, 5, 3), (6, 7, 11; 9, 10, 8).

If n = 13 and u = 8, then  $|S \setminus S_{\mathcal{T}}| = 9$ . Therefore, by Remark 2.1 a partial triple system on U with 9 triangles should be exist, which is impossible because a maximum packing of  $K_8$  with triangles (i.e., a partial  $K_3$ -design of order 8 with the maximum number of blocks) have 8 blocks, and so  $u_{min}(13) \geq 11$ . Since there are two non-isomorphic STS(13)s, in order to prove that  $u_{min}(13) = 11$  we need to embed each STS(13). Firstly, we embed the cyclic one into a 3SS on  $Z_{24}$  as follows:

(0, 1, 4; 13, 18, 14), (1, 2, 5; 13, 23, 14), (2, 3, 6; 13, 18, 14),(3, 4, 7; 13, 15, 14), (4, 5, 8; 13, 15, 14), (5, 6, 9; 13, 18, 19),(6, 7, 10; 13, 15, 14), (7, 8, 11; 13, 15, 14), (8, 9, 12; 13, 20, 15),(9, 10, 0; 13, 15, 14), (10, 11, 1; 13, 16, 14), (11, 12, 2; 13, 16, 14),(12, 0, 3; 13, 15, 23), (0, 2, 7; 16, 22, 21), (1, 3, 8; 15, 19, 16),(2, 4, 9; 15, 16, 14), (3, 5, 10; 14, 16, 17), (4, 6, 11; 17, 15, 18),(5, 7, 12; 17, 16, 14), (6, 8, 0; 16, 17, 18), (7, 9, 1; 17, 15, 16),(8, 10, 2; 18, 16, 17), (9, 11, 3; 16, 15, 17), (10, 12, 4; 18, 17, 19),(11, 0, 5; 17, 19, 18), (12, 1, 6; 18, 17, 19),(0, 17, 20; 21, 6, 1), (1, 19, 21; 22, 2, 3), (2, 16, 18; 20, 3, 4),(3, 15, 20; 22, 13, 4), (4, 21, 22; 23, 2, 0), (5, 19, 20; 21, 7, 6),(6, 21, 23; 22, 8, 0), (7, 18, 20; 22, 9, 8), (8, 19, 22; 23, 10, 5),(9, 17, 21; 22, 13, 10), (10, 20, 22; 23, 11, 12), (11, 19, 23; 21, 12, 1),(12, 20, 21; 23, 13, 14), (13, 14, 16; 18, 15, 17), (13, 22, 23; 19, 11, 5),(14, 17, 19; 18, 15, 16), (14, 20, 23; 22, 16, 7), (15, 16, 21; 19, 23, 13),(15, 18, 22; 23, 19, 16), (17, 18, 23; 22, 21, 9).

A 3SS(24) embedding the non cyclic STS(13) can be obtained from the above one by replacing the 3-suns

(0, 1, 4; 13, 18, 14), (0, 2, 7; 16, 22, 21), (2, 4, 9; 15, 16, 14), (7, 9, 1; 17, 15, 16),

with

 $\begin{array}{ll} (\mathbf{9},\mathbf{1},\mathbf{4};14,18,16), & (\mathbf{9},\mathbf{2},\mathbf{7};15,22,17), \\ (\mathbf{0},\mathbf{2},\mathbf{4};16,15,14), & (\mathbf{0},\mathbf{1},\mathbf{7};13,16,21), \end{array}$ 

In order to prove that for every  $n \equiv 7, 13, 15, 21 \pmod{24}$ ,  $n \neq 7, 13$ ,  $u_{min}(n)$  equals the lower bound of Lemma 2.4, it will be useful the following lemma.

**Lemma 2.6** ([1]) Let M and N be disjoint matchings of a graph G with |M| > |N|. Then there are disjoint matchings M' and N' of G such that |M'| = |M| - 1, |N'| = |N| + 1 and  $M' \cup N' = M \cup N$ .

Now, we determine  $u_{min}(n)$  for every  $n \equiv 7, 13, 15, 21 \pmod{24}$  with the exception of few small orders, which will be settled in Section 3.

