On the Ramsey numbers of tree graphs versus certain generalised wheel graphs

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ON THE RAMSEY NUMBERS OF TREE GRAPHS VERSUS CERTAIN GENERALISED WHEEL GRAPHS

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Abstract

This thesis presents a series of Ramsey results on tree graphs versus generalised wheel graphs, with the focus on the generalised wheel graphs $W_{s,6}$ and $W_{s,7}$ and the wheel graph W_8.

This thesis consists of 7 chapters. In Chapter 1, we give a brief historical introduction to Ramsey theory and Ramsey's Theorem, as well as some brief introduction to the contents of the thesis. Then in Chapter 2, we introduce notation and definitions that will be consistently used throughout the thesis, including some basic knowledge of graph theory which is particularly useful in our discussion.

In Chapter 3, we present Ramsey numbers for tree graphs T_n of order n versus the generalised wheel graphs $W_{s,6}$ and $W_{s,7}$. We determine the Ramsey number $R(T_n, W_{2,6})$ for $n \geq 5$. Then we generalise these results to find $R(T_n, W_{s,6})$ for $s \geq 2$. After that, we also determine the Ramsey number $R(T_n, W_{s,7})$ for $n \geq 5$ and $s \geq 1$. In the last section of Chapter 3, we discuss results on the Ramsey numbers for tree graphs versus generalised wheel graphs, $R(T_n, W_{s,m})$, and propose a conjecture.

Chapters 4 and 5 present the Ramsey numbers T_n for tree graphs of order n versus the wheel graph of order 9, W_8. In Chapter 4, we focus on the tree graphs with maximum degree of at least $n-3$. In Chapter 5, we provide results for the tree graphs with maximum degree of $n-4$ and $n-5$.

In Chapter 6, we present the Ramsey numbers $R(T_n, W_8)$ for the tree graphs with maximum degree of at most $n-6$ where n is sufficiently large.

Chapter 7 concludes the thesis with suggestions for possible future work.

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Chapter 1
Introduction

Ramsey theory is a beautiful but difficult subject, proposed by the British mathematician and philosopher Frank Plumpton Ramsey [44] nearly a century ago. Generally speaking, Ramsey theory shows how, in certain orderly structure, patterns and order can never be completely eradicated by randomness or disarray; in other words, complete randomness is impossible. A typical result in Ramsey theory states that *if some mathematical structure is cut into pieces, then at least one of the parts must attain a given property.* Before Ramsey's death at the age of 26 in 1930, he did seminal work in this area; however, the theory was brought to public attention by the Hungarian mathematician Paul Erdős, who made a huge contribution to combinatorics and graph theory.

The archetypal Ramsey theory result is Ramsey's Theorem [44] which states that *in any edge-colouring of a sufficiently large finite complete graph, one can find some monochromatic complete graph of any given order.* The Ramsey number $N = R(m, n)$ is the minimum integer with the property that the complete graph on N vertices will, whenever its edges are each coloured by one of two given colours, either contain a complete subgraph on m vertices whose edges are each coloured in the first colour, or contain a complete subgraph on n vertices whose edges are each coloured in the second colour. Equivalently, $N = R(m, n)$ is the minimum integer for which each simple undirected graph with N vertices either contains a complete graph of order m or has its graph complement contain a complete graph of order n.

The first lower bound on Ramsey numbers were obtained by Paul Erdős using probabilistic methods [27]. Together with George Szekeres, Paul Erdős also found some upper bounds on these numbers [28].

Over the years, much research had been done to improve these bounds; however, little progress has been made. There are a few interesting results on the lower bound of general Ramsey numbers, which were proposed by Spencer [48] and Alon and Pudlák [2]. The best lower bound up to today was given by Bohman and Keevash [7]:

$$R(m, n) \geq c \frac{n^{\frac{m+1}{2}}}{(\log n)^{\frac{m+1}{2} - \frac{1}{m-2}}}$$

for some positive c. On the other hand, the best upper bound of general Ramsey numbers up to today was proposed by Ajtai, Komlós and Szemerédi [1]:

$$R(m, n) \leq c \frac{n^{m-1}}{(\log n)^{m-2}}$$

for some constant c.

Let consider the case where $m = 3$. This is one of the popular research topics in the area since it is related to the study of triangle-free graphs. In [37], Kim had shown that $R(3, n)$ has order of magnitude $\frac{n^2}{\log n}$. The best-known upper-bound constant is due to Shearer [47], who had shown that

$$R(3, n) \leq \big(1 + o(1)\big) \frac{n^2}{\log n} .$$

On the other hand, Bohman and Keevash [8] had provided a lower bound constant and shown that

$$R(3, n) \geq \Big(\frac{1}{4} - o(1)\Big) \frac{n^2}{\log n} .$$

A similar result was also proved by Pontiveros, Griffiths and Morris; see [42]. This lower bound is within a $4 + o(1)$ factor of the upper bound by Shearer and is currently the best-known lower bound of $R(3, n)$.

Another interesting special type of Ramsey number is called the diagonal Ramsey number, denoted by $R(n, n)$, or just $R(n)$. Trivially, $R(1) = 1$ and $R(2) = 2$. Currently, the only known exact numbers $R(n)$ are $R(3) = 6$ (the famous Party Problem) and $R(4) = 18$ [32]. Even the exact result for $n = 5$ is still unknown, with the currently best known bounds of $43 \leq R(5) \leq 48$; see [3, 29]. In the general case, the first lower bound on $R(n)$ was proposed by Erdős [27] in 1947:

$$R(n) > \frac{1}{e\sqrt{2}} \big(1 + o(n)\big) n 2^{\frac{n}{2}} .$$

This was only improved after 30 years by a factor of 2 by Spencer [49].

On the other hand, the first upper bound of $R(n)$ was from the proof of Erdős and Szekeres [28]:

$$R(n) \leq \binom{2n-2}{n-1} \leq 4^n .$$

Very little progess was made on improving this bound until the mid-1980s. Some improvements were then made by Rödl [30] and Thomason [51]. In 2009, Conlon [12] showed that

$$R(n) \leq n^{-c\frac{\log n}{\log \log n}} \binom{2n-2}{n-1}$$

for some positive c. Very recently, Sah [45] improved this result to

$$R(n) \leq e^{-c(\log n)^2} \binom{2n-2}{n-1} .$$

Another very recent breakthrough result was provided by Campos, Griffiths, Morris and Sahasrabudhe [15]. They gave the first exponential improvement over the upper bound of Erdős and Szekeres and proved that there exists $\epsilon > 0$ such that $R(n) \leq (4 - \epsilon)^n$ for all sufficiently large n ($\epsilon = 2^{-7}$ in their proof).

Looking away from complete graphs, a more general Ramsey number is $R(G, H)$, which is the minimum number of vertices to ensure that, in any graph with that number of vertices, either the graph contains a subgraph G or its complement graph contains a subgraph H.

In this thesis, the Ramsey numbers $R(T_n, W_{s,m})$ have been determined for certain tree graphs T_n and the generalised wheel graph $W_{s,m}$. In [22], Chen et al. determined the Ramsey numbers $R(T_n, W_{1,6})$ and $R(T_n, W_{1,7})$. We extend these results and determine the Ramsey numbers $R(T_n, W_{s,6})$ and $R(T_n, W_{s,7})$ for $s \geq 2$. Next, we proceed with a discussion on the Ramsey numbers $R(T_n, W_{1,8})$. In [18], Chen, Zhang and Zhang conjectured that $R(T_n, W_m) = 2n - 1$ for all tree graphs T_n of order $n \geq m-1$ when m is even and the maximum degree $\Delta(T_n)$ "is not too large"; see also [20, 21, 22]. Later in [33], Hafidh and Baskoro refined this conjecture by specifying the bound $\Delta(T_n) \leq n-m+2$. When n is large compared to m, $\Delta(T_n)$ is not required to be small; indeed, the refined conjecture implies that, for each fixed even integer m, all but a vanishing proportion of the tree graphs $\{T_n : n \geq m-1\}$ satisfy $R(T_n, W_m) = 2n - 1$. One of the main aims of this thesis is to explore and partially verify this conjecture. Very briefly described, our main results provide strong evidence for the conjecture and also show that the conjecture is true for sufficiently large graphs.

The contents of the thesis are as follows. In Chapter 2, we introduce some necessary notation and definitions, including some fundamental graph theory, which will be particularly useful in our discussion. We also introduce some previously known theorems and lemmas which are essential to our discussion.

In Chapter 3, we present Ramsey numbers for tree graphs T_n of order n versus the generalised wheel graphs $W_{s,6}$ and $W_{s,7}$. We determine the Ramsey number $R(T_n, W_{2,6})$ for $n \geq 5$. Then we generalise these results to find $R(T_n, W_{s,6})$ for $s \geq 2$. After that, we also determine the Ramsey number $R(T_n, W_{s,7})$ for $n \geq 5$ and $s \geq 1$. In the last section of the chapter, we discuss results on the Ramsey numbers for tree graphs versus generalised wheel graphs, $R(T_n, W_{s,m})$, and propose a conjecture.

Chapters 4, 5 and 6 present the Ramsey numbers for tree graphs T_n versus the wheel graph W_8 of order 9. In Chapter 4, we focus on the tree graphs with maximum degree of at least $n-3$. There are four types of such graphs, namely S_n, $S_n(1,1)$, $S_n(1,2)$ and $S_n(3)$. In Chapter 5, we present results for the tree graphs with maximum degree of $n-4$ and $n-5$. There are 7 types of tree graphs with maximum degree $n-4$ and 19 types of tree graphs with maximum degree of $n-5$, respectively. In Chapter 6, we discuss the analogous results for the tree graphs with maximum degree of at most $n-6$ where n is sufficiently large.

In Chapter 7, we discuss our results and partially answer our conjecture in Chapter 3. We end our discussion by proposing possible future work on the topic.

Chapter 2
Graph theory

Since graph theory contributes to a major part of our discussion, we will begin the journey with some introductory graph theory.

2.1 Graph theory

In this section, we will present some fundamental graph theory definitions which will be used throughout the thesis.

Definition 2.1.1 (Graph). *A* graph *is a pair of sets $G = (V, E)$ where $V(G) := V$ is a finite non-empty set of elements called* vertices *and $E(G) := E$ is a set of unordered pairs of vertices called* edges*.*

Figure 2.1 shows an example of a graph $G = (V, E)$. It has the vertex set $V = \{s, t, u, v, w\}$ and the edge set $E = \big\{\{s, t\}, \{t, u\}, \{t, v\}, \{u, w\}, \{v, w\}\big\}$.

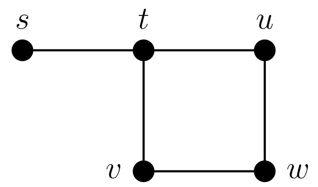


Figure 2.1: A graph G

Definition 2.1.2 (Adjacency). *Two vertices u and v of a graph G are said to be* adjacent *if $\{u, v\}$ is an edge of G. In this case, e is* incident *to u and v.*

In Figure 2.1, vertices s and t are adjacent to each other, while vertex u is not adjacent to vertex v.

Definition 2.1.3 (Neighbourhood and degree). *The* neighbourhood *$N_G(u)$ of a vertex u in graph G is the set of vertices which are adjacent to the vertex u in G. The* degree *of vertex u in G is the number $d_G(u) = |N_G(u)|$ of vertices adjacent to u in G. We use $\Delta(G)$ and $\delta(G)$ to denote the* maximum degree *and* minimum degree *of the vertices in G, respectively.*

In Figure 2.1, $\{s, u, v\}$ forms the neighbourhood $N_G(t)$ of the vertex t, and the degree of vertex t is $d_G(t) = 3$.

Definition 2.1.4 (Chromatic number). *The* chromatic number *$\chi(G)$ of a graph G is the smallest number of colours needed to colour the vertices of graph G so that no two adjacent vertices share the same colour.*

Definition 2.1.5 (Complete graph)**.**
A complete graph *is a graph in which every two vertices are adjacent to each other. A complete graph with n vertices is denoted by K_n.*

Figure 2.2 shows examples of complete graphs.

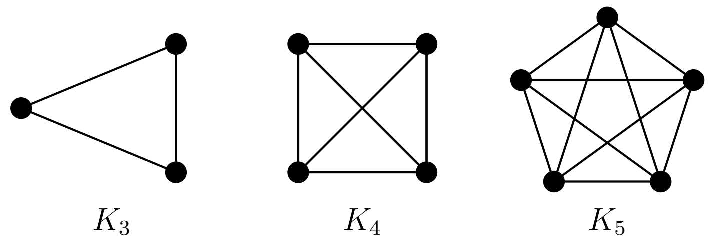


Figure 2.2: Complete graphs

Definition 2.1.6 (Subgraph)**.**
A graph H is a subgraph *of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.*

Figure 2.3 shows an example of a subgraph H of a graph G.

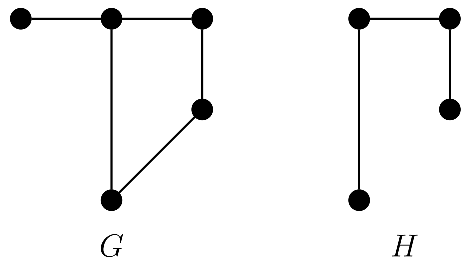


Figure 2.3: H is a subgraph of G

Definition 2.1.7 (Complement of a graph)**.** *The* complement $\overline{G}$ *of a graph G is the graph with vertices $V(\overline{G}) = V(G)$ and edges $E(\overline{G}) = E(K_n) - E(G)$.*

Figure 2.4 shows a graph G and its complement $\overline{G}$.

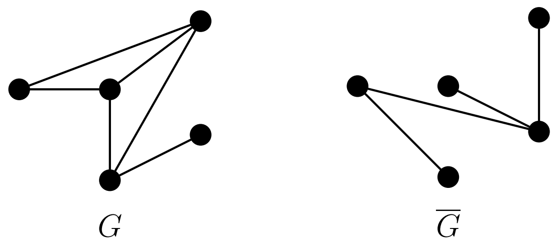


Figure 2.4: A graph G and its complement $\overline{G}$

Definition 2.1.8 (Walk, path and cycle)**.** *A* walk *in a graph G is an alternating sequence of vertices and edges $v_0 e_1 v_1 e_2 v_2 \ldots e_k v_k$ in which the ends of each edge e_i are v_{i-1} and v_i for $i \in [k]$. It is* closed *if $v_0 = v_k$ and is* open *otherwise. A walk in which all vertices $v_0, v_1, \ldots, v_k$ are distinct is called a* path*. A* cycle *is a closed walk in which all vertices $v_0, v_1, \ldots, v_k$ are distinct except for $v_0 = v_k$. The* cycle graph *C_n is the graph consisting of a cycle of order n.*

Definition 2.1.9 (Connected graph)**.** *A graph G is* connected *if there exists a walk between each pair of vertices in G. If G is not connected, then it is* disconnected*.*

Figure 2.5 shows a connected graph G and a disconnected graph H.

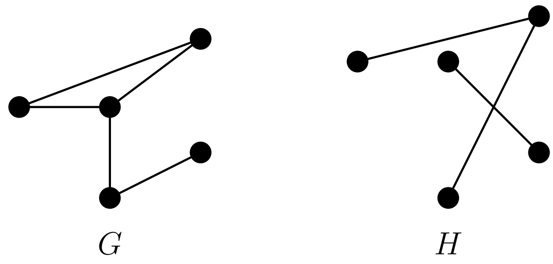


Figure 2.5: A connected graph G and a disconnected graph H

Definition 2.1.10 (Addition of two graphs)**.** *The* addition *of graphs G_1 and G_2, denoted by $G_1 + G_2$, is the graph obtained by adding to the disjoint union of G_1 and G_2 edges between each vertex of G_1 and each vertex of G_2.*

Figure 2.6 shows an example of a graph addition.

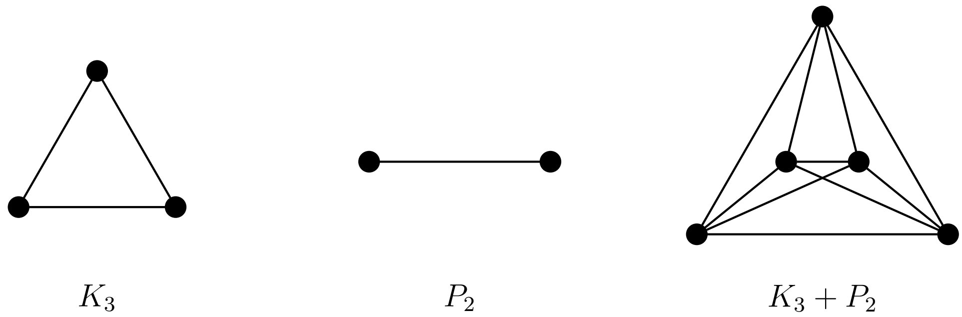


Figure 2.6: Graph addition $K_3 + P_2$

Definition 2.1.11 (Generalised wheel)**.** *The* generalised wheel graph $W_{s,m}$ *is the graph $K_s + C_m$ obtained by adding the graphs K_s and C_m as defined in Definition 2.1.10. If $s = 1$, then $W_{s,m}$ is a* wheel graph *which we also denote by W_m.*

Figure 2.7 shows examples of generalised wheel graphs.

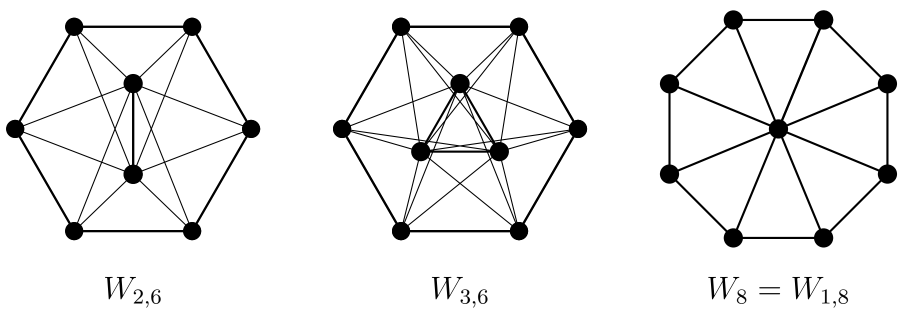


Figure 2.7: Generalised wheel graphs

Definition 2.1.12 (Tree)**.** *A* tree *is a connected graph which has no cycle subgraph. In this thesis, trees with n vertices are denoted by T_n.*

Here, we introduce some of the tree graphs used in our discussions. Let P_n be the *path graph* consisting of a path of order n, and let S_n be the *star graph* of order n consisting of one vertex that is adjacent to $n-1$ vertices which are non-adjacent to each other. Let $S_n(\ell, m)$ be the tree of order n obtained from the star graph $S_{n-\ell\times m}$ by subdividing each of ℓ chosen edges m times. $S_n(\ell)$ is the tree graph of order n obtained by adding an edge joining the centres of two star graphs S_ℓ and $S_{n-\ell}$. $S_n[\ell]$ is the tree graph of order n obtained by adding an edge joining the centre of $S_{n-\ell}$ to a degree-one vertex of S_ℓ.

Figure 2.8 shows examples of these trees. Other tree graphs will be introduced throughout the thesis.

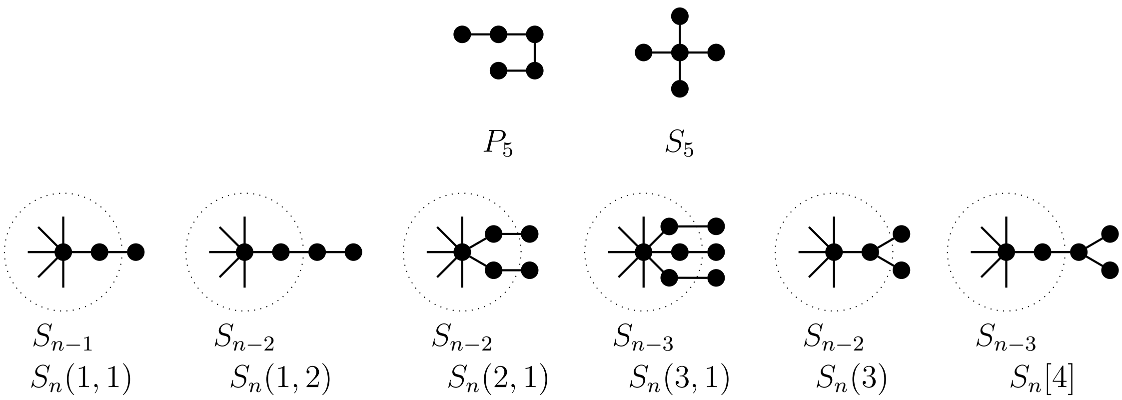


Figure 2.8: Examples of P_n, S_n, $S_n(\ell, m)$, $S_n(\ell)$ and $S_n[\ell]$

Definition 2.1.13 (Multipartite graph)**.** *A k-*partite graph *is a connected graph whose vertex set can be partition into k disjoint subsets containing no edges as subsets; that is, each edge contains a vertex from one subset and a vertex from another subset. A k-partite graph is* complete *if each vertex from one subset is adjacent to every vertex from every other subset. A complete k-partite graph is denoted by $K_{n_1,\ldots,n_k}$ where $n_1, \ldots, n_k$ are the numbers of vertices in each subset, respectively. The graph is* bipartite *if $k = 2$ and* tripartite *if $k = 3$.*

Figure 2.9 shows examples of complete multipartite graphs.

A complete bipartite graph, $K_{3,4}$ A complete tripartite graph, $K_{2,2,2}$

Figure 2.9: Complete multipartite graphs

2.2 Auxiliary results

In this section, we will introduce some previously known results and lemmas which will be particularly useful in our discussions. We do not provide the proofs for these; interested readers are directed to the respective references.

First, we will introduce some known Ramsey theory results relating to the Ramsey numbers of tree graphs versus generalised wheel graphs. These results motivated us into conducting this research work.

In [54], Wang and Chen determined the Ramsey number for tree graphs versus generalised wheel graphs $W_{s,4}$ and $W_{s,5}$. Inspired by their work, we have studied the Ramsey numbers for tree graphs versus generalised wheel graphs $W_{s,6}$ and $W_{s,7}$. We will discuss these numbers in Chapter 3.

Theorem 2.2.1. [54] *If $n \geq 3$ and $s \geq 2$, then $R(T_n, W_{s,4}) = (n-1)(s+1)+1$. Furthermore, if $n \geq 3$ and $s \geq 1$, then $R(T_n, W_{s,5}) = (n-1)(s+2)+1$.*

Now, we introduce some known Ramsey theory results concerning the Ramsey numbers of tree graphs versus the wheel graphs W_m. In [22], Chen, Zhang and Zhang determined the Ramsey numbers $R(T_n, W_6)$ and $R(T_n, W_7)$.

Theorem 2.2.2. [22] *$R(T_n, W_6) = 2n - 1 + \mu$ for $n \geq 5$, where*

(a) *$\mu = 2$, if $T_n = S_n$;*
(b) *$\mu = 1$, if $T_n = S_n(1,1)$ or $T_n = S_n(1,2)$ and $n \equiv 0 \pmod 3$;*
(c) *$\mu = 0$, otherwise.*

Theorem 2.2.3. [22] *$R(T_n, W_7) = 3n - 2$ for $n \geq 6$.*

Next, we introduce results for path and star graphs. Chen, Zhang and Zhang [19] and Zhang [55] determined the Ramsey numbers $R(P_n, W_m)$ for $3 \leq m \leq n+1$ and $n+2 \leq m \leq 2n$, respectively. Combining these results, we have the following theorem.

Theorem 2.2.4. [19, 55] *For $3 \leq m \leq 2n$, we have*

$$R(P_n, W_m) = \begin{cases} 3n-2, & \text{if } m \text{ is odd;} \\ 2n-1, & \text{if } m \text{ is even and } 3 \leq m \leq n+1\,; \\ m+n-2, & \text{if } m \text{ is even and } n+2 \leq m \leq 2n\,. \end{cases}$$

For star graphs, Chen, Zhang and Zhang [17] proved the following result.

Theorem 2.2.5. [17] *$R(S_n, W_m) = 3n - 2$ for m odd and $n \geq m - 1 \geq 2$.*

The exact Ramsey numbers $R(S_n, W_8)$ were determined together in three papers.

Theorem 2.2.6. [56, 57, 58] *For $n \geq 5$, we have*

$$R(S_n, W_8) = \begin{cases} 2n+1, & \text{if } n \text{ is odd;} \\ 2n+2, & \text{if } n \text{ is even}\,. \end{cases}$$

In [11], Burr found an interesting lower bound for the Ramsey number $R(G, H)$ for any pair of graphs G and H, in terms of $|V(G)|, \chi(H)$ and $t(H)$.

Theorem 2.2.7. [11] *Let G be a connected graph of order n, and let H be a graph with parameters $\chi(H)$ and $t(H)$, where $t(H)$ is the minimum number of vertices in any colour class of any vertex-colouring of H with $\chi(H)$ colours and $n \geq t(H)$. Then $R(G, H) \geq (n-1)(\chi(H)-1)+t(H)$.*

Now, we introduce two lemmas that are useful in our discussion.

Lemma 2.2.8 (Handshaking Lemma)**.** *The sum of vertex degrees of a graph G is equal to twice the number of edges in G.*

Lemma 2.2.9. [16] *Let G be a graph with $\delta(G) \geq n-1$. Then G contains all tree graphs of order n.*

Since we are studying the wheel graph, which contains a cycle graph, the following lemmas are particularly useful.

Lemma 2.2.10. [9] *Let G be a graph of order n. If $\delta(G) \geq \frac{n}{2}$, then either G contains C_ℓ for all $3 \leq \ell \leq n$, or n is even and $G = K_{\frac{n}{2},\frac{n}{2}}$.*

Lemma 2.2.11. [36] *Let $G(u, v, k)$ be a simple bipartite graph with bipartition U and V, where $|U| = u \geq 2$ and $|V| = v \geq k$, and each vertex of U has degree at least k. If $G(u, v, k)$ satisfies $u \leq k$ and $v \leq 2k-2$, then it contains a cycle of length $2u$.*

Chapter 3

Ramsey numbers for tree graphs versus certain generalised wheel graphs

In this chapter, we look at the Ramsey numbers for tree graphs versus the generalised wheel graphs $W_{s,6}$ and $W_{s,7}$. The results in this chapter have been published in [23] during my PhD candidature and are joint work with Dr Ta Sheng Tan and Prof. Dr Kok Bin Wong. In this article, I am the main author, in charge of developing and writing the proof of the results, especially those have been incorporated in the chapter. Similar results were also obtained independently by Wang [53].

3.1 Introduction

In [54], Wang and Chen determined the Ramsey numbers for the tree graphs versus $W_{s,4}$ and $W_{s,5}$. This inspires us to study the Ramsey numbers of tree graphs versus generalised wheel graphs beyond $W_{s,4}$ and $W_{s,5}$. We will focus on the results for $W_{s,6}$ and $W_{s,7}$.

Note that $\chi(W_{s,6}) = s+2$ and $t(W_{s,6}) = 1$. By Theorem 2.2.7, we therefore have $R(T_n, W_{s,6}) \geq (s+1)(n-1)+1$. Now, we need to determine the upper bound of $R(T_n, W_{s,6})$ for various types of trees. We will do so in the next few sections. But before that, we want to introduce a useful lemma.

In the paper [11], Burr also established the following definition. Under the condition of Theorem 2.2.7, the graph G is said to be H-good if

$$R(G, H) = (n-1)(\chi(H)-1) + t(H)\,.$$

Lin, Li and Dong [41] proved that, for a tree graph T and a graph G with $t(G) = 1$, if T is G-good, then T is $(K_1 + G)$-good. This leads us to the following lemma whose proof follows that of [41].

Lemma 3.1.1. *Let G be a finite simple graph and T_n be any fixed tree graph of order n. Then $R(T_n, K_1 + G) \leq R(T_n, G) + n - 1$.*

Proof. Let $N = R(T_n, G) + n - 1$. Consider any graph H of order N. Suppose that H does not contain T_n as a subgraph. Let T' be a maximal subtree of H that is (isomorphic to) a subgraph of T_n. Here, the term 'maximal' is in the sense that if a vertex $x \in X := V(H) - V(T')$ and an edge $xu \in E(H)$ for some $u \in V(T')$ are added to T', then the resulting tree is not a subgraph of T_n.

Note that $T' \neq T_n$. This implies that there is a vertex $u \in V(T')$ and a vertex $w \in V(T_n) - V(T')$ such that $uw \in E(T_n)$. So, if u is adjacent to a vertex $x \in X$ in H, then the graph obtained by adding the vertex x and the edge ux to T' is a subtree of H and it also forms a subgraph of T_n. By the maximality of T', this is impossible. Hence, u is not adjacent in H to any vertex $x \in X$.

Since $T' \neq T_n$, the order of T' is at most $n-1$. Therefore, $|X| \geq R(T_n, G)$. Note that $\overline{H}[X]$ must contain G as $H[X]$ does not contain T_n. From the preceding paragraph, $ux \notin E(H)$ for all $x \in X$. This implies that $ux \in E(\overline{H})$ for all $x \in X$. In particular, u is adjacent to all $y \in V(G)$ in $\overline{H}$. Hence, $\overline{H}$ contains $K_1 + G$, and so $R(T_n, K_1 + G) \leq R(T_n, G) + n - 1$. □

Theorem 3.1.2. *Let T_n be any fixed tree graph of order n and $W_{s,m} = K_s + C_m$ be a generalised wheel graph. Then $R(T_n, W_{s,m}) \leq R(T_n, W_m) + (s-1)(n-1)$.*

Proof. Note that $W_{1,m} = W_m$ and for $s \geq 2$, the generalised wheel graph $W_{s,m}$ is $K_1 + W_{s-1,m}$. Hence, by Lemma 3.1.1, it follows that

$$\begin{aligned} R(T_n, W_{s,m}) &\leq R(T_n, W_{s-1,m}) + n - 1 \\ &\leq R(T_n, W_{s-2,m}) + 2(n-1) \\ &\vdots \\ &\leq R(T_n, W_{1,m}) + (s-1)(n-1)\,. \end{aligned}$$

□

3.2 The Ramsey number $R(T_n, W_{2,6})$

In this section, we investigate the Ramsey numbers $R(T_n, W_{2,6})$ for tree graphs T_n of order n versus the generalised wheel graph $W_{2,6}$. As the very first step, we determine the Ramsey number $R(S_n, W_{2,6})$ for the star graph S_n. To do so, we prove the following lemma.

Lemma 3.2.1. *Let G be a graph of order $3n-2$ and $\delta(G) \geq 2n-1$ where $n \geq 5$. Then G contains $W_{2,6}$ as a subgraph.*

Proof. The condition $\delta(G) \geq 2n-1$ implies that $\overline{G}$ does not contain S_n. Let $\omega(G)$ be the number of vertices in a maximum clique of G. By [25], it is known that $R(S_n, K_4) = 3n-2$, so we have $\omega(G) \geq 4$. If $\omega(G) \geq 8$, then G must contain every subgraph of order 8, including $W_{2,6}$. So, we only need to consider the four cases $4 \leq \omega(G) \leq 7$.

Let $\omega = \omega(G)$ and $K = K_\omega \subseteq G$, and define the set $U = V(G) - V(K)$. Then $|U| = 3n - 2 - \omega$. Since $\delta(G) \geq 2n-1$, every vertex in K is adjacent to at least $2n-\omega$ vertices in U. This implies that there are at least $\omega(2n-\omega)$ edges connecting K and U. Now, let

$$\begin{aligned} X &= \{u \in U \,:\, |N_G(u) \cap V(K)| \leq 3\}\,; \\ Y &= \{u \in U \,:\, |N_G(u) \cap V(K)| \geq 4\}\,. \end{aligned}$$

Then $U = X \cup Y$ and $|X| + |Y| = |U| = 3n - 2 - \omega$. Since $K_{\omega+1}$ is not contained in G, each vertex in U is adjacent to at most $\omega - 1$ vertices in K, so we have

$$\omega(2n-\omega) \leq 3|X| + (w-1)|Y|\,. \tag{3.2.1}$$

Case 1: $\omega(G) = 7$.

By substituting $|X| = 3n-9-|Y|$ into Equation (3.2.1), we get $3|Y| \geq 5n-22$. For $n \geq 5$, we have $|Y| \geq 1$. Hence, there must be a vertex in U, say u, that is adjacent to at least 4 vertices in K. Therefore, $G[V(K) \cup \{u\}]$ must contain $W_{2,6}$.

Case 2: $\omega(G) = 6$.

By substituting $|X| = 3n-8-|Y|$ into Equation (3.2.1) and noting that $n \geq 5$, we obtained the inequality $|Y| \geq \frac{3n}{2} - 6 \geq 2$.

Suppose there is a vertex in U, say u_1, that is adjacent to 5 vertices in K. Since $|Y| \geq 2$, there must be another vertex in U, say u_2, that is adjacent to at least 4 vertices in K. As there are only 6 vertices in K, u_1 and u_2 must be adjacent to at least 3 common vertices in K, say k_1, k_2, k_3. Now let $k_4 \in V(K) \cap N_G(u_2) \backslash \{k_1, k_2, k_3\}$, $k_5 \in V(K) \cap N_G(u_1) \backslash \{k_1, \ldots, k_4\}$ and $k_6 \in V(K) \backslash \{k_1, \ldots, k_5\}$. We see that $G[V(K) \cup \{u_1, u_2\}]$ contains $W_{2,6}$ with k_1 and k_2 in the centre and $k_5u_1k_3u_2k_4k_5k_6$ as C_6.

We may therefore assume that every vertex in U is adjacent to at most 4 vertices in K. In this case, we have

$$6(2n-6) \leq 3|X| + 4|Y| = 3|U| + |Y| = 3(3n-8) + |Y|\,,$$

implying that $|Y| \geq 3n-12$ and $|X| \leq 4$. Since $n \geq 5$ and $\delta(G) \geq 2n-1 \geq 9$, we deduce that $G[Y]$ has no isolated vertex.

Let u_1 and u_2 be two adjacent vertices in Y, and note that at least two vertices $k_1, k_2 \in K$ are each adjacent to both u_1 and u_2. Now, let

$$\begin{aligned} & k_3 \in V(K) \cap N_G(u_1) \setminus \{k_1, k_2\}\,, \\ & k_4 \in V(K) \cap N_G(u_2) \setminus \{k_1, k_2, k_3\} \\ \text{and} \quad & \{k_5, k_6\} = V(K) \setminus \{k_1, \ldots, k_4\}\,. \end{aligned}$$

We again see that $G[V(K) \cup \{u_1, u_2\}]$ contains $W_{2,6}$ with k_1 and k_2 in the centre and $k_3u_1u_2k_4k_5k_6k_3$ as C_6.

Case 3: $\omega(G) = 5$.

By substituting $|X| = 3n-7-|Y|$ into Equation (3.2.1), we obtain $|Y| \geq n-4$. We note here that if $|Y| = n-4$, then every vertex in X is adjacent to exactly 3 vertices in K.

Write $V(K) = \{k_1, \ldots, k_5\}$. We can partition Y into five sets $A_1, \ldots, A_5$ where

$$A_i = \{y \in Y \;:\; y \;\text{ is not adjacent to }\; k_i\}\;.$$

Since each vertex in Y is adjacent to exactly 4 vertices in K, we see that each vertex in A_i is adjacent to k_j for $j \in \{1, \ldots, 5\} - \{i\}$.

Note that A_i is an independent set, for we could otherwise find two vertices in A_i, say a_1 and a_2, such that a_1 is adjacent to a_2. Now, $G[S] = K_6$ where $S = \{a_1, a_2, k_j \;:\; j \in \{1, \ldots, 5\} - \{i\}\}$, a contradiction since $\omega(G) = 5$.

Next, note that if any three of the five sets are non-empty, then we have $W_{2,6}$ in G. For illustration purposes, suppose that $A_i \neq \emptyset$ for $i = 1, 2, 3$. Let $a_i \in A_i$. Then $G[V(K) \cup \{a_1, a_2, a_3\}]$ contains $W_{2,6}$ with k_4 and k_5 in the centre and $k_1a_3k_2a_1k_3a_2k_1$ as C_6. Hence, we may assume that $A_i = \emptyset$ for $i = 3, 4, 5$. So, $Y = A_1 \cup A_2$. We also may assume that $|A_1| \geq |A_2|$. Since $|Y| \geq n-4$ and $n \geq 5$, we have $|A_1| \geq 1$.

Case 3.1: Suppose that $|A_1| \geq 2$.

Let $x_1, x_2 \in A_1$ and set $U' = U - \{x_1, x_2\}$. Then $|U'| = 3n - 7 - 2 = 3n - 9$. Also, let

$$X' = \{u \in U' \ : \ |N_G(u) \cap V(K)| \leq 2\} \ ;$$
$$Y' = \{u \in U' \ : \ |N_G(u) \cap V(K)| \geq 3\} \ .$$

Since each x_i is adjacent to 4 vertices in K and $|E_G(U, V(K))| \geq 5(2n-5)$, we have

$$5(2n-5) - 2 \times 4 \leq 2|X'| + 4|Y'| = 2|U'| + 2|Y'| = 2(3n-9) + 2|Y'| \, ,$$

implying that $|Y'| \geq 2n - 7$ and $|X'| \leq n - 2$. Let

$$X_1 = \{u \in U' \ : \ u \text{ is adjacent to } x_1\} \ ;$$
$$X_2 = \{u \in U' \ : \ u \text{ is adjacent to } x_2\} \ .$$

Since x_i is adjacent to 4 vertices in K and x_1 and x_2 are not adjacent to each other, we have $|X_i| \geq 2n - 5$. Therefore, $|X_1 \cap X_2| = |X_1| + |X_2| - |X_1 \cup X_2| \geq 2(2n-5) - (3n-9) = n - 1 > |X'|$, and we deduce that $Y' \cap X_1 \cap X_2 \neq \emptyset$.

Let $u' \in X_1 \cap X_2 \cap Y'$. Note that u' is adjacent to x_1 and x_2, and u' is also adjacent to at least three vertices in K. Therefore, u' must be adjacent to at least two of $k_1, \ldots, k_5$, without loss of generality say k_2 and k_3. Then $G[V(K) \cup \{x_1, x_2, u'\}]$ contains $W_{2,6}$ with k_2 and k_3 in the centre and $x_1 u' x_2 k_4 k_1 k_5 x_1$ as C_6.

Case 3.2: Suppose that $|A_1| = 1$.

Since $n - 4 \leq |Y| = |A_1 \cup A_2| \leq 2$, we must have $|Y| = 2$ with $5 \leq n \leq 6$, or $|Y| = 1$ with $n = 5$.

Case 3.2.1: Suppose that $|Y| = 2$; that is, $|A_1| = |A_2| = 1$.

Let $x_1 \in A_1$ and $x_2 \in A_2$. Recall that every vertex in X is adjacent to at most three vertices in K. If $u \in X$ is adjacent to 3 vertices in K and also adjacent to a vertex in Y, then we may assume $|N_G(u) \cap \{k_3, k_4, k_5\}| = 1$. Suppose otherwise; then without loss of generality, u is adjacent to x_1, k_3, k_4, and another vertex in K. It is then straightforward to check that G contains $W_{2,6}$ with k_3 and k_4 in the centre and C_6 in $G\left[\{k_1, k_2, k_5, u, x_1, x_2\}\right]$.

Now if $n = 6$, then we have equality in Equation (3.2.1), implying that every vertex in X is adjacent to exactly 3 vertices in K. Since $\delta(G) \geq 2n - 1 = 11$, we must have x_1 adjacent to at least 6 vertices in X. Let A be a subset of $N_G(x_1) \cap X$ with $|A| = 6$. We see that every vertex in A is adjacent to both k_1 and k_2. It is straightforward to deduce from the degree conditions that $\delta(G[A]) \geq 3$, implying that $G[A]$ contains C_6 by Lemma 2.2.10. Therefore, G contains $W_{2,6}$.

For the case when $n = 5$, we have $|G| = 13$, $\delta(G) \geq 9$ and $|X| = 6$. By the degree conditions, every vertex in X is adjacent to some vertex in Y. A more refined analysis similar to those used in obtaining Equation (3.2.1) implies that 5 vertices in X are each adjacent to 3 vertices in K, while the remaining vertex $v \in X$ is adjacent to either 2 or 3 vertices in K. Note that every vertex in $X - \{v\}$ is adjacent to both k_1 and k_2.

Suppose that v is adjacent to k_j for some $j \in \{1,2\}$. Then $|N_G(k_j)| = 11$. Since $\overline{G}$ does not contain S_5, and $R(W_6, S_5) = 11$ by Theorem 2.2.2, we deduce that $G[N_G(k_j)]$ contains W_6 which, together with k_j, forms $W_{2,6}$ in G.

The remaining case is, without loss of generality, when $N_G(v) \cap V(K) = \{k_3, k_4\}$. Since $\delta(G) \geq 9$, v is adjacent to both x_1 and x_2. Therefore, $G[V(K) \cup \{v, x_1, x_2\}]$ contains $W_{2,6}$ with k_3 and k_4 in the centre and $k_1x_2vx_1k_2k_5k_1$ as C_6.

Case 3.2.2: Suppose that $|Y| = 1$.

Since $|Y| \geq n-4$, we must have $n = 5$ and equality in (3.2.1). So in this case, the graph G is of order 13 with $\delta(G) \geq 9$ such that, whenever G contains K_5, the following property P on G holds:

> there is exactly one vertex in $V(G) - V(K_5)$ that is adjacent to exactly 4 vertices in K_5 while the remaining vertices are each adjacent to exactly 3 vertices in K_5; and every vertex in $V(K_5)$ has degree exactly 9 in G.

Now let $x \in Y$; then x is adjacent to all vertices except the vertex k_1 in K. Observe that $G[V(K) \cup \{x\} - \{k_1\}]$ is another K_5 in G. Therefore, by property P, x has degree exactly 9 in G. Setting $A = V(G) - (V(K) \cup \{x\})$, we shall now show that there is another K_5 in $G[A]$.

From the above discussion together with property P, it is straightforward to check that $G[V(K) \cup \{x\}]$ has exactly 14 edges, and the number of edges in G from $V(K) \cup \{x\}$ to A is exactly 26, implying that $G[A]$ has at least 19 edges. Since $G[A]$ is a graph of order 7 with at least 19 edges, it is easy to see that $G[A]$ contains K_5, either by deducing from Turan's Theorem [52], or by observing that $G[A]$ can be obtained by deleting at most two edges from K_7.

Suppose that K' is a K_5 subgraph of $G[A]$. From the remaining three vertices in $V(G) - (V(K) \cup V(K'))$, property P implies that there must be a vertex, say y, that is adjacent to exactly three vertices in K and exactly 3 vertices in K'. This implies that y has degree at most 8, which is a contradiction.

Case 4: $\omega(G) = 4$.