In graph theory, the degree of a vertex of a graph is the number of edges that are incident to the vertex; here, we define 2-*degree* of a vertex x of a  $\Gamma$ -design  $\mathcal{D}$ , and denote by  $d_2(x)$ , the number of blocks of  $\mathcal{D}$  containing xas a vertex of degree 2. The 2-*degree sequence* of  $\mathcal{D}$  is the non-decreasing sequence of its vertex 2-degrees.

In what follows, if G is a graph whose vertices belong to  $Z_u$ , then we call *orbit of* B under  $Z_u$  the set  $(G) = \{G + i : i \in Z_u\}$ , where G + i is the graph with  $V(G + i) = \{a + i : a \in V(G)\}$  and  $\mathcal{E}(G + i) = \{\{a + i, b + i\} : \{a, b\} \in V(G)\}$ .

**Lemma 2.7** For any u = 12k + h, h = 5, 8, 9, 12 and  $k \ge 3$ , there exists a {bull, 3-sun}-design of order u whose 2-degree sequence is (2, 3, 3, 3, 3, 4, 4, ..., 4).

Proof. Consider the following orbits under  $Z_u$ : for  $i = 1, 2, 3, \mathcal{B}_i = (B_i)$ , where  $B_1 = (0, 6k - 2, 4k + 3; 3k, 6k - 1), B_2 = (6k, 0, 4k + 1; 6k + 2, 6k + 1),$ and  $B_3 = (0, 6k - 1, 4k + 2; 3k, 6k)$ ; for  $j = 0, 1, \ldots, k - 4, \mathcal{S}_j = (S_j)$ , where  $S_j = (5k + 1 + j, 5k - j, 0; 3k, k, u - 2 - 2j)$ . On  $Z_u$  define the set of graphs  $\mathcal{A} = (\mathcal{B}_1 \cup \mathcal{B}_2^* \cup \mathcal{B}_3^* \cup \mathcal{B}) \cup [(\cup_{j=0}^{k-4}\mathcal{S}_j) \cup \mathcal{S}^* \cup \mathcal{S}]$ , where  $\mathcal{B}_2^* = \mathcal{B}_2 \setminus \{B_2\},$  $\mathcal{B}_3^* = \mathcal{B}_3 \setminus \{B_3 + i : i = 0, 4k + 1, 6k, 6k + 1, 6k + 2\}, \mathcal{S}^* = \{(6k - 1, 4k + 2, 0; 3k, 6k, 4k + 1), (10k, 8k + 3, 4k + 1; 7k + 1, 10k + 1, 6k), (12k - 1, 10k + 2, 6k; 9k, 12k, 0), (12k, 10k + 3, 6k + 1; 9k + 1, 12k + 1, 4k + 1), (12k + 1, 10k + 4, 6k + 2; 9k + 2, 12k + 2, 0)\}$ , while  $\mathcal{B}$  and  $\mathcal{S}$  depend on h. a) h = 5:  $\mathcal{B}$  is the orbit of (6k + 1, 0, 3k; 3k + 2, 6k + 3) under  $Z_u; \mathcal{S} = \emptyset$ .

 $b) h = 8: \ \mathcal{B} = \{(6k+3+i, i, 3k+i; 6k+4+i, 9k+1+i), (9k+5+i, 3k+2+i, 6k+2+i; 6k+4+i, 12k+3+i) : i = 0, 1, \dots, 3k+1, i \in \mathbb{Z}_u\} \cup \{(12k+7+i, 6k+4+i, 9k+4+i; 9k+5+i, 3k-3+i) : i = 0, 1, \dots, 6k+3, i \in \mathbb{Z}_u\}; \\ \mathcal{S} = \{(i, 3k+2+i, 9k+6+i; 3k+1+i, 6k+3+i, 6k+4+i)) : i = 0, 1, \dots, 3k+1, i \in \mathbb{Z}_u\}.$ 

c) h = 9:  $\mathcal{B}$  is the orbit of (6k + 1, 0, 3k; 3k + 3, 9k + 3) under  $Z_u$ ;  $\mathcal{S} = \{(3i, 3k + 2 + 3i, 6k + 4 + 3i; 6k + 5 + 3i, 9k + 7 + 3i, 9k + 6 + 3i)\}$ :  $i = 0, 1, \dots, 4k + 2, i \in Z_u\}$ .

d) h = 12:  $\mathcal{B} = \{(6k+1+i, i, 3k+i; 6k+6+i, 9k+5+i), (9k+4+i, 3k+i)\}$ 

 $\begin{array}{l} 3+i, 6k+3+i; 6k+6+i, 12k+8+i): i=0,1,\ldots, 3k+2, \ i\in Z_u\}\cup\{(12k+7+i, 6k+6+i, 9k+6+i; 12k+9+i, 3k-1+i): i=0,1,\ldots, 6k+5, \ i\in Z_u\};\\ \mathcal{S}=\{(i, 3k+3+i, 9k+9+i; 6k+3+i, 9k+6+i, 6k+6+i)): i=0,1,\ldots, 3k+2, \ i\in Z_u\}\cup\{(3i, 3k+2+3i, 6k+4+3i; 6k+8+3i, 9k+10+3i, 9k+6+3i)): i=0,1,\ldots, 4k+3, \ i\in Z_u\}.\end{array}$ 

 $(Z_u, \mathcal{A})$  is the required design, where  $d_2(6k) = 2$ , the vertices  $d_2(0) = d_2(4k+1) = d_2(6k+1) = d_2(6k+2) = 3$ , and the remaining vertices have 2-degree 4.

**Proposition 2.2** For every  $n \equiv 7, 13, 15, 21 \pmod{24}$ ,  $n \ge 79$ ,  $u_{min}(n) = \frac{n+3}{2}$ .

*Proof.* Let  $(X, \mathcal{T})$  be an STS(n),  $n \equiv 7, 13, 15, 21 \pmod{24}$ ,  $n \geq 79$ , and  $\mathcal{I}$  be its incidence graph.  $\mathcal{E}(\mathcal{I})$  can be partitioned into  $\Delta = \frac{n-1}{2}$  matchings  $M_1, M_2, \ldots, M_{\Delta}$  (see proof of Lemma 2.2). By applying Lemma 2.6 and by using similar arguments as the proof of Theorem 6.3 in [1], it is possible to partition  $\mathcal{E}(\mathcal{I})$  into  $\Delta + 2$  matchings  $M'_1, M'_2, \ldots, M'_{\Delta+2}$ , such that  $M'_i$  covers the vertices of  $X \setminus X_i$ , where  $|X_1| = 2$ ,  $|X_i| = 3$  for i = 2, 3, 4, 5, and  $|X_i| = 4$  for  $i = 6, 7, \ldots, \Delta + 2$  (note that each vertex of X is missing in exctaly two matchings). If  $\mathcal{S}$  denotes the set of 3-suns on  $X \cup \{M'_1, M'_2, \dots, M'_{\Delta+2}\}$  obtained by completing each triple of  $\mathcal{T}$  as in the proof of Lemma 2.2, the pair  $(X \cup \{M'_1, M'_2, \dots, M'_{\Delta+2}\}, S)$  is a partial  $3SS(\frac{3(n+1)}{2})$  embedding  $(X, \mathcal{T})$ . In order to complete the proof it will be sufficient to decompose the graph  $K_{\Delta+2} \cup \mathcal{M}$  into 3-suns, where  $K_{\Delta+2}$  is the complete graph based on  $\{M'_1, M'_2, \ldots, M'_{\Delta+2}\}$  and  $\mathcal{M}$  is the bipartite graph on  $X \cup \{M'_1, M'_2, \dots, M'_{\Delta+2}\}$  such that  $\{x, M'_i\} \in \mathcal{E}(\mathcal{M})$  if and only if  $x \in X$  is missing in  $M'_i$ . By using Lemma 2.7, the complete graph  $K_{\Delta+2}$ can be decomposed into bulls or 3-suns so that  $d_2(M'_1) = 2$ ,  $d_2(M'_i) = 3$  for i = 2, 3, 4, 5, and  $d_2(M'_i) = 4$  for  $i = 6, 7, ..., \Delta + 2$ . To obtain the required decomposition it is sufficient to complete each bull to a 3-sun using the edges of  $\mathcal{M}$ . 