Recall that K is a K_4 subgraph of G and that $U = V(G) - V(K)$. Since $\omega(G) = 4$, we must have $Y = \emptyset$; that is, each vertex in U is adjacent to at most 3 vertices in K. Partition $U = X' \cup Y'$ as follows:

$$X' = \{u \in U \ : \ |N_G(u) \cap V(K)| \leq 2\} \ ;$$
$$Y' = \{u \in U \ : \ |N_G(u) \cap V(K)| = 3\} \ .$$

Since $\delta(G) \geq 2n-1$ and $|U| = 3n-6$, we have

$$4(2n-4) \leq 2|X'| + 3|Y'| = 2|U| + |Y'| = 2(3n-6) + |Y'| \, ,$$

implying that $|Y'| \geq 2n-4$ and $|X'| \leq n-2$. We note here that if $|Y'| = 2n-4$, then every vertex in X' must be adjacent to exactly 2 vertices in K.

Let $V(K) = \{k_1, \ldots, k_4\}$. We can further partition Y' into four sets $A_1, \ldots, A_4$ where

$$A_i = \{\, y \in Y \ : \ y \text{ is not adjacent to } k_i \,\} \, .$$

Since each vertex in Y is adjacent to exactly 3 vertices in K, we see that each vertex in A_i is adjacent to k_j for $j \in \{1,2,3,4\} - \{i\}$. Furthermore, each A_i is an independent set since $\omega(G) = 4$.

Without loss of generality, assume that $|A_1| \geq |A_2| \geq |A_3| \geq |A_4|$. Since $|Y'| \geq 2n - 4 \geq 6$, we have $|A_1| \geq 2$.

Case 4.1: Suppose that $|A_2| \leq 1$.

Then $|A_4| \leq |A_3| \leq 1$. This implies that $|A_1| \geq 2n - 4 - 3 = 2n - 7$. Now, k_1 is not adjacent to any of the vertices in A_1, so k_1 is adjacent to at most

$$\big(|V(G)| - 1\big) - |A_1| \leq \big((3n-2) - 1\big) - (2n-7) = n + 4$$

vertices. Thus, $2n - 1 \leq |N_G(k_1)| \leq n + 4$ which implies that $n \leq 5$. In this scenario, we must have $n = 5$, $|V(G)| = 13$, $|A_1| = 3$, and $|A_2| = |A_3| = |A_4| = 1$; also, k_1 is adjacent to all vertices in $(V(G) - \{k_1\}) - A_1$. Let $A_1 = \{x_1, x_2, x_3\}$, $A_2 = \{x_4\}$, $A_3 = \{x_5\}$ and $A_4 = \{x_6\}$. Since A_1 is independent, x_1 is not adjacent to x_2 or x_3. Now, x_1 is also not adjacent to k_1, so x_1 must be adjacent to all vertices in $V(G) - \{x_2, x_3, k_1\}$, since $\delta(G) \geq 9$. Similarly, x_2 and x_3 are adjacent to all vertices in $V(G) - (A_1 \cup \{k_1\})$. Thus, $|N_G(k_1)| = |N_G(a)| = 9$ for all $a \in A_1$.

Since $|V(G)| = 13$, the Handshaking Lemma implies that one of the vertices in $V(G) - (A_1 \cup \{k_1\})$ must be of degree at least 10. Let $y \in V(G) - (A_1 \cup \{k_1\})$ and $|N_G(y)| \geq 10$. If $|N_G(y)| \geq 11$, Then by Theorem 2.2.2, either $\overline{G}[N_G(y)]$ contains S_5 or $G[N_G(y)]$ contains W_6. If the former holds, then $\overline{G}$ contains S_5, and this contradicts that $\delta(G) \geq 9$. Hence, the latter must hold; that is, $G[N_G(y)]$ contains W_6. Since y is adjacent to all vertices in W_6, $G[V(W_6) \cup \{y\}]$ contains $W_{2,6}$. So, we may assume that $|N_G(y)| = 10$. Then $\big|N_G(y) \cap \big(V(G) - (A_1 \cup \{k_1\})\big)\big| = 6$, since y is adjacent to all vertices in $A_1 \cup \{k_1\}$.

Let $Z = N_G(y) \cap \big(V(G) - (A_1 \cup \{k_1\})\big)$. Then there are only two vertices in $V(G) - (Z \cup A_1 \cup \{y, k_1\})$, say u_1 and u_2. Suppose there is a vertex $z_0 \in Z$ with $|N_G(z_0) \cap Z| \geq 3$. We may assume that z_0 is adjacent to $z_1, z_2, z_3 \in Z$. Then $G[\{k_1, x_1, x_2, z_1, z_2, z_3, z_0, y\}]$ contains $W_{2,6}$ with $k_1 z_1 x_1 z_2 x_2 z_3 k_1$ as C_6 and the vertices y and z_0 in the centre.

Suppose that $|N_G(z) \cap Z| \leq 2$ for all $z \in Z$. Let $z_1 \in Z$; then z_1 is adjacent to all vertices in $A_1 \cup \{k_1, y\}$. Since $\delta(G) \geq 9$, z_1 must be adjacent to u_1 and u_2. In fact, for each $z \in Z$, z is adjacent to u_1 and u_2. Note that Z cannot be an independent set, so let $z_1, z_2 \in Z$ be adjacent to each other. Then $G[\{k_1, x_1, x_2, z_1, z_2, u_1, u_2, y\}]$ contains $W_{2,6}$ with z_1 and z_2 in the centre and $k_1 y x_1 u_1 x_2 u_2 k_1$ as C_6.

Case 4.2: Suppose that $|A_2| \geq 2$.

We first claim that we may assume that there are no two independent edges connecting A_i and A_j for any $i \neq j$. Indeed, if $x_1 y_1$ and $x_2 y_2$ are two independent edges with $x_1, x_2 \in A_i$ and $y_1, y_2 \in A_j$, then we see that G contains $W_{2,6}$ with $V(K) - \{k_i, k_j\}$ in the centre and $k_j x_1 y_1 k_i y_2 x_2 k_j$ as C_6.

Since A_1 and A_2 are independent sets, each of size at least 2, and there are no two independent edges connecting A_1 and A_2, there is an isolated vertex $a \in G[A_1 \cup A_2]$. We consider the case when $a \in A_1$. The other case when $a \in A_2$ is similar.

Recall that $N_G(a) \cap V(K) = \{k_2, k_3, k_4\}$. We have

$$(2n-1) - 3 \leq |N_G(a) \cap U| \leq 3n - 6 - (|A_1| + |A_2|),$$

so $|A_1| + |A_2| \le n - 2$. Since $|Y'| \ge 2n - 4$ and $|A_1| \ge |A_2| \ge |A_3| \ge |A_4|$, this can only happen when $|A_i| = \frac{n}{2} - 1$ for all $1 \le i \le 4$ and n is even.

Note that we now have $\bigl|V(G) - (\{k_1\} \cup A_1 \cup A_2)\bigr| = 2n-1$, and so by the minimum degree condition, a must be adjacent to all vertices in $V(G) - \bigl(\{k_1\} \cup A_1 \cup A_2\bigr)$ and, in particular, to all vertices in $A_3 \cup A_4$. Pick a vertex $b \in A_1 - \{a\}$; then b must be adjacent to at least one vertex in $A_3 \cup A_4$, as we otherwise would have

$$2n - 1 \le |N_G(b)| \le (3n-2) - |\{k_1\} \cup A_1 \cup A_3 \cup A_4| = \frac{3n}{2},$$

giving $n \le 2$, which is a contradiction.

Finally, assume without loss of generality that b is adjacent to a vertex in A_3. Then as $|A_3| \ge 2$ and a is adjacent to all vertices in A_3, we have two independent edges connecting A_1 and A_3. This contradicts the assumption that there are no two independent edges connecting A_i and A_j for any $i \ne j$.

This completes the proof of the lemma. □

Now, we can determine the Ramsey numbers for star graphs versus the generalised wheel graph $W_{2,6}$.

Theorem 3.2.2. *If $n \ge 5$, then $R(S_n, W_{2,6}) = 3n - 2$.*

Proof. From Theorem 2.2.7, we know that $R(S_n, W_{2,6}) \ge (2+1)(n-1)+1 = 3n-2$. From Lemma 3.2.1, we have $R(S_n, W_{2,6}) \le 3n-2$ for $n \ge 5$. We therefore conclude that $R(S_n, W_{2,6}) = 3n - 2$. □

Now, we will look at a similar result for two tree graphs, namely $S_n(1,1)$ and $S_n(1,2)$, versus the generalised wheel graph $W_{2,6}$.

Theorem 3.2.3. *If $n \ge 5$, then $R(T_n, W_{2,6}) = 3n-2$ for $T_n \in \{S_n(1,1),\ S_n(1,2)\}$.*

Proof. From Theorem 2.2.7, we know that $R(T_n, W_{2,6}) \ge (2+1)(n-1)+1 = 3n-2$. We therefore only need to look at the upper bound.

Case 1: Suppose that $T_n = S_n(1,1)$.

Let G be a graph of order $3n-2$ such that $\overline{G}$ does not contain $W_{2,6}$. Then since $R(S_n, W_{2,6}) \le 3n-2$, G must contain S_n. Let T be a S_n subgraph of G, let its centre be v_0, and define $L = N_T(v_0) = \{v_1, \ldots, v_{n-1}\}$. Set $U = V(G) - V(T)$; then $|U| = 2(n-1)$. If G does not contain $S_n(1,1)$, then L must be an independent set and $E(L, U) = \emptyset$.

If $n \ge 6$, then any 3 vertices from L and 3 vertices from U form C_6 in $\overline{G}$ and, with another 2 vertices from L as the centre, give $W_{2,6}$ in $\overline{G}$, a contradiction.

Suppose that $n = 5$. Then G is of order 13 and $|U| = 8$. If $\delta(\overline{G}[U]) \ge 4$, then $\overline{G}[U]$ contains C_6 by Lemma 2.2.10. So, together with any two vertices in L as the centre, we have $W_{2,6}$ in $\overline{G}$, a contradiction. If $\delta(\overline{G}[U]) \le 3$, then $\Delta(G[U]) \ge 4$ and $G[U]$ contains another S_5 disjoint from T, say $T' = S_n$. Let the centre of T' be u_0 and define $L' = N_{T'}(u_0) = \{u_1, \ldots, u_4\}$. If G does not contain $S_5(1,1)$, then L' is an independent set and $E(L, L') = \emptyset$. Then any 8 vertices from $L \cup L'$ form $W_{2,6}$ in $\overline{G}$, a contradiction.

Thus, $R(S_n(1,1), W_{2,6}) \le 3n - 2$.

Case 2: Suppose that $T_n = S_n(1,2)$.

If $n \equiv 1, 2 \pmod 3$, then $R(S_n(1,2), W_6) = 2n - 1$ by Theorem 2.2.2. It follows from Theorem 3.1.2 that $R(S_n(1,2), W_{2,6}) \leq 3n - 2$.

Suppose that $n \equiv 0 \pmod 3$. Then $n \geq 6$. Let G be a graph of order $3n - 2$ such that $\overline{G}$ does not contain $W_{2,6}$. By Case 1, G contains a subgraph $T = S_n(1,1)$. Let $V(T) = \{v_0, \ldots, v_{n-1}\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-2}\} \cup \{v_1v_{n-1}\}$, and define $U = V(G) - V(T)$; then $|U| = 2(n-1)$. If G does not contain $S_n(1,2)$, then neither v_1 nor v_{n-1} are adjacent to any of $v_2, \ldots, v_{n-2}$, and v_{n-1} is not adjacent to any vertex in U. Now, we consider the following two cases.

Case 2.1: $N_G(v_2) \cap U = \emptyset$.

If $\delta(\overline{G}[U] \geq n - 1$, then by Lemma 2.2.10, $\overline{G}[U]$ contains C_6. This C_6 together with v_2 and v_{n-1} as the centre gives $W_{2,6}$ in $\overline{G}$, a contradiction. If $\delta(\overline{G}[U]) \leq n-2$, then $\Delta(G[U]) \geq n - 1$, so $G[U]$ contains a subgraph $T = S_n$. Let u_0 be the centre of T and define $L' = N_{T'}(u_0) = \{u_1, \ldots, u_{n-1}\}$. Suppose that G does not contain $S_n(1,2)$. Then none of $v_1, \ldots, v_{n-1}$ is adjacent to any vertex in L' in G. If L' is an independent set, then $\overline{G}$ contains $W_{2,6}$ with u_1 and u_5 in the centre and $v_2u_2v_3u_3v_4u_4v_2$ as C_6.

Suppose that L' is not an independent set. We may assume that u_1 and u_2 are adjacent to each other. Then u_1 is not adjacent to $u_3, \ldots, u_{n-1}$ since G does not contain $S_n(1,2)$. Furthermore, u_3 is adjacent to at most one vertex in $\{u_4, \ldots, u_{n-1}\}$. We may assume that u_3 is not adjacent to u_4. Then $\overline{G}$ contains $W_{2,6}$ with u_1 and u_3 in the centre and $v_1v_2v_{n-1}v_3u_4v_4v_1$ as C_6.

Case 2.2: v_2 is adjacent to a vertex in U, say b.

Set $U' = V(G) - (V(T) \cup \{b\})$; then $|U'| = 2n - 3$. Suppose that G does not contain $S_n(1,2)$. Then neither v_2 nor b are adjacent to any of $v_1, v_3, v_4, \ldots, v_{n-1}$, and b is not adjacent to any vertex in U'. If $\delta(\overline{G}[U']) \geq n-1$, then by Lemma 2.2.10, $\overline{G}[U']$ contains C_6 which, together with v_{n-1} and b as the centre, gives $W_{2,6}$ in $\overline{G}$, a contradiction. If $\delta(\overline{G}[U']) \leq n - 2$, then $\Delta(G[U']) \geq n - 2$, so $G[U']$ contains a subgraph $T = S_{n-1}$. Let u_0 be the centre of T' and define $L' = N_{T'}(u_0) = \{u_1, \ldots, u_{n-2}\}$. Since G does not contain $S_n(1,2)$, none of $v_1, \ldots, v_{n-1}$ is adjacent to any vertex in L' in G. If L' is an independent set, then $\overline{G}$ contains $W_{2,6}$ with u_1 and u_4 in the centre and $v_2u_2v_3bv_4u_3v_2$ as C_6.

Suppose that L' is not an independent set. We may assume that u_1 and u_2 are adjacent to each other. Then neither u_1 nor u_2 is adjacent to any other vertices in $V(U') - V(T')$ in G. Let $w \in V(U') - V(T')$. Then $\overline{G}$ contain a $W_{2,6}$ with b and v_{n-1} in the centre and $u_1wu_2v_3u_3v_4u_1$ as C_6.

Thus, $R(S_n(1,2), W_{2,6}) \leq 3n - 2$ which completes the proof. □

Next, we will determine the Ramsey numbers $R(T_n, W_{2,6})$ for all other tree graphs T_n versus the generalised wheel graph $W_{2,6}$.

Theorem 3.2.4. *If $n \geq 5$, then $R(T_n, W_{2,6}) = 3n - 2$ where T_n is any tree graph of order n apart from S_n, $S_n(1,1)$ and $S_n(1,2)$.*

Proof. Let T_n be any tree graph of order n apart from S_n, $S_n(1,1)$ and $S_n(1,2)$. By Theorem 2.2.7, $R(T_n, W_{2,6}) \geq (2+1)(n-1)+1 = 3n-2$. Also, by Theorems 2.2.2 and 3.1.2, $R(T_n, W_{2,6}) \leq R(T_n, W_6) + (2-1)(n-1) = (2n-1)+(n-1) = 3n-2$.

We conclude that $R(T_n, W_{2,6}) = 3n - 2$. □

By Theorems 3.2.2, 3.2.3 and 3.2.4, we conclude that $R(T_n, W_{2,6}) = 3n - 2$ for each tree graph T_n of order n. We can now consider the more general Ramsey numbers for the generalised wheel graphs $W_{s,m}$.

3.3 The Ramsey number $R(T_n, W_{s,6})$ and $R(T_n, W_{s,7})$

In this section, we investigate the Ramsey numbers for tree graphs T_n of order n versus the generalised wheel graph $W_{s,6}$ and $W_{s,7}$. We start by considering $W_{s,6}$.

Theorem 3.3.1. *Let $n \geq 5$ and $s \geq 2$. Then $R(T_n, W_{s,6}) = (s+1)(n-1)+1$.*

Proof. By Theorem 3.2.4, $R(T_n, W_{2,6}) = 3n - 2$. By applying Lemma 3.1.1 repeatedly, we see that $R(T_n, W_{s,6}) \leq (s+1)(n-1)+1$. Furthermore, since $\chi(W_{s,6}) = s+2$ and $t(W_{s,6}) = 1$, Theorem 2.2.7 implies that $R(T_n, W_{s,6}) \geq (s+1)(n-1)+1$. Hence, $R(T_n, W_{s,6}) = (s+1)(n-1)+1$. □

Next, we consider $W_{s,7}$.

Theorem 3.3.2. *Let $n \geq 5$ and $s \geq 1$. Then $R(T_n, W_{s,7}) = (s+2)(n-1)+1$.*

Proof. Note that $\chi(W_{s,7}) = s+3$ and $t(W_{s,7}) = 1$. Therefore, Theorem 2.2.7 implies that $R(T_n, W_{s,7}) \geq (s+2)(n-1)+1$ for each tree graph T_n of order n. Also, since $W_{s,7}$ is a subgraph of $W_{s+1,6}$, $R(T_n, W_{s,7}) \leq R(T_n, W_{s+1,6}) = (s+2)(n-1)+1$ by Theorem 3.3.1. Hence, $R(T_n, W_{s,7}) = (s+2)(n-1)+1$. □

3.4 Other results and possible future work

In this section, we state a conjecture.

Conjecture 3.4.1. *Suppose that $m \geq 3$ and $s \geq 2$. Then for sufficiently large n,*

$$R(T_n, W_{s,m}) = \begin{cases} (s+1)(n-1)+1, & \text{if } m \text{ is even;} \\ (s+2)(n-1)+1, & \text{if } m \text{ is odd.} \end{cases}$$

Brennan [10] determined the Ramsey numbers of large trees versus odd cycles.

Theorem 3.4.2. [10] *For all odd $m \geq 3$ and $n \geq 25m$, $R(T_n, C_m) = 2n - 1$.*

Lemma 3.4.3. *Suppose that $\ell \geq 2$, $n \geq \lfloor \frac{m}{2} \rfloor + 1$ and*

$$r(m) = \begin{cases} 2 & \text{, if } m \text{ is odd;} \\ 1 & \text{, if } m \text{ is even.} \end{cases}$$

If $R(T_n, W_{s,m}) \leq \big(s + r(m)\big)(n-1) + \ell$, then

$$R(T_n, W_{s+2,m}) \leq \big(s + 2 + r(m)\big)(n-1) + \ell - 1\,.$$

Proof. Let G be a graph of order $(s+2+r)(n-1)+\ell-1$ where $r = r(m)$. Suppose that G does not contain T_n.

Case 1: Suppose that G has a vertex of degree at most $n-3$, say v_0.

Let $U_1 = \{v_0\} \cup N_G(v_0)$; then $|U_1| \le n-2$. Let $Y_1 = V(G) - U_1$ and consider the graph $G[Y_1]$. Note that $G[Y_1]$ is of order at least

$$|V(G)| - |U_1| \ge \big((s+2+r)(n-1) + \ell - 1\big) - (n-2) = (s+1+r)(n-1) + \ell\,.$$

Since the generalised wheel graph $W_{s+1,m}$ is $K_1 + W_{s,m}$, Lemma 3.1.1 implies that

$$R(T_n, W_{s+1,m}) \le R(T_n, W_{s,m}) + n - 1 \le (s+r+1)(n-1) + \ell\,.$$

Therefore, $\overline{G}[Y_1]$ contains $W_{s+1,m}$. Note that v_0 is adjacent to every vertex of Y_1 in $\overline{G}$. In particular, v_0 is adjacent to every vertex of this $W_{s+1,m}$ in $\overline{G}$. Hence, $\overline{G}$ contains $W_{s+2,m}$.

Case 2: Suppose that each vertex of G has degree at least $n-2$.

Subcase 2.1: Suppose that each component of G is of order at most $n-1$.

Then every component of G is K_{n-1}. This implies that $\overline{G}$ contains a complete $(s+3+r)$-partite graph, where each part has exactly $n-1 \ge \left\lfloor \frac{m}{2} \right\rfloor$ vertices. It is straightforward to see that this complete $(s+3+r)$-partite graph contains $W_{s+2,m}$. Indeed, C_m is a subgraph of the induced subgraph on $r+1$ of the vertex classes, and K_{s+2} is a subgraph of the induced subgraph on the remaining $s+2$ vertex classes.

Subcase 2.2: Suppose that G has a component, say H_0, of order at least n.

Claim: There are two vertices $u, v \in V(H_0)$ such that

(i) u is not adjacent to v in G, and

(ii) $|N_G(u) \cup N_G(v)| \le 2n-5$.

Proof. Suppose that $T_n = S_n$. Then every vertex is of degree $n-2$ in G. Let $u \in V(H_0)$ and consider the graph $G[\{u\} \cup N_G(u)]$. Note that it is of order $n-1$ and that it is a subgraph of H_0. Since H_0 is connected, there is a vertex $v \in V(H_0) - (\{u\} \cup N_G(u))$ that is adjacent to some vertex in $N_G(u)$. Note that u is not adjacent to v and $N_G(u) \cap N_G(v) \ne \emptyset$. Therefore,

$$\begin{aligned}|N_G(u) \cup N_G(v)| &= |N_G(u)| + |N_G(v)| - |N_G(u) \cap N_G(v)| \\ &= (n-2) + (n-2) - |N_G(u) \cap N_G(v)| \le 2n-4-1 = 2n-5\,.\end{aligned}$$

Suppose that $T_n \ne S_n$. Then T_n can be drawn as a rooted tree with one vertex at level 1. Let L_i denote all the vertices at level i. Note that

(i) each vertex at level L_i is adjacent to a unique vertex at level L_{i-1}; and

(ii) no two vertices at level L_i are adjacent to each other.

Since $T_n \ne S_n$, T_n has at least three levels. Since every vertex in H_0 has degree at least $n-2$, H_0 has a subgraph T of order $n-1$, and it is also a subgraph of T_n. Let ℓ be the total levels of T_n. Then $\ell \ge 3$ and there is a vertex in T, say u_0 at level $\ell-1$ such that if a vertex $x \in X = V(H_0) - V(T)$ and an edge $xu_0 \in E(H_0)$ are added to T, then the resulting tree is T_n. This implies that u_0 is not adjacent to any vertex in X. Since u_0 has degree at least $n-2$, it must be of degree exactly $n-2$ and it is adjacent to every vertex in $V(T) - \{u_0\}$ in H_0.

Since H_0 is connected and of order at least n, there is a vertex in X that is adjacent to a vertex in $V(T)$. Let Q be the set of all vertices at level ℓ in T that are adjacent to u_0. Consider the tree $T - Q$. Either there is an edge connecting a vertex in X with a vertex in $T-Q$ or there is no edge connecting a vertex in X with a vertex in $T-Q$. Suppose that the latter holds and let b be a vertex in $T-Q$. Since b has degree at least $n-2$ and is not adjacent to any vertex in X, it must be of degree exactly $n-2$ and is adjacent to every vertex in $V(T)-\{b\}$. This means that $H_0[V(T)-Q]$ is a complete graph and every vertex in Q is adjacent to every vertex in $T-Q$.

Since H_0 is connected, we can find a vertex a in X and a vertex q in Q such that aq is an edge in H_0. Let c be the unique vertex at level $\ell-2$ that is adjacent to u_0. Now, we interchange the nodes c and q in T and consider the resulting graph T'. We can do this because q is adjacent to every vertex in $T-Q$. Note that $V(T)=V(T')$. Let Q' be the set of all the vertices at level ℓ in T' that are adjacent to u_0. Then aq is the edge connecting the vertex a in X with the vertex q in $T'-Q'$. Hence, we may assume from the beginning that there is an edge connecting a vertex in X, say z, with a vertex u in $T-Q$.

Let $u_0u_1\dots u_t=u$ be the unique path in T connecting u_0 to u_t. Note that u_1 is the unique vertex at level $\ell-2$ that is adjacent to u_0. Since u_0 is not adjacent to z, we have $t\geq 1$. We may assume that t is the smallest positive integer such that $N_G(u_t)\cap X\neq\emptyset$ and $N_G(u_i)\cap X=\emptyset$ for $0\leq i\leq t-1$. This implies that each $u_0,\dots,u_{t-1}$ has degree $n-2$ in H_0 and each u_i is adjacent to every vertex in $V(T)-\{u_i\}$ in H_0.

Suppose that z has degree at least $n-1$ in H_0. Then $N_G(z)=N_{H_0}(z)\geq n-1$. Now, we are going to form a new tree T^* which is a subgraph of H_0. Suppose that $t=1$. First, we remove u_0 and all the vertices that are adjacent to u_0 at level ℓ from T. Second, we add the vertex z at level $\ell-1$ and an edge connecting z to u_1. Let the resulting graph be T^*. Note that the graph T^* is of order $|V(T)|-|N_T(u_0)|+1=n-|N_G(u_0)|$. So, $|V(T^*)-\{z\}|=n-|N_T(u_0)|-1$. Now, z has degree at least $n-1$ implies that we can find $|N_T(u_0)|$ vertices in $N_G(z)-(V(T^*)-\{z\})$. By adding these vertices to level ℓ in T^* and edges connecting these vertices to z, the resulting tree is T_n, a contradiction.

Suppose that $t\geq 2$. First, we remove all the vertices that are adjacent to u_0 at level ℓ from T. Note that $|N_T(u_0)|-1$ vertices are removed from T. Let the resulting graph be S. Second, we interchange the node u_t and u_0 in S. This can be done as u_0 is adjacent to every vertex in $V(T)-\{u_0\}$ in H_0 and u_1 is adjacent to u_t (recall that each $u_0,\dots,u_{t-1}$ has degree $n-2$ and is adjacent to every vertex in $V(T)-\{u_i\}$). Let the resulting graph be S'. If u_t has degree at least $n-1$ in H_0, Then following the argument from the previous paragraph, adding some vertices in $N_G(u_t)$ and edges connecting them to u_t into the graph S', we obtain the tree T_n, a contradiction.

So, we may assume that u_t has degree $n-2$. Note also that if u_t is not adjacent to one of the vertices in $V(S)-\{u_t\}$ in H_0, then following the argument as in the previous paragraph, by adding some vertices in $N_G(u_t)$ and edges connecting them to u_t into the graph S', we obtain the tree T_n. So, we may assume that u_t is adjacent to every vertex in $V(S)-\{u_t\}$ in H_0. In this scenario, let's consider the graph T. We interchange the node u_t and u_1 in T. This can be done because u_t is adjacent to all vertices that are adjacent to u_1 in T. Now, we are in the situation as in the previous paragraph with $t=1$. Hence, we may assume that z has degree $n-2$.

Now, let $u = u_{t-1}$ and $v = z$. Then u and v are not adjacent in G and $u_t \in N_G(u) \cap N_G(v)$, which means that $|N_G(u) \cap N_G(v)| \geq 1$. Since u and v are of degree $n-2$, we have $|N_G(u) \cup N_G(v)| \leq 2n-5$.

This completes the proof of the claim. $\square$

Let $u, v \in V(H_0)$ be two vertices satisfying the conditions in the Claim and let $Y_0 = \{u, v\} \cup N_G(u) \cup N_G(v)$. Then

$$|Y_0| \leq |\{u, v\}| + |N_G(u) \cup N_G(v)| \leq 2n - 3\,.$$

Let $Y_1 = V(G) - Y_0$. Note that u and v are not adjacent to any vertices in Y_1. Consider the graph $G[Y_1]$. Note that $G[Y_1]$ is of order at least

$$\begin{aligned}|V(G)| - |Y_0| \geq \big((s+2+r)(n-1) + \ell - 1\big) - (2n-3) &= (s+r)(n-1) + \ell \\ &\geq R(T_n, W_{s,m})\,.\end{aligned}$$

Thus, $\overline{G}[Y_1]$ contains $W_{s,m}$. Now, u and v are adjacent to each other and to each vertex in Y_1 in $\overline{G}$. So, by adding u and v to the hub of $W_{s,m}$, we obtain $W_{s+2,m}$.

This completes the proof of the lemma. $\square$

Theorem 3.4.4. *Let $m \geq 3$. Then*

(a) *If m is odd and $n \geq 25m$, then $R(T_n, W_{s,m}) = (s+2)(n-1) + 1$.*

(b) *If m is even, $n \geq 25(m-1)$ and $s \geq 4n-3$, then $R(T_n, W_{s,m}) = (s+1)(n-1) + 1$.*

Proof. (a) For all odd $m \geq 3$, $\chi(W_{s,m}) = s+3$ and $t(W_{s,m}) = 1$. By Theorem 2.2.7, we have $R(T_n, W_{s,m}) \geq (s+2)(n-1) + 1$ for any tree of order n.

For the upper bound, recall that the wheel graph W_m is the graph $K_1 + C_m$. Therefore, $R(T_n, W_m) \leq R(T_n, C_m) + (n-1) = 3(n-1) + 1$ by Theorem 3.4.2 and Lemma 3.1.1. Therefore, $R(T_n, W_{s,m}) \leq R(T_n, W_m) + (s-1)(n-1) \leq (s+2)(n-1)+1$ by Theorem 3.1.2. Hence, $R(T_n, W_{s,m}) = (s+2)(n-1) + 1$.

(b) Now, m is even implies that $m-1$ is odd and $m-1 \geq 3$. Let G be a graph of order $3n-2$. Suppose that G does not contain T_n. Then G contains a subtree T' that is also a subtree of T_n and is maximal in the sense that it cannot be extended to a larger tree in T_n. Note that $T' \neq T_n$. Thus, T' is at most of order $n-1$. This implies that there is a vertex $u \in V(T')$ such that if a new vertex z' and a new edge uz' are added to T', then it is a larger subtree of T_n. Thus, u is not adjacent to any vertex in $X = V(G) - V(T')$ in G.

We now consider the graph $G[X]$. It is of order at least $3n-2-(n-1) = 2n-1$. By Theorem 3.4.2, $\overline{G}[X]$ contains C_{m-1}, say $a_1a_2\ldots a_{m-1}a_1$. Since u is adjacent to every vertex of X in $\overline{G}$, $a_1a_2\ldots a_{m-1}ua_1$ forms C_m in $\overline{G}$. Thus, $R(T_n, C_m) \leq 3n-2$. By Lemma 3.1.1, $R(T_n, W_m) \leq R(T_n, C_m) + (n-1) \leq 2(n-1) + 2n-1$. By Lemma 3.4.3, $R(T_n, W_{3,m}) \leq 4(n-1) + 2n-2$ and then $R(T_n, W_{5,m}) \leq 6(n-1) + 2n-3$. Continuing this way, we see that $R(T_n, W_{2(2n-1)-1,m}) \leq ((2(2n-1)-1)+1)(n-1)+1$. So, $R(T_n, W_{s,m}) \leq (s+1)(n-1)+1$ for all $s \geq 2(2n-1)-1 = 4n-3$ by Lemma 3.1.1 and induction.

For the lower bound, $\chi(W_{s,m}) = s+2$ and $t(W_{s,m}) = 1$. By Theorem 2.2.7, $R(T_n, W_{s,m}) \geq (s+1)(n-1)+1$, so $R(T_n, W_{s,m}) = (s+1)(n-1)+1$. $\square$

Chapter 4

Ramsey numbers for tree graphs with maximum degree of $n-1$, $n-2$ and $n-3$ versus the wheel graph of order 9

In this chapter, we will look at the Ramsey numbers for tree graphs T_n of order n versus the wheel graph W_8 of order 9, focusing on tree graphs with maximum degree of at least $n-3$. Similar results have been determined independently by Hafidh and Baskoro [33].

4.1 Introduction

In 2006, Chen, Zhang and Zhang [22] determined $R(T_n, W_6)$ and showed that this number is not generally $2n-1$, especially when T_n is one of the graphs S_n, $S_n(1,1)$ or $S_n(1,2)$. So as the first step to analyse the Ramsey numbers for tree graphs of order n versus the wheel graphs W_8 of order 9, we first look at these trees. So, in this chapter, we will present results for tree graphs T_n with maximum degree of $n-1$, $n-2$ and $n-3$ or, more specifically, on S_n, $S_n(1,1)$, $S_n(1,2)$ and $S_n(3)$.

4.2 Ramsey numbers for tree graphs with maximum degree of $n-1$ and $n-2$ versus the wheel graph of order 9

In this section, we investigate the Ramsey numbers for tree graphs with maximum degree of $n-1$ and $n-2$ versus the wheel graph of order 9. There are only two types of graph need to be studied, namely S_n and $S_n(1,1)$. In a series of papers [56, 57, 58], Zhang et al. determined the Ramsey numbers $R(S_n, W_8)$ for the star graph S_n versus the wheel graph W_8, as stated in Theorem 2.2.6. Now, we only need to consider $S_n(1,1)$.

Theorem 4.2.1. *For $n \geq 5$,*

$$R\big(S_n(1,1), W_8\big) = \begin{cases} 2n+1 & \textit{if } n \textit{ is odd,} \\ 2n & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. Consider the graph $G = K_{n-1} \cup H$ where

$$\overline{H} = \begin{cases} \frac{n-5}{4} K_4 \cup K_{3,3} & \text{if } n \equiv 1 \pmod 4; \\ \frac{n+1}{4} K_4 & \text{if } n \equiv 3 \pmod 4; \\ 2K_4 & \text{if } n = 8; \\ C_n & \text{if } n \text{ is even and } n \neq 8. \end{cases}$$

Note that G is a graph of order $2n$ when n is odd and of order $2n-1$ when n is even. Also, G does not contain $S_n(1,1)$ since K_{n-1} does not contain $S_n(1,1)$ and since H

is $(n-3)$-regular when $n \neq 8$ and 4-regular when $n = 8$. Assume that $\overline{G}$ contains W_8 with hub x. Then $x \notin V(K_{n-1})$ as $\overline{H}$ does not contain C_8, and so $x \in V(H)$. Since x is adjacent to at most 3 vertices in $\overline{H}$, at least 5 vertices in $V(\overline{K_{n-1}})$ are vertices of a cycle C_8 in $\overline{G}$, a contradiction since $\overline{K_{n-1}}$ has no edges. Therefore, $\overline{G}$ does not contain W_8, so $R(S_n(1,1), W_8) \geq |V(G)| + 1 = 2n + (n \bmod 2)$.

Now let G be a graph that does not contain $S_n(1,1)$ and assume that $\overline{G}$ does not contain W_8. Let $n \geq 5$ be odd and suppose that G has order $2n+1$. By Theorem 2.2.6, $R(S_n, W_8) = 2n+1$, so G contains S_n. Let v be a vertex in G that is adjacent to all vertices in a set L of $n-1 \geq 4$ vertices. Since G does not contain $S_n(1,1)$, L must be an independent set and no vertex in L is adjacent to any vertex in $U = V(G) - (\{v\} \cup L)$. Now $|U| = n+1$ and $G[U]$ does not contain $S_n(1,1)$, so, by Lemma 2.2.9, some vertex u_1 in U is not adjacent to at least two other vertices in U, say u_0 and u_2. Let u_3 and u_4 be two other vertices in U and consider any vertices $v_1, \ldots, v_4 \in L$. Then $L \cup \{u_0, \ldots, u_4\}$ spans W_8 in $\overline{G}$ with hub v_1 and rim $v_2u_0u_1u_2v_3u_3v_4u_4v_2$, a contradiction. Therefore, $R(S_n(1,1), W_8) \leq 2n+1$.

For even $n \geq 6$, suppose that G has order $2n$. If G has a vertex v that is adjacent to all vertices in a set L of $n-1 \geq 5$ vertices, Then as above, $\overline{G}$ must contain W_8, a contradiction. Therefore, $\Delta(G) \leq n-2$. By Theorem 2.2.6, $R(S_{n-1}, W_8) = 2n-1$, so G contains a vertex-disjoint star S_{n-1}. Let u be its centre vertex. Since $G - \{u\}$ is of order $2n-1$, it must contain another star S_{n-1}. These two stars are vertex-disjoint since $\Delta(G) \leq n-2$ and G does not contain $S_n(1,1)$. Let X_1 and X_2 be the vertex sets of these two stars. Then for each $i \in \{1,2\}$, no vertex of X_i is adjacent to any vertex outside X_i. Therefore, $\overline{G}$ contains W_8 with a vertex $x \in V(G) - (X_1 \cup X_2)$ as hub and its C_8 rim spanned by $X_1 \cup X_2$, a contradiction. Therefore, $R(S_n(1,1), W_8) \leq 2n$. □

4.3 Ramsey numbers for tree graphs with maximum degree of $n-3$ versus the wheel graph of order 9

In this section, we study the Ramsey numbers $R(T_n, W_8)$ for tree graphs T_n with maximum degree of $n-3$ versus the wheel graph W_8 of order 9. There are three types of graph to be studied, namely $S_n(1,2)$ and $S_n(3)$ and $S_n(2,1)$. Before we continue, there are several observations and lemmas have to be introduced.

First note two very simple observations for the existence of $S_n(1,2)$ in a graph and the existence of W_8 in the complement of a graph. These observations will be used repeatedly in deriving the exact Ramsey numbers for $S_n(1,2)$ versus W_8.

Observation 4.3.1. *If a graph G contains S_{n-1} and there is a vertex $v \in V(G) - V(S_{n-1})$ such that v is adjacent to at least two leaves of S_{n-1}, then G contains $S_n(1,2)$.*

Observation 4.3.2. *If $G = H_1 \cup H_2$ is the disjoint union of graphs H_1 and H_2, where $\overline{H_1}$ contains S_5 and H_2 is a graph of order at least 4, then $\overline{G}$ contains W_8.*

Lemma 4.3.3. *Let $n \geq 6$. If H is a graph of order $n+1$ with $\delta(H) \geq n-3$, then either H contains $S_n(1,2)$, or $n \equiv 3 \pmod 4$ and $\overline{H}$ is the disjoint union of $\frac{n+1}{4}$ copies of K_4; i.e., $\overline{H} = \frac{n+1}{4}K_4$.*

Proof. Suppose that some vertex in H has degree at least $n-2$; then H contains S_{n-1}. Since $\delta(H) \geq n-3 \geq 3$, the two vertices in $V(H) - V(S_{n-1})$ are either adjacent and must each be adjacent to at least one leaf of S_{n-1}, or they are not adjacent and must each be adjacent to at least two leaves of S_{n-1}. In either case, H contains $S_n(1,2)$.

Now suppose that H is $(n-3)$-regular and let v_0 be any vertex of H. The set $U = V(H) - N_H(v_0)$ has exactly 3 vertices, each with degree $n-3 \geq 3$ and each therefore adjacent to at least one vertex in $N_H(v_0)$. If $H[U]$ has an edge, then H contains $S_n(1,2)$; otherwise, U is an independent set, and so $\{v_0\} \cup U$ is an independent set of size 4. Furthermore, $N_H(u) = N_H(v_0)$ for all $u \in U$, as every vertex has degree $n-3$. Hence, $\overline{H}[\{v_0\} \cup U] = K_4$ and is a component in $\overline{H}$. Applying the above arguments to each vertex $v_0 \in V(H)$ shows that either that H contains $S_n(1,2)$ or that $\overline{H}$ is the disjoint union of $\frac{n+1}{4}$ copies of K_4, in which case $n \equiv 3 \pmod 4$. □

Lemma 4.3.4. *Let H_1 be a graph whose complement $\overline{H_1}$ contains S_4, and let H_2 be a graph of order $m \geq 5$. If $G = H_1 \cup H_2$, then either $\overline{G}$ contains W_8, or H_2 is K_m or $K_m - e$, where e is an edge in K_m.*

Proof. If $\overline{H_2}$ has at most one edge, then H_2 is the complete graph K_m or the graph $K_m - e$ obtained from removing an edge e from K_m. Suppose now that $\overline{H_2}$ has at least two edges. Consider a star S_4 in $\overline{H_1}$ and let v_0 be its centre and v_1, v_2, v_3 its leaves. Note that each v_i is adjacent to each $a \in V(H_2)$ in $\overline{G}$. Choose 5 vertices $a, b, c, d, e \in V(H_2)$ such that either ab and cd are independent edges, or abc is a path, in $\overline{H_2}$. In both cases, $\overline{G}$ contains W_8 with hub v_0. In the former case, $v_1abv_2cdv_3ev_1$ forms the C_8 rim; in the latter, $v_1abcv_2dv_3ev_1$ forms the C_8 rim. □

The following lemmas provides sufficient conditions for a graph or its complement to contain C_8.

Lemma 4.3.5. *Suppose that $U = \{u_1, \ldots, u_4\}$ and $V = \{v_1, \ldots, v_4\}$ are two disjoint subsets of vertices of a graph G for which $|N_{G[V]}(u)| \leq 1$ for each $u \in U$ and $|N_{G[U]}(v)| \leq 2$ for each $v \in V$. Then $\overline{G}[U \cup V]$ contains C_8.*

Proof. Suppose that $N_{G[U]}(v) \leq 1$ for each $v \in V$. Then $\overline{G}[U \cup V]$ contains a subgraph obtained by removing a matching from $K_{4,4}$ and therefore contains C_8. Suppose now that $N_{G[U]}(v_1) = \{u_1, u_2\}$, and assume without loss of generality that $v_3 \notin N_{G[V]}(u_3)$ and $v_4 \notin N_{G[V]}(u_4)$. Neither u_1 nor u_2 is adjacent to v_2, v_3 or v_4, so $v_1u_3v_3u_1v_2u_2v_4u_4v_1$ forms C_8 in $\overline{G}[U \cup V]$. □

Lemma 4.3.6. *Let $U = \{u_1, u_2, u_3\}$ and $V = \{a, b, c, d, e, f\}$ be disjoint sets of vertices of a graph G. Suppose that, for each $v \in V$, either v is adjacent to all vertices in U, or v is adjacent to exactly two vertices in U and every vertex in $V - \{v\}$. If $G[V]$ has at least two edges, then $G[U \cup V]$ contains C_8.*

Proof. Consider the set $X = \{v \in V : v$ is not adjacent to some vertex in $U\}$.

Case 1: Suppose that $|X| = 0$. The graph $G[V]$ contains either a path of length two, say abc, or two disjoint edges, say ab and cd. Then either $eu_1abcu_2du_3e$ or $eu_1abu_2cdu_3e$ forms C_8 in G.

Case 2: Suppose that $1 \leq |X| \leq 4$. Without loss of generality, assume that $e, f \in V - X$ and $a \in X$. Then a is adjacent to each vertex in $V - \{a\}$. Now, b is adjacent to some vertex in U, say u_1, and c is adjacent to at least one other vertex in U, say u_2. Then $u_1bacu_2eu_3fu_1$ forms C_8.

Case 3: Suppose that $|X| = 5$. Then $V - X$ contains a single vertex, say f, and $G[V - \{f\}] = K_5$. Without loss of generality, a is adjacent to u_1 and e is adjacent to u_2. Then $u_1abcdeu_2fu_1$ forms C_8.

Case 4: Suppose that $|X| = 6$. Then $G[V] = K_6$. Each vertex in V is adjacent to 2 vertices in U, so 12 edges connect the 3 vertices in U with the 6 vertices in V. Thus, some vertex u_i is adjacent to at least 4 vertices in V and some other vertex u_j is adjacent to at least 3 vertices in V. Suppose that u_i is adjacent to a and b, and that u_j is adjacent to c and d. Then au_ibcu_jdefa forms C_8.

In each case, $G[U \cup V]$ contains C_8. □

The next two lemmas consider a graph of order $2n$ obtained from the disjoint union of two graphs whose orders differ by at most two.

Lemma 4.3.7. *Let $G = H_1 \cup H_2$, where H_1 and H_2 are graphs of order $n \geq 6$. Then either G contains $S_n(1,2)$ or $\overline{G}$ contains W_8.*

Proof. Suppose that G does not contain $S_n(1,2)$. Then neither H_1 nor H_2 is K_n or $K_n - e$, where e is an edge in K_n. By Lemma 4.3.4, neither $\overline{G}[H_1]$ nor $\overline{G}[H_2]$ contains S_4, so each vertex in $\overline{G}$ has degree at most two; hence, each vertex in G has degree at least $n-3$. If some vertex in G has degree at least $n-2$, then H_1 or H_2 contains $S_n(1,2)$, a contradiction.

Therefore, G is $(n-3)$-regular. Then $\overline{G}[H_1]$ and $\overline{G}[H_2]$ are 2-regular graphs and must each be a union of cycles. Since $|V(H_1)| = |V(H_2)| = n \geq 6$, there are vertex-disjoint paths of length two in $\overline{G}[H_1]$, say abc and def, and a path xyz in $\overline{G}[H_2]$. Now, as every vertex in $V(H_2)$ is adjacent to every vertex in $V(H_1)$ in $\overline{G}$, the graph $\overline{G}$ contains W_8 with hub y and rim $xabczdefx$. □

Lemma 4.3.8. *For $n \geq 6$, let $G = H_1 \cup H_2$, where H_1 and H_2 are graphs of order $n-1$ and $n+1$, respectively. If G does not contain $S_n(1,2)$ and $\overline{G}$ does not contain W_8, then $n \equiv 3 \pmod 4$ and $H_1 = K_{n-1}$ or $H_1 = K_{n-1} - e$ where e is an edge in K_{n-1}, while $\overline{H}_2 = \frac{n+1}{4}K_4$.*

Proof. The graph $\overline{H_2}$ does not contain S_5 since $\overline{G}$ would otherwise contain W_8. Each vertex of H_2 therefore has degree at least $n-3$ in H_2 (and in G). By Lemma 4.3.3, $n \equiv 3 \pmod 4$ and $\overline{H_2}$ is the disjoint union of $\frac{n+1}{4}$ copies of K_4. Therefore, $\overline{H_2}$ contains S_4, and since H_1 has order $n-1 \geq 5$, Lemma 4.3.4 implies that $H_1 = K_{n-1}$ or $K_{n-1} - e$ where e is an edge in K_{n-1}. □

The following theorem implies that, for most graphs G of order $2n$, either G contains $S_n(1,2)$ or $\overline{G}$ contains W_8.

Theorem 4.3.9. *For $n \geq 6$, let G be a graph of order $2n$. Suppose that G does not contain $S_n(1,2)$ and $\overline{G}$ does not contain W_8. Then $n \equiv 3 \pmod 4$ and $G = H_1 \cup H_2$ where $H_1 = K_{n-1}$ or $H_1 = K_{n-1} - e$ where e is an edge in K_{n-1}, and $\overline{H}_2 = \frac{n+1}{4}K_4$.*

Proof. Since $n - 1 \geq 5$, G has a subgraph $T = S_{n-1}(1,1)$ by Theorem 4.2.1. Let $V(T) = \{a, v_0, \ldots, v_{n-3}\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-3}, v_1a\}$.

Assume that v_0 is adjacent to a vertex v_{n-2} in $V(G) - V(T)$. Then the graph $T_1 = S_n(1,1)$ is obtained from T by adding the vertex v_{n-2} and the edge v_0v_{n-2}. Since G does not contain $S_n(1,2)$, a is not adjacent to any vertex in $V(G) - \{v_0, v_1\}$. Let $U = V(G) - V(T_1)$ and note that $|U| = n \geq 6$. If each vertex of U has degree at least $n-2$ in $G[U]$, then $G[U]$ contains $S_n(1,2)$, a contradiction. There is then a vertex of U with degree at most $n-3$ in $G[U]$, so $\overline{G}[U]$ contains a path of length two. Since G does not contain $S_n(1,2)$, each vertex $u \in U$ is adjacent to at most one vertex in $\{v_1, \ldots, v_{n-2}\}$ and if u is adjacent to one of these vertices, then u is not adjacent to any vertex in U. Let $Y_1 = \{v_2, v_3, v_4\}$ and $Y_2 \subset U$ be a set of six vertices such that $\overline{G}[Y_2]$ contains a path of length two. Then the graph $\overline{G}[Y_1 \cup Y_2]$ satisfies the conditions in Lemma 4.3.6 and therefore contains C_8 which, with a as hub, forms W_8, a contradiction.

Hence, v_0 is not adjacent to any vertex in $V(G) - V(T)$. Let $G = H_1 \cup H_2$, where H_1 is the component of G containing T and where $V(H_2)$ may be empty. Set $U = V(H_1) - V(T)$ and note that a is not adjacent to any vertex in U since G does not contain $S_n(1,2)$. If $G[U]$ contains an edge uv, then since H_1 is connected, either u or v is adjacent to v_i for some $1 \leq i \leq n-3$, and G contains $S_n(1,2)$, a contradiction. Therefore, U is independent; indeed, $\{v_0\} \cup U$ and $\{a\} \cup U$ are two independent sets in G. Assume that $|U| \geq 3$. Since $|U \cup V(H_2)| = n+1 \geq 7$, there are at least 3 vertices $b, c, d \in U$ and 4 vertices $x, y, z, w \in U \cup V(H_2) - \{b, c, d\}$. Together with v_0 and a, these vertices span W_8 in $\overline{G}$ with hub b and rim $axv_0yczdwa$, a contradiction. Therefore, $|U| \leq 2$, so the orders of H_1 and H_2 differ by at most two, and the theorem follows from Lemmas 4.3.7 and 4.3.8. $\square$

We are now ready to determine the exact Ramsey number for $S_n(1,2)$ versus W_8.

Theorem 4.3.10. *For $n \geq 6$,*

$$R(S_n(1,2), W_8) = \begin{cases} 2n+1 & \textit{if } n \equiv 3 \pmod 4, \\ 2n & \textit{otherwise.} \end{cases}$$

Proof. For the upper bound, Theorem 4.3.9 implies that $R(S_n(1,2), W_8) \leq 2n$ unless $n \equiv 3 \pmod 4$. Suppose that $n \equiv 3 \pmod 4$, and let G be a graph of order $2n+1$ such that $\overline{G}$ does not contain W_8. Then G contains S_n by Theorem 2.2.6. For any vertex $a \notin V(S_n)$, the graph $G_1 = G - \{a\}$ has order $2n$ and contains a vertex of degree at least $n-1$, so G_1 cannot equal $H_1 \cup H_2$ for $H_1 = K_{n-1}$ or $H_1 = K_{n-1} - e$ and $H_2 = \overline{\frac{n+1}{4} K_4}$. By Theorem 4.3.9, G_1 and thus G contains $S_n(1,2)$.

For the lower bound, let m and ℓ be any non-negative integers with $4m + 3\ell = n$; such integers exist since $n \geq 6$. Consider the graph $G = K_{n-1} \cup H$, where $\overline{H} = \frac{n+1}{4} K_4$ if $n \equiv 3 \pmod 4$ and $\overline{H} = mK_4 \cup \ell K_3$ otherwise. Now, K_{n-1} does not contain $S_n(1,2)$; nor does H, since each vertex v of H has degree at most $n-3$ and the set of vertices in H that are not adjacent to v is an independent set in G. Thus, G does not contain $S_n(1,2)$. Assume that $\overline{G}$ contains W_8 with hub x. Then $x \notin V(K_{n-1})$ since $\overline{H}$ does not contain C_8, so $x \in V(H)$. Since x is adjacent to at most 3 vertices in $\overline{G}[V(H)]$, at least 5 vertices in $V(K_{n-1})$ are vertices of C_8 subgraph of $\overline{G}$, a contradiction since $\overline{K_{n-1}}$ has no edges. Therefore, $\overline{G}$ does not contain W_8, completing the proof of the theorem. $\square$

Theorem 4.3.11. *If $n \geq 6$, then*

$$R(S_n(3), W_8) = \begin{cases} 2n-1 & \text{, for odd } n \geq 9\,; \\ 2n & \text{, otherwise}\,. \end{cases}$$

Proof. First, consider the case where $n \geq 9$ is odd. The graph $2K_{n-1}$ does not contain $S_n(3)$ and its complement does not contain W_8, so $R(S_n(3), W_8) \geq 2n-1$. To prove that $R(S_n(3), W_8) \leq 2n-1$, let G be any graph of order $2n-1$ and assume that G does not contain $S_n(3)$ and that $\overline{G}$ does not contain W_8. By Theorem 2.2.6, G contains S_{n-2}. Let v_0 be the centre of S_{n-2} and let $L = \{v_1, \ldots, v_{n-3}\}$ be its leaves. Set $U = V(G) - V(S_{n-2})$; then $|U| = n+1$. Since G does not contain $S_n(3)$, $v_1, \ldots, v_{n-3}$ are each adjacent to at most one vertex in U.

Claim 1: If some vertex in U is adjacent in G to at least 4 vertices in L, then $\overline{G}$ contains W_8.

Proof of Claim 1. Let v_1, v_2, v_3 and v_4 be vertices in L that are adjacent in G to some vertex $u \in U$. Set $U' = U - \{u\}$ and write $U' = \{u_1, \ldots, u_n\}$. Then v_1, v_2, v_3 and v_4 are not adjacent in G to any vertex of U'. Assume that $\Delta(\overline{G}[U']) \leq 3$; then $\delta(G[U']) \geq n-4$. Since n is odd, the Handshaking Lemma implies that $\Delta(G[U']) \geq n-3$, so some vertex of U', say u_1, must be adjacent in G to at least other $n-3$ vertices of U, say $u_2, \ldots, u_{n-2}$. Note that u_{n-1} and u_n are both adjacent to at least $n-6$ vertices of $\{u_2, \ldots, u_{n-2}\}$ in G. If $n \geq 11$, then at least one of $u_2, \ldots, u_{n-2}$ is adjacent to both u_{n-1} and u_n, forming $S_n(3)$, a contradiction. Suppose that $n = 9$. The vertices u_8 and u_9 cannot both be adjacent in G to some vertex in $\{u_2, \ldots, u_7\}$ since that would form $S_n(3)$; therefore, u_8 and u_9 are adjacent to each other as well as to u_1; also, u_8 is adjacent to three of the vertices $u_2, \ldots, u_7$ and u_9 is adjacent to other the three, again forming $S_9(3)$ in G, a contradiction.

Therefore, $\Delta(\overline{G}[U']) \geq 4$ and, by Observation 4.3.2, $\overline{G}$ contains W_8. □

Claim 2: If each vertex in U is non-adjacent in G to at least 5 vertices of L, then $\overline{G}$ contains W_8.

Proof of Claim 2. Assume that $\Delta(\overline{G}[U]) \leq 3$. Then $\delta(G[U]) \geq n-3$. Write $U = \{u_1, \ldots, u_{n+1}\}$. Without loss of generality, u_1 is adjacent in G to every vertex of $U' = \{u_2, \ldots, u_{n-2}\}$. Now, u_{n-1}, u_n and u_{n+1} are each adjacent to at least $n-6$ vertices of U'. Since $n \geq 9$, at least two of u_{n-1}, u_n and u_{n+1} are adjacent to some vertex in U', forming $S_n(3)$ in G, a contradiction.

Therefore, $\Delta(\overline{G}[U]) \geq 4$. Then some vertex $u \in U$ is adjacent in $\overline{G}$ to at least 4 other vertices of U, say $u_1, \ldots, u_4$. Let $v_1, \ldots, v_5$ be 5 vertices of L that are not adjacent to u in G. If any of $u_1, \ldots, u_4$ is adjacent in G to 4 vertices from $\{v_1, \ldots, v_5\}$, then $\overline{G}$ contains W_8 by Claim 1. Otherwise, $u_1, \ldots, u_4$ are each adjacent in $\overline{G}$ to at least two of $v_1, \ldots, v_5$. Since each vertex v_i is adjacent to at most one vertex in U, it is adjacent in $\overline{G}$ to at least 3 vertices from $\{u_1, \ldots, u_4\}$. Then 4 vertices v_i together with $u_1, \ldots, u_4$ form C_8 in $\overline{G}$, and thus W_8 with vertex u as hub, a contradiction. □

Proof of Theorem 4.3.11 (continued). For $n \geq 11$, $|L| \geq 8$. By Claim 1, each vertex in U is adjacent in G to at most 3 vertices of L. Then by Claim 2, $\overline{G}$ contains W_8, a contradiction.

Suppose that $n = 9$; then $|L| = 6$. By Claim 1, each vertex in U is adjacent in G to at most 3 vertices in L. Therefore, by Claim 2, at least one vertex $u \in U$ must be adjacent in G to either 2 or 3 vertices of L. Assume that u is adjacent in G to exactly 3 vertices of L, say v, v' and v''. Set $U' = U - \{u\}$ and note that no vertex in U' is adjacent in G to v, v' or v''. If each vertex in U' is adjacent to at most two vertices in L, then every vertex in U' is non-adjacent to at least 4 vertices in L. If $\Delta(\overline{G}[U']) \geq 4$, then some vertex $u' \in U'$ is non-adjacent to at least 4 vertices of L and 4 vertices of U' in G. Since 3 of the vertices in L are non-adjacent to each vertex in U' and $d_{U'}(v) \leq 1$ for all $v \in L$, these 8 vertices form C_8 in $\overline{G}$ which, with u' as hub, forms W_8 in $\overline{G}$, a contradiction. If $\Delta(\overline{G}[U']) \leq 3$, then $\delta(G[U']) \geq 5$. By a similar argument to that in the proof of Claim 1, G contains $S_9(3)$, a contradiction. Therefore, suppose that some vertex in U' is non-adjacent to exactly 3 vertices of L in G. Let u' and u'' be the two vertices that are adjacent to exactly 3 vertices of L in G. Note that no vertex of L is adjacent in G to the vertices in $U - \{u', u''\}$. If $\Delta(\overline{G}[L]) \geq 4$, then $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction. Therefore, $\Delta(\overline{G}[L] \leq 3$ and so $\delta(G[L]) \geq 2$. Since $S_9(3) \nsubseteq G$, v_0 is not adjacent in G to any vertex of U. Now, if $\delta(G[U]) \geq 6$, by the similar argument to that in the proof of Claim 2, G contains $S_9(3)$, a contradiction. On the other hand, suppose that $\delta(G[U]) \leq 5$. Then $\Delta(\overline{G}[U]) \geq 4$, so some vertex $u \in U$ is adjacent in $\overline{G}$ to at least 4 other vertices of U. Together with v_0 and 3 other vertices from L, these 5 vertices from U form W_8 in $\overline{G}$ with u as hub, a contradiction.

Now, consider the case where $u \in U$ is adjacent in G to exactly two vertices of L, say v and v'. Set $U' = U - \{u\}$ and note that every vertex in U is adjacent in G to at most two vertices of L; for otherwise, relabel the vertex u and apply the previous case. If u is non-adjacent to at least 4 vertices in U', then since $d_{G[U']}(w) \leq 1$ for all $w \in L$, these 4 vertices and the remaining 4 vertices of L form C_8 in $\overline{G}$ by Lemma 4.3.5 and, with u as hub, form W_8, a contradiction. Therefore, u is adjacent to at least 6 vertices of U' in G. Then neither v and v' are adjacent to the remaining 4 vertices in L, since G does not contain $S_9(3)$. Then 4 vertices of U' and the 4 vertices of L form C_8 in $\overline{G}$ by Lemma 4.3.5 and, with v as hub, form W_8, a contradiction.

Hence, $R(S_n(3), W_8) \leq 2n - 1$, so $R(S_n(3), W_8) = 2n - 1$ for all odd $n \geq 9$.

Now, consider the cases in which $n = 7$ and $n \geq 6$ is even. Define the graph $G = K_{n-1} \cup H$, where H is as shown in Figure 4.1 if $n = 7$; $\overline{H} = \frac{n}{4}K_4$ if $n \equiv 0 \pmod 4$; and $\overline{H} = \frac{n-6}{4}K_4 \cup 2K_3$ if $n \equiv 2 \pmod 4$. Since G has no $S_n(3)$ subgraph and $\overline{G}$ does not contain W_8, $R(S_n(3), W_8) \geq 2n$.

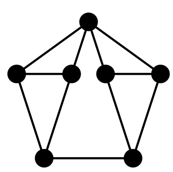


Figure 4.1: The graph H when $n = 7$.

For the upper bound, let G be any graph of order $2n$. Suppose to the contrary that G does not contain $S_n(3)$ and $\overline{G}$ does not contain W_8. By Theorem 2.2.6, G has a subgraph $T = S_{n-1}$. Let v_0 be the centre of T and $L = N_T(v_0) = \{v_1, \ldots, v_{n-2}\}$. Set $U = V(G) - V(T)$; then $|U| = n + 1$.

Case 1: $E_G(L, U) \neq \emptyset$.

Without loss of generality, assume that v_1 is adjacent to $u \in U$, and set $U' = U - \{u\}$. Since G does not contain $S_n(3)$, $N_G(v_1) = \{v_0, u\}$ and $d_{U'}(v_i) \leq 1$ for $2 \leq i \leq n-2$. Then for $n \geq 7$, there are 4 vertices from $L - \{v_1\}$ and 4 vertices from U' that together form C_8 in $\overline{G}$ and, with v_1 as hub, form W_8 in $\overline{G}$, a contradiction.

Suppose that $n = 6$. If $\Delta(\overline{G}[U']) \geq 3$, then some vertex $u' \in U'$ is adjacent in $\overline{G}$ to at least 3 other vertices of U', say u_1, u_2, u_3. Since $d_{U'}(v_i) \leq 1$ for $2 \leq i \leq n-2$, each v_i is adjacent in $\overline{G}$ to at least two of u_1, u_2, u_3, and so $\overline{G}$ contains W_8. To illustrate this, suppose that v_2 is adjacent to u_1. Since v_3 is adjacent to two of u_1, u_2, u_3 in $\overline{G}$, v_3 must be adjacent to another vertex other than u_1, say u_2, in $\overline{G}$. Let u_4 and u_5 be the two remaining vertices of U'. Then $v_2u_1u'u_2v_3u_4v_4u_5v_2$ and v_1 W_8 in $\overline{G}$, a contradiction. Therefore, $\Delta(\overline{G}[U']) \leq 2$, and $\delta(G[U']) \geq 3$. Let $U' = \{u_1, \ldots, u_6\}$. Suppose that U' has a vertex, say u_1, that is adjacent in G to at least 4 other vertices, say u_2, u_3, u_4, u_5. Then u_6 is adjacent to u_i and u_i is adjacent to u_j for some $2 \leq i \neq j \leq 5$, so $G[U']$ contains $S_6(3)$, a contradiction. Therefore, $G[U']$ is 3-regular. Suppose that u_1 is adjacent to u_2, u_3 and u_4. Since u_5 and u_6 are adjacent to at least two of u_2, u_3, u_4, u_i is adjacent to u_5 and u_6 for some $2 \leq i \leq 4$. Then $G[U']$ contains $S_6(3)$, a contradiction.

Case 2: $E_G(L, U) = \emptyset$.

If n is even, then $R(S_n(1,1), W_8) = 2n$ by Theorem 4.2.1, and Case 1 applies. Hence, it suffices to consider $n = 7$. If $\Delta(\overline{G}[U]) \geq 4$, then some vertex $u \in U$ is adjacent in $\overline{G}$ to at least 4 vertices of U. Together with any 4 vertices from L, these vertices form W_8, with u as hub, in $\overline{G}$, a contradiction. Suppose that $\Delta(\overline{G}[U]) \leq 3$. Then $\delta(G[U]) \geq 4$. Write $U = \{u_1, \ldots, u_8\}$ where u_1 is adjacent to $\{u_2, \ldots, u_5\}$. Since $\delta(G[U]) \geq 4$, each of the vertices u_6, u_7, u_8 is adjacent to at least one of $u_2, \ldots, u_5$. If u_1 is not adjacent in G to u_6, u_7 or u_8 in G, then one of $u_2, \ldots, u_5$ is adjacent to at least two of these 3 vertices and G therefore contains $S_7(3)$, a contradiction. Now, suppose that u_1 is adjacent to one of u_6, u_7, u_8, say u_6. Since $\delta(G[U]) \geq 4$, u_7 is adjacent to at least two vertices of $u_2, \ldots, u_6$, say u_2 and u_3. Since $\delta(G[U]) \geq 4$, u_2 is adjacent to another vertex from $u_3, \ldots, u_6$. Then G therefore contains $S_7(3)$, a contradiction.

In either case, $R(S_n(3), W_8) \leq 2n$ for $n = 7$ and even $n \geq 6$. □

Theorem 4.3.12. *If $n \geq 6$, then*

$$R(S_n(2,1), W_8) = \begin{cases} 2n - 1 & , \textit{if } n \textit{ is odd.} \\ 2n & , \textit{otherwise.} \end{cases}$$

Proof. When n is odd, note that $G = 2K_{n-1}$ has no $S_n(2,1)$ subgraph and $\overline{G}$ does not contain W_8. Hence, $R(S_n(2,1), W_8) \geq 2n-1$. When n is even, define $H = \overline{\frac{n}{4}K_4}$ if $n \equiv 0 \pmod 4$ and $H = \overline{\frac{n-6}{4}K_4 \cup 2K_3}$ if $n \equiv 2 \pmod 4$; then $G = K_{n-1} \cup H$ does not contain $S_n(2,1)$ and $\overline{G}$ does not contain W_8. Hence, $R(S_n(2,1), W_8) \geq 2n$.

Now let G be a graph of order $n + 2\lfloor n/2 \rfloor$ and assume that G does not contain $S_n(2,1)$ and that $\overline{G}$ does not contain W_8. Suppose that $n \geq 8$. Then by Theorem 4.3.11, G has a subgraph $T = S_n(3)$. Let $V(T) = \{v_0, \ldots, v_{n-1}\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-3}, v_1v_{n-2}, v_1v_{n-1}\}$. Set $U = V(G) - V(T)$ and $U' = \{v_{n-2}, v_{n-1}\} \cup U$; then $|U| = 2\lfloor n/2 \rfloor$. Since $S_n(2,1) \nsubseteq G$, none of $v_2, \ldots, v_{n-3}$ is adjacent to any vertex in U'. Then $\Delta(\overline{G}[U']) \leq 3$ by Observation 4.3.2. This implies that $\delta(G[U']) \geq |U'| - 4 \geq n - 3$. Choose a $S_{|U'|-3}$ subgraph in $G[U']$ and note that each of the remaining 3 vertices in U' must be adjacent to at least two leaves of this $S_{|U'|-3}$, forming $S_n(2,1)$, a contradiction.

Suppose now that $n = 7$. Then G is a graph of order 13. Two cases are now considered.

Case 1a: Suppose that $\Delta(G) \geq 5$.

Let T be an S_6 subgraph in G with centre v_0 and leaves $L = \{v_1, \ldots, v_5\}$. Set $U = V(G) - V(T)$. Since $G[U]$ does not contain $S_7(2,1)$, it is straightforward to verify that $\delta(G[U]) \leq 2$. Therefore, $\Delta(\overline{G}[U]) \geq 4$. If at least 4 vertices in L are not adjacent to any vertex in U, then $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction. Since G does not contain $S_7(2,1)$, the only possible case avoiding the above scenario is when two of the vertices in L, say v_1 and v_2, are adjacent to a common vertex $u \in U$. Again as G does not contain $S_7(2,1)$, v_5 is not adjacent to any vertex in $L - \{v_5\}$, and no vertex in L is adjacent to any vertex in $U - \{u\}$. Then $\overline{G}$ contains W_8 with hub v_5 and C_8 formed by $L - \{v_5\}$ and any 4 vertices in $U - \{u\}$, a contradiction.

Case 1b: Suppose that $\Delta(G) \leq 4$.

By Theorem 4.2.1, G has a subgraph $T = S_6(1,1)$. Let $V(T) = \{v_0, \ldots, v_5\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_4, v_1v_5\}$. Set $U = V(G) - V(T)$. As in Case 1a, $\Delta(\overline{G}[U]) \geq 4$. Since $\Delta(G) \leq 4$, v_0 is not adjacent to any vertex in U, and none of the vertices v_2, v_3, v_4 is adjacent to any vertex in U since G does not contain $S_7(2,1)$. Again, $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

In either case, $R(S_n(2,1), W_8) \leq 2n - 1$. Hence, $R(S_n(2,1), W_8) = 2n - 1$ for all odd $n \geq 7$.

Suppose that $n = 6$. If some vertex $u \in U$ is adjacent to v_1 in G, then since G does not contain $S_6(2,1)$, neither v_5 nor u is adjacent to v_2, v_3, v_4 or any vertex in U. Then v_3, v_4, v_5, u and any other 4 vertices of U form C_8 in $\overline{G}$ which, with v_2 as hub, forms W_8, a contradiction.

Suppose then that v_1 is not adjacent in G to any vertex of U. Consider the following two cases.

Case 2a: Suppose that v_1 is not adjacent to v_2, v_3 or v_4.

Let $U = \{u_1, \ldots, u_6\}$. If $\Delta(\overline{G}[U]) \geq 2$, then some vertex in U, say u_1, is adjacent to another two vertices in U, say u_2 and u_3, in $\overline{G}$. Then $u_2u_1u_3v_1u_4v_2u_5v_3u_2$ and v_4 form W_8 in $\overline{G}$, a contradiction. If $\Delta(\overline{G}[U]) \leq 1$, then $\delta(G[U]) \geq 4$. Suppose that u_1 is adjacent to $u_2, \ldots, u_5$ in G. Since u_5 and u_6 are each adjacent to at least two vertices of $\{u_2, u_3, u_4\}$, $G[U]$ contains $S_n(2,1)$, a contradiction.

Case 2b: v_1 is adjacent to another vertex of T other than v_0 and v_5 in G.

Without loss of generality, suppose that v_1 is adjacent to v_2 in G. Since G does not contain $S_6(2,1)$, v_5 is not adjacent to v_3, v_4 or any vertex in U. Let $U = \{u_1, \ldots, u_6\}$. If $\Delta(\overline{G}[U]) \geq 2$, then some vertex in U, say u_1, is adjacent in $\overline{G}$

to another two vertices in U, say u_2 and u_3, so $u_2u_1u_3v_5u_4v_2u_5v_3u_2$ and v_4 form W_8 in $\overline{G}$, a contradiction. Thus, $\Delta(\overline{G}[U]) \leq 1$, and $\delta(G[U]) \geq 4$. As in Case 1, $G[U]$ must contain $S_n(2,1)$, a contradiction.

In either case, $R(S_n(2,1), W_8) \leq 2n$. Thus, $R(S_n(2,1), W_8) = 2n$ for all even $n \geq 6$. $\square$

Chapter 5

Ramsey numbers for tree graphs with maximum degree of $n-4$ and $n-5$ versus the wheel graph of order 9

In this chapter, we will continue to look at the Ramsey numbers for tree graphs of order n versus the wheel graph W_8 of order 9, focusing on tree graphs T_n with maximum degree $n-4$ and $n-5$.

5.1 Introduction

Before we start to look into the Ramsey results, in this section, we introduce the trees that will appear in our discussion. First, we introduce all tree graphs T_n of order $n \geq 6$ with $\Delta(T_n) = n-4$. For $n = 6$, there is just one such graph, namely the path graph $T_6 = P_6$. Theorem 2.2.4 provides the Ramsey number $R(P_6, W_8) = 12$. For $n = 7$, there are 5 tree graphs with $\Delta(T_7) = 7-4 = 3$, which are A, B, C, D and E shown in Figure 5.1.

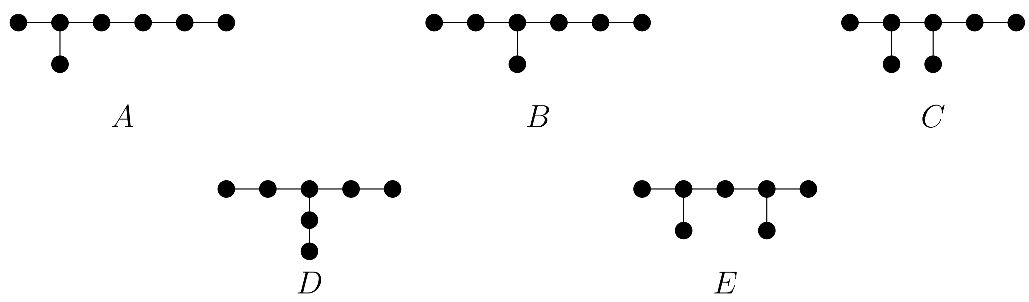


Figure 5.1: Tree graphs of order 7

For $n \geq 8$, there are 7 tree graphs T_n of order n with $\Delta(T_n) = n-4$, namely $S_n(4)$, $S_n[4]$, $S_n(1,3)$, $S_n(3,1)$ as defined in Definition 2.1.12, as well as $T_A(n)$, $T_B(n)$ and $T_C(n)$ shown in 5.2.

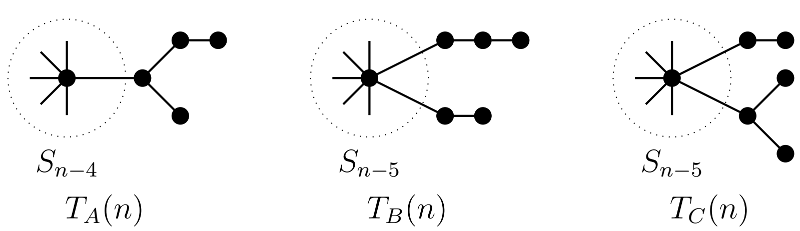


Figure 5.2: Three tree graphs with $\Delta(T_n) = n-4$.

Next, we introduce all the tree graphs T_n of order $n \geq 7$ with maximum degree of $n-5$. For $n = 7$, there is just one such graph, namely the path graph $T_7 = P_7$.

Theorem 2.2.4 provides the Ramsey number $R(P_7, W_8) = 13$. For $n \geq 8$, there are 19 tree graphs T_n of order n with $\Delta(T_n) = n - 5$, namely $S_n(1,4)$, $S_n(5)$, $S_n[5]$, $S_n(2,2)$, $S_n(4,1)$ and the tree graphs shown in Figure 5.3.

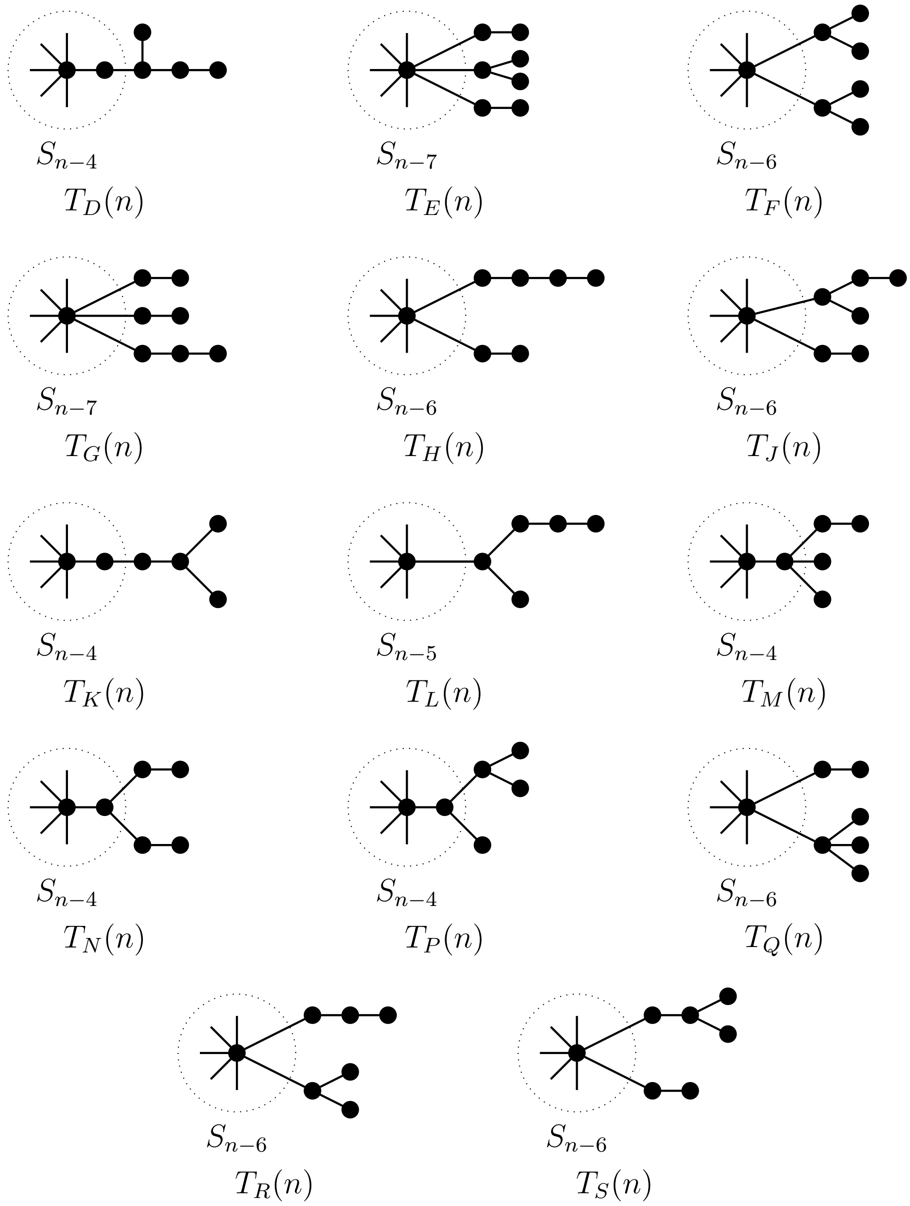


Figure 5.3: Tree graphs T_n with $\Delta(T_n) = n - 5$.

5.2 Ramsey numbers for tree graphs with maximum degree of $n - 4$ versus the wheel graph of order 9

In this section, we discuss the Ramsey numbers for tree graphs with maximum degree of $n - 4$ versus the wheel graph of order 9. We will start by looking at the results for tree graph of order 7. As mentioned in previous section, there will be 5 tree graphs to be discussed, which are A, B, C, D and E as shown in Figure 5.1.

Theorem 5.2.1. $R(T, W_8) = 13$ *for* $T \in \{A, B, C\}$.

Proof. Note that $G = 2K_6$ does not contain A, B or C and that $\overline{G}$ does not contain W_8. Therefore, $R(T, W_8) \geq 13$ for $T = A, B, C$.

Let G be a graph of order 13 whose complement $\overline{G}$ does not contain W_8. By Theorem 4.3.12, G has a subgraph $T = S_7(2,1)$. Label $V(T)$ as in Figure 5.4. Set $U = V(G) - V(T)$; then $|U| = 6$.

First suppose that $A \nsubseteq G$. Then v_1 is not adjacent to v_2 or v_6, and v_2 and v_5 are not adjacent.

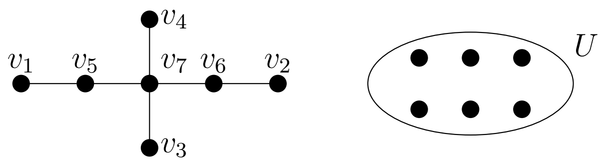


Figure 5.4: $S_7(2,1)$ and U in G.

Case 1a: There is a vertex in U, say u, that is adjacent to v_1.

Since A is not contained in G, v_1 is not adjacent to v_3, v_4 or any vertex of U other than u. Let $W = \{v_2, v_3, v_4, v_6, u_1, \ldots, u_4\}$ for any 4 vertices $u_1, \ldots, u_4$ in U other than u. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 2.2.10 and, together with v_1 as hub, forms W_8, a contradiction. Thus, $\delta(\overline{G}[W]) \leq 3$ and $\Delta(G[W]) \geq 4$. Note that $|N_{G[\{u_1,\ldots,u_4,v_i\}]}(v_i)| \leq 1$ for $i = 2, 3, 4, 6$ since G does not contain A. It is now straightforward to check that v_2, v_3, v_4 and v_6 cannot be the vertex with degree at least 4. Without loss of generality, assume that u_1 has degree at least 4 in $G[W]$. Then u_1 is adjacent to at least one of v_2, v_3, v_4, v_6, so G contains A, a contradiction.

Case 1b: v_1 is not adjacent to any vertex in U.

By arguments similar to those in Case 1a, v_2 is not adjacent to any vertex in U. Let $W = \{v_2, v_6\} \cup U$. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 2.2.10 which, with v_1 as hub, forms W_8 in $\overline{G}[W]$, a contradiction. Thus, $\delta(\overline{G}[W]) \leq 3$ and $\Delta(G[W]) \geq 4$. Since v_2 is not adjacent to any vertex in U, there are only three subcases to be considered.

Subcase 1b.1: $d_{G[W]}(v_6) \geq 4$.

Label $U = \{u_1, \ldots, u_6\}$ so that v_6 is adjacent to u_1, u_2 and u_3 in $G[W]$. Since G does not contain A, vertices u_1, u_2, u_3, v_2 are not adjacent to v_3 or v_4 in G. Note that by arguments as in Case 1a, u_1, u_2 and u_3 are isolated vertices in $G[U]$. Then $v_1u_4u_2v_3v_2u_5u_3u_6v_1$ and u_1 form W_8 in $\overline{G}$, a contradiction.

Subcase 1b.2: $d_{G[W]}(v_6) \leq 3$ and v_6 is adjacent to some $u \in U$ with $d_{G[W]}(u) \geq 4$.

The graph G contains A, with u as the vertex of degree 3 in A, a contradiction.

Subcase 1b.3: $d_{G[W]}(v_6) \leq 3$ and v_6 is not adjacent to any vertex $u \in U$ with $d_{G[W]}(u) \geq 4$.

Label $V(U) = \{u_1, \ldots, u_6\}$ so that u_6 is adjacent to u_2, u_3, u_4 and u_5 in G. Since $A \nsubseteq G$, none of $v_1, \ldots, v_7$ is adjacent in G to any of $u_2, \ldots, u_5$. If v_1 is not adjacent in G to any two of the vertices v_3, v_4, v_7, then $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction. Therefore, $N_{G[v_3,v_4,v_7]}(v_1) \geq 2$ and, similarly, $N_{G[v_3,v_4,v_7]}(v_2) \geq 2$. Hence, one of v_3, v_4, v_7 is adjacent in G to both v_1 and v_2. If v_3 or v_4 is adjacent to both v_1 and v_2, then G contains A, with v_7 as vertex of degree 3, a contradiction.

Finally, if both v_1 and v_2 are adjacent in G to v_7 and each of them is adjacent to a different vertex in v_3 and v_4, then G also contains A, where either v_1 or v_2 is the vertex of degree 3, a contradiction.

Therefore, $R(A, W_8) \leq 13$, so $R(A, W_8) = 13$.

Now, suppose that $B \nsubseteq G$. Then v_1, v_2, v_5, v_6 are not adjacent to v_3 or v_4 in G, and v_1 and v_2 are not adjacent to U in G. Label the vertices $U = \{u_1, \ldots, u_6\}$ and let $W = \{v_3, v_4\} \cup U$. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 2.2.10 which, with v_1 as hub, forms W_8, a contradiction. Therefore, $\delta(\overline{G}[W]) \leq 3$ and $\Delta(G[W]) \geq 4$. If v_3 or v_4 is adjacent to the vertex of degree at least 4 in $G[W]$, then B is contained in G, with v_7 as the vertex of degree 3. Hence, only two cases need to be considered.

Case 2a: v_3 or v_4 is the vertex of degree at least 4 in $G[W]$.

Without loss of generality, assume that v_3 is the vertex of degree at least 4 in $G[W]$. As previously shown, v_3 is not adjacent to v_4. Therefore, it may be assumed that v_3 is adjacent to u_1, u_2, u_3 and u_4 in G. Since $B \nsubseteq G$, $u_1, \ldots, u_4$ are independent in G and are not adjacent to $\{v_1, v_2, v_4, v_5, v_6\}$. Also, v_1 is not adjacent to v_6 and v_2 is not adjacent to v_5. Then $v_1v_6u_2v_2v_5u_3v_4u_4v_1$ and u_1 form W_8 in $\overline{G}$, a contradiction.

Case 2b: One of the vertices in U, say u_1, is the vertex of degree at least 4 in $G[W]$.

As above, u_1 is not adjacent to v_3 or v_4 in G. It may then be assumed that u_1 is adjacent to u_2, u_3, u_4 and u_5. Since $B \nsubseteq G$, $v_1, \ldots, v_7$ are not adjacent to $\{u_2, \ldots, u_5\}$. Note that v_3 is not adjacent to $\{v_1, v_2, v_5, v_6\}$. By Observation 4.3.2, $\overline{G}$ contains W_8, a contradiction.

Therefore, $R(B, W_8) \leq 13$.

Lastly, suppose that $C \nsubseteq G$. Then v_5 and v_6 are not adjacent in G to each other or to v_3, v_4 or U. Furthermore, v_5 is not adjacent to v_2 and v_6 is not adjacent to v_1. Label the vertices $U = \{u_1, \ldots, u_6\}$ and let $W = \{v_3, v_4, v_6, u_1, \ldots, u_5\}$. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 2.2.10 which, with v_5 as hub, forms W_8, a contradiction. Then $\delta(\overline{G}[W]) \leq 3$ and $\Delta(G[W]) \geq 4$. Since v_6 is not adjacent to v_3, v_4 or U, v_6 is not the vertex of degree at least 4 in $G[W]$ and is not adjacent to that vertex. Note that if v_3 or v_4 is the vertex of degree 4, then G contains C, with v_3 or v_4 and v_7 as the vertices of degree 3. Thus, one of the vertices in U, say u_1, is the vertex of degree at least 4 in $G[W]$. Now, consider the following three cases.

Case 3a: Both v_3 and v_4 are adjacent to u_1 in $G[W]$.

Suppose that u_1 is also adjacent to u_2 and u_3 in $G[W]$. Since $C \nsubseteq G$, v_3 is not adjacent in G to v_4 and neither v_3 nor v_4 is adjacent to $\{v_1, v_2, v_5, v_6, u_2, \ldots, u_6\}$. Note that $|N_{G[\{v_1,v_2,u_i\}]}(u_i)| \leq 1$ for $i = 2, 3$ since $C \nsubseteq G$. If v_1 is adjacent to u_2 and u_3 in $\overline{G}$, then $v_1u_2v_5u_4v_3u_5v_6u_3v_1$ and v_4 form W_8 in $\overline{G}$, a contradiction. Therefore, v_1 is adjacent in G to at least one of u_2 and u_3. Similarly, v_2 is adjacent to at least one of u_2 and u_3. Since $|N_{G[\{v_1,v_2,u_i\}]}(u_i)| \leq 1$ for $i = 2, 3$, v_1 is adjacent to u_2 and v_2 is adjacent to u_3, or vice versa. Then neither u_2 nor u_3 is adjacent in G to u_4, u_5, u_6, since $C \nsubseteq G$. Therefore, $v_1v_3v_2v_5u_2u_4u_3v_6v_1$ and v_4 form W_8 in $\overline{G}$, a contradiction.