#### 3 Cases left

To determine  $u_{min}(n)$  for the remaining orders  $n \in \{15, 21, 31, 37, 39, 45, 55, 61, 63, 69\}$ , we will start from an  $STS(n)(X, \mathcal{T})$ , with  $X = \{x_1, x_2, \ldots, x_n\}$ , and prove that  $(X, \mathcal{T})$  can be embedded in a 3-sun system  $(X \cup Z_{\frac{n+3}{2}}, \mathcal{S})$  by taking the following steps.

Step 1. Partition the edges of the complete graph on  $Z_{\frac{n+3}{2}}$  into a set  $\mathcal{A}$  of

triangles, kites, bulls or 3-suns so that  $|\mathcal{A}| = |\mathcal{S} \setminus \mathcal{T}| = (n^2 + 20n + 3)/48$  and  $\sum_{i=0}^{(n+1)/2} d_2(i) = 2n$ . For later convenience (see *Step* 4.), give  $\mathcal{A}$  partitioned into suitable subsets  $\mathcal{A}_j$ ,  $j \in J$ , such that for every  $j \in J$  and for every vertex  $i \in \mathbb{Z}_{\frac{n+3}{2}}$ , the number of blocks of  $\mathcal{A}_j$  containing i as a vertex of degree 2 is at most 1.

Step 2. Partition the edge-set of the incidence graph  $\mathcal{I}$  of  $(X, \mathcal{T})$  into  $\frac{n+3}{2}$  matchings  $M_0, M_1, \ldots, M_{\frac{n+1}{2}}$  such that, denoted by  $X_i$  the set of vertices of X not satured by  $M_i, |X_i| = d_2(i)$  for each  $i = 0, 1, \ldots, \frac{n+1}{2}$ .

Step 3. Complete each triple of  $\mathcal{T}$  as in the proof of Lemma 2.2 and obtain a partial 3-sun system  $(X \cup \{M_0, M_1, \ldots, M_{\frac{n+1}{2}}\}, \mathcal{S})$  embedding  $(X, \mathcal{T})$ .

Step 4. Call missing graph the bipartite graph  $\mathcal{M}$  on  $X \cup \{M_0, M_1, \ldots, M_{\frac{n+1}{2}}\}$  consisting of all the edges  $\{x, M_i\}$  such that  $x \in X_i$  and, for the sake of simplicity, for every  $i = 0, 1, \ldots, \frac{n+1}{2}$  identify  $M_i$  with  $i \in \mathbb{Z}_{\frac{n+3}{2}}$ .

Step 5. Partition the edges of the missing graph into suitable matchings  $M'_j$ ,  $j \in J$ , such that for every  $j \in J$  the edges of  $M'_j$  can be used to complete the blocks of  $\mathcal{A}_j$  so to obtain a 3-sun system of order  $\frac{3(n+1)}{2}$  embedding  $(X, \mathcal{T})$ .

To begin with, we give an alternative solution for  $n \equiv 15 \pmod{24}$ (which settles the orders v = 15, 39, 63 as well) by means of a technique used in [7] and involving the concepts of parallel classes and resolution of an STS.

A parallel class of an STS(n) is a set of  $\frac{n}{3}$  triples such that no two triples in the set share an element; a partition of all triples of an STS(n)into parallel classes is a *resolution* and the STS is said to be *resolvable*. An STS(n) together with a resolution of its triples is a *Kirkman triple system*, KTS(n), and exists if and only if  $n \equiv 3 \pmod{6}$  (see [2]).