Case 3b: One of v_3 and v_4, say v_3, is adjacent to u_1 in $G[W]$.

Suppose that u_1 is adjacent to u_2, u_3 and u_4 in $G[W]$. Then v_3 is not adjacent to $v_1, v_2, v_4, v_5, v_6, u_2, u_3, u_4$ in G and $|N_{G[\{v_4,u_2,u_3,u_4\}]}(v_4)| \leq 1$. Without loss of generality, assume that v_4 is not adjacent to u_2 or u_3 in G. Now, suppose that v_4 is adjacent to u_4 in G. Since $C \nsubseteq G$, u_4 is not adjacent to v_1 or v_2 in G. Then $v_1u_4v_2v_5u_2v_4u_3v_6v_1$ and v_3 form W_8 in $\overline{G}$, a contradiction. Otherwise, suppose that v_4 is not adjacent to u_4 in G. Then $|N_{G[\{u_i,v_1,v_2\}]}(u_i)| \leq 1$ for $i = 2, 3, 4$ and at least two of u_2, u_3 and u_4 are not adjacent to v_1 or v_2 in G. Without loss of generality, assume that u_2 and u_3 are not adjacent to v_1 in G. In this case, $v_1u_2v_4u_4v_5u_5v_6u_3v_1$ and v_3 form W_8 in $\overline{G}$, again a contradiction.

Case 3c: v_3 and v_4 are both non-adjacent in $G[W]$ to u_1.

Assume that u_1 is adjacent to each of $u_2, \ldots, u_5$ in $G[W]$. Since $C \nsubseteq G$, $|N_{G[\{v_1,\ldots,v_7,u_i\}]}(u_i)| \leq 1$ for $i = 2, \ldots, 5$, and $|N_{G[\{u_2,\ldots,u_5,v_j\}]}(v_j)| \leq 1$ for $j = 3, 4$. Since $|N_{G[\{v_1,v_2,u_i\}]}(u_i)| \leq 1$ for $i = 2, \ldots, 5$, one of v_1 and v_2, say v_1, satisfies $|N_{G[\{u_2,\ldots,u_5,v_1\}]}(v_1)| \leq 2$. By Lemma 4.3.5, $\overline{G}[v_1, v_3, v_4, v_5, u_2, \ldots, u_5]$ contains C_8 which, with hub v_6, forms W_8 in $\overline{G}$.

Therefore, $R(C, W_8) \leq 13$. This completes the proof of the theorem. □

Theorem 5.2.2. $R(D, W_8) = 14$.

Proof. Let $G = K_6 \cup H$ where H is the graph shown in Figure 4.1 in the proof of Theorem 4.3.11. Since G does not contain D and $\overline{G}$ does not contain W_8, $R(D, W_8) \geq 14$.

Now, let G be any graph of order 14. Suppose neither G contains D as a subgraph, nor $\overline{G}$ contains W_8 as a subgraph. By Theorem 5.2.1, $B \subseteq G$. Label the vertices of B as shown in Figure 5.5 and set $U = \{u_1, \ldots, u_7\} = V(G) - V(B)$. Since $D \nsubseteq G$, v_7 is non-adjacent to v_6 and U, and v_4 is non-adjacent to v_1 and v_2.

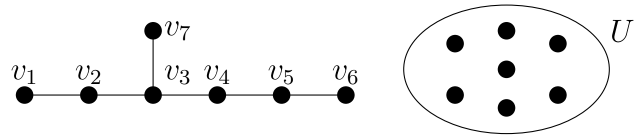


Figure 5.5: $B \subseteq G$

Let $W = \{v_6\} \cup U$. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 2.2.10 which, with v_7 as hub, forms W_8, a contradiction. Thus, $\delta(\overline{G}[W]) \leq 3$ and $\Delta(G[W]) \geq 4$. Three cases will now be considered.

Case 1: v_6 is the vertex of degree at least 4 in $G[W]$.

Assume that v_6 is adjacent to u_1, u_2, u_3 and u_4 in $G[W]$. Then v_5 is adjacent to v_1 and v_2 in $\overline{G}$ and v_3 is adjacent in $\overline{G}$ to v_6, u_1, u_2, u_3 and u_4.

Subcase 1.1: $E_G(\{u_1, \ldots, u_4\}, \{u_5, u_6, u_7\}) \neq \emptyset$.

Without loss of generality, assume that u_1 is adjacent to u_5 in G. Since $D \nsubseteq G$, $\{u_2, u_3, u_4\}$ is independent in G and is adjacent to v_1, v_2, u_6 and u_7 in $\overline{G}$; v_6 is adjacent in $\overline{G}$ to v_1 and v_2; v_4 and v_5 are adjacent in $\overline{G}$ to u_1 and u_5; and v_3 is adjacent in $\overline{G}$ to u_5. If v_4 is adjacent to u_2 in G, then v_5 is adjacent in $\overline{G}$ to u_3 and u_4, so $v_1v_5v_2u_2u_6v_7u_7u_3v_1$ and u_4 form W_8 in $\overline{G}$, a contradiction. Thus, v_4 is adjacent to u_2 in $\overline{G}$, and $v_1v_4v_2u_4u_6v_7u_7u_3v_1$ and u_2 form W_8 in $\overline{G}$, again a contradiction.

Subcase 1.2: $\{u_1, \ldots, u_4\}$ is not adjacent to $\{u_5, u_6, u_7\}$ in $G[W]$.

Suppose that v_5 is adjacent in G to v_7; then v_7 is not adjacent to v_1 or v_2. If $|N_{G[\{u_1,\ldots,u_4,v_2\}]}(v_2)| \leq 2$, then $\overline{G}[u_1, \ldots, u_7, v_2]$ contains C_8 by Lemma 4.3.5 which with v_7 forms W_8 in $\overline{G}$, a contradiction. Thus, $|N_{G[\{u_1,\ldots,u_4,v_2\}]}(v_2)| \geq 3$, so v_1 is not adjacent to $u_1, \ldots, u_4$ in G. By Lemma 4.3.5, $\overline{G}[u_1, \ldots, u_7, v_1, v_7]$ contains W_8, a contradiction.

Hence, v_5 is not adjacent to v_7 in G. Now, if $|N_{G[\{u_1,\ldots,u_4,v_5\}]}(v_5)| \leq 2$, then $\overline{G}[u_1, \ldots, u_7, v_5]$ contains C_8 by Lemma 4.3.5 which with v_7 forms W_8 in $\overline{G}$, a contradiction. Thus $|N_{G[\{u_1,\ldots,u_4,v_5\}]}(v_5)| \geq 3$, so v_4 is not adjacent to $\{u_1, \ldots, u_4\}$ in G, or else G will contain D with v_4 be the vertex of degree 3. By Lemma 4.3.5, $\overline{G}[u_1, \ldots, u_7, v_1]$ contains C_8. If v_4 is not adjacent to v_7 in G, then $\overline{G}$ contains W_8, a contradiction. Thus, v_4 is adjacent to v_7, and since $D \nsubseteq G$, v_1 is not adjacent to v_7. If $|N_{G[\{u_1,\ldots,u_4,v_1\}]}(v_1)| \leq 2$, then $\overline{G}[u_1, \ldots, u_7, v_1]$ contains C_8 by Lemma 4.3.5 which with v_7 forms W_8, a contradiction, so $|N_{G[\{u_1,\ldots,u_4,v_1\}]}(v_1)| \geq 3$. Thus, $|N_{G[\{u_1,\ldots,u_4,v_1\}]}(v_1) \cap N_{G[\{u_1,\ldots,u_4,v_5\}]}(v_5)| \geq 2$, and G contains D with v_5 as the vertex of degree 3, a contradiction.

Case 2: u_1 is the vertex of degree at least 4 in $G[W]$ and v_6 is adjacent to u_1.

Without loss of generality, suppose that u_1 is adjacent to u_2, u_3 and u_4 in $G[W]$. If v_5 is adjacent to u_1, then Case 1 applies with v_6 replaced by u_1. Suppose then that v_5 is not adjacent to u_1. Since $D \nsubseteq G$, v_1 and v_2 are not adjacent in G to v_4, v_5 or v_6; v_3 is not adjacent to $v_6, u_1, \ldots, u_4$; and v_4 is not adjacent to $u_1, \ldots, u_4$.

Subcase 2.1: $E_G(\{u_2, u_3, u_4\}, \{u_5, u_6, u_7\}) \neq \emptyset$.

Without loss of generality, assume that u_2 is adjacent to u_5 in G. Then u_3 and u_4 are not adjacent to each other or to v_1, v_2, u_6, u_7. Also, u_1 is not adjacent to v_1 or v_2, and neither u_2 nor u_5 is adjacent to v_3, v_4, v_5, v_6.

Suppose that v_7 is adjacent to v_4 in G. If u_1 is adjacent to v_1, u_5, u_6 or u_7, then Case 1 can be applied through a slight adjustment of the vertex labelings. Suppose that u_1 is not adjacent to any of these vertices. Since $D \nsubseteq G$, v_7 is not adjacent to v_1. If v_6 is not adjacent to u_6, then $v_1u_1u_5v_6u_6u_3u_7u_4v_1$ and v_7 form W_8 in $\overline{G}$, a contradiction. Similarly, $\overline{G}$ contains W_8 if v_6 is not adjacent to u_7, a contradiction. Therefore, v_6 is adjacent to both u_6 and u_7 in G. Since $D \nsubseteq G$, u_6 is not adjacent to u_7, and neither u_6 nor u_7 is adjacent to u_2. Then $v_1u_1u_5v_6u_2u_6u_7u_3v_1$ and v_7 form W_8 in $\overline{G}$, a contradiction.

Suppose now that v_7 is not adjacent to v_4 in G. If v_7 is adjacent to v_5, then v_7 is not adjacent to v_1 or v_2, and v_4 is not adjacent to v_6, u_6 or u_7. Then $v_1u_1v_2u_3u_6v_4u_7u_4v_1$ and v_7 form W_8 in $\overline{G}$, a contradiction. Therefore, v_7 is not adjacent to v_5 in G. If v_6 is not adjacent to u_3, then $u_3v_6u_2v_5u_5v_4u_4u_6u_3$ and v_7 form W_8 in $\overline{G}$, a contradiction. Similarly, $\overline{G}$ contains W_8 if v_6 is not adjacent to u_4, a contradiction. Then v_6 is adjacent to both u_3 and u_4 in G, so v_6 is not adjacent to u_6 and u_7, or else Case 1 applies. Hence, $v_4u_2v_5u_5v_6u_6u_3u_4v_4$ and v_7 form W_8 in $\overline{G}$, a contradiction.

Subcase 2.2: $\{u_2, u_3, u_4\}$ is not adjacent to $\{u_5, u_6, u_7\}$ in $G[W]$.

If $|N_{G[\{u_2,u_3,u_4,v_6\}]}(v_6)| \geq 3$ or $|N_{G[\{u_5,u_6,u_7,v_6\}]}(v_6)| \geq 3$, then Case 1 applies, so $|N_{G[\{u_2,u_3,u_4,v_6\}]}(v_6)| \leq 2$ and $|N_{G[\{u_5,u_6,u_7,v_6\}]}(v_6)| \leq 2$. Without loss of generality, assume that v_6 is not adjacent in G to u_2 or u_5.

Suppose that v_4 is not adjacent to v_7 in G. If u_5 is adjacent to u_6 or u_7, say u_6, then v_4 is not adjacent to u_5 or u_6, so $v_4u_2v_6u_5u_3u_7u_4u_6v_4$ and v_7 form W_8 in $\overline{G}$, a contradiction. If u_5 is not adjacent to u_6 or u_7, then $v_4u_2v_6u_5u_6u_3u_7u_4v_4$ and v_7 form W_8 in $\overline{G}$, a contradiction. Suppose that v_4 is adjacent to v_7 in G. By similar arguments to those in Subcase 2.1, u_1 is not adjacent to v_1, u_5, u_6 or u_7, and v_7 is not adjacent to v_1. Then $v_1v_6u_5u_2u_6u_3u_7u_1v_1$ and v_7 form W_8 in $\overline{G}$, a contradiction.

Case 3: u_1 is the vertex of degree at least 4 in $G[W]$ and v_6 is not adjacent to u_1.

Assume that u_1 is adjacent to u_2, u_3, u_4 and u_5 in $G[W]$. Since $D \nsubseteq G$, v_3 and v_4 are not adjacent to u_1, u_2, u_3, u_4 or u_5 in G. If either v_1 or v_5 are adjacent to u_1 in G, then Case 1 applies, so suppose that v_1 and v_5 are not adjacent to u_1. In addition, v_1 and v_5 is not adjacent to u_2, u_3, u_4 or u_5 in G, or else Case 2 applies.

Subcase 3.1: $N_{G[u_2,\ldots,u_5]}(v_6) \neq \emptyset$.

Assume that v_6 is adjacent to u_2 in G. Note that v_4 is not adjacent to v_6, v_7, u_6 or u_7 in G, and v_3 is not adjacent to v_5 in G, or else Case 2 applies by slight adjustment of vertex labels. Since $D \nsubseteq G$, v_1 and v_2 are not adjacent in G to v_5, v_6 or u_2, and v_3 is not adjacent to v_6 in G.

If u_2 and u_6 are not adjacent in G, then $v_1u_1v_6v_2u_2u_6v_7u_3v_1$ and v_4 form W_8 in $\overline{G}$, a contradiction. A similar contradiction arises if u_2 and u_7 not adjacent. Therefore, u_2 is adjacent to both u_6 and u_7 in G, and u_3, u_4 and u_5 are not adjacent to u_6 or u_7 in G since $D \nsubseteq G$. Then $v_1u_1v_6v_2u_2v_7u_6u_3v_1$ and v_4 form W_8 in $\overline{G}$, a contradiction.

Subcase 3.2: $N_{G[u_2,\ldots,u_5]}(v_6) = \emptyset$.

Suppose that v_1 is adjacent to v_7 in G. Then v_2 is not adjacent to v_5, v_6 or U since $D \nsubseteq G$. If $|N_{G[\{u_2,\ldots,u_6\}]}(u_6)| \leq 2$, then Lemma 4.3.5 implies that $\overline{G}[u_2, u_3, u_4, u_5, v_4, v_5, v_6, u_6]$ contains C_8 in $\overline{G}$ which with v_2 forms W_8, a contradiction. Therefore, $|N_{G[\{u_2,\ldots,u_6\}]}(u_6)| \geq 3$. Similarly, $|N_{G[\{u_2,\ldots,u_5,u_7\}]}(u_7)| \geq 3$. By the Inclusion-exclusion Principle, $|N_{G[\{u_2,\ldots,u_6\}]}(u_6) \cap N_{G[\{u_2,\ldots,u_5,u_7\}]}(u_7)| \geq 2$. Without loss of generality, u_6 is adjacent to u_2, u_3 and u_4 in G, and u_7 is adjacent to u_3 and u_4, and $G[u_1, \ldots, u_7]$ contains D with u_3 or u_4 being the vertex of degree 3, a contradiction.

Now suppose that v_1 is not adjacent to v_7 in G. If v_7 is adjacent to v_4 in G, then v_2 is not adjacent to any of $u_1, \ldots, u_5$ in G, or else either Case 1 or Case 2 applies. Also, $|N_{G[\{v_2,v_5,v_7\}]}(v_7)| \leq 1$ since $D \subseteq G$. Assume that v_7 is not adjacent to v_2 in G. If $|N_{G[\{u_2,\ldots,u_6\}]}(u_6)| \leq 2$, then Lemma 4.3.5 implies that $\overline{G}[u_2, u_3, u_4, u_5, v_1, v_2, v_6, u_6]$ contains C_8 which with v_7 forms W_8, a contradiction. Thus, $|N_{G[\{u_2,\ldots,u_6\}]}(u_6)| \geq 3$. Similarly, $|N_{G[\{u_2,\ldots,u_5,u_7\}]}(u_7)| \geq 3$, so $|N_{G[\{u_2,\ldots,u_6\}]}(u_6) \cap N_{G[\{u_2,\ldots,u_5,u_7\}]}(u_7)| \geq 2$. By arguments similar to those in the previous paragraph, G will contain a subgraph D, a contradiction.

Thus, $R(D, W_8) \leq 14$ which completes the proof of the theorem. □

Theorem 5.2.3. $R(E, W_8) = 15$.

Proof. The graph $G = K_6 \cup K_{4,4}$ does not contain E and $\overline{G}$ does not contain W_8. Thus, $R(E, W_8) \geq 15$. For the upper bound, let G be any graph of order 15. Suppose that G does not contain E and that $\overline{G}$ does not contain W_8. By Theorem 4.3.11, G contains $T = S_7(3)$ subgraph. Label the vertices of this subgraph as in Figure 5.6 and set $U = V(G) - V(T)$. Note that $|U| = 8$.

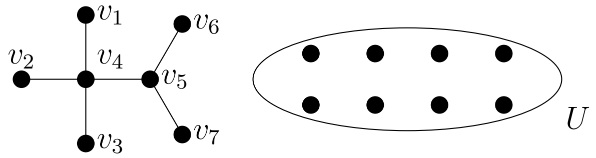


Figure 5.6: $S_7(3)$ and U in G.

Case 1: Some vertex u in U is adjacent to v_6.

Since $E \nsubseteq G$, v_6 is not adjacent to v_1, v_2, v_3, v_7 or any vertex of U other than u. Let $W = \{v_1, v_2, v_3, v_7, u_1, \ldots, u_4\}$, for any vertices $u_1, \ldots, u_4$ in U other than u. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 2.2.10 which with v_6 forms W_8, a contradiction. Therefore, $\delta(\overline{G}[W]) \leq 3$ and $\Delta(G[W]) \geq 4$. Since $E \nsubseteq G$, $N_{G[\{u_1,u_2,u_3,u_4,v_1,v_7\}]}(v_7) \leq 1$ and $N_{G[\{u_1,u_2,u_3,u_4,v_7,v_i\}]}(v_i) \leq 1$ for $i = 1, 2, 3$, so none of v_1, v_2, v_3, v_7 has degree at least 4. Without loss of generality, assume that u_1 has degree at least 4. If u_1 is adjacent to v_7, then G contains E with u_1 and v_5 as the vertices of degree 3, a contradiction. Similarly, if u_1 is adjacent to v_1, v_2 or v_3, then G contains E with u_1 and v_4 as the vertices of degree 3, a contradiction. Therefore, u_1 is not adjacent to v_1, v_2, v_3 or v_7. However, then u_1 has degree at most 3 in $G[W]$, a contradiction.

Case 2: v_6 is not adjacent to any vertex in U.

If v_7 is adjacent to some vertex in U, then Case 1 applies with v_7 replacing v_6, so suppose that v_7 is not adjacent to any vertex in U. Now, if $\delta(\overline{G}[U]) \geq 4$, then $\overline{G}[U]$ contains C_8 by Lemma 2.2.10 which, with v_6 or v_7, forms W_8, a contradiction. Thus, $\delta(\overline{G}[U]) \leq 3$ and $\Delta(G[U]) \geq 4$. Let $V(U) = \{u_1, \ldots, u_8\}$. Without loss of generality, assume that u_1 is adjacent to u_2, u_3, u_4 and u_5. Since $E \nsubseteq G$, v_4 is not adjacent in G to any of $u_1, \ldots, u_5$; v_5 is not adjacent to any of $v_1, v_2, v_3, u_1, \ldots, u_5$; and u_1 is not adjacent to v_1, v_2 or v_3. Furthermore, $|N_{G[\{u_2,\ldots,u_5,v_i\}]}(v_i)| \leq 1$ for $i = 1, 2, 3$ and $|N_{G[\{v_1,v_2,v_3,u_j\}]}(u_j)| \leq 1$ for $j = 2, \ldots, 5$.

Now, suppose that $N_{G[\{v_5,u_6,u_7,u_8\}]}(v_5) = \emptyset$. If $|N_{G[\{u_2,\ldots,u_6\}]}(u_6)| \leq 1$, then $\overline{G}[u_2, \ldots, u_5, v_1, v_2, v_3, u_6]$ contains C_8 by Lemma 4.3.5 which with v_5 forms W_8, a contradiction. Therefore, $|N_{G[\{u_2,\ldots,u_6\}]}(u_6)| \geq 2$. Similarly, $|N_{G[\{u_2,\ldots,u_5,u_7\}]}(u_7)| \geq 2$ and $|N_{G[\{u_2,\ldots,u_5,u_8\}]}(u_8)| \geq 2$. By the Inclusion-Exclusion Principle, u_2, u_3, u_4 or u_5 is adjacent in G to at least two of u_6, u_7, u_8. Without loss of generality, assume that u_2 is adjacent to u_6 and u_7. Then u_2 is not adjacent to u_3, u_4 or u_5, Therefore, Lemma 4.3.5 implies that $\overline{G}[u_1, u_3, u_4, u_5, v_1, v_2, v_3, u_2]$ contains C_8 which with v_5 forms W_8, a contradiction.

On the other hand, if $N_{G[u_6,u_7,u_8]}(v_5) \neq \emptyset$, then without loss of generality assume that u_6 is adjacent to v_5 in G. Since $E \nsubseteq G$, v_4 is not adjacent to v_6, v_7 or u_6 in G. Also, $\{v_1, v_2, v_3\}$ and $\{v_6, v_7, u_6\}$ are independent in G, and $v_1, v_2, v_3, v_6, v_7, u_6 \notin N_G(u_i)$ for $i = 1, \ldots, 5, 7, 8$, or else Case 1 applies with vertex label adjustments. Now, if u_1 is not adjacent to both u_7 and u_8 in G, then $v_1v_2v_3u_7v_6v_7u_6u_8v_1$ and u_1 form W_8 in $\overline{G}$, a contradiction. Therefore, $N_{G[\{u_1,u_7,u_8\}]}(u_1) \neq \emptyset$. Without loss of generality, assume that u_1 is adjacent to u_7 in G. Note that for $E \nsubseteq G$, $|N_{G[\{v_4,v_5,u_8\}]}(u_8)| \leq 1$. Now, suppose that u_8 is not adjacent to v_4 in G. If $|N_{G[\{u_2,\ldots,u_5,u_8\}]}(u_8)| \leq 3$, then assume without loss of generality that u_8 is not adjacent to u_2 or u_3 in G. Then $v_6u_4v_7u_5u_6u_2u_8u_3v_6$ and v_4 form W_8 in $\overline{G}$, a contradiction. Similar arguments work if u_8 is not adjacent to v_5 in G, by replacing v_4 with v_5 and v_6, v_7, u_6 with v_1, v_2, v_3, respectively, so $|N_{G[\{u_2,\ldots,u_5,u_7,u_8\}]}(u_8)| \geq 4$. However, G then contains E with u_1 and u_8 of degree 3, a contradiction.

Thus, $R(E, W_8) \leq 15$. This completes the proof of the theorem. $\square$

Next, we will proceed to the results for the tree graphs T_n with $n \geq 8$. There are 7 types of tree graphs to be discussed, namely $S_n(4)$, $S_n[4]$, $S_n(1,3)$, $S_n(3,1)$, $T_A(n)$, $T_B(n)$ and $T_C(n)$ as shown in Figure 5.2.

Lemma 5.2.4. *Let $n \geq 8$. Then for each tree graph $T_n \in \{S_n(4), S_n(3,1), T_C(n)\}$, $R(T_n, W_8) \geq 2n-1$. Also, for each tree graph $T_n \in \{S_n[4], S_n(1,3), T_A(n), T_B(n)\}$, $R(T_n, W_8) \geq 2n-1$ if $n \not\equiv 0 \pmod 4$ and $R(T_n, W_8) \geq 2n$ otherwise.*

Proof. The graph $G = 2K_{n-1}$ does not contain any tree graphs of order n, and $\overline{G}$ does not contain W_8. Finally, if $n \equiv 0 \pmod 4$, then the graph $G = K_{n-1} \cup K_{4,\ldots,4}$ of order $2n-1$ does not contain $S_n[4]$, $S_n(1,3)$, $T_A(n)$ or $T_B(n)$; nor does the complement $\overline{G}$ contain W_8. $\square$

Theorem 5.2.5. *If $n \geq 8$, then*

$$R(S_n(4), W_8) = \begin{cases} 2n-1 & \textit{if } n \geq 9\,; \\ 16 & \textit{if } n = 8\,. \end{cases}$$

Proof. By Lemma 5.2.4, $R(S_n(4), W_8) \geq 2n-1$ for $n \geq 8$. For $n = 8$, observe that the graph $G = K_7 \cup H_8$, where H_8 is the graph of order 8 as shown in Figure 5.7 does not contain $S_8(4)$ and its complement $\overline{G}$ does not contain W_8. Therefore, for $n = 8$, we have a better bound of $R(S_8(4), W_8) \geq 16$.

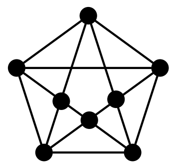


Figure 5.7: The graphs H_8.

For the upper bound, let G be any graph of order $2n-1$ if $n \geq 9$, and of order 16 if $n = 8$. Assume that G does not contain $S_n(4)$ and that $\overline{G}$ does not contain W_8.

If $n \geq 9$ is odd or $n = 8$, then G has a subgraph $T = S_n(3)$ by Theorem 4.3.11. Let $V(T) = \{v_0, \ldots, v_{n-3}, w_1, w_2\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-3}, v_1w_1, v_1w_2\}$. Also, let $V = \{v_2, \ldots, v_{n-3}\}$ and $U = V(G) - V(T)$; then $|V| = n-4 \geq 5$ and $|U| = n-1 \geq 8$ if n is odd, while $|U| = 8$ if $n = 8$. Since $S_n(4) \nsubseteq G$, v_1 is not adjacent in G to any vertex of $U \cup V$ in G. Furthermore, for each $2 \leq i \leq n-3$, v_i is adjacent to at most two vertices of U in G. By Corollary 5.3.1, $\overline{G}[U \cup V]$ contains C_8, and together with v_1, gives us W_8 in $\overline{G}$, a contradiction.

For the remaining case when $n \geq 10$ is even, $S_{n-1} \subseteq G$ by Theorem 2.2.6. Let v_0 be the centre of S_{n-1} and set $L = N_{S_{n-1}}(v_0) = \{v_1, \ldots, v_{n-2}\}$ and $U = V(G) - V(S_{n-1})$. Then $|U| = n$. Since G does not contain $S_n(4)$, each vertex of L is adjacent to at most two vertices of U. We consider two cases here.

Case 1: $E(L, U) = \emptyset$.

If $\Delta(\overline{G}[U]) \geq 4$, then some vertex u in U is adjacent to at least four vertices in $\overline{G}[U]$. These four vertices and any four vertices from L form C_8 in $\overline{G}$ which, with hub u, form W_8, a contradiction. Therefore, $\Delta(\overline{G}[U]) \leq 3$ and $\delta(G[U]) \geq n-4$. Suppose that $\delta(G[U]) = n - 4 - \ell$ for some $\ell \geq 0$, and let u_0 be a vertex in U with minimum degree in $G[U]$. Label the remaining vertices in U as $u_1, \ldots, u_{n-1}$ such that $U_A = \{u_1, \ldots, u_{n-4}\} \subseteq N_G(u_0)$, and let $U_B = \{u_{n-3}, u_{n-2}, u_{n-1}\}$. Since $S_n(4) \nsubseteq G$, each vertex in U_A is adjacent to at most two vertices in U_B, and so $|E_G(U_A, U_B)| \leq 2(n-4)$. On the other hand, noting that u_0 is adjacent to exactly ℓ vertices in U_B and letting $e_B \leq 3$ be the number of edges in $G[U_B]$, we see that $|E_G(U_A, U_B)| \geq 3\delta(G[U]) - \ell - 2e_B = 3(n-4-\ell) - \ell - 2e_B$. Therefore, $2(n-4) \geq |E_G(U_A, U_B)| \geq 3n - 12 + 2\ell - 2e_B$, implying that $n + 2\ell \leq 4 + 2e_B \leq 10$, which is only possible when $n = 10$, $\ell = 0$, $e_B = 3$, and $|E_G(U_A, U_B)| = 2(n-4) = 12$. For such scenario where $n = 10$, noting that u_0 was an arbitrary vertex with minimum degree in $G[U]$, it is straightforward to deduce that the only possible edge set of $G[U]$ (up to isomorphism) with $S_{10}(4) \nsubseteq G[U]$ is

$$\begin{aligned} &\{u_1u_0, \ldots, u_6u_0\} \cup \{u_1u_7, \ldots, u_4u_7\} \cup \{u_1u_8, u_2u_8, u_5u_8, u_6u_8\} \cup \{u_3u_9, \ldots, u_6u_9\} \\ &\cup \{u_1u_2, u_3u_4, u_5u_6\} \cup \{u_1u_3, u_1u_5, u_3u_5\} \cup \{u_2u_4, u_2u_6, u_4u_6\} \cup \{u_7u_8, u_7u_9, u_8u_9\}. \end{aligned}$$

Observe now that $\overline{G}[U]$ contains C_8, which forms a W_8 in $\overline{G}$ with any vertex in L as hub, a contradiction.

Case 2: $E(L, U) \neq \emptyset$.

Without loss of generality, assume that v_1 is adjacent to u_1 in G. Since $S_n(4) \nsubseteq G$, v_1 is adjacent to at most one vertex of $U \cup L \setminus \{u_1\}$ in G. Therefore, we can find a 4-vertex set $V' \subseteq V \setminus \{v_1\}$ and an 8-vertex set $U' \subseteq U \setminus \{u_1\}$ such that v_1 is not adjacent in G to any vertex of $U' \cup V'$. Note that each vertex of V' is adjacent to at most two vertices of U' in G, so $|E(V', U')| \leq 8$. This implies that there are four vertices in U' that are each adjacent in G to at most one vertex of V', and so $\overline{G}$ contains C_8 by Lemma 4.3.5 and, with v_1 as hub, form W_8, a contradiction.

Thus, $R(S_n(4), W_8) \leq 2n - 1$ when $n \geq 9$ and $R(S_n(4), W_8) \leq 16$ when $n = 8$. This completes the proof of the theorem. □

Lemma 5.2.6. *Let H be a graph of order $n \geq 8$ with minimum degree $\delta(H) \geq n-4$. Then either H contains $S_n[4]$ and $T_A(n)$, or $n \equiv 0 \pmod 4$ and $\overline{H}$ is the disjoint union of $\frac{n}{4}$ copies of K_4, i.e., $\overline{H} = \frac{n}{4}K_4$.*

Proof. Let $V(H) = \{u_0, \ldots, u_{n-1}\}$. We first consider the case that H has a vertex of degree at least $n-3$, which we may assume without loss of generality that this vertex is u_0, and that $\{u_1, \ldots, u_{n-3}\} \subseteq N_H(u_0)$.

Suppose that u_{n-2} is adjacent to u_{n-1} in H. Since $\delta(H) \geq n-4$, u_{n-2} is adjacent to at least $n-6 \geq 2$ vertices of $\{u_1, \ldots, u_{n-3}\}$, say u_1 and u_2, and so H contains $S_n[4]$. Furthermore, also by the minimum degree condition, u_1 is adjacent to at least $n-7 \geq 1$ vertices of $\{u_1, \ldots, u_{n-3}\}$, and so H contains $T_A(n)$.

Suppose now that u_{n-2} is not adjacent to u_{n-1} in H. Then by the minimum degree condition, there is a vertex in $\{u_1, \ldots, u_{n-3}\}$, say u_1, that is adjacent to both

u_{n-2} and u_{n-1}. The vertices u_1 and u_{n-2} must also each be adjacent to a vertex of $\{u_2, \ldots, u_{n-3}\}$, and so H contains both $S_n[4]$ and $T_A(n)$.

For the remaining case, suppose that H is $(n-4)$-regular and that $N_H(u_0) = \{u_1, \ldots, u_{n-4}\}$. Let $U = \{u_{n-3}, u_{n-2}, u_{n-1}\}$ and suppose that $H[U]$ has an edge, say $u_{n-3}u_{n-2}$. Since u_{n-3} must be adjacent in H to some vertex of $N_H(u_0)$, it follows that H contains $S_n[4]$ if u_{n-3} or u_{n-2} is adjacent to u_{n-1}. Suppose then that neither u_{n-3} nor u_{n-2} is adjacent to u_{n-1}. Then u_{n-1} is adjacent to every vertex of $N_H(u_0)$. Note that $d_{H[N_H(u_0)\cup\{u_{n-3}\}]}(u_{n-3}) = n-5$ and let u be the vertex of $N_H(u_0)$ that is not adjacent in H to u_{n-3}. Since $d_H(u) = n-4$, u is adjacent in H to some vertex in $N_H(u_{n-3})$, so H contains $S_n[4]$. Also, note that u_{n-3} is adjacent in H to at least $n-6$ vertices of $N_H(u_0)$. If u_{n-1} is adjacent to some vertex of $N_{H[N_H(u_0)\cup\{u_{n-3}\}]}(u_{n-3})$, then H contains $T_A(n)$. Note that this will always happen for $n \geq 9$. For $n = 8$, there is a case where $|N_{H[N_H(u_0)\cup\{u_{n-3}\}]}(u_{n-3})| = |N_{H[N_H(u_0)\cup\{u_{n-1}\}]}(u_{n-1})| = 2$ and $N_{H[N_H(u_0)\cup\{u_{n-3}\}]}(u_{n-3}) \cap N_{H[N_H(u_0)\cup\{u_{n-1}\}]}(u_{n-1}) = \emptyset$, so u_{n-1} is adjacent to u_{n-3} and u_{n-2}, giving $T_A(n)$ in H.

Now, suppose that $H[U]$ contains no edge. Then $U_1 = U \cup \{u_0\}$ is an independent set in H. Furthermore, $N_H(u) = \{u_1, \ldots, u_{n-4}\}$ for every $u \in U$, as every vertex has degree $n-4$. Therefore, $\overline{H}[U_1]$ is a K_4 component in $\overline{H}$. Repeating the above proof for each vertex u of H shows that either u is contained in a K_4 component of $\overline{H}$, or H contains both $S_n[4]$ or $T_A(n)$. In other words, either H contains both $S_n[4]$ and $T_A(n)$, or $\overline{H}$ is the disjoint union of $\frac{n}{4}$ copies of K_4, and so $n \equiv 0 \pmod 4$. $\square$

Theorem 5.2.7. *If $n \geq 8$, then*

$$R(S_n[4], W_8) = \begin{cases} 2n-1 & \text{if } n \not\equiv 0 \pmod 4\,; \\ 2n & \text{otherwise}\,. \end{cases}$$

Proof. Lemma 5.2.4 provides the lower bounds, so it remains to prove the upper bounds. Now let G be a graph that does not contain $S_n[4]$ and assume that $\overline{G}$ does not contain W_8.

We first suppose that G has order $2n$ if $n \equiv 0 \pmod 4$ and G has order $2n-1$ if n is odd. By Theorem 4.3.11, G has a subgraph $T = S_n(3)$. Let $V(T) = \{v_0, \ldots, v_{n-3}, w_1, w_2\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-3}\} \cup \{v_1w_1, v_1w_2\}$. Set $U = V(G) - V(T)$ and $V = \{v_2, \ldots, v_{n-3}\}$. Then $|U| = n-j$, for $j = 0$ if $n \equiv 0 \pmod 4$ and $j = 1$ if n is odd, and $|V| = n-4$. Since G does not contain $S_n[4]$, v_1 is not adjacent to any vertex of V in G, and each vertex of V is adjacent to at most $n-6$ vertices of $U \cup V$ in G. Noting also that w_1 and w_2 each is adjacent to at most one vertex of $\{w_1, w_2\} \cup U$ in G, we consider two cases.

Case 1: At least one of w_1 and w_2 is not an isolated vertex in $G[\{w_1, w_2\} \cup U]$.

Without loss of generality, assume that w_1 is adjacent to some vertex $u \in \{w_2\} \cup U$ in G. Let $Z = \big(V \cup U \cup \{w_2\}\big) \setminus \{u\}$ and note that $|Z| = 2n-4-j$. Since $S_n[4] \nsubseteq G$, w_1 is not adjacent to any vertex of Z in G. If $\delta(\overline{G}[Z]) \geq \lceil \frac{2n-4-j}{2} \rceil$, then $\overline{G}[Z]$ contains C_8 by Lemma 2.2.10 which with w_1, forms W_8 in $\overline{G}$, a contradiction. Therefore, $\delta(\overline{G}[Z]) \leq \lceil \frac{2n-4-j}{2} \rceil - 1$ and $\Delta(G[Z]) \geq \lfloor \frac{2n-4-j}{2} \rfloor = n-2-j$. Since each v of V is adjacent to at most $n-6$ vertices of $U \cup V$ in G, and w_2 is adjacent to at most one vertex of U in G, a vertex with maximum degree in $G[Z]$ must be a vertex of $U \setminus \{u\}$. So let u_2 be a vertex of U with $d_{G[Z]}(u_2) \geq n-2$. As $S_n[4] \nsubseteq G$, observe that $N_{G[Z]}(u_2) \subseteq U$; each vertex of V is adjacent to at most one vertex of $N_{G[Z]}(u_2)$ in G; and each vertex of $N_{G[Z]}(u_2)$ is adjacent to at most one vertex of V in G. Then by Lemma 4.3.5, any four vertices from V and any four vertices from $N_{G[Z]}(u_2)$ form C_8 in $\overline{G}$ which with w_1 forms W_8 in $\overline{G}$, a contradiction.

Case 2: w_1 and w_2 are isolated vertices in $G[\{w_1, w_2\} \cup U]$.

If $\delta(\overline{G}[U]) \geq \frac{n-j}{2}$, then $\overline{G}[U]$ contains C_8 by Lemma 2.2.10 which with w_1 forms W_8, a contradiction. Thus, $\delta(\overline{G}[U]) \leq \frac{n-j}{2} - 1$, and $\Delta(G[U]) \geq \frac{n-j}{2}$. Let u_1 be a vertex of U with $d_{G[U]} \geq \frac{n-j}{2}$. Since $S_n[4] \nsubseteq G$, v_0 is not adjacent to any vertex of $N_{G[U]}(u_1)$ in G. Now, if v_1 is adjacent to some vertex u of $N_{G[U]}(u_1)$ in G, then apply Case 1 with w_1 and u interchanged. So we may assume that v_1 is not adjacent to any vertex of $N_{G[U]}(u_1)$ in G.

If $E(V, N_{G[U]}(u_1)) = \emptyset$ in G, then any four vertices of V and any four vertices of $N_{G[U]}(u_1)$ form C_8 in $\overline{G}$, and with v_1, form W_8 in $\overline{G}$, a contradiction. So without loss of generality, assume that v_2 is adjacent to some vertex u_2 of $N_{G[U]}(u_1)$ in G. Since $S_n[4] \nsubseteq G$, u_2 is not adjacent to any vertex of $U \setminus \{u_1\}$. Then v_0, v_1, w_1, w_2 and any four vertices from $U \setminus \{u_1, u_2\}$, at least three of which are from $N_{G[U]}(u_1) \setminus \{u_2\}$, form C_8 in $\overline{G}$ and, with u_2, form W_8 in $\overline{G}$, a contradiction.

In either case, $R(S_n[4], W_8) \leq 2n$ for $n \equiv 0 \pmod 4$ and $R(S_n[4], W_8) \leq 2n-1$ for odd n.

Next, suppose that $n \equiv 2 \pmod 4$ and G has order $2n-1$. If G contains $S_n(3)$, then we can use the previous arguments to show that $R(S_n[4], W_8) \leq 2n-1$. Hence, we only need to consider the case where G does not contain $S_n(3)$. Now, by Theorem 5.2.5, G has a subgraph $T = S_n(4)$. Let $V(T) = \{v_0, \dots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \dots, v_0v_{n-4}, v_1w_1, v_1w_2, v_1w_3\}$. Let $U = V(G) - V(T)$; then $|U| = n-1$. Since G does not contain $S_n(3)$ and $S_n[4]$, v_0 is not adjacent in G to w_1, w_2, w_3 or U. Now, set $U' = N_{G[U\cup\{w_1\}]}(w_1) \cup N_{G[U\cup\{w_2\}]}(w_2) \cup N_{G[U\cup\{w_3\}]}(w_3)$. Then $|U'| \leq 3$ and w_1, w_2 and w_3 are not adjacent in G to any vertex of $U \setminus U'$. By Lemma 4.3.4, $G[U \setminus U']$ is either $K_{n-1-|U'|}$ or $K_{n-1-|U'|} - e$. If $d_{\overline{G}[U\setminus U']}(u') \geq 2$ for some vertex u' in U', then at least two vertices of $U \setminus U'$ are not adjacent to u' in G. Let X be a set containing these two vertices and any other two vertices in $U \setminus U'$, and set $Y = \{w_1, w_2, w_3, u'\}$. Note that $\overline{G}[X \cup Y]$ contains C_8 by Lemma 4.3.5 which, with v_0 as hub, forms W_8, a contradiction. Therefore, every vertex of U' is adjacent in G to at least $n-2-|U'|$ vertices of $U \setminus U'$. Hence, $\delta(G[U]) \geq n-5$, and since $S_n[4] \nsubseteq G$, $E_G(T, U) = \emptyset$. Now, if $\overline{G}[V(T)]$ contains S_5, then $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction. Thus, $\delta(G[V(T)]) \geq n-4$. By Lemma 5.2.6, G contains $S_n[4]$, a contradiction. Hence, $R(S_n[4], W_8) \leq 2n-1$ for $n \equiv 2 \pmod 4$.

This completes the proof. $\square$

Theorem 5.2.8. *If $n \geq 8$, then*

$$R(S_n(1,3), W_8) = \begin{cases} 2n-1 & \text{if } n \not\equiv 0 \pmod 4\,; \\ 2n & \text{otherwise}\,. \end{cases}$$

Proof. Lemma 5.2.4 provides the lower bounds. It therefore remains to prove the upper bounds. Let G be any graph of order $2n$ if $n \equiv 0 \pmod 4$ and of order

$2n-1$ if $n \not\equiv 0 \pmod 4$. Assume that G does not contain $S_n(1,3)$ and that $\overline{G}$ does not contain W_8. By Theorem 5.2.7, G has a subgraph $T = S_n[4]$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, w_1v_1, w_1w_2, w_1w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$. Since $S_n(1,3) \not\subseteq G$, w_2 and w_3 are not adjacent to each other, or to any vertex in $U \cup V$. Since $C_8 \not\subseteq \overline{G}[U \cup V]$ as $W_8 \not\subseteq \overline{G}$, Lemma 2.2.10 implies that $G[U \cup V]$ has a vertex u of degree at least $n-3$ in $G[U \cup V]$. Since $S_n(1,3) \not\subseteq G$, $u \in U$ and u is not adjacent to any vertex in V. Furthermore, $E(V, N_{G[U]}(u)) = \emptyset$. Finally, note that w_3, any 3 vertices in V and any 4 vertices in $N_{G[U]}(u)$ form C_8 in $\overline{G}$ which, with w_2 as hub, form W_8, a contradiction. □

Theorem 5.2.9. *If $n \geq 8$, then*

$$R(T_A(n), W_8) = \begin{cases} 2n-1 & \text{if } n \not\equiv 0 \pmod 4\,; \\ 2n & \text{otherwise}\,. \end{cases}$$

Proof. Lemma 5.2.4 provides the lower bounds, so it remains to prove the upper bounds. Let G be any graph of order $2n$ if $n \equiv 0 \pmod 4$ and of order $2n-1$ if $n \not\equiv 0 \pmod 4$. Assume that G does not contain $T_A(n)$ and that $\overline{G}$ does not contain W_8.

Suppose that G has a subgraph $T = S_n(3)$. Let $V(T) = \{v_0, \ldots, v_{n-3}, w_1, w_2\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-3}, v_1w_1, v_1w_2\}$. Set $V = \{v_2, \ldots, v_{n-3}\}$ and $U = V(G) - V(T)$. Since G does not contain $T_A(n)$, w_1 and w_2 are not adjacent to any vertex of $U \cup V$ in G. Let V' be the set of any $n-5$ vertices in V, and U' be the set of any $n-1$ vertices in U. If $\delta(\overline{G}[U' \cup V']) \geq n-3$, then $\overline{G}[U' \cup V']$ contains C_8 by Lemma 2.2.10 which, with w_1 as hub, form W_8, a contradiction. Therefore, $\delta(\overline{G}[U' \cup V']) \leq n-4$ and $\Delta(G[U' \cup V']) \geq n-3$. Since $T_A(n) \not\subseteq G$, $d_{G[U' \cup V']}(v) \leq n-6$ for each $v \in V'$. Hence, some vertex $u \in U'$ satisfies $d_{G[U' \cup V']}(u) \geq n-3$, which also implies that u is adjacent to at least two vertices of U.

Since $T_A(n) \not\subseteq G$, each vertex of V is adjacent to at most one vertex of $N_{G[U]}(u)$. If $|N_{G[U]}(u)| \geq n-4$, then we also have that each vertex of $N_{G[U]}(u)$ is adjacent to at most one vertex of V, and so $\overline{G}[V \cup N_{G[U]}(u)]$ contains C_8 by Lemma 2.2.10 which, with w_1 as hub, form W_8, a contradiction. Thus, at least three vertices of V' (and so of V), v_2, v_3, and v_4, are adjacent to u in G. Let a and b be any two vertices in $N_{G[U]}(u)$. As $T_A(n) \not\subseteq G$, each of v_2, v_3, v_4 is not adjacent to any vertex of $V(G) \setminus \{u, v_0\}$. Then $w_1v_5w_2v_3av_1bv_4w_1$ and v_2 form W_8 in $\overline{G}$, a contradiction.

By Theorem 4.3.11, we have shown that $R(S_n(3), W_8) \leq 2n$ for $n \equiv 0 \pmod 4$. So we may now assume that G has order $2n-1$ with $n \not\equiv 0 \pmod 4$, and that G does not contain $S_n(3)$. By Theorem 5.2.5, G has a subgraph $T = S_n(4)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2, v_1w_3\}$. Then $U = V(G) - V(T)$ and $|U| = n-1$. Since $T_A(n) \not\subseteq G$, w_1, w_2, w_3 are not adjacent to each other in G or to any vertex of U. Since $S_3(n) \not\subseteq G$, v_0 is not adjacent any vertex of $U \cup \{w_1, w_2, w_3\}$. By Lemma 4.3.4, $G[U]$ is K_{n-1} or $K_{n-1} - e$. Since $T_A(n) \not\subseteq G$, each vertex of T is not adjacent to any vertex of U in G, and so $\delta(G[V(T)]) \geq n-4$ by Observation 4.3.2, which in turn implies that $G[V(T)]$ contains $T_A(n)$ by Lemma 5.2.6, a contradiction.

This completes the proof of the theorem. □

Theorem 5.2.10. *If $n \geq 8$, then*

$$R(T_B(n), W_8) = \begin{cases} 2n-1 & \text{if } n \not\equiv 0 \pmod 4\,; \\ 2n & \text{otherwise}\,. \end{cases}$$

Proof. Lemma 5.2.4 provides the lower bounds, so it remains to prove the upper bounds. Let G be a graph with no $T_B(n)$ subgraph whose complement $\overline{G}$ does not contain W_8.

Suppose that $n \equiv 0 \pmod 4$ and that G has order $2n$. By Theorem 5.2.7, G has a subgraph $T = S_n[4]$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, w_1w_2, w_1w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n$. Since $T_B(n) \nsubseteq G$, $E_G(U, V) = \emptyset$ and neither w_2 nor w_3 is adjacent in G to V. Suppose that $n \geq 12$. If w_2 is non-adjacent to some 4 vertices from U, then these 4 vertices and any 4 vertices from V form C_8 in $\overline{G}$ that with w_2 forms W_8, a contradiction. Otherwise, w_2 must be adjacent to at least $n-3$ vertices of U in G. Since $T_B(n) \nsubseteq G$, w_3 must not be adjacent to these $n-3$ vertices; then any 4 vertices from these $n-3$ vertices and 4 vertices from V form C_8 in $\overline{G}$ and, with w_3 as hub, form W_8, again a contradiction. For $n = 8$, $|V| = 3$ and $|U| = 8$. If w_2 is not adjacent to any vertex of U in G, then by Lemma 4.3.4, $G[U]$ is K_8 or $K_8 - e$ which contains $T_B(8)$, a contradiction. Otherwise, suppose that w_2 is adjacent to $u \in U$. Since $T_B(8) \nsubseteq G$, w_1 must not be adjacent to $(U \cup V) \setminus \{u\}$ in G. Now, if w_3 is not adjacent to v_0 in G, then by Observation 4.3.2, $\overline{G}$ contains W_8, a contradiction. Else, u is not adjacent to $V \cup \{w_3\}$, and again by Observation 4.3.2, $\overline{G}$ contains W_8, another contradiction. Thus, $R(T_B(n), W_8) \leq 2n$ for $n \equiv 0 \pmod 4$.

Next, suppose that $n \not\equiv 0 \pmod 4$ and that G has order $2n-1$. By Theorem 5.2.7, G has a subgraph $T = S_n[4]$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, w_1w_2, w_1w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n-1$. Since $T_B(n) \nsubseteq G$, $E_G(U, V) = \emptyset$ and neither w_2 nor w_3 is adjacent in G to V. For $n \geq 9$, if w_2 is non-adjacent to some 4 vertices from U, then these 4 vertices and any 4 vertices from V form C_8 in $\overline{G}$ and, with w_2 as hub, form W_8, a contradiction. Otherwise, w_2 is adjacent to at least $n-4$ vertices of U in G. Since $T_B(n) \nsubseteq G$, w_3 is not adjacent to these $n-4$ vertices, so any 4 vertices from these $n-4$ vertices and 4 vertices from V form C_8 in $\overline{G}$ that, with w_3, form W_8, again a contradiction. Therefore, $R(T_B(n), W_8) \leq 2n-1$ for $n \not\equiv 0 \pmod 4$.

This completes the proof. □

Theorem 5.2.11. *For $n \geq 8$, $R(T_C(n), W_8) = 2n-1$.*

Proof. Lemma 5.2.4 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$ and assume that G does not contain $T_C(n)$ and that $\overline{G}$ does not contain W_8.

Suppose first that there is a subset $X \subseteq V(G)$ of size n with $\delta(G[X]) \geq n-4$. If $\delta(G[X]) = n-4$, then let $x \in X$ be such that $d_{G[X]}(x) = n-4$, and set $Y = X \setminus (\{x\} \cup N_{G[X]}(x))$ where $|Y| = 3$. Noting that $3(n-6) > n-4$ for $n \geq 8$, there must be two vertices of Y that are adjacent to a common vertex of $N_{G[X]}(x)$

in G, say to $x' \in N_{G[X]}(x)$. Then the remaining vertex of Y is not adjacent to any vertex of $N_{G[X]}(x) \setminus \{x'\}$ as $T_C(n) \nsubseteq G$, a contradiction to $\delta(G[X]) \geq n-4$. Thus, $\delta(G[X]) \geq n-3$. Pick any vertex $x \in X$ and pick a subset $X' \subseteq N_{G[X]}(x)$ of size $n-3$. Set $Y = X \setminus (\{x\} \cup X')$ where $|Y| = 2$. As $2(n-5) > n-3$ for $n \geq 8$, the two vertices of Y must be adjacent to a common of X' in G, say to x'. Then $G[X' \setminus \{x'\}]$ is an empty graph since $T_C(n) \nsubseteq G$, a contradiction to $\delta(G[X]) \geq n-3$.

We may now assume that $\delta(G[X]) \leq n-5$ whenever $X \subseteq V(G)$ is of size n. By Theorem 4.3.11, G has a subgraph $T = S_{n-1}(3)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n$. Since $T_C(n) \nsubseteq G$, $E_G(U, V) = \emptyset$.

For the case $n = 8$ such that v_1 is not adjacent to any vertex of U in G, or the case $n \geq 9$, there are four vertices of $V(T)$ that are not adjacent to any vertex of U in G. Since $\delta(G[U]) \leq n-5$, $\overline{G}[U]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

For the final case $n = 8$ with v_1 adjacent to some vertex u of U in G, observe that since $T_C(8) \nsubseteq G$, the vertex u is not adjacent to any vertex of $\{v_2, v_3, v_4\} \cup U$. By Lemma 4.3.4, $G[U \setminus \{u\}]$ is K_7 or $K_7 - e$, which implies that every vertex of $V(T) \cup \{u\}$ is not adjacent to any vertex of $U \setminus \{u\}$ in G as $T_C(8) \nsubseteq G$. Since $\delta(G[V(T) \cup \{u\}]) \leq n-5$, $\overline{G}[V(T) \cup \{u\}]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

This completes the proof of the theorem. □

Theorem 5.2.12. *For $n \geq 8$, $R(S_n(3,1), W_8) = 2n-1$.*

Proof. Lemma 5.2.4 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $S_n(3,1)$ and that $\overline{G}$ does not contain W_8.

Suppose first that there is a subset $X \subseteq V(G)$ of size n with $\delta(G[X]) \geq n-4$. Let x_0 be any vertex of X, and pick a subset $X' \subseteq N_{G[X]}(x_0)$ of size $n-4$. Set $Y = X \setminus (\{x_0\} \cup X')$, and so $|Y| = 3$. Since $\delta(G[X]) \geq n-4$, each vertex of Y is adjacent to at least $n-7$ vertices of X' in G. For $n \geq 10$, it is straightforward to see that there is a matching from Y to X' in G; hence, G contains $S_n(3,1)$, a contradiction. For $n = 9$, if $d_{G[X]}(x_0) = n-4 = 5$, we can similarly deduce the contradiction that G contains $S_9(3,1)$, since in this case, each vertex of Y is adjacent to at least $n-6 = 3$ vertices of X' in G. As x_0 was arbitrary, we may assume for the case when $n = 9$, we have $\delta(G[X]) \geq n-3 = 6$, which again leads to the contradiction that G contains $S_9(3,1)$.

Now for $n = 8$, suppose that $d_{G[X]}(x_0) = 4$. Let $X' = \{x_1, x_2, x_3, x_4\}$ and $Y = \{x_5, x_6, x_7\}$. Noting that $\delta(G[X]) \geq n-4$ and $S_8(3,1) \nsubseteq G$, we deduce that $G[Y]$ is K_3; all three vertices of Y are adjacent to exactly two common vertices of X' in G, say to x_1 and x_2; and each of x_3 and x_4 is not adjacent to any vertex of Y in G. By the minimum degree condition, x_3 and x_4 are then adjacent in G, and each of them is also adjacent to both x_1 and x_2. This implies that G contains $S_8(3,1)$, with x_1 being the vertex with degree four, a contradiction. As x_0 was arbitrary, we may assume for the case when $n = 8$, we have $\delta(G[X]) \geq 5$, which again leads to the contradiction that G contains $S_8(3,1)$.

We may now assume that $\delta(G[X]) \leq n-5$ whenever $X \subseteq V(G)$ is of size n. Recall that G has order $2n-1$, so by Theorem 4.3.12, G has a subgraph $T = S_{n-1}(2,1)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_2w_2\}$. Set $V = \{v_3, v_4, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-6$ and $|U| = n$. Since $S_n(3,1) \nsubseteq G$, $E_G(U,V) = \emptyset$. Now as $\delta(G[U]) \leq n-5$, $\overline{G}[U]$ contains S_5, and so for $n \geq 10$, $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

For $n = 9$, Theorem 4.3.12 shows that G has a subgraph $T = S_9(2,1)$, so without loss of generality, assume that v_0 is adjacent to some vertex u in U. Since $S_9(3,1) \nsubseteq G$, $G[V \cup \{u\}]$ is an empty graph and u is not adjacent to any vertex of U in G. By Lemma 4.3.4, $G[U \setminus \{u\}]$ is K_8 or $K_8 - e$, which implies that each vertex of $V(T) \cup \{u\}$ is not adjacent to any vertex of $U \setminus \{u\}$ in G since $S_9(3,1) \nsubseteq G$. Since $\delta(G[V(T) \cup \{u\}]) \leq n-5$, $\overline{G}[V(T) \cup \{u\}]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

Finally for $n = 8$, recall that G has order 15, and so G has a subgraph $T' = S_7$ by Theorem 2.2.6. Let $V(T') = \{v'_0, \ldots, v'_6\}$ and $E(T') = \{v'_0v'_1, \ldots, v'_0v'_6\}$. Set $V' = \{v'_1, \ldots, v'_6\}$ and $U' = V(G) - V(T')$, then $|U'| = 8$. Suppose that v'_2 and v'_3 are adjacent to a common vertex u of U' in G, while v'_1 is adjacent to another vertex $u' \neq u$ of U' in G. Then as $S_8(3,1) \nsubseteq G$, every vertex of $\{v'_4, v'_5, v'_6\} \cup (U' \setminus \{u, u'\})$ is not adjacent to any vertex of $V' \setminus \{v'_1\}$ in G. Now $\overline{G[V' \setminus \{v'_1\}]}$ contains S_5 and $|U' \setminus \{u, u'\}| = 6$, and so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction. Similar arguments lead to the same contradiction when the roles of v'_1, v'_2, and v'_3 are replaced by any three vertices of V'. So we may assume that no two vertices of V' are adjacent to a common vertex of U' in G while a third vertex of V' is adjacent to another vertex of U' in G.

For $i = 1, \ldots, 6$, let $d_i = |E_G(\{v'_i\}, U)|$ be the number of vertices of U' that are adjacent to v'_i. Without loss of generality, assume that $d_1 \geq d_2 \geq \cdots \geq d_6$. Since $\delta(G[U']) \leq 3$ and so $S_5 \subseteq \overline{G}[U']$, Observation 4.3.2 implies that $d_3 \geq 1$. If $d_1 \geq 3$ and $d_2 \geq 2$, then it is trivial that G contains $S_8(3,1)$, a contradiction. By our assumption on the adjacencies of vertices in V' to vertices of U' in G, it is also clear that when (d_1, d_2, d_3) is of the form $(2,2,1)$, $(2,2,2)$ or $(k,1,1)$ for $k \geq 3$, there is a matching from $\{v'_1, v'_2, v'_3\}$ to U' in G, as v'_2 and v'_3 are adjacent to different vertices of U' in G. Then G contains $S_8(3,1)$, a contradiction. If $(d_1, d_2, d_3) = (2,1,1)$, then we similarly have that v'_2 and v'_3 are adjacent to different vertices of U' in G, say to u and u', respectively, which in turn implies that v'_1 is adjacent to two vertices in $U' \setminus \{u, u'\}$. So G contains $S_8(3,1)$, again a contradiction.

For the final case when $d_1 = d_2 = d_3 = 1$, our assumption implies that v'_1, v'_2 and v'_3 are adjacent to a common vertex $u \in U'$ in G to avoid a matching from $\{v'_1, v'_2, v'_3\}$ to U' in G. Furthermore, none of v'_4, v'_5, v'_6 is adjacent to any vertex of $U' \setminus \{u\}$ in G. Now if $S_5 \subseteq \overline{G}[V']$, then $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction. So $\delta(G[V']) \geq 2$, and in particular, v'_4 is adjacent to some vertex of V' in G. Without loss of generality, v_4 is adjacent to either v_1 or v_5 in G. Since $S_8(3,1) \nsubseteq G$, $\overline{G}[\{v'_5, v'_2, v'_3, v'_6\}]$ contains S_4 if v'_4 is adjacent to v'_1 in G, while $\overline{G}[\{v'_6, v'_1, v'_2, v'_3\}]$ contains S_4 if v'_4 is adjacent to v'_5 in G. By Lemma 4.3.4, $G[U' \setminus \{u\}]$ is K_7 or $K_7 - e$, which implies that every vertex of $V(T') \cup \{u\}$ is not adjacent to any vertex of $U' \setminus \{u\}$ in G since $S_8(3,1) \nsubseteq G$. Since $\delta(G[V(T') \cup \{u\}]) \leq 3$, $\overline{G}[V(T) \cup \{u\}]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

Thus, $R(S_n(3,1), W_8) \leq 2n-1$ for $n \geq 8$ which completes the proof. $\square$

5.3 Ramsey numbers for tree graphs with maximum degree of $n-5$ versus the wheel graph of order 9

In this section, we discuss the Ramsey numbers $R(T_n, W_8)$ for tree graphs T_n with maximum degree of $n-5$ versus the wheel graph of order 9. As introduced in the previous section, there will be 19 tree graphs to be discussed, which are $S_n(1,4)$, $S_n(5)$, $S_n[5]$, $S_n(4,1)$ and all the tree graphs shown in Figure 5.3. Before that, we introduce two corollaries about the existence of the cycle graph C_8.

Corollary 5.3.1. *Suppose that U and V are two disjoint subsets of vertices of a graph G for which $|N_{G[V\cup\{u\}]}(u)| \leq 2$ for each $u \in U$. If $|U| \geq 4$ and $|V| \geq 6$, then $\overline{G}[U \cup V]$ contains C_8.*

Proof. Since $|U| \geq 4$ and $|V| \geq 6$, we can choose any 4 vertices from U to form U' and any 6 vertices from V to form V'. We have that $N_{G[V'\cup\{u\}]}(u) \leq 2$ for each $u \in U'$. Then each vertex of U' is adjacent to at least 4 vertices of V' in $\overline{G}$ and $\overline{G}[U' \cup V']$ contains a graph with the properties of $G(4,6,4)$ in Lemma 2.2.11. Hence by that lemma, $\overline{G}[U \cup V]$ must contain C_8. □

Corollary 5.3.2. *Suppose that U and V are two disjoint subsets of vertices of a graph G for which $|N_{G[V\cup\{u\}]}(u)| \leq 3$ for each $u \in U$. If $|U| \geq 4$ and $|V| \geq 8$, then $\overline{G}[U \cup V]$ contains C_8.*

Proof. Since $|U| \geq 4$ and $|V| \geq 8$, we can choose any 4 vertices from U to form U' and any 8 vertices from V to form V'. We have that $N_{G[V'\cup\{u\}]}(u) \leq 3$ for each $u \in U'$. Then each vertex of U' is adjacent to at least 5 vertices of V' in $\overline{G}$ and $\overline{G}[U' \cup V']$ contains a graph with the properties of $G(4,8,5)$ in Lemma 2.2.11. Hence by that lemma, $\overline{G}[U \cup V]$ must contain C_8. □

We are now ready to present the Ramsey numbers for tree graphs with maximum degree of $n-5$ versus the wheel graph of order 9.

Lemma 5.3.3. *Let $n \geq 8$. Then $R(T_n, W_8) \geq 2n-1$ for each $T_n \in \{S_n(1,4), S_n(5), S_n[5], S_n(4,1), T_D(n), \ldots, T_S(n)\}$. Also, $R(T_n, W_8) \geq 2n$ if $n \equiv 0 \pmod 4$ and $T_n \in \{S_n(1,4), T_D(n), S_n(2,2), T_N(n)\}$ or if $T_n \in \{T_E(8), T_F(8)\}$.*

Proof. The graph $G = 2K_{n-1}$ clearly does not contain any tree graphs of order n, and $\overline{G}$ does not contain W_8. Furthermore, if $n \equiv 0 \pmod 4$, then the graph $G = K_{n-1} \cup K_{4,\ldots,4}$ of order $2n-1$ does not contain $S_n(1,4)$, $T_D(n)$ or $S_n(2,2)$; nor does the complement $\overline{G}$ contain W_8. Finally, the graph $G = K_7 \cup K_{4,4}$ does not contain $T_E(8)$ or $T_F(8)$ and $\overline{G}$ does not contain W_8. □

Theorem 5.3.4. *If $n \geq 8$, then*

$$R(S_n(1,4), W_8) = \begin{cases} 2n-1 & \text{if } n \not\equiv 0 \pmod 4\,; \\ 2n & \text{otherwise}\,. \end{cases}$$

Proof. Lemma 5.3.3 provides the lower bound, so it remains to prove the upper bound. Let G be a graph with no $S_n(1,4)$ subgraph whose complement $\overline{G}$ does not contain W_8. Suppose that G has order $2n$ if $n \equiv 0 \pmod 4$ and that G has order

$2n-1$ if $n \not\equiv 0 \pmod 4$. By Theorem 5.2.8, G has a subgraph $T = S_n(1,3)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, w_1w_2, w_2w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = j$ where $j = n$ if $n \equiv 0 \pmod 4$ and $j = n-1$ if $n \not\equiv 0 \pmod 4$. Since $S_n(1,4) \nsubseteq G$, w_3 is not adjacent in G to any vertex of $U \cup V$ and $d_{G[U\cup V]}(v_i) \leq n-7$ for each $v_i \in V$. If $\delta(\overline{G}[U \cup V]) \geq \lceil \frac{n-5+j}{2} \rceil \geq \frac{n-5+j}{2}$, then $\overline{G}[U \cup V]$ contains C_8 by Lemma 2.2.10 and thus W_8 with w_3 as hub, a contradiction. Therefore, $\delta(\overline{G}[U \cup V]) \leq \lceil \frac{n-5+j}{2} \rceil - 1$ and $\Delta(G[U \cup V]) \geq n-5+j - \lceil \frac{n-5+j}{2} \rceil = \lfloor \frac{n-5+j}{2} \rfloor \geq n-3$. Since $d_{G[U\cup V]}(v_i) \leq n-7$ for each $v_i \in V$, $d_{G[U\cup V]}(u) \geq n-3$ for some vertex $u \in U$. Since $S_n(1,4) \nsubseteq G$, no vertex of V is adjacent to $\{u\} \cup N_{G[U\cup V]}(u)$ in G.

For $n \geq 9$, any 4 vertices from V and any 4 vertices from $\{u\} \cup N_{G[U\cup V]}(u)$ form C_8 in $\overline{G}$ and, with w_3 as hub, form W_8, a contradiction. Suppose that $n = 8$; then $V = \{v_2, v_3, v_4\}$. Let $\{u_1, \ldots, u_4\}$ be 4 vertices in $N_{G[U\cup V]}(u)$. Since $S_8(1,4) \nsubseteq G$, w_1 is not adjacent to $N_{G[U\cup V]}(u)$. If w_1 is not adjacent to w_3, then $w_1u_1v_2u_2v_3u_3v_4u_4w_1$ and w_3 form W_8 in $\overline{G}$, a contradiction. Therefore, w_1 is adjacent to w_3 in G. Then w_2 is not adjacent to any vertex of $U \cup V$ in G. Since $d_{G[V]}(v_i) \leq 1$ for $i = 2,3,4$, one of the vertices of V, say v_2, is not adjacent to the other two vertices of V. Then $u_1w_2u_2w_3u_3v_3u_4v_4u_1$ and v_2 form W_8 in $\overline{G}$, a contradiction. Thus, $R(S_n(1,4), W_8) \leq 2n$ for $n \equiv 0 \pmod 4$ and $R(S_n(1,4), W_8) \leq 2n-1$ for $n \not\equiv 0 \pmod 4$.

This completes the proof. □

Theorem 5.3.5. *If $n \geq 9$, then $R(S_n(5), W_8) = 2n-1$.*

Proof. Lemma 5.3.3 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $S_n(5)$ and that $\overline{G}$ does not contain W_8. By Theorem 5.2.5, G has a subgraph $T = S_n(4)$. Let $V(T) = \{v_0,\ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1,\ldots, v_0v_{n-4}, v_1w_1, v_1w_2, v_1w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n-1$. Since $S_n(5) \nsubseteq G$, v_1 is not adjacent to any vertex of $U \cup V$ in G. Furthermore, for each v_i in V, v_i is adjacent to at most three vertices of U in G.

For $n \geq 9$, we have $|V| \geq 4$ and $|U| \geq 8$. By Corollary 5.3.2, $\overline{G}[U \cup V]$ contains C_8 which together with v_1 gives W_8 in $\overline{G}$, a contradiction. Thus, $R(S_n(5), W_8) \leq 2n-1$ which completes the proof. □

Theorem 5.3.6. *If $n \geq 9$, then $R(S_n[5], W_8) = 2n-1$.*

Proof. Lemma 5.3.3 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $S_n[5]$ and that $\overline{G}$ does not contain W_8. By Theorem 5.3.5, G has a subgraph $T = S_n(5)$. Let $V(T) = \{v_0, \ldots, v_{n-5}, w_1, \ldots, w_4\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-5}, v_1w_1, \ldots, v_1w_4\}$. Set $V = \{v_2, \ldots, v_{n-5}\}$ and $U = V(G) - V(T)$; then $|V| = n-6$ and $|U| = n-1$. Since $S_n[5] \nsubseteq G$, v_0 is not adjacent to $w_1, \ldots, w_4$ in G and $w_1, \ldots, w_4$ are each adjacent to at most two vertices of U in G. Now, suppose that v_0 is non-adjacent to at least six vertices of U in G. By Corollary 5.3.1, six of these vertices together with $w_1, \ldots, w_4$ contain C_8 in $\overline{G}$ which with v_0 gives W_8 in $\overline{G}$, a contradiction. Then suppose that v_0 is adjacent to at least $n-6$ vertices of U in G. Choose a set U'

of $n-6$ of these vertices. Since $S_n[5] \nsubseteq G$, v_1 is not adjacent to any vertex of $V \cup U'$ in G. If $\delta(\overline{G}[V \cup U']) \geq n-6$, then by Lemma 2.2.10, $\overline{G}[V \cup U']$ contains C_8 which with v_1 gives W_8 in $\overline{G}$, a contradiction. Therefore, $\delta(\overline{G}[V \cup U']) \leq n-7$ and $\Delta(G[V \cup U']) \geq n-6$. However, this gives $S_n[5]$ in G with u and v_1 as the centre of S_{n-5} and S_5, respectively, where u is a vertex in $V \cup U'$ with $d_{G[V \cup U']}(u) \geq n-6$, a contradiction. Thus, $R(S_n[5], W_8) \leq 2n-1$ which completes the proof. □

Theorem 5.3.7. *If $n \geq 8$, then*

$$R(S_n(2,2), W_8) = \begin{cases} 2n-1 & \text{if } n \not\equiv 0 \pmod 4; \\ 2n & \text{otherwise}. \end{cases}$$

Proof. Lemma 5.3.3 provides the lower bound, so it remains to prove the upper bound. Assume that G is a graph with no $S_n(2,2)$ subgraph whose complement $\overline{G}$ does not contain W_8. Suppose that $n \equiv 0 \pmod 4$ and that G has order $2n$. By Theorem 5.2.10, G has a subgraph $T = T_B(n)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, w_1w_2, v_2w_3\}$. Set $V = \{v_3, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-6$ and $|U| = n$. Since $S_n(2,2) \nsubseteq G$, w_3 is not adjacent in G to $U \cup V$ and v_2 is not adjacent to V. If $\delta(\overline{G}[U \cup V]) \geq \frac{2n-6}{2} = n-3$, then $\overline{G}[U \cup V]$ contains C_8 by Lemma 2.2.10 which with w_2 forms W_8, a contradiction. Therefore, $\delta(\overline{G}[U \cup V]) \leq n-4$, and $\Delta(G[U \cup V]) \geq n-3$. Now, there are two cases to be considered.

Case 1a: One of the vertices of V, say v_3, is a vertex of degree at least $n-3$ in $G[U \cup V]$.

Note that in this case, there are at least 4 vertices from U, say $u_1, \ldots, u_4$, that are adjacent to v_3 in G. Since $S_n(2,2) \nsubseteq G$, these 4 vertices are independent and are not adjacent to any other vertices of U. Since $n \geq 8$, U contains at least 4 other vertices, say $u_5, \ldots, u_8$, so $u_1u_5u_2u_6u_3u_7u_4u_8u_1$ and w_3 forms W_8 in $\overline{G}$, a contradiction.

Case 1b: Some vertex $u \in U$ has degree at least $n-3$ in $G[U \cup V]$.

Since $S_n(2,2) \nsubseteq G$, u is not adjacent to any vertex of V in G. Therefore, u must be adjacent to at least $n-3$ vertices of U in G. Without loss of generality, suppose that $u_1, \ldots, u_{n-3} \in N_{G[U]}(u)$. Note that V is not adjacent to $N_{G[U]}(u)$, or else there will be $S_n(2,2)$ in G, a contradiction. If $n \geq 12$, then any 4 vertices from $N_{G[U]}(u)$ and any 4 vertices from V form C_8 in $\overline{G}$ which, with w_3 as hub, forms W_8, a contradiction. Suppose that $n = 8$ and let the remaining two vertices be u_6 and u_7. If $|N_{G[\{u_1,\ldots,u_5,u_i\}}(u_i)| \leq 1$ for $i = 6, 7$, then let $X = \{u_1, \ldots, u_4\}$ and $Y = \{v_3, v_4, u_6, u_7\}$. By Lemma 4.3.5, $\overline{G}[X \cup Y]$ contains C_8 and, with w_3 as hub, forms W_8 in $\overline{G}$, a contradiction. Therefore, one of u_6 and u_7, say u_6, is adjacent to at least two of $u_1, \ldots, u_5$, say u_1 and u_2. Since $S_8(2,2) \nsubseteq G$, u_7 is adjacent in $\overline{G}$ to at least two of u_3, u_4, u_5, say u_3 and u_4, and $v_0, \ldots, v_4, w_1$ are not adjacent in G to $u, u_1, \ldots, u_6$. Now, if w_3 is not adjacent to some vertex $a \in \{v_0, v_1, w_1\}$, then $u_1v_3u_2v_4u_3u_7u_4au_1$ and w_3 form W_8 in $\overline{G}$, a contradiction. Hence, w_3 is adjacent to v_0, v_1 and w_1 in G. Similarly, v_2 is not adjacent to u_7 and v_2 is adjacent to v_1 and w_1. Since $S_8(2,2) \nsubseteq G$, w_2 is not adjacent to $U \cup V$, and w_1 is not adjacent to V. Then $u_1v_2u_2w_1u_3w_2u_4w_3u_1$ and v_3 forms W_8 in $\overline{G}$, a contradiction.

In either case, $R(S_n(2,2), W_8) \leq 2n$.

Suppose that $n \not\equiv 0 \pmod 4$ and that G has order $2n-1$. By Theorem 5.2.10, G has a subgraph $T = T_B(n)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, w_1w_2, v_2w_3\}$. Set $V = \{v_3, \ldots, v_{n-4}\}$ and $U = V(G)-V(T)$; then $|V| = n-6$ and $|U| = n-1$. Since $S_n(2,2) \nsubseteq G$, w_3 is not adjacent in G to $U \cup V$. If $\delta(\overline{G}[U \cup V]) \geq \lceil \frac{2n-5}{2} \rceil$, then $\overline{G}[U \cup V]$ contains C_8 by Lemma 2.2.10 which with w_3 forms W_8, a contradiction. Therefore, $\delta(\overline{G}[U \cup V]) \leq \lceil \frac{2n-5}{2} \rceil - 1 = n-3$, and $\Delta(G[U \cup V]) \geq n-3$. Again, there are two cases to be considered.

Case 2a: A vertex of V, say v_3, has degree at least $n-3$ in $G[U \cup V]$.

There must be at least 4 vertices from U, say $u_1, \ldots, u_4$ that are adjacent to v_3 in G. Since $S_n(2,2) \nsubseteq G$, $u_1, \ldots, u_4$ are independent and are not adjacent to any other vertex of U. Since $n \geq 9$, there are at least 4 other vertices of U, say $u_5, \ldots, u_8$, and $u_1u_5u_2u_6u_3u_7u_4u_8u_1$ and w_3 form W_8 in $\overline{G}$, a contradiction.

Case 2b: A vertex $u \in U$ has degree at least $n-3$ in $G[U \cup V]$.

Since $S_n(2,2) \nsubseteq G$, no vertex of V is adjacent to u or to $N_{G[U]}(u)$. Then u is adjacent to at least $n-3$ vertices of U in G; suppose without loss of generality that $u_1, \ldots, u_{n-3} \subseteq N_{G[U]}(u)$. If $n \geq 10$, then any 4 vertices from $N_{G[U]}(u)$, any 4 vertices from V and w_3 form W_8 in $\overline{G}$, a contradiction. Suppose that $n = 9$ and let u_7 be the vertex in $U \setminus \{u, u_1, \ldots, u_{n-3}\}$. If u_7 is adjacent in $\overline{G}$ to at least two of $u_1, \ldots, u_6$, say u_1 and u_2, then $u_1u_7u_2v_3u_3v_4u_4v_5u_1$ and w_3 form W_8 in $\overline{G}$, a contradiction. Therefore, u_7 is adjacent in G to at least 5 of the vertices $u_1, \ldots, u_6$, say $u_1, \ldots, u_5$. Since $S_9(2,2) \nsubseteq G$, U is not adjacent in G to $\{v_0, v_1, v_2, w_1\} \cup V$ and w_2 is not adjacent to u or u_7. If w_3 is not adjacent to some vertex $a \in \{v_0, v_1, w_1, w_2\}$, then $uv_3u_1v_4u_2v_5u_7au$ and w_3 form W_8 in $\overline{G}$, a contradiction. Hence, w_3 is adjacent to v_0, v_1, w_1 and w_2 in G. Similarly, v_2 is adjacent to v_1, w_1 and w_2. Since $S_9(2,2) \nsubseteq G$, w_2 is non-adjacent to at least one of v_3, v_4, v_5, say v_3 without loss of generality. If v_1 is also not adjacent to v_3, then $uw_2u_7v_1u_1v_2u_2w_3u$ and w_3 form W_8 in $\overline{G}$, a contradiction. Thus, v_1 is adjacent to v_3, then v_3 is not adjacent to both v_4 and v_5, or else G contains $S_9(2,2)$. Without loss of generality, assume that v_3 is not adjacent to v_4 in G. Then $uw_2u_7v_4u_1v_2u_2w_3u$ and w_3 form W_8 in $\overline{G}$, a contradiction.

In either case, $R(S_n(2,2), W_8) \leq 2n-1$ for $n \not\equiv 0 \pmod 4$. □

Theorem 5.3.8. *If $n \geq 9$, then $R(S_n(4,1), W_8) = 2n-1$.*

Proof. Lemma 5.3.3 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $S_n(4,1)$ and that $\overline{G}$ does not contain W_8.

Suppose first that there is a subset $X \subseteq V(G)$ of size n with $\delta(G[X]) \geq n-4$. Let x_0 be any vertex of X, and pick a subset $X' \subseteq N_{G[X]}(x_0)$ of size $n-5$. Set $Y = X \setminus (\{x_0\} \cup X')$, and so $|Y| = 4$. Since $\delta(G[X]) \geq n-4$, each vertex of Y is adjacent to at least $n-8$ vertices of X' in G and each vertex of X' is adjacent to at least one vertex of Y in G. Hence, for $n \geq 11$, it is straightforward to see that there is a matching from Y to X' in G; hence, G contains $S_n(4,1)$, a contradiction.

For $n = 10$ and $\delta(G[X]) \geq n-4 = 6$, let $X = \{x_0, \ldots, x_9\}$ and $\{x_1, \ldots, x_6\} \subseteq N_{G[X]}(x_0)$. Since $\delta(G[X]) \geq 6$, vertices x_7, x_8 and x_9 must each be adjacent to at least 3 vertices of $x_1, \ldots, x_6$. It is straightforward to see that there is a matching from $\{x_7, x_8, x_9\}$ to $\{x_1, \ldots, x_6\}$ in G; without loss of generality, assume that x_i is adjacent to x_{i+6} in G for $i = 1, 2, 3$. Now, if there is any edge in $G[\{x_4, x_5, x_6\}]$, then $S_{10}(4, 1) \subseteq G$, a contradiction. Otherwise, $G[\{x_4, x_5, x_6\}]$ is independent and each of x_4, x_5, x_6 must be adjacent to at least two vertices of x_7, x_8, x_9 in G. Without loss of generality, assume that x_4 is adjacent to x_7 and x_8 in G. Since $S_{10}(4, 1) \nsubseteq G$, x_5 cannot be adjacent to x_1 and x_2 in G, but this is impossible since $\delta(G[X]) \geq 6$.

Now for $n = 9$, suppose that $d_{G[X]}(x_0) = n-4 = 5$. Let $N_{G[X]}(x_0) = \{x_1, \ldots, x_5\}$ and $Y = \{x_6, x_7, x_8\}$. Then three vertices of Y are each adjacent to at least $n-6 = 3$ vertices of $N_{G[X]}(x_0)$ in G. Without loss of generality, assume that x_1 is adjacent to x_6, x_2 is adjacent to x_7 and x_3 is adjacent to x_8, respectively. Now, if x_4 is adjacent to x_5, then G contains $S_9(4, 1)$, a contradiction. Otherwise, x_4 and x_5 must each be adjacent to at least one of x_6, x_7 and x_8. Assume that x_4 is adjacent to x_6. Then x_5 is not adjacent to x_1 and x_4 in G, or else G contains $S_9(4, 1)$. If x_5 is adjacent to x_6, then x_1, x_4, x_5 must be independent in G, and they are each adjacent to x_7 or x_8 in G; assume that x_1 is adjacent to x_7. Then x_4 and x_5 are not adjacent to x_2 in G, and since $\delta(G[X]) \geq 5$, they are adjacent to x_7 and x_8 in G, and G contains $S_9(4, 1)$, a contradiction. If x_5 is not adjacent to x_6, then since $d_{G[X]}(v_0) \geq 5$, x_5 is adjacent to x_2, x_3, x_7 and x_8 in G. Then x_4 is not adjacent to x_2 and x_3 in G, and x_4 is adjacent to x_1, x_6, x_7 and x_8 in G, and this gives us $S_9(4, 1)$ in G, a contradiction. As x_0 was arbitrary, assume for the case when $n = 9$ that $\delta(G[X]) \geq n - 3 = 6$, which again leads to the contradiction that G contains $S_9(4, 1)$.

Now assume that $\delta(G[X]) \leq n - 5$ whenever $X \subseteq V(G)$ is of size n. Recall that G has order $2n - 1$, and so by Theorem 5.2.12, G has a subgraph $S_n(3, 1)$ and thus a subgraph $T = S_{n-1}(3, 1)$. Let $V(T) = \{v_0, \ldots, v_{n-5}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-5}, v_1w_1, v_2w_2, v_3w_3\}$. Set $V = \{v_4, \ldots, v_{n-5}\}$ and $U = V(G) - V(T) = \{u_1, \ldots, u_n\}$; then $|V| = n - 8$ and $|U| = n$. Since $S_n(4, 1) \nsubseteq G$, V is not adjacent to any vertex of U in G. Now as $\delta(G[U]) \leq n - 5$, $\overline{G}[U]$ contains S_5, and so for $n \geq 12$, $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

Suppose that $n = 11$. If v_0 is not adjacent to any vertex of U in G, then $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction. Assume that v_0 is adjacent to some vertex $u \in U$. Since $S_{11}(4, 1) \nsubseteq G$, $G[V \cup \{u\}]$ is an empty graph and u is not adjacent to any vertex of U in G. By Lemma 4.3.4, $G[U \setminus \{u\}]$ is K_{10} or $K_{10} - e$, so no vertex of $V(T) \cup \{u\}$ is adjacent to any vertex of $U \setminus \{u\}$ in G, as $S_{11}(4, 1) \nsubseteq G$. Since $\delta(G[V(T) \cup \{u\}]) \leq n - 5$, $\overline{G}[V(T) \cup \{u\}]$ contains S_5, so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

Now, suppose that $n = 10$. Then G has order 19, and by Theorem 5.2.12, G has a subgraph $T' = S_{10}(3, 1)$. Let $V(T') = \{v'_0, \ldots, v'_6, w'_1, w'_2, w'_3\}$ and $E(T') = \{v'_0v'_1, \ldots, v'_0v'_6, v'_1w'_1, v'_2w'_2, v'_3w'_3\}$. Set $V' = \{v'_4, v'_5, v'_6\}$ and $U' = V(G) - V(T') = \{u'_1, \ldots, u'_9\}$. Since $S_{10}(4, 1) \nsubseteq G$, V' must be independent in G and is not adjacent to any vertex of U' in G. If v'_0 is adjacent to some vertices in U' in G, say u'_1. Since $S_{10}(4, 1) \nsubseteq G$, u'_1 is not adjacent to any vertex of V' or $U' \setminus \{u'_1\}$ in G. Then by Lemma 4.3.4, $G[U' \setminus \{u'_1\}]$ is K_8 or $K_8 - e$, so no vertex of $V(T')$ is adjacent to any vertex of $U' \setminus \{u'_1\}$ in G, as $S_{10}(4, 1) \nsubseteq G$. Since $\delta(G[V(T')]) \leq 5$, $\overline{G}[V(T')]$ contains S_5, so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction. Now, suppose that v'_0 is not adjacent to any vertex of U' in G. Note that $|U' \cup \{w'_1\}| = n$; therefore, $\delta(G[U' \cup \{w'_1\}]) \leq 5$, and so $\overline{G}[U' \cup \{w'_1\}]$ contains S_5. If w'_1 is not adjacent to any vertex from $V' \cup \{v'_0\}$, then by Observation 4.3.2, $\overline{G}$ contains W_8, a contradiction. Otherwise, there are two cases to be considered.

Case 1a: w'_1 is adjacent to some vertices of V' in G.

Without loss of generality, assume that w'_1 is adjacent to v'_4 in G. In this case, v'_1 is not adjacent to $U' \cup \{v'_5, v'_6\}$. Then by Lemma 4.3.4, $G[U']$ is K_9 or $K_9 - e$, so no vertex of $V(T')$ is adjacent to any vertex of U' in G, as $S_{10}(4,1) \nsubseteq G$. Since $\delta(G[V(T')]) \leq 5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

Case 1b: w'_1 is non-adjacent to each vertex of V' in G.