**Proposition 3.1** For every  $n \equiv 15 \pmod{24}$ ,  $u_{min}(n) = \frac{n+3}{2}$ .

*Proof.* Let  $(X, \mathcal{T})$  be an STS(n), n = 24k + 15,  $k \ge 0$ . Consider a resolution  $P_i$ ,  $i = 1, 2, \ldots, 6k + 4$  of a KTS on  $\mathbb{Z}_{\frac{n+3}{2}}$ . Without loss of generality, assume that  $P_1$  contains the triangle t = (0, 1, 2). Construct a set  $\mathcal{K}$  of kites obtained by attaching the edges of t to the triangles  $t_1, t_2, t_3$  of  $P_2$  containing 0, 1, 2, respectively, and the set  $\mathcal{A}_0$  of 3-suns obtained from the parallel classes  $P_i$ ,  $i = 5, 6, \ldots, 6k + 4$  by using the technique in Lemma 3.8 of [7]. The set  $\mathcal{A} = \cup_{j=0}^4 \mathcal{A}_j$ , where  $\mathcal{A}_1 = P_1 \setminus \{t\}$ ,  $\mathcal{A}_2 = (P_2 \setminus \{t_1, t_2, t_3\}) \cup \mathcal{K}$  and  $\mathcal{A}_j = P_j$  for j = 3, 4, is a partition of  $\mathcal{E}(K_{\frac{n+3}{2}})$  such that  $\sum_{i=0}^{(n+1)/2} d_2(i) = 2n$ . After applying Step 2. , Step 3. and Step 4. proceed as follows. It is easy to

see that the missing graph admits two matchings  $M'_1$  and  $M'_2$  both saturing the vertices  $3, 4, \ldots, \frac{n+1}{2}$ ; while, the edges of  $\mathcal{M}$  not in  $M'_1$  and  $M'_2$  form a subgraph with maximum degree 2 and so can be partitioned into two matchings  $M'_3$  and  $M'_4$  both saturing all the vertices of  $Z_{\frac{n+3}{2}}$ . For every j = 1, 2, 3, 4, complete the blocks of  $\mathcal{A}_j$  by using the edges of  $M'_j$ .  $\Box$ 

**Proposition 3.2** For every  $n \in \{21, 31, 37, 45, 55, 61, 69\}$ ,  $u_{min}(n) = \frac{n+3}{2}$ .

*Proof.* Let  $(X, \mathcal{T})$  be an STS(n).

For n = 21, partition the edges of the complete graph on  $Z_{12}$  into the following set  $\mathcal{A}$ :

$$\begin{aligned} \mathcal{A}_1 &= \{ (1,2,0;11), (3,7,2;5), (0,4,3;9) \} \} \\ \mathcal{A}_2 &= \{ (0,5,6), (1,8,11), (7,4,10), (2,9,8;10), (3,1,5;10,8) \} \\ \mathcal{A}_3 &= \{ (0,9,10), (3,6,8), (5,7,11), (2,4,11;6), (1,7,9;6,11) \} \\ \mathcal{A}_4 &= \{ (0,7,8), (3,10,11), (5,9,4;8), (1,4,6;9), (2,6,10;5) \} \end{aligned}$$

where  $d_2(i) = 3$  for  $i \in \{5, 6, 8, 9, 10, 11\}$  and  $d_2(i) = 4$  for  $i \in \{0, 1, 2, 3, 4, 7\}$ . After applying *Step* 2., *Step* 3. and *Step* 4. proceed as follows. Since  $\mathcal{M}$  has maximum degree 4, it is easy to see that  $\mathcal{M}$  admits a matching  $M'_1$  saturing  $\{0, 1, 2, 3, 4, 7\}$ . Use  $M'_1$  to complete the kites in  $\mathcal{A}_1$ . The graph obtained from  $\mathcal{M}$  by deleting the edges of  $M'_1$  is a bipartite graph such that all the vertices in  $Z_{12}$  has degree 3 and so its edges can be partitioned into three matchings  $M'_2$ ,  $M'_3$  and  $M'_4$ , each of which satures the vertices of  $Z_{12}$ . For every j = 2, 3, 4, use the edges of  $M'_j$  to complete the blocks of  $\mathcal{A}_j$ .