In this case, w'_1 is adjacent to v'_0 in G. Note that w'_1 is not adjacent to U', since this would revert to the case where v'_0 is adjacent to some vertex of U'. Then again by Lemma 4.3.4, $G[U']$ is K_9 or $K_9 - e$, so no vertex of $V(T')$ is adjacent to any vertex of U' in G, as $S_{10}(4,1) \nsubseteq G$. Since $\delta(G[V(T')]) \leq 5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

Finally, suppose that $n = 9$. Then G has order 17, and so G has a subgraph $T' = S_9(2,1)$ by Theorem 4.3.12. Let $V(T') = \{v'_0, \ldots, v'_6, w'_1, w'_2\}$ and $E(T') = \{v'_0v'_1, \ldots, v'_0v'_6, v'_1w'_1, v'_2w'_2\}$. Set $V' = \{v'_3, \ldots, v'_6\}$ and $U' = V(G) - V(T') = \{u'_1, \ldots, u'_8\}$.

Now, suppose that $E_G(V', U') \neq \emptyset$. Without loss of generality, assume that v'_3 is adjacent to u'_1 in G. Since $S_9(4,1) \nsubseteq G$, v'_4, v'_5, v'_6 are independent and not adjacent to any vertex of $U' \setminus \{u'_1\}$ in G.

Suppose that v'_0 is adjacent to some vertex of $U' \setminus \{u'_1\}$, say u'_2. Then u'_2 is non-adjacent to $\{v'_4, v'_5, v'_6\} \cup U' \setminus \{u'_1, u'_2\}$ in G. Since $\delta(G[\{w'_1, w'_2\} \cup U' \setminus \{u'_2\}]) \leq n - 5$, $\overline{G}[\{w'_1, w'_2\} \cup U' \setminus \{u'_2\}]$ contains S_5. If v'_4, v'_5, v'_6 and u'_2 are not adjacent to w'_1, w'_2 or u'_1 in G, then $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction. Assume that v'_4 is adjacent to w'_1 in G. In this case, v'_1 is not adjacent to $\{v'_5, v'_6\} \cup U' \setminus \{u'_1\}$ in G, and $v'_1u'_3v'_4u'_4v'_6u'_7u'_2u'_8v'_1$ and v'_5 form W_8 in $\overline{G}$, a contradiction. Similar contradictions occur if we assume that v'_5, v'_6 or u'_2 are adjacent to w'_1, w'_2 or u'_1 in G.

Thus, v'_0 is not adjacent to any vertex of $U' \setminus \{u'_1\}$ in G. Since $\delta(G[\{w'_1, w'_2\} \cup U' \setminus \{u'_1\}]) \leq n - 5$, $\overline{G}[\{w'_1, w'_2\} \cup U' \setminus \{u'_1\}]$ contains S_5. If v'_0, v'_4, v'_5 and v'_6 are not adjacent to w'_1 or w'_2 in G, then $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction. There are two cases to be considered.

Case 2a: v'_0 is adjacent to w'_1 or w'_2 in G.

Without loss of generality, assume that v'_0 is adjacent to w'_1 in G. Note that v'_1 and w'_1 are not adjacent to $U' \setminus \{u'_1\}$, since this would revert to the case where v'_0 is adjacent to some vertex of $U' \setminus \{u'_1\}$. Again, since $\delta(G[\{w'_2\} \cup U']) \leq n - 5$, $\overline{G}[\{w'_2\} \cup U'\}]$ contains S_5. If v'_1, v'_4, v'_5 and v'_6 are not adjacent to w'_2 and u'_1 in G, then $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

Suppose that v'_1 is adjacent to w'_2 or u'_1, say w'_2, in G. If w'_1 is not adjacent to v'_4, v'_5 or v'_6, then by Lemma 4.3.4, $G[U' \setminus \{u'_1\}]$ is K_7 or $K_7 - e$, so no vertex of $V(T') \cup \{u'_1\}$ is adjacent to any vertex of $U' \setminus \{u'_1\}$ in G, as $S_9(4,1) \nsubseteq G$. Since $\delta(G[V(T')]) \leq n - 5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction. Otherwise, w'_1 is adjacent to at least one of v'_4, v'_5, v'_6 in G, say v'_4. Then v'_2 is not adjacent to $\{v'_5, v'_6\} \cup U' \setminus \{u'_1\}$, since G does not contain $S_9(4,1)$. Similarly, by Lemma 4.3.4, $G[U' \setminus \{u'_1\}]$ is K_7 or $K_7 - e$, so no vertex of $V(T') \cup \{u'_1\}$ is adjacent to any vertex of $U' \setminus \{u'_1\}$ in G, as $S_9(4,1) \nsubseteq G$. Again, since $\delta(G[V(T')]) \leq n - 5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

Now suppose that v_1' is non-adjacent to both w_2' and u_1' in G. Then one of v_4', v_5', v_6' is adjacent to w_2' or u_1' in G. Without loss of generality, assume that v_4' is adjacent to w_2' in G. In this case, v_2' is not adjacent to $\{v_5', v_6'\} \cup U' \setminus \{u_1'\}$. Then again, by Lemma 4.3.4, $G[U' \setminus \{u_1'\}]$ is K_7 or $K_7 - e$, so no vertex of $V(T') \cup \{u_1'\}$ is adjacent to any vertex of $U' \setminus \{u_1'\}$ in G, as $S_9(4,1) \nsubseteq G$. Since $\delta(G[V(T')]) \leq n-5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

Case 2b: v_0' is non-adjacent to both w_1' and w_2' in G.

In this case, one of v_4', v_5', v_6' is adjacent to w_1' or w_2' in G, say v_4' to w_1' in G. Since $S_9(4,1) \nsubseteq G$, v_1' is not adjacent to $\{v_5', v_6'\} \cup U' \setminus \{u_1'\}$ in G. By Lemma 4.3.4, $G[U' \setminus \{u_1'\}]$ is K_7 or $K_7 - e$, so no vertex of $V(T') \cup \{u_1'\}$ is adjacent to any vertex of $U \setminus \{u_1'\}$ in G, as $S_9(4,1) \nsubseteq G$. Since $\delta(G[V(T')]) \leq n-5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

Now suppose that $E_G(V', U') = \emptyset$. If $\delta(G[V']) = 0$, then by Lemma 4.3.4, $G[U']$ is K_8 or $K_8 - e$, and no vertex of $V(T')$ is adjacent to any vertex of U' in G, as $S_9(4,1) \nsubseteq G$. Since $\delta(G[V(T')]) \leq n-5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction. Hence, $\delta(G[V']) \geq 1$, and since $S_9(4,1) \nsubseteq G$, one of the vertices in V' is adjacent to other three in G. Without loss of generality, assume that v_3' is adjacent to v_4', v_5' and v_6' in G. Since G does not contain $S_9(4,1)$, v_4', v_5', v_6' are independent in G. Furthermore, v_0' is not adjacent to U' in G or else this reverts to the case where v_3' is adjacent to u_1' and v_0' is adjacent to any vertex of $U' \setminus \{u_1'\}$. Since $\delta(G[\{w_1'\} \cup U']) \leq n-5$, $\overline{G}[\{w_1'\} \cup U']$ contains S_5. If v_0', v_4', v_5' and v_6' are non-adjacent to w_1' in G, then $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction. Again, there are two cases to be considered.

Case 3a: v_0' is adjacent to w_1' in G.

Note that v_1' and w_1' are not adjacent to U', or else this reverts to the case where v_3' is adjacent to u_1' and v_0' is adjacent to any vertex of $U' \setminus \{u_1'\}$. Now, since $\delta(G[\{w_2'\} \cup U']) \leq n-5$, $\overline{G}[\{w_2'\} \cup U'\}]$ contains S_5. If v_0', v_4', v_5' and v_6' are non-adjacent to w_2' in G, then $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

Suppose that v_0' is adjacent to w_2' in G. Again, v_2' and w_2' are non-adjacent to U', or else else this reverts to the case where v_3' is adjacent to u_1' and v_0' is adjacent to any vertex of $U' \setminus \{u_1'\}$. Now, $E_G(V(T'), U') = \emptyset$, and since $\delta(G[V(T')]) \leq n-5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

Therefore, w_2' is adjacent to at least one of v_4', v_5' and v_6' in G, say v_4'. Then v_2' is not adjacent to v_5', v_6' or U', as $S_9(4,1) \nsubseteq G$, a contradiction. By Lemma 4.3.4, $G[U']$ is K_8 or $K_8 - e$, so no vertex of $V(T')$ is adjacent to any vertex of U' in G, as $S_9(4,1) \nsubseteq G$. Again, since $\delta(G[V(T')]) \leq n-5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

Case 3b: v_0' is not adjacent to w_1' in G.

In this case, one of v_4', v_5', v_6' is adjacent to w_1' in G, say v_4'. Since $S_9(4,1) \nsubseteq G$, v_1' is not adjacent to v_5', v_6' or U' in G. By Lemma 4.3.4, $G[U']$ is K_8 or $K_8 - e$, so no vertex of $V(T') \cup \{u_1'\}$ is adjacent to any vertex of U' in G, as $S_9(4,1) \nsubseteq G$. Since $\delta(G[V(T')]) \leq n-5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3.2, a contradiction.

Thus, $R(S_n(4,1), W_8) \leq 2n-1$ for $n \geq 9$ which completes the proof. $\square$

Theorem 5.3.9. *If $n \geq 8$, then*

$$R(T_D(n), W_8) = \begin{cases} 2n-1 & \text{if } n \not\equiv 0 \pmod 4\,; \\ 2n & \text{otherwise}\,. \end{cases}$$

Proof. Lemma 5.3.3 provides the lower bound, so it remains to prove the upper bound. Let G be a graph with no $T_D(n)$ subgraph whose complement $\overline{G}$ does not contain W_8. Suppose that $n \equiv 0 \pmod 4$ and that G has order $2n$. By Theorem 5.2.7, G has a subgraph $T = S_n[4]$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, w_1w_2, w_1w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n$. Since $T_D(n) \nsubseteq G$, neither w_2 nor w_3 is adjacent in G to $U \cup V$.

Suppose that $n = 8$. Since G does not contain $T_D(n)$, V must be independent and non-adjacent to U in G. Then for any vertices $u_1, \ldots, u_4$ in U, $v_3u_1v_4u_2w_2u_3w_3u_4v_3$ and v_2 form W_8 in $\overline{G}$, a contradiction. Suppose that that $n \geq 12$. Then $|U \cup V| = 2n-5$. If $\delta(\overline{G}[U \cup V]) \geq \lceil \frac{2n-5}{2} \rceil$, then $\overline{G}[U \cup V]$ contains C_8 by Lemma 2.2.10 which, with w_2 as hub, forms W_8, a contradiction. Thus, $\delta(\overline{G}[U \cup V]) \leq \lceil \frac{2n-5}{2} \rceil - 1 = n-3$, and $\Delta(G[U \cup V]) \geq n-3$. Now, there are two cases to consider.

Case 1: One of the vertices of V, say v_2, is a vertex of degree at least $n-3$ in $G[U \cup V]$.

Since $T_D(n) \nsubseteq G$, v_1 is not adjacent in G to w_2, w_3 or $U \cup V \setminus \{v_2\}$. Let $U' = \{w_2, w_3\} \cup U \cup V \setminus \{v_2\}$; then $|U'| = 2n-4$. Now, if $\delta(\overline{G}[U']) \geq \frac{2n-4}{2} = n-2$, then $\overline{G}[U']$ contains C_8 by Lemma 2.2.10 which, with v_1 as hub, forms W_8, a contradiction. Hence, $\delta(\overline{G}[U']) \leq n-3$, and $\Delta(G[U']) \geq n-2$. Note that neither w_2 nor w_3 have degree $\Delta(G[U'])$. Therefore, $d_{G[U']}(u') \geq n-2$ for some vertex $u' \in U \cup V \setminus \{v_2\}$. By the Inclusion-Exclusion Principle, some vertex $a \in U \cup V \setminus \{v_2\}$ is adjacent in G to both u' and v_2. Then G has a subgraph $T_D(n)$ in which u' is the vertex of degree $n-5$ and v_2 is the vertex of degree 3, a contradiction.

Case 2: Some vertex $u \in U$ has degree at least $n-3$ in $G[U \cup V]$.

Suppose that there is at least one vertex in V that is adjacent to u in G, say v_2. Then G has a subgraph $T_D(n)$ in which u is the vertex of degree $n-5$ and v_0 is the vertex of degree 3, a contradiction. Similarly, no other vertex of V is adjacent to u. Now, since $T_D(n) \nsubseteq G$, we must have $d_{G[N_{G[U]}(u) \cup \{v\}]}(v) \leq 1$ and $d_{G[V \cup \{x\}]}(x) \leq 1$, for any $v \in V$ and $x \in N_{G[U]}(u)$. Then by Lemma 4.3.5, $\overline{G}[V \cup N_{G[U]}(u)]$ must contain C_8, which with w_2 as hub, forms W_8 in $\overline{G}$, a contradiction.

Now, suppose that $n \not\equiv 0 \pmod 4$ and that G has order $2n-1$. By Theorem 5.2.7, G has a subgraph $T = S_n[4]$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, w_1w_2, w_1w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n-1$. Since $T_D(n) \nsubseteq G$, neither w_2 nor w_3 is adjacent to $U \cup V$ in G. If $\delta(\overline{G}[U \cup V]) \geq \frac{2n-6}{2} = n-3$, then $\overline{G}[U \cup V]$ contains C_8 by Lemma 2.2.10 which, with w_2 as hub, forms W_8 in $\overline{G}$, a contradiction. Thus, $\delta(\overline{G}[U \cup V]) \leq n-4$, and $\Delta(G[U \cup V]) \geq n-3$. The arguments of the preceding cases then lead to contradictions.

Thus, $R(T_D(n), W_8) \leq 2n$, which completes the proof. □

Lemma 5.3.10. *Each graph H of order $n \geq 8$ with minimal degree at least $n-4$ contains $T_E(n)$ unless $n=8$ and $H = K_{4,4}$.*

Proof. Let $V(H) = \{u_0, \ldots, u_{n-1}\}$. First, suppose that $\Delta(H) \geq n-3$ and assume without loss of generality that $u_1, \ldots, u_{n-3} \in N_H(u_0)$. Suppose that u_{n-2} and u_{n-1} are adjacent in H. Since $\delta(H) \geq n-4$, $N_H(u_0) \cap N_H(u_{n-2}) \neq \emptyset$, so assume without loss of generality that u_1 is adjacent to u_{n-2} in H. Furthermore, u_1 must be adjacent to at least $n-7$ vertices from $\{u_2, \ldots, u_{n-3}\}$ in H. Without loss of generality, assume that u_1 is adjacent to $u_2, \ldots, u_{n-6}$ in H. Now, if any vertex of $\{u_2, \ldots, u_{n-6}\}$ is adjacent to u_{n-5}, u_{n-4} or u_{n-3} in H, then we have $T_E(n)$ in H. Suppose that is not the case; then each vertex of $\{u_2, \ldots, u_{n-6}\}$ must be adjacent to each other and to u_0, u_1, u_{n-2} and u_{n-1} in H. Since $d_H(u_{n-3}) \geq n-4$, u_{n-3} is adjacent to at least one of u_1, u_{n-2} and u_{n-1} in H, so H contains $T_E(n)$, a contradiction.

Suppose that u_{n-2} is not adjacent to u_{n-1} in H. Since $\delta(H) \geq n-4$, u_{n-2} and u_{n-1} are each adjacent to at least $n-5$ vertices in $N_H(u_0)$, so at least one vertex of $N_H(u_0)$, say u_1, is adjacent in H to both u_{n-2} and u_{n-1}. If $H[\{u_2, \ldots, u_{n-3}\}]$ contains subgraph $2K_2$, then H contains subgraph $T_E(n)$. Note that this will always happens for $n \geq 11$, since $\delta(H) \geq n-4$.

Suppose that $n=10$. Since $\delta(H) \geq 6$, u_2 must be adjacent in H to at least two vertices of $u_3, \ldots, u_7$, without loss of generality say u_3 and u_4. If $H[\{u_4, \ldots, u_7\}]$ contains any edge, then H contains $T_E(10)$. Otherwise, $\{u_4, \ldots, u_7\}$ must be independent in H and each of these vertices must be adjacent to u_0, u_1, u_2, u_3, u_8 and u_9; this also gives a subgraph $T_E(10)$ in H.

Similarly, for $n=9$, u_2 must be adjacent to at least one of $u_3, \ldots, u_6$, say u_3, in H. If $H[\{u_4, u_5, u_6\}]$ contains any edge, then H contains $T_E(9)$. Otherwise, $\{u_4, u_5, u_6\}$ is independent in H and since $\delta(H) \geq 5$, u_4 is adjacent to at least one of u_2 and u_3, and u_5 is adjacent to at least one of u_7 and u_8. Again, this gives a subgraph $T_E(9)$ in H.

For $n=8$, if $u_2, \ldots, u_5$ are independent in H, then they are each adjacent to u_0, u_1, u_6 and u_7 in H, which gives $T_E(8)$ in H. Otherwise, we can assume that u_2 is adjacent to u_3 in H. If u_4 is adjacent to u_5 in H, we will have $T_E(8)$ in H; otherwise, assume that u_4 is not adjacent to u_5. Now, suppose that u_4 is adjacent to u_2 or u_3 in H. If u_5 is adjacent to u_6 or u_7 in H, then H contains $T_E(8)$. Otherwise, u_5 must be adjacent to u_0, u_1, u_2 and u_3 since $\delta(H) \geq 4$. However, this also gives $T_E(8)$ in H. On the other hand, suppose that u_4 is adjacent to neither u_2 nor u_3 in H. Similarly, u_5 is not adjacent to u_2 or to u_3 in H. Since $\delta(H) \geq 4$, both u_4 and u_5 are adjacent to u_0, u_1, u_6 and u_7 in H, and this also gives $T_E(8)$ in H.

Suppose that H is $(n-4)$-regular and that $N_H(u_0) = \{u_1, \ldots, u_{n-4}\}$. By the Handshaking Lemma, this only happens when n is even.

Suppose that $n \geq 10$. Note that u_{n-3}, u_{n-2} and u_{n-1} are each adjacent to at least $n-6$ vertices of $N_H(u_0)$ in H. By the Inclusion-Exclusion Principle, at least one of $u_1, \ldots, u_{n-4}$ is adjacent to two of $u_{n-3}, u_{n-2}, u_{n-1}$ in H, say u_1 to u_{n-3} and u_{n-2}, and there must be another vertex, say u_2, that is adjacent to u_{n-1} in H. Now, if there is any edge in $H[\{u_3, \ldots, u_{n-4}\}]$, then $T_E(n) \subseteq H$, and this always happens for $n \geq 12$. For $n=10$, since $d_H(u_1) = 6$, u_1 is non-adjacent in H to at least one of $u_3, \ldots, u_6$, say u_3. Since $d_H(u_3) = 6$, u_3 is adjacent to one of u_4, u_5, u_6, giving $T_E(10)$ in H.

Now suppose that $n = 8$. If u_5, u_6 and u_7 are independent in H, then $H = K_{4,4}$. Otherwise, we can assume that u_5 is adjacent to u_6 in H. If u_5 is also adjacent to u_7 in H, then u_5 is adjacent in H to two vertices of $N_H(u_0)$, say u_1 and u_2. Suppose that u_6 is adjacent to u_1 or u_2, say u_1, in H. Since $d_H(u_6) = 4$, u_6 is also adjacent to at least one of u_2, u_3, u_4, u_7, so $T_E(8) \subseteq H$. Otherwise, suppose that neither u_6 nor u_7 is adjacent to u_1 or u_2 in H. Since H is a 4-regular graph, u_6 and u_7 are both adjacent to u_3 and u_4 in H, and u_1 is adjacent to at least one of u_3 and u_4 in H. This gives $T_E(8)$ in H. On the other hand, suppose that u_5 is not adjacent to u_7 in H. Then similarly, u_6 is not adjacent to u_7 in H, so u_7 is adjacent to u_1, u_2, u_3 and u_4 in H, and H contains $T_E(8)$. □

Theorem 5.3.11. *For $n \geq 8$,*

$$R(T_E(n), W_8) = \begin{cases} 2n-1 & \textit{if } n \geq 9\,; \\ 16 & \textit{if } n = 8\,. \end{cases}$$

Proof. Lemma 5.3.3 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$ if $n \geq 9$ and of order 16 if $n = 8$. Assume that G does not contain $T_E(n)$ and that $\overline{G}$ does not contain W_8.

By Theorem 5.2.12, G has a subgraph $T = S_n(3,1)$. Let

$$\begin{aligned} & V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\} \\ \text{and} \quad & E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_2w_2, v_3w_3\}\,. \end{aligned}$$

Set $V = \{v_4, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$. Then $|V| = n-7$ and $|U| \geq n-1$. Since $T_E(n) \nsubseteq G$, each of v_1, v_2, v_3 is not adjacent to any vertex of $V \cup U$ in G, and each vertex of V is adjacent to at most one vertex of U in G. Let W be a set of $n-2$ vertices of U that are not adjacent to v_4 in G. By Lemma 4.3.4, $G[W]$ is K_{n-2} or $K_{n-2} - e$. Since $T_E(n) \nsubseteq G$, every vertex of T is not adjacent to any vertex of W, and so $\delta(G[V(T)]) \geq n-4$ by Observation 4.3.2.

Now Lemma 5.3.10 implies that $G[V(T)]$ contains $T_E(n)$ if $n \geq 9$, which is a contradiction, and so we must have $n = 8$ and $G[V(T)] = K_{4,4}$. Observe now that $|U| = 8$, and as $T_E(8) \nsubseteq G$, no vertex of U is adjacent to any vertex of $G[V(T)]$. So again by Lemma 4.3.4, $G[U]$ is K_8 or $K_8 - e$, which clearly contains $T_E(8)$, a contradiction.

Therefore, $R(T_E(n), W_8) \leq 2n-1$ when $n \geq 9$ and $R(T_E(n), W_8) \leq 16$ when $n = 8$. This completes the proof of the theorem. □

Lemma 5.3.12. *Each graph H of order $n \geq 8$ with minimal degree at least $n-4$ contains $T_F(n)$ unless $n = 8$ and $H = K_{4,4}$.*

Proof. Let $V(H) = \{u_0, u_1 \ldots, u_{n-1}\}$ with $d(u_0) = \delta(H)$ and $V := \{u_1, \ldots, u_{n-4}\} \subseteq N(u_0)$. Set $U = \{u_{n-3}, u_{n-2}, u_{n-1}\}$. By the minimum degree condition, every vertex of U is adjacent to at least $n-6$ vertices of V. It is straightforward to see that some pair of vertices in U has a common neighbour in V, and moreover for $n \geq 9$, every pair of vertices in U has a common neighbour in V.

We assume without loss of generality that u_1 is adjacent to both u_{n-3} and u_{n-2}, and that u_2 is adjacent to u_{n-1}. If u_2 is adjacent to a vertex of $V \setminus \{u_1\}$, which is the case when $n \geq 10$, then H contains $T_F(n)$. We may assume now that $n \leq 9$ and that u_2 is not adjacent to any vertex of $V \setminus \{u_1\}$.

For the case when $n = 9$, we know u_{n-1} is adjacent to at least $n-6 = 3$ vertices of V, and so it is adjacent to another vertex, say to u_3. As above, we may assume that u_3 is not adjacent to any vertex of $V \setminus \{u_1\}$. By the minimum degree condition, each of u_2 and u_3 is adjacent to every vertex of $\{u_1\} \cup U$, giving $T_F(9)$ in H.

For the final case when $n = 8$, the minimum degree condition implies that u_2 is adjacent to at least two of u_1, u_5, u_6. If u_2 is adjacent to u_1, H contains $T_F(8)$. Thus, we are left with the case in which u_2 is not adjacent to u_1 but is adjacent to both u_5 and u_6. Exchanging the roles of u_1 and u_2, we may further assume that u_1 is adjacent to u_7 but not to any vertex of V. From the minimum degree condition on u_3 and u_4, it is easy to see that either H contains $T_F(8)$ or $H = K_{4,4}$. □

Theorem 5.3.13. *For $n \geq 8$,*

$$R(T_F(n), W_8) = \begin{cases} 2n-1 & \text{if } n \geq 9\,; \\ 16 & \text{if } n = 8\,. \end{cases}$$

Proof. Lemma 5.3.3 provides the lower bound, so it remains to prove the upper bound. Let G be a graph with no $T_F(n)$ subgraph whose complement $\overline{G}$ does not contain W_8. Suppose that $n = 8$ and that G has order 16. By Theorem 5.2.11, G has a subgraph $T = T_C(8)$. Let $V(T) = \{v_0, \ldots, v_4, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_4, v_1w_1, v_2w_2, v_2w_3\}$. Set $U = V(G)–V(T) = \{u_1, \ldots, u_8\}$; then $|U| = 8$. Since $T_F(8) \nsubseteq G$, v_1 is not adjacent in G to $\{v_2, v_3, v_4\} \cup U$, and $d_{G[U]}(v) \leq 1$ for $v = v_3, v_4, w_2, w_3$.

Suppose that v_1 is adjacent to w_2 or w_3, without loss of generality say w_2. Since $T_F(8) \nsubseteq G$, v_2 is not adjacent to $\{v_3, v_4\} \cup U$. If neither v_3 nor v_4 are adjacent to U, then by Lemma 4.3.4, $G[U]$ is K_8 or $K_8 - e$, so $G[U]$ contains $T_F(8)$, a contradiction. Suppose that only one of the vertices v_3 and v_4 is adjacent to U in G, say v_3. By Lemma 4.3.4, $G[U \setminus \{u_1\}]$ is K_7 or $K_7 - e$, and $G[V(T) \cup \{u_1\}]$ is not adjacent to $G[U \setminus \{u_1\}]$. By Observation 4.3.2, $\delta(G[V(T) \cup \{u_1\}]) \geq 5$, and by Lemma 5.3.12, $G[V(T) \cup \{u_1\}]$ contains $T_F(9)$ and hence $T_F(8)$, a contradiction. Suppose that both v_3 and v_4 are adjacent to U in G and assume that v_3 is adjacent to u_1 and that v_4 is adjacent to u_2. By Lemma 4.3.4, $G[U \setminus \{u_1, u_2\}]$ is K_6 or $K_6 - e$. At most one vertex from $G[V(T) \cup \{u_1, u_2\}]$ is adjacent to $G[U \setminus \{u_1, u_2\}]$ or else G will contain $T_F(8)$. Therefore, 9 vertices from $G[V(T) \cup \{u_1, u_2\}]$ form a vertex set W that is not adjacent to $U \setminus \{u_1, u_2\}$. By Observation 4.3.2, $\delta(G[W]) \geq 5$, and by Lemma 5.3.12, $G[W]$ contains $T_F(9)$ and hence $T_F(8)$, a contradiction.

Suppose then that v_1 is not adjacent to w_2 or w_3. Since $d_{G[U]}(v) \leq 1$ for $v = v_3, v_4, w_2, w_3$, there are 4 vertices from U that are not adjacent to $\{v_3, v_4, w_2, w_3\}$. These 8 vertices form C_8 in $\overline{G}$ and thus, with v_1 as hub, W_8, a contradiction.

Thus, $R(T_F(8), W_8) \leq 16$.

Now, suppose that $n \geq 9$ and that G has order $2n-1$. By Theorem 5.2.11, G has a subgraph $T = T_C(n)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, v_4, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_2w_2, v_2w_3\}$. Set $V = \{v_3, \ldots, v_{n-4}\}$ and $U = V(G) - V(T) = \{u_1, \ldots, u_{n-1}\}$; then $|V| = n-6$ and $|U| = n-1$. Since $T_F(n) \nsubseteq G$, v_1 is not adjacent in G to any vertex of $U \cup V$, and $d_{G[U]}(v) \leq 1$ for $v \in V$. Since $n \geq 10$, there are 4 vertices from U, 4 vertices from V and v_1 that form W_8 in $\overline{G}$, a contradiction. Thus, $R(T_F(n), W_8) \leq 2n-1$ for $n \geq 10$.

Suppose that $n = 9$ and let m be the number of vertices of U that are adjacent in G to at least one vertex of V. Since $d_{G[U]}(v) \leq 1$ for $v \in V$, $0 \leq m \leq 3$. If $m = 0$, then $G[U]$ is K_8 or $K_8 - e$ by Lemma 4.3.4, so $G[V(T)]$ is not adjacent to $G[U]$. By Observation 4.3.2, $\delta(G[V(T)]) \geq 5$, and $G[V(T)]$ contains $T_F(9)$ by Lemma 5.3.12, a contradiction. Suppose that $m = 1$. Assume without loss of generality that u_1 is adjacent to some vertex of V, and that $E_G(V, U \setminus \{u_1\}) = \emptyset$. By Lemma 4.3.4, $G[U \setminus \{u_1\}]$ is K_7 or $K_7 - e$, and at most one vertex from $G[V(T) \cup \{u_1\}]$ is adjacent to $G[U \setminus \{u_1\}]$ or else G contains $T_F(9)$. There are then 9 vertices from $G[V(T) \cup \{u_1\}]$ that form a vertex set W_1 that is not adjacent to $U \setminus \{u_1\}$. By Observation 4.3.2, $\delta(G[W_1]) \geq 5$, and $G[W_1]$ contains $T_F(9)$ by Lemma 5.3.12, a contradiction. Suppose that $m = 2$. Assume that u_1 and u_2 are adjacent to some vertices of V and that $E_G(V, U \setminus \{u_1, u_2\}) = \emptyset$. By Lemma 4.3.4, $G[U \setminus \{u_1, u_2\}]$ is K_6 or $K_6 - e$. If at least three vertices in $U \setminus \{u_1, u_2\}$ are adjacent to $V(T) \cup \{u_1\}$, then $T_F(9) \subseteq G$. If at most two vertices in $U \setminus \{u_1, u_2\}$ are adjacent to $V(T) \cup \{u_1\}$, then there are 4 vertices in $U \setminus \{u_1, u_2\}$ that are not adjacent to $V(T)$. Then by Observation 4.3.2, $\delta(G[V(T)]) \geq 5$, and $G[V(T)]$ contains $T_F(9)$ by Lemma 5.3.12, a contradiction. Suppose that $m = 3$. Assume that u_1, u_2, u_3 are each adjacent to some vertex of V and that $E_G(V, U \setminus \{u_1, u_2, u_3\}) = \emptyset$. Without loss of generality, assume that u_i is adjacent to v_{i+2} for $i = 1, 2, 3$. By Lemma 4.3.4, $G[U \setminus \{u_1, u_2, u_3\}]$ is K_5 or $K_5 - e$. Since $T_F(9) \nsubseteq G$, $\{v_1, v_3, v_4, v_5\}$ is independent and $V(T) \setminus \{w_1\}$ is not adjacent to $U \setminus \{u_1, u_2, u_3\}$. Then by Observation 4.3.2, $\delta(G[V(T) \setminus \{w_1\}]) \geq 4$, and v_1, v_3, v_4 and v_5 are each adjacent to v_2, w_2 and w_3 in G. This gives $T_F(9)$ in G. Therefore, $T_F(9) \leq 17 = 2n-1$. □

Theorem 5.3.14. *If $n \geq 8$, then $R(T_G(n), W_8) = 2n-1$.*

Proof. Lemma 5.3.3 provides the lower bound; it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $T_G(n)$ and that $\overline{G}$ does not contain W_8. By Theorem 5.2.12, G has a subgraph $T = S_n(3,1)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_2w_2, v_3w_3\}$. Set $V = \{v_4, v_5, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-7$ and $|U| = n-1$. Since $T_G(n) \nsubseteq G$, w_1, w_2, w_3 are not adjacent to $U \cup V$ in G, and v_1, v_2, v_3 are not adjacent to V.

Suppose that $n \geq 9$; then $|U| \geq 8$. If $\delta(\overline{G}[U]) \geq \frac{n-1}{2}$, then $\overline{G}[U]$ contains C_8 by Lemma 2.2.10 which, with w_2 as hub, forms W_8, a contradiction. Therefore, $\delta(\overline{G}[U]) < \frac{n-1}{2}$, and $\Delta(G[U \cup V]) \geq \frac{n-1}{2} \geq 4$. Therefore, some vertex $u \in U$ satisfies $|N_{G[U]}(u)| \geq 4$. Since $T_G(n) \nsubseteq G$, $N_{G[U]}(u)$ is not adjacent in G to $N_{G[V(T)]}(v_0)$. Hence, 4 vertices from $N_{G[U]}(u)$, v_1, v_2, v_3, w_1 and any vertex from V form W_8 in $\overline{G}$, a contradiction. Thus, $R(T_G(n), W_8) \leq 2n-1$ for $n \geq 9$.

Suppose that $n=8$ and let $U=\{u_1,\ldots,u_7\}$ and $W=\{v_4\}\cup U$. If $\delta(\overline{G}[W])\geq 4$, then $\overline{G}$ contains C_8 by Lemma 2.2.10 and thus W_8 with w_1 as hub, a contradiction. Hence, $\delta(\overline{G}[W]) \leq 3$, and $\Delta(G[W]) \geq 4$. Suppose that $d_{G[W]}(v_4) \geq 4$. Then without loss of generality, assume that $u_1,\ldots,u_4 \in N_G(v_4)$. Then $u_1,\ldots,u_4,w_1,w_2,w_3$ are independent and are not adjacent to u_5, u_6 or u_7, giving W_8, a contradiction. On the other hand, suppose that some vertex in U, say u_1, satisfies $d_{G[W]}(u_1) \geq 4$. Then v_4 is not adjacent to u_1; therefore, assume that $u_2,\ldots,u_5 \in N_G(u_1)$. Then $v_1,\ldots,v_4$ are not adjacent to $\{u_1,\ldots,u_5\}$, so $v_1u_1v_2u_2v_3u_3w_1u_4v_1$ and v_4 form W_8 in $\overline{G}$, a contradiction. Thus, $R(T_G(8),W_8) \leq 15$. □

Lemma 5.3.15. *Each graph H of order $n \geq 8$ with minimal degree at least $n-4$ contains $T_H(n)$, $T_K(n)$ and $T_L(n)$.*

Proof. Let $V(H) = \{u_0,\ldots,u_{n-1}\}$ where $u_1,\ldots,u_{n-4} \in N_H(u_0)$. Suppose that u_{n-3}, u_{n-2} or u_{n-1}, say u_{n-3}, is adjacent in H to the two others.

Since $\delta(H) \geq n-4$, u_{n-3} is adjacent to at least one of $u_1,\ldots,u_{n-4}$, say u_1. If u_1 is adjacent to another vertex in $\{u_2,\ldots,u_{n-4}\}$, then H contains $T_K(n)$. Note that this always happens for $n \geq 9$. Suppose that $n = 8$ and that u_1 is not adjacent to any of u_2,u_3,u_4. Then u_1 is adjacent to u_6 and u_7. Since $\delta(H) \geq n-4$, u_2 is adjacent to at least one of u_5,u_6,u_7, giving $T_K(n)$ in H.

Similarly, since $\delta(H) \geq n-4$, u_{n-2} is adjacent to at least $n-7$ vertices of $\{u_1,\ldots,u_{n-4}\}$. Suppose that u_{n-2} is adjacent to u_1. If $n \geq 10$, then at least two of $u_2,\ldots,u_{n-4}$ are adjacent, so H contains $T_H(n)$. If $n \geq 9$, then u_1 is adjacent to at least one of $u_2,\ldots,u_{n-4}$, so H contains $T_L(n)$. Now suppose that $n = 9$. If any of $u_2,\ldots,u_5$ are adjacent to each other, then H contains $T_H(9)$. Otherwise, $u_2,\ldots,u_5$ are each adjacent to u_6, u_7 and u_8, and so H contains $T_H(9)$. Finally, suppose that $n = 8$. If any two of u_2,u_3,u_4 are adjacent, then H contains $T_H(8)$; otherwise, they are each adjacent to u_6 or u_7. Now, if u_1 is adjacent to any of u_2,u_3,u_4, then H contains $T_H(8)$. Otherwise, $u_1,\ldots,u_4$ are each adjacent to u_5, u_6 and u_7, and H also contains $T_H(8)$. Furthermore, if u_1 is adjacent to u_2, u_3 or u_4, then H contains $T_L(8)$. If u_1 is not adjacent to u_2, u_3 or u_4, then u_6,u_7,u_8 are adjacent to u_2,u_3,u_4, and then H contains $T_L(8)$. Now if u_{n-2} is adjacent to some $u_2,\ldots,u_{n-4}$, say u_2, then similar arguments apply by interchanging u_1 and u_2.

Suppose now that none of u_{n-3},u_{n-2},u_{n-1} is adjacent to both of the others. Then one of these, say u_{n-3}, is adjacent to neither of the others. Since $\delta(H) \geq n-4$, u_{n-3} is adjacent to at least $n-5$ of the vertices $u_1,\ldots,u_{n-4}$. Without loss of generality, assume that $u_1,\ldots,u_{n-5} \in N_H(u_{n-3})$. Then u_{n-2} is adjacent to at least $n-7$ of the vertices $u_1,\ldots,u_{n-5}$ including, without loss of generality, the vertex u_1. Also, u_{n-1} is adjacent to at least one of $u_2,\ldots,u_{n-4}$, so H contains $T_H(n)$. If u_{n-2} is adjacent to u_{n-1}, then H also contains $T_L(n)$. If u_{n-2} is not adjacent to u_{n-1}, then u_{n-2} is adjacent to at least $n-6$ vertices of $u_1,\ldots,u_{n-5}$, so H contains $T_L(n)$. Now, suppose that $n \geq 9$. Then u_{n-2} and u_{n-1} are each adjacent to at least 3 of $u_1,\ldots,u_5$, and one of those vertices must be adjacent to both u_{n-2} and u_{n-1}; thus, H contains $T_K(n)$. Finally, suppose that $n = 8$. If u_6 and u_7 are each adjacent to at least two of the vertices u_1,u_2,u_3, then one of those vertices must be adjacent to both u_6 and u_7; thus, H contains $T_K(8)$. Otherwise, u_6 or u_7, say u_6, is non-adjacent to at least two of u_1,u_2,u_3, say u_1 and u_2. Then u_6 is adjacent to u_0, u_3, u_4 and u_7, and so H contains $T_K(8)$. □

Theorem 5.3.16. *If $n \geq 8$, then $R(T_H(n), W_8) = 2n - 1$.*

Proof. Lemma 5.3.3 provides the lower bound; it remains to prove the upper bound. Let G be any graph of order $2n-1$ and assume that G does not contain $T_H(n)$ and that $\overline{G}$ does not contain W_8. By Theorem 5.3.14, G has a subgraph $T = T_G(n)$. Let $V(T) = \{v_0,..., v_{n-5}, w_1,..., w_4\}$ and $E(T) = \{v_0v_1,..., v_0v_{n-5}, v_1w_1, v_2w_2, v_3w_3, w_3w_4\}$. Set $U = \{u_1, \ldots, u_{n-1}\} = V(G) - V(T)$; then $|U| = n - 1$. Since $T_G(n) \nsubseteq G$, $E_G(\{w_1, w_2\}, \{w_3, w_4\}) = \emptyset$ and w_4 is not adjacent to U. Now, let $W = \{w_1\} \cup U$; then $|W| = n$. If $\delta(\overline{G}[W]) \geq \frac{n}{2}$, then $\overline{G}[W]$ contains C_8 by Lemma 2.2.10 which, with w_4 as hub, forms W_8, a contradiction. It follows that $\delta(\overline{G}[W]) < \frac{n}{2}$, and $\Delta(G[W]) \geq \lfloor \frac{n}{2} \rfloor \geq 4$.

First, suppose that w_1 is a vertex with degree at least $\frac{n}{2}$ in $G[W]$. Assume without loss of generality that $u_1, \ldots, u_4 \in N_{G[W]}(w_1)$. Since $T_H(n) \nsubseteq G$, $u_1, \ldots, u_4$ are independent and are not adjacent to $\{w_2, u_5, \ldots, u_{n-1}\}$ in G. Then $w_2, u_1, \ldots, u_4, w_4$ and any 3 vertices from $\{u_5, \ldots, u_{n-1}\}$ form W_8 in $\overline{G}$, a contradiction.

Hence, $d_{G[W]}(u') \geq \frac{n}{2}$ for some vertex $u' \in U$, say $u' = u_1$. Note that w_1 is not adjacent to u_1, or else G contains $T_H(n)$. Without loss of generality, suppose that $u_2, \ldots, u_5 \in N_{G[W]}(u_1)$. Since $T_H(n) \nsubseteq G$, $u_2, \ldots, u_5$ are not adjacent to $V(T) \setminus \{v_0\}$ in G. Now, if v_0 is not adjacent to $\{u_2, \ldots, u_5\}$ in G, then by Observation 4.3.2, $\delta(G[V(T)]) \geq n - 4$, or else $\overline{G}$ contains W_8. By Lemma 5.3.15, $G[V(T)]$ contains $T_H(n)$, a contradiction. On the other hand, suppose that v_0 is adjacent to at least one of $u_2, \ldots, u_5$, say u_2. Then u_3, u_4, u_5 are independent in G and are not adjacent to u_6 and u_7 in G. Furthermore, w_4 is not adjacent to v_1 or v_2. Then $v_1u_3v_2u_4u_6w_1u_7u_5v_1$ and w_4 form W_8 in $\overline{G}$, a contradiction.

Thus, $R(T_H(n), W_8) \leq 2n - 1$. □

Theorem 5.3.17. *If $n \geq 8$, then $R(T_J(n), W_8) = 2n - 1$.*

Proof. Lemma 5.3.3 provides the lower bound; it remains to prove the upper bound. Let G be any graph of order $2n-1$ and assume that G does not contain $T_J(n)$ and that $\overline{G}$ does not contain W_8. By Theorem 5.2.11, G has a subgraph $T = T_C(n)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2, v_2w_3\}$. Set $V = \{v_3, \ldots, v_{n-4}\}$ and $U = V(G) - V(T) = \{u_1, \ldots, u_{n-1}\}$. Since $T_J(n) \nsubseteq G$, neither w_1 nor w_2 is adjacent in G to any vertex from $U \cup V$.

Let $W = \{v_3\} \cup U$; then $|W| = n$. If $\delta(\overline{G}[W]) \geq \lceil \frac{n}{2} \rceil \geq \frac{n}{2}$, then $\overline{G}[W]$ contains C_8 by Lemma 2.2.10 which with w_1 forms W_8, a contradiction. Thus, $\delta(\overline{G}[W]) < \lceil \frac{n}{2} \rceil$, and $\Delta(G[W]) \geq \lfloor \frac{n}{2} \rfloor \geq 4$.

Suppose that $d_{G[W]}(v_3) \geq \lfloor \frac{n}{2} \rfloor \geq 4$. Without loss of generality, assume that $u_1, \ldots, u_4 \in N_G(v_3)$. Since $T_J(n) \nsubseteq G$, $u_1, \ldots, u_4$ is independent in G and is not adjacent to any remaining vertices from U in G. Then $u_2w_1u_3u_5u_4u_6w_2u_7u_2$ and u_1 form W_8 in $\overline{G}$, a contradiction. Hence, there is a vertex in U, say u_1, such that $d_{G[W]}(u_1) \geq \lfloor \frac{n}{2} \rfloor \geq 4$.