For n = 31, partition the edges of the complete graph on  $Z_{17}$  into the following set  $\mathcal{A}$ :

$$\mathcal{A}_{1} = \{(0, 4, 1; 7) + i : i = 2, 3, 4, 5, 11, 12, 13, 14, i \in Z_{17}\} \cup \{(10, 12, 0; 3, 7)\}$$

$$\mathcal{A}_{2} = \{(0, 4, 1; 7) + i : i = 0, 1, 6, 7, 8, 9, 15, 16, i \in Z_{17}\} \cup \{(14, 7, 9; 2, 0)\}$$

$$\mathcal{A}_{3} = \{(0, 7, 2; 10) + i : i = 1, 4, 13, 15, 16, i \in Z_{17}\} \cup \{(10, 14, 11; 0), (9, 4, 2; 12, 0), (12, 2, 14; 10, 5)\}$$

$$\mathcal{A}_{4} = \{(0, 7, 2; 10) + i : i = 3, 5, 6, 8, 9, 11, 14, i \in Z_{17}\}$$

where  $d_2(i) = 2$  for  $i \in \{0, 2, 7\}$  and  $d_2(i) = 4$  for  $i \in Z_{17} \setminus \{0, 2, 7\}$ . After applying *Step* 2., *Step* 3. and *Step* 4. proceed as follows. Consider a subgraph  $\mathcal{M}'$  of the missing graph such that each vertex in  $Z_{17}$  has degree 2. Partition the edges of  $\mathcal{M}'$  into two matchings  $M'_1$  and  $M'_2$  and use them to complete the kites in  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , respectively. After deleting the edges of  $M'_1$  and  $M'_2$  the remaining edges of  $\mathcal{M}$  can be partitioned into two matchings  $M'_3$  and  $M'_4$ , each of which satures the vertices in  $Z_{17} \setminus \{0, 2, 7\}$  and can be used to complete the kites in  $\mathcal{A}_3$  and  $\mathcal{A}_4$ , respectively.

By similar arguments it is possible to settle the remaining cases n = 37, 45, 55, 61, 69, for which we refer to Appendix where we give the sets  $A_{js}$ , which automatically determine the matchings  $M'_{js}$ .

#### 4 Main result and conclusion

Combining Lemmas 2.1, 2.4, and Propositions 2.1, 2.2, 3.1, 3.2 gives our main result.

#### Main Theorem

- (i) If  $n \equiv 1, 3, 9, 19 \pmod{24}$ , then  $u_{min}(n) = \frac{n-1}{2}$  for every  $n \neq 3, 9$ ,  $u_{min}(3) = 6$ , and  $u_{min}(9) = 7$ .
- (ii) If  $n \equiv 7, 13, 15, 21 \pmod{24}$ , then  $u_{min}(n) = \frac{n-1}{2} + 2$  for every  $n \neq 7, 13, u_{min}(7) = 6$ , and  $u_{min}(13) = 11$ .

In [13] a complete solution to the Doyen-Wilson problem for 3-sun systems is given and it is proved that any 3SS(n) can be embedded in a 3SS(m)if and only if  $m \ge \frac{7}{5}n + 1$  or m = n. For every integer  $v \in \Sigma(K_3)$ , combining Main Theorem with the above result gives an integer  $m_v$  such that any STS(v) can be embedded in a 3SS(m) for every admissible  $m \ge m_v$ . A question to be asked is the following.

**Open Problem** Can one embed any STS(v) in a 3SS(m) for every admissible *m* such that  $v + u_{min}(v) < m < m_v$ ?

## Appendix

$$\begin{split} n &= 37: \\ \mathcal{A}_1 &= \{(4,11,0;8) + 2i: i = 0,1,\ldots,9, \ i \in Z_{20}\} \\ \mathcal{A}_2 &= \{(5,12,1;9) + 2i: i = 0,1,\ldots,9, \ i \in Z_{20}\} \\ \mathcal{A}_3 &= \{(14,16,13;2,19), (4,6,3;16,9)\} \\ \mathcal{A}_4 &= \{(1,3,0;6) + i: i = 0,12,17, \ i \in Z_{20}\} \cup \\ &\{(7,12,2;17), (16,17,19;14), (5,7,4;17,10), (6,8,5;18,11), \\ &(8,10,7;15,13), (11,9,8;4,14), (10,12,9;17,15), (2,4,1;14,16)\} \\ \mathcal{A}_5 &= \{(1,3,0;6) + i: i = 14,15, \ i \in Z_{20}\} \cup \\ &\{(5,10,0;15), (8,13,3;18), (0,2,19;4), (3,5,2;15,8), (7,9,6;19,12), \\ &(11,13,10;18,16), (12,14,11;19,17), (1,18,19;4,5), (6,11,1;16,7)\} \end{split}$$

$$n = 45:$$

$$\mathcal{A}_{1} = \{(1, 13, 7; 19) + i : i = 0, 1, 2, 3, 4, i \in \mathbb{Z}_{24}\} \cup \{(8, 16, 0; 22, 12), (9, 17, 1; 23, 19), (10, 18, 2; 6, 20), (19, 11, 3; 0, 21), (20, 12, 4; 6, 22), (21, 13, 5; 19, 23), (22, 14, 6; 20, 0), (7, 15, 23; 21, 12)\}$$

$$\mathcal{A}_{2} = \{(0, 1, 5) + 3i : i = 0, 1, \dots, 7, i \in \mathbb{Z}_{24}\}$$

$$\mathcal{A}_{3} = \{(1, 2, 6) + 3i : i = 0, 1, \dots, 7, i \in \mathbb{Z}_{24}\}$$

$$\mathcal{A}_{4} = \{(2, 3, 7) + 3i : i = 0, 1, \dots, 7, i \in \mathbb{Z}_{24}\}$$

$$\mathcal{A}_{5} = \{(1, 3, 10; 12, 6, 20) + i : i \in \mathbb{Z}_{24} \setminus \{11, 23\}\} \cup \{(0, 2, 9; 18, 5, 19), (12, 14, 21; 18, 17, 7)\}$$

$$n = 55:$$

$$\begin{aligned} \mathcal{A}_1 &= \{ (13, 27, 0; 25) + i : i = 3, 4, \dots, 13, \ i \in \mathbb{Z}_{29} \} \cup \\ &\{ (15, 0, 2; 25, 27), (12, 14, 27; 10, 13), (28, 13, 15; 0, 11), (14, 0, 16; 27, 12), \\ \mathcal{A}_2 &= \{ (13, 27, 0; 25) + i : i = 1, 17, 18, \dots, 28, \ i \in \mathbb{Z}_{29} \} \\ \mathcal{A}_3 &= \{ (0, 10, 11; 2, 6) + i : i \in \mathbb{Z}_{29} \} \\ \mathcal{A}_4 &= \{ (0, 9, 12; 2, 6) + i : i \in \mathbb{Z}_{29} \} \end{aligned}$$

n = 61:

$$\begin{aligned} \mathcal{A}_1 &= \{(0, 10, 29; 9) + i : i = 0, 1, 2, 3, 6, 17, 18, 19, 20, i \in Z_{32}\} \cup \\ &\{(14, 22, 6; 30)\} \cup \{(23, 10, 13; 22, 29) + i : i = 0, 1, 2, i \in Z_{32}\} \cup \\ &\{(4, 23, 26; 3, 18) + i : i = 0, 1, 3, 4, 5, i \in Z_{32}\} \cup \{(26, 13, 16; 25, 24), \\ &(21, 18, 31; 30, 7)\} \end{aligned}$$

$$\mathcal{A}_2 &= \{(8, 16, 0; 24) + i : i = 0, 1, \dots, 5, i \in Z_{32}\} \cup \{(4, 14, 1; 13), \\ &(0, 22, 19; 31), (2, 24, 21; 1)\} \cup \\ &\{(1, 20, 23; 0, 15) + i : i = 0, 2, 5, i \in Z_{32}\} \cup \{(5, 2, 15; 14, 7)\} \end{aligned}$$

$$\mathcal{A}_3 &= \{(17, 4, 7; 16, 23) + i : i = 0, 1, 2, 3, 4, 5, i \in Z_{32}\} \\ \mathcal{A}_4 &= \{(9, 0, 2; 11, 17) + i : i \in Z_{32}\} \end{aligned}$$