Now, suppose that v_3 is adjacent to u_1 in $G[W]$. Then u_1 is adjacent to at least 3 other vertices of U in G, say u_2, u_3 and u_4. Since $T_J(n) \nsubseteq G$, v_3 is not adjacent to $v_1, v_2, v_4, \ldots, v_{n-4}, w_1, w_2, w_3, u_2, u_3, u_4$ and neither v_1 nor v_2 is adjacent to u_2, u_3 or u_4 in G. Then $v_2u_2v_1u_3w_1v_4w_2u_4v_2$ and v_3 form W_8 in $\overline{G}$, a contradiction.

Thus, v_3 is not adjacent to u_1 in G. Note that u_1 is not adjacent to any other vertices of V in G or else previous arguments apply. Similarly, v_0 is not adjacent to $N_{G[W]}(u_1)$ in G. Since $T_J(n) \nsubseteq G$, neither v_1 nor v_2 is adjacent to u_1 or $N_{G[W]}(u_1)$ in G, and so $d_{N_{G[W]}(u_1)}(v) \leq 1$ for all $v \in V$.

Suppose that $n \geq 10$; then $|V| \geq 4$ and $|N_{G[W]}(u_1)| \geq 5$. If $d_{G[V]}(u) \leq 2$ for each $u \in N_{G[W]}(u_1)$, then $\overline{G}[V \cup N_{G[W]}(u_1)]$ contains C_8 by Lemma 4.3.5 which, with w_1 as hub, forms W_8 in $\overline{G}$, a contradiction. Thus, $d_V(u') \geq 3$ for some vertex $u' \in N_{G[W]}(u_1)$. Then any 4 vertices from V, of which at least 3 are in $N_{G[V]}(u')$, and any 4 vertices from $N_{G[W]}(u_1) \setminus \{u'\}$ satisfy the condition in Lemma 4.3.5, so $\overline{G}[V \cup N_{G[W]}(u_1)]$ contains C_8 which with w_1 forms W_8, a contradiction.

Suppose that $n = 9$; then $V = \{v_3, v_4, v_5\}$. Assume that $u_2, \ldots, u_5 \in N_{G[W]}(u_1)$. Suppose that w_1 is not adjacent to w_2 in G. Let $X = \{v_3, v_4, v_5, w_2\}$ and $Y = \{u_2, \ldots, u_5\}$ and note that $d_{G[Y]}(x) \leq 1$ for each $x \in X$. If $d_{G[X]}(y) \leq 2$ for each $y \in Y$, then $\overline{G}[X \cup Y]$ contains C_8 by Lemma 4.3.5 which, with w_1 as hub, forms W_8, a contradiction. Thus, $d_{G[X]}(u') \geq 3$ for some $u' \in Y$, say $u' = u_2$, so X is not adjacent to $Y \setminus \{u_2\}$. Hence, $v_3u_1v_4u_3v_5u_4w_2u_5v_3$ and w_1 form W_8 in $\overline{G}$, a contradiction.

Thus, w_1 is adjacent to w_2 in G. Then v_1 is not adjacent to $\{v_3, v_4, v_5\} \cup U$ and suppose that v_1 is not adjacent to v_2. Set $X = \{v_2, \ldots, v_5\}$ and $Y = \{u_2, \ldots, u_5\}$. If $d_{G[X]}(y) \leq 2$ for each $y \in Y$, then $\overline{G}[X \cup Y]$ contains C_8 by Lemma 4.3.5 which, with v_1 as hub, forms W_8, a contradiction. Thus, $d_{G[X]}(u') \geq 3$ for some $u' \in Y$, say $u' = u_2$, so X is not adjacent to $Y \setminus \{u_2\}$, and $v_2u_1v_3u_3v_4u_4v_5u_5v_2$ and v_1 form W_8 in $\overline{G}$, a contradiction. Thus, v_1 is adjacent to v_2 in G. Then V is independent and is not adjacent to U in G. Since $W_8 \nsubseteq \overline{G}$, $G[U]$ is K_{n-1} or $K_{n-1} - e$ by Lemma 4.3.4. Since $T_J(9) \nsubseteq G$, T is not adjacent to U and, by Observation 4.3.2, $\delta(G[V(T)]) \geq 5$. However, this cannot be since V is independent and is not adjacent to v_1, w_1 or w_2.

Finally, suppose that $n = 8$; then $V = \{v_3, v_4\}$. Assume that $u_2, \ldots, u_5 \in N_{G[W]}(u_1)$. If v_3 is adjacent to any vertex of $\{u_2, \ldots, u_5\}$, say u_2, then v_3 is not adjacent to $\{v_1, v_2, v_4, w_3\} \cup U \setminus \{u_2\}$, so $v_1u_1v_2u_3w_1u_4w_2u_5v_1$ and v_3 form W_8 in $\overline{G}$, a contradiction. Thus, v_3 is not adjacent to $\{u_2, \ldots, u_5\}$. Similarly, v_4 is not adjacent to $\{u_2, \ldots, u_5\}$. Now, if w_3 is adjacent to any of the vertices $u_2, \ldots, u_5$, say u_2, then v_2 is not adjacent to $\{w_1, w_2, v_3, v_4\}$, so $v_3u_1v_4u_2w_1u_3w_2u_4v_3$ and v_2 form W_8 in $\overline{G}$, a contradiction. Thus, w_3 is not adjacent to $\{u_2, \ldots, u_5\}$. By Observation 4.3.2, $\delta(G[V(T)]) \geq 4$. Suppose that v_2 is adjacent to w_1. Since $T_J(8) \nsubseteq G$, neither v_3 nor v_4 is adjacent to w_3. Since $\delta(G[V(T)]) \geq 4$, v_3 and v_4 are adjacent to v_1 and v_2, and $\{w_1, w_2, w_3\}$ is not independent. However, then $T_J(8) \subseteq G[V(T)]$, a contradiction. Thus, v_2 is not adjacent to w_1 and, similarly, v_2 is not adjacent to w_2. Since $\delta(G[V(T)]) \geq 4$, w_1 and w_2 are adjacent to each other and to w_3. Since $T_J(8) \nsubseteq G$, neither v_3 nor v_4 is adjacent to v_1 or v_2; however, this contradicts $\delta(G[V(T)]) \geq 4$.

In each case, $R(T_J(8), W_8) \leq 2n - 1$ which completes the proof of the theorem. □

Theorem 5.3.18. *If $n \geq 8$, then $R(T_K(n), W_8) = 2n - 1$.*

Proof. Lemma 5.3.3 provides the lower bound, so it remains to prove the upper bound. Let G be a graph of order $2n - 1$ and assume that G does not contain $T_K(n)$ and that $\overline{G}$ does not contain W_8.

Suppose that $n \not\equiv 0 \pmod 4$. By Theorem 5.2.8, G has a subgraph $T = S_n(1,3)$. Let $V(T) = \{v_0,\dots,v_{n-4},w_1,w_2,w_3\}$ and $E(T) = \{v_0v_1,\dots,v_0v_{n-4},v_1w_1,w_1w_2,w_2w_3\}$. Set $V = \{v_2,\dots,v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n-1$. Since $T_K(n) \nsubseteq G$, w_2 is not adjacent in G to any vertex of $U \cup V$. Now, if $\delta(G[U]) \geq \frac{n-1}{2}$, then $\overline{G}[U]$ contains C_8 by Lemma 2.2.10 which, with v_1 as hub, forms W_8, a contradiction. Therefore, $\delta(\overline{G}[U]) < \frac{n-1}{2}$, and $\Delta(G[U]) \geq \lfloor\frac{n-1}{2}\rfloor$. Let $U = \{u_1,\dots,u_{n-1}\}$ and assume without loss of generality that $d_{G[U]}(u_1) \geq \lfloor\frac{n-1}{2}\rfloor \geq 4$. Since $T_K(n) \nsubseteq G$, $E_G(V, N_{G[U]}(u_1)) = \emptyset$, so any 4 vertices from V, any 4 vertices from $N_{G[U]}(u_1)$ and w_2 form W_8 in $\overline{G}$, a contradiction. Therefore, $R(T_K(n), W_8) \leq 2n-1$ for $n \not\equiv 0 \pmod 4$.

Let $n = 8$. By Theorem 5.3.16, G has a subgraph $T = T_H(8)$. Let $V(T) = \{v_0,\dots,v_3,w_1,\dots,w_4\}$ and $E(T) = \{v_0v_1,v_0v_2,v_0v_3,v_1w_1,w_1w_2,w_2w_3,v_2w_4\}$. Set $U = V(G)-V(T) = \{u_1,\dots,u_7\}$; then $|U| = 7$. Since $T_K(8) \nsubseteq G$, w_2 is not adjacent to $\{w_4\}\cup U$. Let $W = \{w_4\}\cup U$; then $|W| = 8$. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 2.2.10 which, with w_2 as hub, forms W_8, a contradiction. Therefore, $\delta(\overline{G}[W]) < 3$, and $\Delta(G[W]) \geq 4$.

Now, suppose that $d_{G[W]}(w_4) \geq 4$ and assume without loss of generality that w_4 is adjacent to u_1, u_2, u_3 and u_4. Then v_1 is not adjacent to $\{v_3,w_2,w_3\}\cup U$ and neither v_2 nor v_3 is adjacent to $\{u_1,\dots,u_4\}$, since $T_K(8) \nsubseteq G$. Now, suppose that $E_G(\{u_1,\dots,u_4\},\{u_5,u_6,u_7\}) \neq \emptyset$ and assume that u_1 is adjacent to u_5. Then u_1 is not adjacent to $\{w_1,w_2,w_3,u_2,\dots,u_7\}$ in G, and $v_1u_2v_2u_3v_3u_4w_2u_6v_1$ and u_1 form W_8 in $\overline{G}$, a contradiction. Thus, $E_G(\{u_1,\dots,u_4\},\{u_5,u_6,u_7\}) = \emptyset$, so $u_1u_5u_2u_6u_3u_7u_4v_3u_1$ and v_1 form W_8 in $\overline{G}$, a contradiction.

Now suppose that $d_{G[W]}(u') \geq 4$ for some vertex $u' \in U$, say $u' = u_1$. Since, $T_K(8) \nsubseteq G$, w_4 is not adjacent to u_1. Then without loss of generality, suppose that $u_2,\dots,u_5 \in N_G(u_1)$. Since $T_K(8) \nsubseteq G$, $E_G(\{v_1,v_2,v_3\},\{u_2,\dots,u_5\}) = \emptyset$. If u_2 is adjacent to w_1, then u_2 is not adjacent to $\{u_3,\dots,u_7\}$ and v_1 is not adjacent to u_6. Then $w_2u_3v_2u_4v_3u_5v_1u_6w_2$ and u_2 form W_8 in $\overline{G}$, a contradiction. Thus, u_2 is not adjacent to w_1. Similarly, u_3, u_4 and u_5 are not adjacent to w_1. If u_2 is adjacent to v_0, then v_2 is not adjacent to $\{v_1,v_3,w_1,w_2,w_3,u_2,\dots,u_7\}$, and $v_1u_2v_3u_3w_1u_4w_2u_5v_1$ and v_2 form W_8 in $\overline{G}$, a contradiction. Thus, u_2 is not adjacent to v_0. Similarly, u_3, u_4 and u_5 are not adjacent to v_0. By similar arguments, u_3, u_4 and u_5 are not adjacent to w_3 or w_4.

Hence, $u_2,\dots,u_5$ are not adjacent to $V(T)$ in G, so $\delta(G[V(T)]) \geq 4$ by Observation 4.3.2. By Lemma 5.3.15, $G[V(T)]$ contains $T_K(8)$, a contradiction. Thus, $R(T_K(8), W_8) \leq 15$.

Now suppose that $n \equiv 0 \pmod 4$ and that $n \geq 12$. If G has an $S_n(1,3)$ subgraph, then the arguments above lead to contradictions. Thus, G does not contain $S_n(1,3)$ as a subgraph. Now, by Theorem 5.3.16, G has a subgraph $T = T_H(n)$. Let $V(T) = \{v_0,\dots,v_{n-5},w_1,\dots,w_4\}$ and $E(T) = \{v_0v_1,\dots,v_0v_{n-5},v_1w_1,w_1w_2,w_2w_3,v_2w_4\}$. Set $V = \{v_3,\dots,v_{n-5}\}$ and let $U = V(G) - V(T) = \{u_1,\dots,u_{n-1}\}$. Then $|V| = n-7$ and $|U| = n-1$. Since $T_K(n) \nsubseteq G$, w_2 is not adjacent in G to $\{w_4\}\cup U$. Since $S_n(1,3) \nsubseteq G$, v_0 is not adjacent to $\{w_4\}\cup U$.

If $\delta(\overline{G}[U]) \geq \frac{n-1}{2}$, then $\overline{G}[U]$ contains C_8 by Lemma 2.2.10 which, with w_2, forms W_8, a contradiction. Thus, $\delta(\overline{G}[U]) < \frac{n-1}{2}$, and $\Delta(G[U]) \geq \lfloor\frac{n-1}{2}\rfloor \geq 5$. Without loss of generality, assume that $u_2, \ldots, u_6 \in N_G(u_1)$. Since $T_K(n) \nsubseteq G$, v_1, v_2 and V are not adjacent to $\{u_2, \ldots, u_6\}$, and w_1 and w_2 are not adjacent to u_1.

Now, if u_2 is adjacent to w_1, then u_2 is not adjacent to $\{w_3, w_4\} \cup U \setminus \{u_1\}$, since $T_K(n) \nsubseteq G$, so $v_0u_3v_1u_4v_2u_5v_3u_6v_0$ and u_2 form W_8 in $\overline{G}$, a contradiction. Thus, u_2 is not adjacent to w_1. Similarly, $u_3, \ldots, u_6$ are not adjacent to w_1. If u_2 is adjacent to w_3 in G, then v_0 is not adjacent to w_1, w_2, w_3, and $d_{G[U\setminus\{u_1,u_2\}]}(u_i) \leq n-6$ for $i = 3, \ldots, 6$, since $S_n(1,3) \nsubseteq G$. Since $T_K(n) \nsubseteq G$, w_3 is not adjacent to w_1 or w_4. Since $d_{G[U\setminus\{u_1,u_2\}]}(u_3) \leq n-6$ and $d_{G[U\setminus\{u_1,u_2\}]}(u_4) \leq n-6$, u_3 and u_4 are adjacent in $\overline{G}$ to at least 2 vertices in $\{u_7, \ldots, u_{n-1}\}$. Without loss of generality, assume that u_3 is adjacent in $\overline{G}$ to u_7 and that u_4 is adjacent to u_8. Then $u_3u_7w_2u_8u_4w_1w_3w_4u_3$ and v_0 form W_8 in $\overline{G}$, a contradiction. Thus, u_2 is not adjacent to w_3. Similarly, $u_3, \ldots, u_6$ are not adjacent to w_4.

Thus, $u_2, \ldots, u_6$ are not adjacent to $V(T)$. By Observation 4.3.2, $\delta(G[V(T)]) \geq 4$, so $G[V(T)]$ contains $T_K(n)$ by Lemma 5.3.15, a contradiction.

Hence, $R(T_K(n), W_8) \leq 2n-1$ for $n \equiv 0 \pmod 4$. This completes the proof. □

Theorem 5.3.19. *If $n \geq 8$, then $R(T_L(n), W_8) = 2n-1$.*

Proof. Lemma 5.3.3 provides the lower bound, so it remains to prove the upper bound. Let G be a graph with no $T_L(n)$ subgraph whose complement $\overline{G}$ does not contain W_8. Suppose that $n \not\equiv 0 \pmod 4$ and that G has order $2n-1$. By Theorem 5.2.8, G has a subgraph $T = S_n(1,3)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, w_1w_2, w_2w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n-1$. Since $T_L(n) \nsubseteq G$, v_1 is not adjacent to $U \cup V$, and $d_{G[U]}(v_i) \leq n-7$ for each $v_i \in V$. Now, if $\delta(G[U]) \geq \frac{n-1}{2}$, then $\overline{G}[U]$ contains C_8 by Lemma 2.2.10 which, with v_1, forms W_8, a contradiction. Thus, $\delta(\overline{G}[U]) < \frac{n-1}{2}$, and $\Delta(G[U]) \geq \lfloor \frac{n-1}{2} \rfloor$.

Let $U = \{u_1, \ldots, u_{n-1}\}$ and without loss of generality assume that $d_{G[U]}(u_1) \geq \lfloor \frac{n-1}{2} \rfloor \geq 4$ and that $u_2, \ldots, u_5 \in N_{G[U]}(u_1)$. Now if $E_G(V, N_{G[U]}(u_1)) = \emptyset$, then 4 vertices from V, 4 vertices from $N_{G[U]}(u_1)$ and v_1 form W_8 in $\overline{G}$, a contradiction. Thus, $E_G(V, N_{G[U]}(u_1)) \neq \emptyset$. Assume without loss of generality that v_2 is adjacent to u_2. Since $T_L(n) \nsubseteq G$, v_2 is not adjacent to $U \setminus \{u_1, u_2\}$. Since $d_{G[U]}(v_i) \leq n-7$ for each $v_i \in V$, v_5 is non-adjacent to at least one of $u_6, \ldots, u_{n-1}$, say u_6. Now if $E_G(\{v_3, v_4, v_5\}, \{u_3, u_4, u_5\}) = \emptyset$, then $v_2u_3v_3u_4v_4u_5v_5u_6v_2$ and v_1 form W_8 in $\overline{G}$, a contradiction. Thus assume, say, that v_3 is adjacent to u_3 in G; then v_3 is not adjacent to $U \setminus \{u_1, u_3\}$. Again, if $E_G(\{v_4, v_5\}, \{u_4, u_5\}) = \emptyset$, then $v_2u_7v_3u_4v_4u_5v_5u_6v_2$ and v_1 form W_8 in $\overline{G}$, a contradiction. Thus assume, say, that v_4 is adjacent to u_4, then v_4 is not adjacent to $U \setminus \{u_1, u_4\}$. If v_5 is not adjacent to u_5, then $v_2u_7v_3u_2v_4u_5v_5u_6v_2$ and v_1 form W_8 in $\overline{G}$, a contradiction. Thus, v_5 is adjacent to u_5, so v_5 is not adjacent to $U \setminus \{u_1, u_5\}$, and $v_2u_7v_3u_2v_4u_3v_5u_6v_2$ and v_1 form W_8 in $\overline{G}$, a contradiction.

Hence, $R(T_L(n), W_8) \leq 2n-1$ for $n \not\equiv 0 \pmod 4$.

Now, suppose that $n \equiv 0 \pmod 4$ and that G has order $2n-1$. Suppose first that $n = 8$. By Theorem 5.3.16, G has a subgraph $T = T_H(8)$. Let $V(T) = \{v_0, \ldots, v_3, w_1, \ldots, w_4\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_3, v_1w_1, w_1w_2, w_2w_3, v_2w_4\}$. Set $U = V(G) - V(T) = \{u_1, \ldots, u_7\}$; then $|U| = 7$. Since $T_L(8) \nsubseteq G$, neither v_1 nor v_2 are adjacent to U, and $d_{G[U]}(v_3) \leq 1$. Furthermore, v_1 is not adjacent to w_4, and

v_2 is not adjacent to w_1 or w_3. Let $W = w_4 \cup U$; then $|W| = 8$. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 2.2.10 which, with v_1, forms W_8, a contradiction. Thus, $\delta(\overline{G}[W]) < 3$ and $\Delta(G[W]) \geq 4$.

Now, suppose that $d_{G[W]}(w_4) \geq 4$ and assume without loss of generality that $u_1, \ldots, u_4 \in N_G(w_4)$. Then v_2 is not adjacent to v_1, v_3, w_1, w_2 and $d_{G[U]}(u_i) \leq 1$ for $1 \leq i \leq 4$, or else $T_L(8) \subseteq G$, a contradiction. Since $d_{G[U]}(v_3) \leq 1$, assume without loss of generality that v_3 is not adjacent to u_3 or u_4. Now, suppose that $E_G(\{u_1, \ldots, u_4\}, \{u_5, u_6, u_7\}) \neq \emptyset$ and assume, say, that u_1 is adjacent to u_5. Then u_1 is not adjacent to $\{v_3, w_1, w_2, w_3, u_2, \ldots, u_7\}$. Since $T_L(8) \nsubseteq G$, at least one of w_1 and w_2 is adjacent in $\overline{G}$ to u_2, u_3 and u_4, say w_1, so $v_1u_2w_1u_3v_3u_4v_2u_6v_1$ and u_1 form W_8 in $\overline{G}$, a contradiction. Thus, $E_G(\{u_1, \ldots, u_4\}, \{u_5, u_6, u_7\}) = \emptyset$. Then $u_1u_5u_2u_6u_3u_7u_4v_2u_1$ and v_1 form W_8 in $\overline{G}$, a contradiction. Therefore, $d_{G[W]}(u') \geq 4$ for some vertex of $u' \in U$, say $u' = u_1$.

Suppose that w_4 is adjacent to u_1. Then without loss of generality, we assume that u_1 is adjacent to u_2, u_3 and u_4. Since $T_L(8) \nsubseteq G$, neither v_0 nor w_4 is adjacent to w_1 or w_2, and w_4 is not adjacent to $\{v_1, v_3\} \cup U \setminus \{u_1\}$. If $E_G(\{u_2, u_3, u_4\}, \{u_5, u_6, u_7\}) \neq \emptyset$, Then say, u_2 is adjacent to u_5 and is thus not adjacent to $\{v_0, v_3, w_1, w_2, w_3, u_3, u_4, u_6, u_7\}$, so $w_1v_0w_2w_4u_3v_1u_4v_2w_1$ and u_2 form W_8 in $\overline{G}$, a contradiction. Thus $E_G(\{u_1, \ldots, u_4\}, \{u_5, u_6, u_7\} = \emptyset$. Let $X = \{v_1, u_2, u_3, u_4\}$ and $Y = \{v_3, u_5, u_6, u_7\}$. Since $d_{G[U]}(v_3) \leq 1$, $\overline{G}[X \cup Y]$ contains C_8 by Lemma 4.3.5 which, with w_4, forms W_8, a contradiction.

Thus, u_1 is not adjacent to w_4 so we can assume without loss of generality that $u_2, \ldots, u_5 \in N_G(u_1)$. Since G does not contain $T_L(8)$, $d_{G[V(T)]}(u_i) \leq 1$ for $2 \leq i \leq 5$. If u_2 is adjacent to w_4, then u_2 is not adjacent to $V(G) \setminus \{u_1, w_4\}$ in G. Since $d_{G[U]}(v_3) \leq 1$, that v_3 is not adjacent to, say, u_3 or u_4. Since $d_{G[V(T)]}(u_i) \leq 1$ for $2 \leq i \leq 5$, u_4 and u_5 are each adjacent in $\overline{G}$ to at least 2 of w_1, w_2, w_3, so some $w_i \in \{w_1, w_2, w_3\}$ is adjacent in $\overline{G}$ to both u_4 and u_5. Therefore, $u_3v_3u_4w_iu_5v_2u_6v_1u_3$ and u_2 form W_8 in $\overline{G}$, a contradiction. Thus, u_2 is not adjacent to w_4. Similarly, u_3, u_4, u_5 are not adjacent to w_4. Similar arguments show that $u_2, \ldots, u_5$ are not adjacent to w_1 or w_2.

Now, if u_2 is adjacent to any other vertex of $V(T)$, then u_2 is not adjacent to $\{u_3, u_4, u_5\}$, so $u_3w_1u_4w_4u_5v_2u_6v_1u_3$ and u_2 form W_8 in $\overline{G}$, a contradiction. Hence, u_2 is not adjacent to $V(T)$ and, similarly, u_3, u_4, u_5 are not adjacent to $V(T)$. Therefore, by Observation 4.3.2, $\delta(G[V(T)]) \geq 4$. By Lemma 5.3.15, $G[V(T)]$ contains $T_L(8)$, a contradiction. Thus, $R(T_L(8), W_8) \leq 15$.

Now suppose that $n \geq 12$. If G contains $S_n(1, 3)$, then the previous arguments above lead to contradictions. Thus, G does not contain $S_n(1, 3)$. By Theorem 5.2.11, G has a subgraph $T = T_C(n)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_2w_2, v_2w_3\}$. Set $U = V(G) - V(T) = \{u_1, \ldots, u_{n-1}\}$; then $|U| = n - 1$.

Suppose that w_2 is not adjacent to U. If $\delta(\overline{G}[U]) \geq \frac{n-1}{2}$, then G contains C_8 by Lemma 2.2.10 and, with w_2 as hub, forms W_8, a contradiction. Therefore, $\delta(\overline{G}[U]) < \frac{n-1}{2}$ and so $\Delta(G[U]) \geq \lfloor \frac{n-1}{2} \rfloor \geq 5$. Without loss of generality, assume that $u_2, \ldots, u_6 \in N_G(u_1)$. Since $S_n(1, 3) \nsubseteq G$, $u_2, \ldots, u_6$ are not adjacent to $V(T) \setminus \{v_0\}$. If u_2 is adjacent to v_0, then since $S_n(1, 3) \nsubseteq G$, $u_3, \ldots, u_6$ are not adjacent to $\{u_7, \ldots, u_{n-1}\}$, so $u_3u_7u_4u_8u_5u_9u_6u_{10}u_3$ and w_2 form W_8 in $\overline{G}$, a contradiction. Thus, u_2 is not adjacent to v_0 and, similarly, $u_3, \ldots, u_6$ are also not adjacent to

v_0. Hence, $u_2, \ldots, u_6$ are not adjacent to $V(T)$. Therefore, by Observation 4.3.2, $\delta(G[V(T)]) \geq n-4$, so $G[V(T)]$ contains $T_L(n)$ by Lemma 5.3.15, a contradiction.

Thus some vertex of U, say u_{n-1}, is adjacent to w_2. Set $U' = U \setminus \{u_{n-1}\}$; then $|U'| = n-2$. Since $T_L(n) \nsubseteq G$, u_{n-1} is not adjacent to U' in G. Now, if $\delta(\overline{G}[U']) \geq \frac{n-2}{2}$, then $\overline{G}[U']$ contains C_8 by Lemma 2.2.10 which, with u_{n-1}, forms W_8, a contradiction. Thus, $\delta(\overline{G}[U']) \leq \frac{n-2}{2} - 1$, and $\Delta(G[U']) \geq \frac{n-2}{2} \geq 5$. Without loss of generality, assume that $u_2, \ldots, u_6 \in N_G(u_1)$ and repeat the above arguments to prove that $u_2, \ldots, u_6$ are not adjacent to $V(T)$. Therefore, $\delta(G[V(T)]) \geq n-4$ by Observation 4.3.2, so $G[V(T)]$ contains $T_L(n)$ by Lemma 5.3.15, a contradiction.

Thus, $R(T_L(n), W_8) \leq 2n-1$ for $n \equiv 0 \pmod 4$ which completes the proof. □

Theorem 5.3.20. *If $n \geq 9$, then $R(T_M(n), W_8) = 2n-1$.*

Proof. Lemma 5.3.3 provides the lower bound; it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $T_M(n)$ and that $\overline{G}$ does not contain W_8. By Theorem 5.2.5, G has a subgraph $T = S_n(4)$. Now, let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2, v_1w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T) = \{u_1, \ldots, u_{n-1}\}$; then $|V| = n-5$ and $|U| = n-1$. Since $T_M(n) \nsubseteq G$, w_1, w_2 and w_3 are not adjacent to any vertex of $U \cup V$ in G.

Now, suppose that some vertex in V is adjacent to at least 4 vertices of U in G, say v_2 to $u_1, \ldots, u_4$. Then $u_1, \ldots, u_4$ are not adjacent to other vertices in U. Then $u_1u_5u_2u_6u_3u_7u_4u_8u_1$ and w_1 form W_8 in $\overline{G}$, a contradiction. Therefore, each vertex in V is adjacent to at most three vertices of U in G. Choose any 8 vertices of U. By Corollary 5.3.2, $\overline{G}[U \cup V]$ contains C_8 which together with w_1 gives W_8 in $\overline{G}$, a contradiction.

Thus, $R(T_M(n), W_8) \leq 2n-1$ for $n \geq 9$. This completes the proof. □

Theorem 5.3.21. *If $n \geq 9$, then*

$$R(T_N(n), W_8) = \begin{cases} 2n-1 & \textit{if } n \not\equiv 0 \pmod 4; \\ 2n & \textit{otherwise}. \end{cases}$$

Proof. Lemma 5.3.3 provides the lower bound; it remains to prove the upper bound. Let G be any graph of order $2n$ if $n \equiv 0 \pmod 4$ and of order $2n-1$ if $n \not\equiv 0 \pmod 4$. Assume that G does not contain $T_N(n)$ and that $\overline{G}$ does not contain W_8. By Theorem 5.2.9, G has a subgraph $T = T_A(n)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2, w_1w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T) = \{u_1, \ldots, u_j\}$, where $j = n-1$ if $n \not\equiv 0 \pmod 4$ and $j = n$ otherwise. Since $T_N(n) \nsubseteq G$, w_2 is not adjacent to $U \cup V$ in G. If each $v_i \in V$ is adjacent to at most three vertices of U in G, then by Corollary 5.3.2, $\overline{G}[U \cup V]$ contains C_8 which with w_2 gives W_8 in $\overline{G}$, a contradiction. Therefore, some vertex in V, say v_2, is adjacent to at least four vertices of U in G, say $u_1, \ldots, u_4$. If none of these is adjacent to other vertices of U in G, then $u_1u_5u_2u_6u_3u_7u_4u_8u_1$ and w_2 form W_8 in $\overline{G}$, a contradiction.

Therefore, assume that u_1 is adjacent to u_5 in G. Since $T_N(n) \nsubseteq G$, u_2, u_3, u_4 are not adjacent to $\{u_6, \ldots, u_j\}$ in G. For $n = 9$ and $n = 10$, $\{v_3, \ldots, v_{n-4}\}$ is not adjacent to $\{u_5, \ldots, u_{n-1}\}$ or else G will contain $T_N(n)$ with v_2 and v_0 being the vertices of degree $n-5$ and 3, respectively. However, $v_3u_5v_4u_6u_2u_7u_3u_8v_3$ and w_2 form W_8 in $\overline{G}$, a contradiction. For $n \geq 11$, if v_2 is not adjacent to $\{u_6, \ldots, u_j\}$ in G, then $v_2u_6u_2u_7u_3u_8u_4u_9v_2$ and w_2 form W_8 in $\overline{G}$, a contradiction. Therefore, assume that v_2 is adjacent to u_6 in G. Then u_6 is not adjacent to $\{u_7, \ldots, u_j\}$ in G, and $u_2u_7u_3u_8u_4u_9u_6u_{10}u_2$ and w_2 form W_8 in $\overline{G}$, again a contradiction.

Thus, $R(T_N(n), W_8) \leq 2n$ for $n \equiv 0 \pmod 4$ and $R(T_N(n), W_8) \leq 2n-1$ for $n \not\equiv 0 \pmod 4$. $\square$

Theorem 5.3.22. *If $n \geq 9$, then $R(T_P(n), W_8) = 2n-1$.*

Proof. Lemma 5.3.3 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $T_P(n)$ and that $\overline{G}$ does not contain W_8. Suppose that $n \not\equiv 0 \pmod 4$. By Theorem 5.2.9, G has a subgraph $T = T_A(n)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2, w_1w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n-1$. Since $T_P(n) \nsubseteq G$, w_1 is not adjacent to any vertex of $U \cup V$ in G. If each v_i in V is adjacent to at most three vertices of U in G, then by Corollary 5.3.2, $\overline{G}[U \cup V]$ contains C_8 which with w_1 gives W_8 in $\overline{G}$, a contradiction. Therefore, some vertex in V, say v_2, is adjacent to at least four vertices of U in G, say $u_1, \ldots, u_4$. For $n = 9$ and $n = 10$, G contains $T_P(9)$ and $T_P(10)$ with edge set $\{u_1v_2, u_2v_2, u_3v_2, v_2v_0, v_0v_1, v_0v_3, v_1w_1, v_1w_2\}$ and $\{u_1v_2, u_2v_2, u_3v_2, u_4v_2, v_2v_0, v_0v_1, v_0v_3, v_1w_1, v_1w_2\}$, respectively. For $n \geq 11$, each of $u_1, \ldots, u_4$ is adjacent to at most two remaining vertices in U. Then by Corollary 5.3.1, $\overline{G}[U]$ contains C_8 which with w_1 gives W_8 in $\overline{G}$, a contradiction.

On the other hand, suppose that $n \equiv 0 \pmod 4$. By Theorem 5.3.20, G contains a subgraph $T = T_M(n)$. Now, we let $V(T) = \{v_0, \ldots, v_{n-5}, w_1, \ldots, w_4\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-5}, v_1w_1, v_1w_2, v_1w_3, w_1w_4\}$. Let $V = \{v_2, \ldots, v_{n-5}\}$ and $U = V(G) - V(T)$; then $|V| = n-6$ and $|U| = n-1$. Since $T_P(n) \nsubseteq G$, w_1 is not adjacent to $\{v_0, w_2, w_3\} \cup U$ in G, and so $d_{G[U]}(w_2) \leq 1$, $d_{G[U]}(w_3) \leq 1$ and $d_{G[U]}(v) \leq n-7$ for any vertex $v \in V$. Now, if G contains a subgraph $T_A(n)$, then we can use arguments similar to those used for the case $n \not\equiv 0 \pmod 4$ above. Therefore, G does not contain $T_A(n)$. Then v_0 is not adjacent to $\{w_2, w_3\} \cup U$ in G.

Suppose that some vertex $v \in V$ is not adjacent to w_1 in G. Let X be any four vertices in U that are not adjacent to v in G and set $Y = \{v, v_0, w_2, w_3\}$. By Lemma 4.3.5, $\overline{G}[X \cup Y]$ contains C_8 which with w_1 gives W_8 in $\overline{G}$, a contradiction. Therefore, each vertex of V is adjacent to w_1 in G. Since $T_P(n) \nsubseteq G$, w_4 is adjacent to at most $n-7$ vertices of U in G. Since $T_A(n) \nsubseteq G$, w_2 and w_3 are not adjacent in G. Now, if w_4 is adjacent to both w_2 and w_3 in G, then w_4 is not adjacent to v_0 in G since $T_P(n) \nsubseteq G$. Let X be any four vertices of U that are not adjacent to w_4 in G and let $V = \{w_1, \ldots, w_4\}$. By Lemma 4.3.5, $\overline{G}[X \cup Y]$ contains C_8 which with w_1 gives W_8 in $\overline{G}$, a contradiction. Therefore, w_4 is non-adjacent to either w_2 or w_3 in G, say w_2. Since $d_{G[U]}(w_2) \leq 1$ and $d_{G[U]}(w_4) \leq n-7$, there is a set X of four vertices in U that are not adjacent to both w_2 and w_4 in G. Let $Y = \{v_0, w_1, w_3, w_4\}$. By Lemma 4.3.5, $\overline{G}[X \cup Y]$ contains C_8 which with w_1 gives W_8 in $\overline{G}$, again a contradiction.

In either case, $R(T_P(n), W_8) \leq 2n-1$ for $n \geq 9$ and this completes the proof. $\square$

Theorem 5.3.23. *If $n \geq 9$, then $R(T_Q(n), W_8) = 2n - 1$.*

Proof. Lemma 5.3.3 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $T_Q(n)$ and that $\overline{G}$ does not contain W_8. By Theorem 5.2.5, G has a subgraph $T = S_n(4)$. We let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2, v_1w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n-1$. Since $T_Q(n) \nsubseteq G$, $G[V]$ are independent vertices and not adjacent to U.

Suppose that $n \geq 10$. Then $|V| \geq 5$ and $|U| \geq 9$, so by Observation 4.3.2, $\overline{G}$ contains W_8, a contradiction. If $n = 9$, then $|V| = 4$ and $|U| = 8$. By Lemma 4.3.4, $G[U]$ is K_8 or $K_8 - e$. Since $T_Q(9) \nsubseteq G$, T is not adjacent to U, and $\delta(G[V(T)] \geq 5$. As $v_2, \ldots, v_5$ are independent in G, they are each adjacent to all other vertices in $G[V(T)]$, Hence, $G[V(T)]$ contains $T_Q(9)$ with v_2 and v_0 as the vertices of degree 4, a contradiction.

Thus, $R(T_Q(n), W_8) \leq 2n - 1$ for $n \geq 9$ which completes the proof. $\square$

Theorem 5.3.24. *If $n \geq 9$, then $R(T_R(n), W_8) = 2n - 1$.*

Proof. Lemma 5.3.3 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $T_R(n)$ and that $\overline{G}$ does not contain W_8. By Theorem 5.2.11, G has a subgraph $T = T_C(n)$. Now, let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_2w_2, v_2w_3\}$. Set $V = \{v_3, \ldots, v_{n-4}\}$ and $U = V(G) - V(T) = \{u_1, \ldots, u_{n-1}\}$; then $|V| = n-6$ and $|U| = n-1$. Since $T_R(n) \nsubseteq G$, w_1 is not adjacent in G to any vertex of $U \cup V$. If $\delta(\overline{G}[U \cup V]) \geq \lceil \frac{2n-7}{2} \rceil$, then $\overline{G}[U \cup V]$ contains C_8 by Lemma 2.2.10 which, with w_3 as hub, forms W_8, a contradiction. Therefore, $\delta(\overline{G}[U \cup V]) \leq \lceil \frac{2n-7}{2} \rceil - 1$, and $\Delta(G[U \cup V]) \geq \lfloor \frac{2n-7}{2} \rfloor = n-4$. Now, there are two cases to be considered.

Case 1: One of the vertices of V, say v_3, is a vertex of degree at least $n-4$ in $G[U \cup V]$.

Note that in this case, there are at least 3 vertices from U, say $u_1, \ldots, u_3$, that are adjacent to v_3 in G. Suppose that v_3 is also adjacent to a in G, where a can be a vertex in U or V. Since $T_R(n) \nsubseteq G$, these 4 vertices are independent and are not adjacent to any other vertices of U. Since $n \geq 9$, U contains at least 4 other vertices, say $u_5, \ldots, u_8$, so $u_1u_5u_2u_6u_3u_7au_8u_1$ and w_3 forms W_8 in $\overline{G}$, a contradiction.

Case 2: Some vertex $u \in U$ has degree at least $n-4$ in $G[U \cup V]$.

Since $T_R(n) \nsubseteq G$, u is not adjacent to any vertex of V in G. Therefore, u must be adjacent to at least $n-4$ vertices of U in G. Without loss of generality, suppose that $u_1, \ldots, u_{n-4} \in N_{G[U]}(u)$. Note that V is not adjacent to $N_{G[U]}(u)$, or else it will form $T_R(n)$ in G, a contradiction. If $n \geq 10$, then any 4 vertices from $N_{G[U]}(u)$ and any 4 vertices from V form C_8 in $\overline{G}$ which, with w_3 as hub, forms W_8, a contradiction. Suppose that $n = 9$ and let the remaining two vertices be u_6 and u_7. If either u_6 or u_7 is not adjacent to any two vertices of $\{u_1, \ldots, u_5\}$ in G, say u_6 is not adjacent to u_1 or u_2 in G, then $u_1u_6u_2v_3u_3v_4u_4v_5u_1$ and w_3 forms W_8 in $\overline{G}$, a contradiction. So, both u_6 and u_7 is adjacent to at least 4 vertices of $\{u_1, \ldots, u_5\}$ in G. Since $T_R(9) \nsubseteq G$, T cannot be adjacent to U, and $\delta(G[V(T)] \geq 5$. As both v_2

and w_3 are not adjacent to v_3, v_4 and v_5 in G, they is adjacent to all other vertices in $G[V(T)]$. Similarly, since v_3 does not adjacent to v_2 and w_3 in G, v_3 is adjacent to w_1 or w_2 in G, Without loss of generality, we assume that v_3 is adjacent to w_1. Then $G[V(T)]$ contains $T_R(9)$ with edge set $\{v_2w_2, v_2v_1, v_2v_0, v_0v_4, v_0v_5, v_2w_3, v_2w_1, w_1v_3\}$, a contradiction.

In either case, $R(T_R(n), W_8) \leq 2n - 1$. $\square$

Theorem 5.3.25. *If $n \geq 9$, then $R(T_S(n), W_8) = 2n - 1$.*

Proof. Lemma 5.3.3 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $T_S(n)$ and that $\overline{G}$ does not contain W_8. Suppose that $n \not\equiv 0 \pmod 4$. By Theorem 5.2.7, G has a subgraph $T = S_n[4]$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, w_1w_2, w_1w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G)-V(T)$; then $|V| = n - 5$ and $|U| = n - 1$. Since $T_S(n) \nsubseteq G$, $G[V]$ are independent vertices and are not adjacent to U. If $n \geq 10$, then $|V| \geq 5$ and $|U| \geq 9$, so by Observation 4.3.2, $\overline{G}$ contains W_8, a contradiction. Suppose that $n = 9$. Then $|V| = 4$ and $|U| = 8$. By Lemma 4.3.4, $G[U]$ is K_8 or $K_8 - e$. Since $T_S(9) \nsubseteq G$, T is not adjacent to U, and $\delta(G[V(T)] \geq 5$. As $v_2, \ldots, v_5$ are independent in G, they are adjacent to all other vertices in $G[V(T)]$, and so $G[V(T)]$ contains $T_S(9)$ with edge set $\{v_0v_1, v_0v_2, v_1v_4, v_1v_5, v_2w_1, v_2w_2, v_2w_3, v_3w_1\}$.

On the other hand, suppose that $n \equiv 0 \pmod 4$. By Theorem 5.2.7, G has a subgraph $T = S_{n-1}[4]$. Let $V(T) = \{v_0, \ldots, v_{n-5}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-5}, v_1w_1, w_1w_2, w_1w_3\}$. Set $V = \{v_2, \ldots, v_{n-5}\}$ and $U = V(G)-V(T)$; then $|V| = n - 6$ and $|U| = n$. Since $T_S(n) \nsubseteq G$, $G[V]$ is not adjacent to U. Since $|V| = n - 6 > 4$, by Observation 4.3.2, $\Delta(\overline{G}[U]) \leq 3$ and $\delta(G[U]) \geq n - 4$ since $W_8 \nsubseteq \overline{G}$. By Lemma 5.2.6, either $G[U]$ is $K_{4,\ldots,4}$ and contains $T_S(n)$ or $G[U]$ contains $S_n[4]$ and the arguments from the $n \not\equiv 0 \pmod 4$ case above lead to a contradiction.

Thus, $R(T_S(n), W_8) \leq 2n - 1$ for $n \geq 9$ which completes the proof. $\square$

Chapter 6

Ramsey numbers for large tree graphs versus the wheel graphs of order 9

In this chapter, we provide some insight on the Ramsey numbers for tree graphs of order n versus the wheel graph W_8 of order 9, focusing on the tree graphs with maximum degree at most $n-6$ for large values of n.