n = 69:

$$\begin{aligned} \mathcal{A}_1 &= \{(4,2,0;6,34) + 3i: i = 5,6,7,8,9,10, \ i \in Z_{36} \} \\ \mathcal{A}_2 &= \{(4,2,0;6,34) + 3i: i = 0,1,2,3,4,11, \ i \in Z_{36} \} \cup \\ &\{(24,12,0;30,18) + i,(30,18,6;24) + i: i = 0,1,2,3,4,5, \ i \in Z_{36} \} \\ \mathcal{A}_3 &= \{(0,7,15;1) + 2i: i = 0,1,\ldots,17, \ i \in Z_{36} \} \\ \mathcal{A}_4 &= \{(1,8,16;2) + 2i: i = 0,1,\ldots,17, \ i \in Z_{36} \} \\ \mathcal{A}_5 &= \{(9,20,0;3,13) + i: i \in Z_{36} \} \end{aligned}$$

$$\mathcal{A}_6 = \{ (0, 6, 1; 10, 32, 11) + 9i, (1, 7, 2; 5, 11, 12) + 9i, (2, 8, 3; 5, 9, 13) + 9i, (3, 9, 4; 6, 14, 8) + 9i, (4, 10, 5; 1, 7, 8) + 9i, (5, 11, 6; 35, 8, 9) + 9i, (6, 12, 7; 16, 9, 17) + 9i, (7, 13, 8; 4, 23, 18) + 9i : i = 0, 1, 2, 3, i \in \mathbb{Z}_{36} \}$$

## References

- J.A. Bondy, U.S.R. Murty, Graph theory with applications, North Holland, 1976.
- [2] C.J. Colbourn, *Triple Systems*, in: C. J. Colbourn and J. H. Dinitz (eds.), CRC Handbook of Combinatorial Designs, Second Edition, Chapman and Hall/CRC, Boca Raton, FL, 2007, pp. 58–71.
- C.J. Colbourn, A.C.H. Ling, G. Quattrocchi, Minimum embedding of Steiner triple systems into (K<sub>4</sub>-e)-designs I, Discrete Math., 308 (2008) 5308-5311.
- [4] C.J. Colbourn, A.C.H. Ling, G. Quattrocchi, Minimum embedding of Steiner triple systems into (K<sub>4</sub> - e)-designs II, Discrete Math., 309 (2009) 400-411.
- [5] C.J. Colbourn, G. Quattrocchi, V.R. Syrotiuk, Grooming for two-period optical networks, Networks 52 (2008), 307-324.
- [6] J. Doyen and R.M. Wilson, *Embeddings of Steiner triple systems*, Discrete Math. 5 (1973), 229-239.
- [7] C.M. Fu, N.H. Jhuang, Y.L. Lin, S.W. Lo, and H.M. Sung, From Steiner triple systems to 3-sun systems, Taiwan. J. Math. 16 (2012), 531-543.
- [8] C.M. Fu, Y.L. Lin, S.W. Lo, Y.F. Hsu, and W.C. Huang, The Doyen-Wilson theorem for bull designs, Discrete Math. 313 (2013), 498-507.
- [9] M. Gionfriddo, G. Quattrocchi, G. Ragusa, Minimum embedding of STSs into  $(K_3 + e)$ -systems, Discrete Math., 313 (2013) 1419-1428.
- [10] D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, Math. Ann. 77(4) (1916), 453–465.
- [11] G. Lo Faro and A. Tripodi, *Embeddings of*  $\lambda$ -fold kite systems,  $\lambda \geq 2$ , Australas. J. Combin. 36 (2006), 143-150.

- [12] G. Lo Faro and A. Tripodi, The Doyen-Wilson theorem for kite systems, Discrete Math. 306 (2006), 2695-2701.
- [13] G. Lo Faro and A. Tripodi, The Doyen-Wilson theorem for 3-sun systems, Ars Math. Contemp. 16 (2019), 119–139.
- [14] M. Meszka, A. Rosa, Embedding Steiner triple systems into Steiner systems S(2, 4, v), Discrete Math., 274 (2004) 199-212.