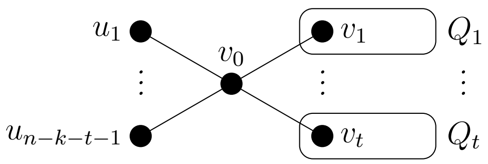
6.1 Introduction

Before looking into the Ramsey numbers, we define a particular tree as follows.

Definition 6.1.1. *Let $Q_1, \ldots, Q_t$ be disjoint trees with $|V(Q_1)|, \ldots, |V(Q_t)| \geq 2$. Define $k = |V(Q_1)| + \cdots + |V(Q_t)| - t$, and let $v_i \in V(Q_i)$ for each $i = 1, \ldots, t$. Finally, let $T = T_{n,k}(v_1, \ldots, v_t; Q_1, \ldots, Q_t)$ be the tree on n vertices with*

$$V(T) = \{v_0, u_1, \ldots, u_{n-k-t-1}\} \cup V(Q_1) \cup \cdots \cup V(Q_t)\,;$$
$$E(T) = \{v_0u_1, \ldots, v_0u_{n-k-t-1}\} \cup \{v_0v_1, \ldots, v_0v_t\} \cup E(Q_1) \cup \cdots \cup E(Q_t)\,,$$

as illustrated below:

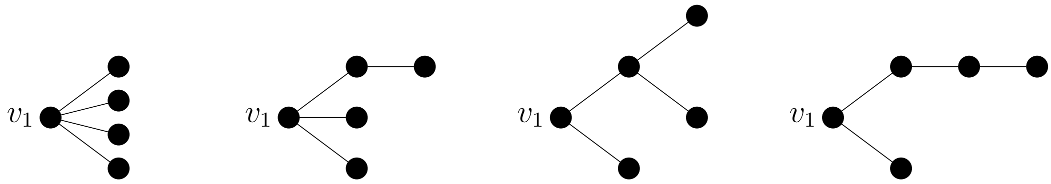


$T_{n,k}(v_1, \ldots, v_t; Q_1, \ldots, Q_t)$

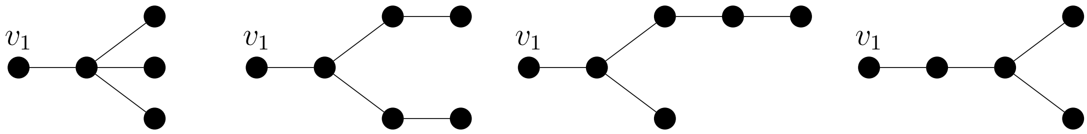
6.2 Some lemmas

In this section, we introduce some lemmas that are helpful in our discussion on the Ramsey numbers for large trees T_n with maximum degree at most $n-6$ versus the wheel graph W_8 of order 9.

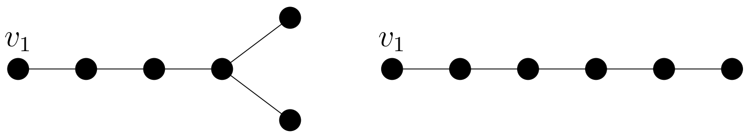
Lemma 6.2.1. *Suppose that $k \geq 5$ and that $T = T_{n,k}(v_1; Q)$ for some tree Q with $|V(Q)| = k+1$. Then Q has at least one of the following graphs as a subgraph:*



Z_1 Z_2 Z_3 Z_4



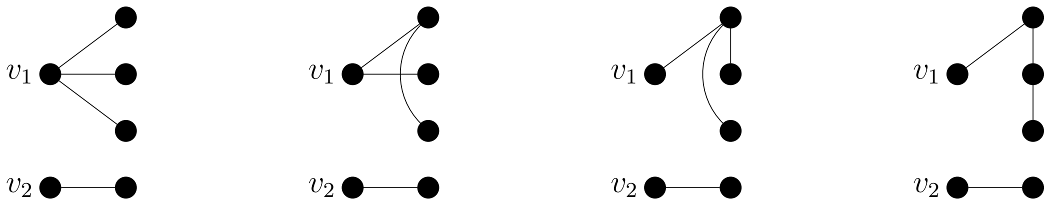
Z_5 Z_6 Z_7 Z_8



Z_9 Z_{10}

Proof. Note that Q contains v_1 and has at least 6 vertices. If $\deg_T(v_1) \geq 4$, then Q contains Z_1. If $\deg_T(v_1) = 3$, then Q contains Z_2. If $\deg_T(v_1) = 2$, then Q contains Z_3 or Z_4. If $\deg_T(v_1) = 1$, then Q contains Z_5, Z_6, Z_7, Z_8, Z_9 or Z_{10}. $\square$

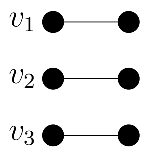
Lemma 6.2.2. *Suppose that $k \geq 5$ and that $T = T_{n,k}(v_1, v_2; Q_1, Q_2)$ for trees Q_1 and Q_2 with $|V(Q_1)| + |V(Q_2)| = k+2$. If $|V(Q_1)| \geq |V(Q_2)|$, then $Q_1 \cup Q_2$ contains at least one of the following graphs as subgraph:*



Z_{11} Z_{12} Z_{13} Z_{14}

Proof. Note that $Q_1 \cup Q_2$ contains $\{v_1, v_2\}$ and has $|V(Q_1)| + |V(Q_2)| = k + 2 \geq 7$ vertices. Suppose that $|V(Q_1)| \geq |V(Q_2)|$; then Q_1 has at least 4 vertices. If $\deg_T(v_1) \geq 3$, then $Q_1 \cup Q_2$ contains Z_{11}. If $\deg_T(v_1) = 2$, then $Q_1 \cup Q_2$ contains Z_{12}. Finally, if $\deg_T(v_1) = 1$, then $Q_1 \cup Q_2$ contains Z_{13} or Z_{14}. $\square$

Lemma 6.2.3. *Suppose that $k \geq 5$ and that $T = T_{n,k}(v_1, \ldots, v_t; Q_1, \ldots, Q_t)$ for trees $Q_1, \ldots, Q_t$ for which $|V(Q_1)| + \cdots + |V(Q_t)| = k+t$. If $t \geq 3$, then $Q_1 \cup Q_2 \cup Q_3$ contains the subgraph*



Z_{15}

Proof. Based on Definition 6.1.1, each v_i in Q_i has degree at least 1. $\square$

Lemma 6.2.4. *Let G be a graph, let $U \subseteq V(G)$ with $|U| = m$ and let $y_1, y_2, y_3 \in V(G) \setminus U$. If $|N_U(y_i)| \geq m - \ell$ for all i, then*

(a) *for all $1 \leq i < j \leq 3$, $|N_U(y_i) \cap N_U(y_j)| \geq m - 2\ell$;*
(b) $|N_U(y_1) \cap N_U(y_2) \cap N_U(y_3)| \geq m - 3\ell$.

Proof. (a) $|N_U(y_i) \cap N_U(y_j)| = |N_U(y_i)| + |N_U(y_j)| - |N_U(y_i) \cup N_U(y_j)| \geq 2(m-\ell) - |U| = m - 2\ell$. (b) By part (a), $|N_U(y_1) \cap N_U(y_2) \cap N_U(y_3)| \geq |N_U(y_1) \cap N_U(y_2)| + |N_U(y_3)| - |U| \geq m - 3\ell$. $\square$

Lemma 6.2.5. *Let G be a graph with $V(G) = \{x_1, \ldots, x_{n-t}, y_1, y_2, y_3\}$. Suppose that each vertex in G has degree at least $n - t - \ell$. Let $Z_1, \ldots, Z_{10}$ be defined as in Lemma 6.2.1. If $n \geq t + 3\ell + 7$, then for each $i \in \{1, \ldots, 10\}$, there are $x_{i1}, x_{i2}, x_{i3} \in \{x_1, \ldots, x_{n-t}\}$ such that $G[\{x_{i1}, x_{i2}, x_{i3}, y_1, y_2, y_3\}]$ contains a subgraph U_i which is isomorphic to Z_i. Furthermore, the isomorphism can be chosen so that x_{i1} is mapped to v_1 in Z_i.*

Proof. Let $X = \{x_1, \ldots, x_{n-t}\}$ and note that $|N_X(y_j)| \geq n - t - \ell - 2$ for $j = 1, 2, 3$. Also, define $d = n - t - \ell - 3$ and note that $d \geq 2\ell + 4 \geq 4$. Finally, define $G' = G[\{x_{i1}, x_{i2}, x_{i3}, y_1, y_2, y_3\}]$. By Lemma 6.2.4(b), $|N_X(y_1) \cap N_X(y_2) \cap N_X(y_3)| \geq n - t - 3(\ell + 2) \geq 1$, so $N_X(y_1) \cap N_X(y_2) \cap N_X(y_3)$ is non-empty.

Case $i = 1$. Let $x_{i1} \in N_X(y_1) \cap N_X(y_2) \cap N_X(y_3)$. Since x_{i1} is adjacent to at least d vertices in $V(G) \setminus \{y_1, y_2, y_3\}$, it is adjacent to some $x_{i2} \in X \setminus \{x_{i1}\}$. Choose $x_{i3} \in X \setminus \{x_{i1}, x_{i2}\}$; then G' has a subgraph isomorphic to Z_i and x_{i1} is mapped to v_1 by this isomorphism.

Case $i = 2$. Let $x_{i1} \in N_X(y_1) \cap N_X(y_2)$. Since y_1 is adjacent to at least d vertices in $V(G) \setminus \{x_{i1}, y_2, y_3\}$, it is adjacent to a vertex $x_{i2} \in X \setminus \{x_{i1}\}$. Choose $x_{i3} \in X \setminus \{x_{i1}, x_{i2}\}$; then G' has a subgraph isomorphic to Z_i and x_{i1} is mapped to v_1 by this isomorphism.

Case $i = 3$. Let $x_{i2} \in N_X(y_1) \cap N_X(y_2)$ and let $X' = X \setminus \{x_{i2}\}$. Then $|N_{X'}(x_{i2})| \geq d$ and $|N_{X'}(y_3)| \geq d$. By Lemma 6.2.4(a), $|N_{X'}(y_3) \cap N_{X'}(x_{i2})| \geq n - t - 2(\ell + 3) \geq 1$, so there is some $x_{i1} \in N_{X'}(y_3) \cap N_{X'}(x_{i2})$. Choose $x_{i3} \in X' \setminus \{x_{i1}\}$; then G' has a subgraph isomorphic to Z_i and x_{i1} is mapped to v_1 by this isomorphism.

Case $i = 4$. Let $x_{i2} \in N_X(y_1) \cap N_X(y_2)$ and let $X' = X \setminus \{x_{i2}\}$. Then $|N_{X'}(y_1)| \geq d$ and $|N_{X'}(y_3)| \geq d$. By Lemma 6.2.4(a), $|N_{X'}(y_1) \cap N_{X'}(y_3)| \geq n - t - 2(\ell + 3) \geq 1$, so there is some $x_{i1} \in N_{X'}(y_1) \cap N_{X'}(y_3)$. Choose $x_{i3} \in X \setminus \{x_{i1}, x_{i2}\}$; then G' has a subgraph isomorphic to Z_i and x_{i1} is mapped to v_1 by this isomorphism.

Case $i = 5$. Let $x_{i2} \in N_X(y_1) \cap N_X(y_2) \cap N_X(y_3)$ and let $X' = X \setminus \{x_{i2}\}$. Since $|N_{X'}(x_{i2})| \geq d \geq 1$, some $x_{i1} \in X'$ is adjacent to x_{i2}. Choose $x_{i3} \in X \setminus \{x_{i1}, x_{i2}\}$;

then G' has a subgraph isomorphic to Z_i and x_{i1} is mapped to v_1 by this isomorphism.

Case $i = 6$**.** As in Case $i = 4$, there is some $x_{i2} \in N_X(y_1) \cap N_X(y_2)$ and $|N_{X'}(y_1) \cap N_{X'}(y_3)| \geq 1$ where $X' = X \setminus \{x_{i2}\}$. Let $x_{i3} \in N_{X'}(y_1) \cap N_{X'}(y_3)$ and set $X'' = X \setminus \{x_{i2}, x_{i3}\}$. Since $|N_{X''}(y_1)| \geq d - 1 \geq 1$, some $x_{i1} \in X''$ is adjacent to y_1. Thus, G' has a subgraph isomorphic to Z_i and x_{i1} is mapped to v_1 by this isomorphism.

Case $i = 7$**.** As in Case $i = 6$, there is some $x_{i2} \in N_X(y_1) \cap N_X(y_2)$ and some $x_{i3} \in N_{X'}(y_1) \cap N_{X'}(y_3)$ where $X' = X \setminus \{x_{i2}\}$. Let $X'' = X \setminus \{x_{i2}, x_{i3}\}$. Since $|N_{X''}(x_{i2})| \geq d - 1 \geq 1$, some $x_{i1} \in X''$ is adjacent to x_{i2}, so G' has a subgraph isomorphic to Z_i and x_{i1} is mapped to v_1 by this isomorphism.

Case $i = 8$**.** Let $x_{i2} \in N_X(y_1) \cap N_X(y_2) \cap N_X(y_3)$ and let $X' = X \setminus \{x_{i2}\}$. Since $|N_{X'}(y_1)| \geq d \geq 1$, some vertex $x_{i1} \in X'$ is adjacent to y_1. Choose $x_{i3} \in X \setminus \{x_{i1}, x_{i2}\}$; then G' has a subgraph isomorphic to Z_i and x_{i1} is mapped to v_1 by this isomorphism.

Case $i = 9$**.** Let $x_{i2} \in N_X(y_1) \cap N_X(y_2) \cap N_X(y_3)$ and let $X' = X \setminus \{x_{i2}\}$. Since $|N_{X'}(y_1)| \geq d \geq 1$, some $x_{i3} \in X'$ is adjacent to y_1. Let $X'' = X \setminus \{x_{i2}\}$. Since $|N_{X''}(x_{i3})| \geq d - 1 \geq 1$, x_{i3} is adjacent to some $x_{i1} \in X''$. Thus, G' has a subgraph isomorphic to Z_i and x_{i1} is mapped to v_1 by this isomorphism.

Case $i = 10$**.** As in Case $i = 6$, there is some $x_{i2} \in N_X(y_1) \cap N_X(y_2)$ and some $x_{i3} \in N_{X'}(y_1) \cap N_{X'}(y_3)$ where $X' = X \setminus \{x_{i2}\}$. Let $X'' = X \setminus \{x_{i2}, x_{i3}\}$. Since $|N_{X''}(y_2)| \geq d - 1 \geq 1$, some $x_{i1} \in X''$ is adjacent to y_2, so G' has a subgraph isomorphic to Z_i and x_{i1} is mapped to v_1 by this isomorphism.

This completes the proof of the lemma. □

Lemma 6.2.6. *Let G be a graph with $V(G) = \{x_1, \ldots, x_{n-t}, y_1, y_2, y_3\}$ in which each vertex has degree at least $n - t - \ell$. For $11 \leq i \leq 14$, let Z_i be defined as in Lemma 6.2.2. If $n \geq t + 3\ell + 7$, then for each $i \in \{11, \ldots, 14\}$, there are $x_{i,1}, x_{i,2}, x_{i,3} \in \{x_1, \ldots, x_{n-t}\}$ such that $G[\{x_{i,1}, x_{i,2}, x_{i,3}, y_1, y_2, y_3\}]$ contains a subgraph U_i which is isomorphic to Z_i. Furthermore, the isomorphism can be chosen so that $x_{i,1}$ is mapped to v_1 and $x_{i,2}$ is mapped to v_2 in Z_i.*

Proof. Let $X = \{x_1, \ldots, x_{n-t}\}$ and note that $|N_X(y_j)| \geq n - t - \ell - 2$ for $j = 1, 2, 3$. Also, define $d = n - t - \ell - 3$ and note that $d \geq 2\ell + 4 \geq 4$. Finally, define $G' = G[\{x_{i,1}, x_{i,2}, x_{i,3}, y_1, y_2, y_3\}]$. By Lemma 6.2.4(b), $|N_X(y_1) \cap N_X(y_2) \cap N_X(y_3)| \geq n - t - 3(\ell + 2) \geq 1$, so $N_X(y_1) \cap N_X(y_2) \cap N_X(y_3) \neq \emptyset$.

Case $i = 11$**.** Let $x_{11,j_1} \in N_X(y_1) \cap N_X(y_2) \cap N_X(y_3)$ and $x_{11,j_2} \in X \setminus \{x_{11,j_1}\}$. Since x_{11,j_2} is adjacent to at least $d - 1$ vertices in $V(G) \setminus \{x_{11,j_1}, y_1, y_2, y_3\}$, it is adjacent to some $x_{11,j_3} \in X \setminus \{x_{11,j_1}\}$. Thus, G' has a subgraph isomorphic to Z_{11}, and x_{11,j_1} is mapped to v_1 and x_{11,j_2} is mapped to v_2.

Case $i = 12$**.** Note that y_1 is adjacent to some $x_{12,j_3} \in X$. Let $X' = X \setminus \{x_{12,j_3}\}$; then $|N_{X'}(y_2)| \geq d$ and $|N_{X'}(x_{12,j_3})| \geq d$. By Lemma 6.2.4(a), $|N_{X'}(y_2) \cap N_{X'}(x_{12,j_3})| \geq n - t - 1 - 2(\ell + 2) \geq 1$, so there is some $x_{12,j_1} \in X'$. Let $X'' = X \setminus \{x_{12,j_1}, x_{12,j_3}\}$. Since $|N_{X''}(y_3)| \geq d - 1 \geq 1$, some $x_{12,j_2} \in X''$ is adjacent to y_3. Hence, G' has a subgraph isomorphic to Z_{12}, and x_{12,j_1} is mapped to v_1 and x_{12,j_2} is mapped to v_2.

Case $i = 13$. Let $x_{13,j_3} \in N_X(y_1) \cap N_X(y_2)$ and let $X' = X \setminus \{x_{13,j_3}\}$. Since $|N_{X'}(x_{13,j_3})| \geq d-1$, some $x_{13,j_1} \in X'$ is adjacent to x_{13,j_3}. Let $X'' = X \setminus \{x_{13,j_1}, x_{13,j_3}\}$. Since $|N_{X''}(y_3)| \geq d-1 \geq 1$, some $x_{13,j_2} \in X''$ is adjacent to y_3. Thus, G' has a subgraph isomorphic to Z_{13}, and x_{13,j_1} is mapped to v_1 and x_{13,j_2} is mapped to v_2.

Case $i = 14$. Let $x_{14,j_3} \in N_X(y_1) \cap N_X(y_2)$ and let $X' = X \setminus \{x_{14,j_3}\}$. Since $|N_{X'}(y_1)| \geq d \geq 1$, some $x_{14,j_1} \in X'$ is adjacent to y_1. Let $X'' = X \setminus \{x_{14,j_1}, x_{14,j_3}\}$. Since $|N_{X''}(y_3)| \geq d-1 \geq 1$, some $x_{14,j_2} \in X''$ is adjacent to y_3. Thus, G' has a subgraph isomorphic to Z_{14}, and x_{14,j_1} is mapped to v_1 and x_{14,j_2} is mapped to v_2.

This completes the proof of the lemma. □

Lemma 6.2.7. *Let G be a graph with $V(G) = \{x_1, \ldots, x_{n-t}, y_1, y_2, y_3\}$ in which each vertex has degree at least $n-t-\ell$. Let Z_{15} be defined as in Lemma 6.2.3. If $n \geq t+\ell+5$, then there are $x_{i_1}, x_{i_2}, x_{i_3} \in \{x_1, \ldots, x_{n-t}\}$ such that $G[\{x_{i_1}, x_{i_2}, x_{i_3}, y_1, y_2, y_3\}]$ contains a subgraph U which is isomorphic to Z_{15}. Furthermore, the isomorphism can be chosen so that x_{i_1} is mapped to v_1, x_{i_2} is mapped to v_2 and x_{i_3} is mapped to v_3 in Z_{15}.*

Proof. Let $X = \{x_1, \ldots, x_{n-t}\}$; then $|N_X(y_1)| \geq n-t-\ell-2 \geq 1$, so y_1 is adjacent to some $x_{i_1} \in X$. Let $X' = X \setminus \{x_{i_1}\}$. Since $|N_{X'}(y_2)| \geq n-t-\ell-3 \geq 1$, y_2 is adjacent to some $x_{i_2} \in X'$. Let $X'' = X \setminus \{x_{i_1}, x_{i_2}\}$. Since $|N_{X''}(y_3)| \geq n-t-\ell-4 \geq 1$, y_3 is adjacent to some $x_{i_3} \in X''$. Hence, $G[\{x_{i_1}, x_{i_2}, x_{i_3}, y_1, y_2, y_3\}]$ has a subgraph isomorphic to Z_{15} and x_{i_1} is mapped to v_1, x_{i_2} is mapped to v_2 and x_{i_3} is mapped to v_3 in Z_{15}. □

Lemma 6.2.8. *Let G be a graph with $V(G) = Z_1 \cup Z_2$ for sets Z_1 and Z_2 with $|Z_2| \geq n-1$ where $n \geq 5n_1+5$ for some positive integer n_1. If each vertex in Z_1 is adjacent in G to at most n_1 vertices in Z_2 and $\overline{G}[Z_1]$ contains the star graph S_5, then $\overline{G}$ contains W_8.*

Proof. Suppose that $\overline{G}[Z_1]$ contains S_5 and write $V(S_5) = \{z_0, \ldots, z_4\}$ and $E(S_5) = \{z_0z_1, \ldots, z_0z_4\}$. Since each vertex in Z_1 is adjacent in G to at most n_1 vertices in Z_2, $Z_2 \setminus \big(N_{Z_2}(z_0) \cup \cdots \cup N_{Z_2}(z_4)\big)$ contains at least $n-1-5n_1 \geq 4$ vertices, so choose four such vertices, say $a_1, \ldots, a_4$. Then $\overline{G}$ contains W_8 with hub z_0 and $z_1a_1z_2a_2z_3a_3z_4a_4z_1$ as C_8. □

Lemma 6.2.9. *Suppose that k is a fixed positive integer and let T_1 be a tree graph $T_{n,k}(v_1, \ldots, v_t; Q_1, \ldots, Q_t)$ of order n as defined in Definition 6.1.1. Suppose that $|V(Q_1)| \geq 2$ and that $q \in V(Q_1) \setminus \{v_1\}$ has degree 1 in Q_1. Let Q_1' be the tree obtained from Q_1 by removing q and its incident edge. Let $T_2 = T_{n,k-1}(v_1, \ldots, v_t; Q_1', Q_2, \ldots, Q_t)$. There is a positive integer $n_0(k)$ such that, for each integer $n \geq n_0(k)$, if G is a graph with $2n-1$ vertices that contains T_2 but whose complement $\overline{G}$ does not contain W_8, then G contains T_1.*

Proof. Let q_0 be the vertex in $V(Q_1)$ adjacent to q. Note that q_0 is also a vertex in $V(Q'_1)$. Let $\mathcal{T}_k$ be the family of non-isomorphic forests with at most k vertices. Set

$$n_1(k) = \max_{T \in \mathcal{T}_k} R(T, W_8)\,.$$

Suppose that G is a graph on $2n-1$ vertices, that T_2 is a subgraph of G, and that $\overline{G}$ does not contain W_8. Let $V(T_2) = \{v_0\} \cup U_1 \cup V(Q'_1) \cup V(Q_2) \cup \cdots \cup V(Q_t)$ where $U_1 = \{u_1, \ldots, u_{n-k-t}\}$ and

$$E(T_2) = \{v_0v_1, \ldots, v_0v_t\} \cup \{v_0u_1, \ldots, v_0u_{n-k-t}\} \cup E(Q'_1) \cup E(Q_2) \cup \cdots \cup E(Q_t)\,.$$

Note that $u_1, \ldots, u_{n-k-t}$ each has degree 1 in T_2. Let $U_2 = V(G) \setminus V(T_2)$; then $|U_2| = n-1$.

If q_0 is adjacent to a vertex in $U_1 \cup U_2$, then G contains T_1. Therefore, assume that q_0 is not adjacent to any vertex in $U_1 \cup U_2$. Note that Q_1 is a tree with $|V(Q_1)| \le k+1$. Now, $Q_1 - v_1$ is a forest $Q_{11} \cup \cdots \cup Q_{1\ell}$ of ℓ disjoint trees for some $\ell \ge 1$. Clearly, $R(Q_1 - v_1, W_8)$ is at most $n_1(k)$.

Suppose that u_1 is adjacent in G to at least $n_1(k)$ vertices in U_2. Since $\overline{G}$ does not contain W_8, the subgraph $G[N_{U_2}(u_1)]$ contains $Q_1 - v_1 = Q_{11} \cup \cdots \cup Q_{1\ell}$. Now, u_1 is adjacent to each vertex in $Q_1 - v_1$. Adding all of these vertices to T_2 gives the subgraph T_1 in G. Therefore, assume that u_1 is adjacent to at most $n_1(k) - 1$ vertices in U_2. Similarly, assume that u_j is adjacent to at most $n_1(k) - 1$ vertices in U_2 for $j = 2, 3, 4$.

Let $Z_1 = \{q_0, u_1, \ldots, u_4\}$. Since q_0 is not adjacent to $u_1, \ldots, u_4$, $\overline{G}[Z_1]$ contains S_5. Now, each vertex in Z_1 is adjacent in G to at most $n_1(k) - 1$ vertices in U_2. By Lemma 6.2.8, $\overline{G}$ contains W_8, provided that $n \ge 5n_1(k)$. This is not possible as $\overline{G}$ does not contain W_8. Hence, G contains T_1. □

Corollary 6.2.10. *Let k be a fixed positive integer and let T_1 be a tree graph $T_{n,k}(v_1, \ldots, v_t; Q_1, \ldots, Q_t)$ of order n as defined in Definition 6.1.1. Suppose that $0 \le k' < k$ and $1 \le t' \le t$. Let*

$$T_2 = T_{n,k'}(v'_1, \ldots, v'_{t'}; Q'_1, \ldots, Q'_{t'})$$

where, for each $i \in \{1, \ldots, t'\}$, Q'_i is isomorphic to a subgraph of Q_i where $v'_i \in V(Q'_i)$ is mapped to $v_i \in V(Q_i)$ under the isomorphism. There is a positive integer $n_0(k)$ such that, for each integer $n \ge n_0(k)$, if G is a graph with $2n-1$ vertices that contains T_2 but whose complement $\overline{G}$ does not contain W_8, then G contains T_1.

Proof. Without loss of generality, assume that $|V(Q_1)| \ge |V(Q_2)| \ge \cdots \ge |V(Q_t)|$. By Definition 6.1.1, $|V(Q_t)| \ge 2$. Now, by repeatedly adding vertices to $Q'_{t'}$ to obtain $Q_{t'}$ and then applying Lemma 6.2.9, we can conclude that G contains $T_{n,k''}(v'_1, \ldots, v_{t'}; Q'_1, \ldots, Q_{t'})$ where

$$k'' = \big(|V(Q'_1)| + \cdots + |V(Q'_{t'-1})|\big) + |V(Q_{t'})| - t'\,.$$

Repeat the same process to each Q'_j, by adding vertices to obtain Q_j. Then G contains the subgraph $T_3 = T_{n,k'''}(v_1, \ldots, v_{t'}; Q_1, \ldots, Q_{t'})$ where

$$k''' = \big(|V(Q_1)| + \cdots + |V(Q_{t'})|\big) - t' .$$

If $t' = t$, then G contains $T_3 = T_1$. Suppose that $t' < t$. Now,

$$V(T_3) = \{v_0, u_1, \ldots, u_{n-k'''-t'-1}\} \cup V(Q_1) \cup \cdots \cup V(Q_{t'}) ;$$
$$E(T_3) = \{v_0u_1, \ldots, v_0u_{n-k'''-t'-1}\} \cup \{v_0v_1, \ldots, v_0v_{t'}\} \cup E(Q_1) \cup \cdots \cup E(Q_{t'}) .$$

Since $|Q_t| \geq 2$, we have $t \leq k$. Let $\mathcal{T}_k$ be the family of non-isomorphic forests with at most $2k$ vertices. Set

$$n_0 = \max_{T \in \mathcal{T}_k} R(T, W_8) .$$

Now, $n - k''' - t' - 1 \geq n - 2k - 1$. If $n - 2k - 1 \geq n_0$, then $G[\{u_1, \ldots, u_{n-k'''-t'-1}\}]$ contains the forest $Q_{t'+1} \cup \cdots \cup Q_t$ which with T_3 gives the subgraph T_1 in G. □

Lemma 6.2.11. *Let G be a graph with $V(G) = \{v_1, \ldots, v_4\} \cup U$ where $|U| = n$ and none of $v_1, \ldots, v_4$ is adjacent to any vertex in U. Let $Z_1, \ldots, Z_{15}$ be defined as in Lemmas 6.2.1-6.2.3. For sufficiently large n, if $\overline{G}$ does not contain W_8, then*
(a) *$G[U]$ contains $T_{n,|V(Z_i)|-1}(v_1; Z_i)$ for each $i = 1, \ldots, 10$;*
(b) *$G[U]$ contains $T_{n,4}(v_1, v_2; X_{i1}, X_{i2})$ for each $i = 11, \ldots, 14$ with $X_{i1} \cup X_{i2} = Z_i$;*
(c) *$G[U]$ contains $T_{n,3}(v_1, v_2, v_3; X_1, X_2, X_3)$ where $X_1 \cup X_2 \cup X_3 = Z_{15}$.*

Proof. Suppose that $\overline{G}[U]$ contains S_5, and write $V(S_5) = \{z_0, \ldots, z_4\}$ and $E(S_5) = \{z_0z_1, \ldots, z_0z_4\}$. Then $\overline{G}$ contains W_8 with hub z_0 and $z_1v_1z_2v_2z_3v_3z_4v_4z_1$ as C_8. Therefore, assume that $\overline{G}[U]$ does not contain S_5; then every vertex in $\overline{G}[U]$ has degree at most 3. Thus, each vertex in $G[U]$ has degree at least $n - 4$. Write $U = \{a_0, \ldots, a_{n-4}, b_1, b_2, b_3\}$ so that each of $a_0a_1, \ldots, a_0a_{n-4}$ is an edge of $G[U]$. Now, consider the graph $G[U \setminus \{a_0\}]$. Every vertex in $G[U \setminus \{a_0\}]$ has degree at least $n - 5$.

(a) By Lemma 6.2.5, there are elements $a_{i1}, a_{i2}, a_{i3} \in \{a_1, \ldots, a_{n-4}\}$ such that $G[\{a_{i1}, a_{i2}, a_{i3}, b_1, b_2, b_3\}]$ contains a subgraph U'_i isomorphic to Z_i. Furthermore, the isomorphism can be chosen so that a_{i1} is mapped to v_1 in Z_i. Therefore, G contains $T_{n,|V(Z_i)|-1}(v_1; Z_i)$.

(b) By Lemma 6.2.6, there are elements $a_{i1}, a_{i2}, a_{i3} \in \{a_1, \ldots, a_{n-4}\}$ such that $G[\{a_{i1}, a_{i2}, a_{i3}, b_1, b_2, b_3\}]$ contains a subgraph U'_i isomorphic to Z_i. Furthermore, the isomorphism can be chosen so that a_{i1} is mapped to v_1 and a_{i2} is mapped to v_2 in Z_i. Therefore, G contains $T_{n,4}(v_1, v_2; X_{i1}, X_{i2})$.

(c) By Lemma 6.2.7, there are elements $a_{j_1}, a_{j_2}, a_{j_3} \in \{a_1, \ldots, a_{n-4}\}$ such that $G[\{a_{j_1}, a_{j_2}, a_{j_3}, b_1, b_2, b_3\}]$ contains a subgraph U isomorphic to Z_{15}. Furthermore, the isomorphism can be chosen so that a_{j_1} is mapped to v_1, a_{j_2} is mapped to v_2 and a_{j_3} is mapped to v_3 in Z_{15}. Therefore, G contains $T_{n,3}(v_1, v_2, v_3; X_1, X_2, X_3)$. □

Lemma 6.2.12. *Let $Z_1, \dots, Z_{15}$ be defined as in Lemmas 6.2.1-6.2.3. For each $i = 11, \dots, 14$, let $Z_i = X_{i1} \cup X_{i2}$ where X_{i1} is a tree and X_{i2} is an edge disjoint from X_{i2}. Let $Z_{15} = X_1 \cup X_2 \cup X_3$ where X_1, X_2, X_3 are disjoint edges. Then*

(a) *$R(T_{n,|V(Z_i)|-1}(v_1; Z_i), W_8) = 2n - 1$ when n is sufficiently large;*
(b) *$R(T_{n,4}(v_1, v_2; X_{i1}, X_{i2}), W_8) = 2n - 1$ when n is sufficiently large;*
(c) *$R(T_{n,3}(v_1, v_2, v_3; X_1, X_2, X_3), W_8) = 2n - 1$ when n is sufficiently large.*

Proof. The union of two complete graphs $G' = K_{n-1} \cup K_{n-1}$ does not contain $T_{n,|V(Z_i)|-1}(v_1; Z_i)$ and $\overline{G'}$ does not contain W_8, so $R(T_{n,|V(Z_i)|-1}(v_1; Z_i), W_8) \geq 2n-1$. Similarly, we are able to prove that $R(T_{n,4}(v_1, v_2; X_{i1}, X_{i2}), W_8) \geq 2n - 1$ and that $R(T_{n,3}(v_1, v_2, v_3; X_1, X_2, X_3), W_8) \geq 2n - 1$.

Let G be a graph with $2n - 1$ vertices such that $\overline{G}$ does not contain W_8. By Theorem 2.2.6, G contains S_{n-2}. If G contains S_n, then by Corollary 6.2.10, G contains $T_{n,|V(Z_i)|-1}(v_1; Z_i)$ for each $i \in \{1, \dots, 10\}$, $T_{n,4}(v_1, v_2; X_{i1}, X_{i2})$ for each $i \in \{11, \dots, 14\}$ and $T_{n,3}(v_1, v_2, v_3; X_1, X_2, X_3)$. Therefore, assume that G does not contain S_n. We consider two cases.

Case 1. G contains S_{n-1}.
Write $V(S_{n-1}) = \{x_0, \dots, x_{n-2}\}$ and $E(S_{n-1}) = \{x_0x_1, \dots, x_0x_{n-2}\}$, and let $U_2 = V(G) \setminus V(S_{n-1})$. Since G does not contain S_n, x_0 is not adjacent to any vertex in U_2. If x_1 is adjacent to a vertex in U_2, then G contains $T_{n,2}(x_1; P_2)$ where P_2 is a path with two vertices and $x_1 \in V(P_2)$. Clearly, for each $i = 1, \dots, 10$, P_2 is isomorphic to a subgraph of Z_i and x_1 is mapped to $v_1 \in V(Z_i)$ by this isomorphism. By Corollary 6.2.10, G contains $T_{n,|V(Z_i)|-1}(v_1; Z_i)$. For each $i = 11, \dots, 14$, P_2 is isomorphic to a subgraph of X_{i1} and x_1 is mapped to $v_1 \in V(X_{i1})$ by this isomorphism. Again by Corollary 6.2.10, G contains $T_{n,4}(v_1, v_2; X_{i1}, X_{i2})$. Similarly, G contains $T_{n,3}(v_1, v_2, v_3; X_1, X_2, X_3)$. Therefore, assume that x_1 is not adjacent to any vertex in U_2. Similarly, assume that none of $x_2, \dots, x_{n-2}$ is adjacent to any vertex in U_2.

Now $|U_2| = n$ and $x_1, \dots, x_4$ are not adjacent to any vertex in U_2. It follows from Lemma 6.2.11 that $G[U_2]$ contains $T_{n,|V(Z_i)|-1}(v_1; Z_i)$ for $i = 1, \dots, 10$, $T_{n,4}(v_1, v_2; X_{i1}, X_{i2})$ for $i = 11, \dots, 14$ and $T_{n,3}(v_1, v_2, v_3; X_1, X_2, X_3)$.

Case 2. G contains S_{n-2} but does not contain S_{n-1}.
Write $V(S_{n-2}) = \{x_0, \dots, x_{n-3}\}$ and $E(S_{n-2}) = \{x_0x_1, \dots, x_0x_{n-3}\}$, and let $U_2 = V(G) \setminus V(S_{n-2})$. Then $|U_2| = n + 1$ and x_0 is not adjacent to any vertex in U_2. Let $u \in U$ and suppose that there are vertices $x_{l_1}, x_{l_2}, x_{l_3} \in \{x_1, \dots, x_{n-3}\}$ that are not adjacent to any vertex in $U_2 \setminus \{u\}$. Since $|U_2 \setminus \{u\}| = n$ and x_0 is also not adjacent to any vertex in $U_2 \setminus \{u\}$, it follows from Lemma 6.2.11 that $G[U_2 \setminus \{u\}]$ contains $T_{n,|V(Z_i)|-1}(v_1; Z_i)$ for $1 \leq i \leq 10$, $T_{n,4}(v_1, v_2; X_{i1}, X_{i2})$ for $11 \leq i \leq 14$ and $T_{n,3}(v_1, v_2, v_3; X_1, X_2, X_3)$. Therefore, assume that for each $u \in U_2$ and all subsets $Y \subseteq \{x_1, \dots, x_{n-3}\}$ with $|Y| = 3$, at least one vertex of $U_2 \setminus \{u\}$ is adjacent of some vertex of Y.

Let $\mathcal{T}_5$ be the family of non-isomorphic forests with at most 5 vertices. Set

$$n_0 = \max_{T \in \mathcal{T}_5} R(T, W_8)\,.$$

and note that $n_0 \geq 2$. Suppose that x_1 is adjacent to at least $n_0 + 1$ vertices in U_2 and let $i \in \{1, \dots, 10\}$. Since $\overline{G}$ does not contain W_8 and $Z_i - v_1$ is a forest of size at most 5, the subgraph $G[N_{U_2}(x_1)]$ contains $Z_i - v_1$. Hence, G contains $T_{n,|V(Z_i)|-1}(v_1; Z_i)$.

Next, we show that G contains $T_{n,4}(v_1, v_2; X_{i1}, X_{i2})$. At least one of x_2, x_3, x_4 is adjacent to some vertex $u_2 \in U_2$, without loss of generality say x_2. Let $U_2' = U_2 \setminus \{u_2\}$. Now, x_1 is adjacent to at least n_0 vertices in U_2'. Since $\overline{G}$ does not contain W_8 and $X_{i1} - v_1$ is a forest of size 3, the subgraph $G[N_{U_2'}(x_1)]$ contains $X_{i1} - v_1$. Thus, G contains $T_{n,4}(v_1, v_2; X_{i1}, X_{i2})$ where X_{i2} is the path x_2u_2.

As above, we can assume that x_2 is adjacent to a vertex $u_2 \in U_2$. Also, at least one of x_3, x_4, x_5 is adjacent to some vertex in $u_3 \in U_2$, without loss of generality, say x_3. Since x_1 is adjacent to at least $n_0 - 1$ vertices in $U_2 \setminus \{u_2, u_3\}$, there is a vertex $u_1 \in U_2 \setminus \{u_2, u_3\}$ for which $x_1u_1 \in E(G)$. Thus, G contains $T_{n,3}(v_1, v_2, v_3; X_1, X_2, X_3)$.

Thus, we may assume that x_1 is adjacent to at most n_0 vertices in U_2. Similarly, we may assume that each of $x_2, \ldots, x_{n-3}$ is adjacent to at most n_0 vertices in U_2.

By Lemma 6.2.8, we may assume that $\overline{G}[V(S_{n-2})]$ does not contain S_5. Each vertex of $\overline{G}[V(S_{n-2})]$ therefore has degree at most 3. Thus, each vertex of $G[V(S_{n-2})]$ has degree at least $n - 6$.

At least one of x_1, x_2, x_3 is adjacent to some vertex $w_1 \in U_2$, say x_1. Recall that x_1 is adjacent to at least $n - 6$ vertices in $G[V(S_{n-2})]$, say $b_1, \ldots, b_{n-6}$. Suppose that w_1 is adjacent to at least n_0 vertices in $U_2 \setminus \{w_1\}$. Since $\overline{G}$ does not contain W_8 and $Z_i - v_1$ is a forest of size at most 5, the subgraph $G[N_{U_2\setminus\{w_1\}}(w_1)]$ contains $Z_i - v_1$. Let $U_3 \subseteq N_{U_2\setminus\{w_1\}}(w_1)$ be such that $G[U_3]$ contains the forest $Z_i - v_1$. Then $G[U_3 \cup \{b_1, \ldots, b_{n-6}, x_1, w_1\}]$ contains $T_{n,|V(Z_i)|-1}(v_1; Z_i)$.

Next, recall that w_1 is adjacent to at least n_0 vertices in $U_2 \setminus \{w_1\}$. Since $\overline{G}$ does not contain W_8 and $X_{i1} - v_1$ is a forest of size 3, the subgraph $G[N_{U_2\setminus\{w_1\}}(w_1)]$ contains $X_{i1} - v_1$. Choose an element $c \in V(S_{n-2}) \setminus \{x_1, b_1, \ldots, b_{n-6}\}$. Since c has degree at least $n - 6$ in $G[V(S_{n-2})]$, it is adjacent to at least $n - 9$ vertices in $\{b_1, \ldots, b_{n-6}\}$, including, say, b_1. Thus, G contains $T_{n,4}(v_1, v_2; X_{i1}, X_{i2})$ where X_{i2} is the path cb_1.

Now note that w_1 is adjacent to a vertex in $U_2 \setminus \{w_1\}$. Choose two elements $c_1, c_2 \in V(S_{n-2}) \setminus \{x_1, b_1, \ldots, b_{n-6}\}$. Since each c_i has degree at least $n - 6$ in $G[V(S_{n-2})]$, there are two vertices $d_1, d_2 \in \{b_1, \ldots, b_{n-6}\}$ such that c_1d_1 and c_2d_2 are edges in G. Hence, G contains $T_{n,3}(v_1, v_2, v_3; X_1, X_2, X_3)$.

We may therefore assume that w_1 is adjacent to at most $n_0 - 1$ vertices in $U_2 \setminus \{w_1\}$. Consider the graph $\overline{G}[V(S_{n-2}) \cup \{w_1\}]$. Now, each vertex in $V(S_{n-2}) \cup \{w_1\}$ is adjacent in G to at most n_0 vertices in $U_2 \setminus \{w_1\}$. By Lemma 6.2.8, we may assume that $\overline{G}[V(S_{n-2}) \cup \{w_1\}]$ does not contain S_5. Thus, each vertex in $\overline{G}[V(S_{n-2}) \cup \{w_1\}]$ has degree at most 3, so each vertex in $G[V(S_{n-2}) \cup \{w_1\}]$ has degree at least $n - 5$.

Now, $|U_2 \setminus \{w_1\}| = n$. Choose a vertex $a_0 \in V(S_{n-2}) \cup \{w_1\}$ and write $V(S_{n-2}) \cup \{w_1\} = \{a_0, \ldots, a_{n-5}, c_1, c_2, c_3\}$ so that each of $a_0a_1, \ldots, a_0a_{n-5}$ is an edge in $G[V(S_{n-2}) \cup \{w_1\}]$. Each vertex in $G[\{a_1, \ldots, a_{n-5}, c_1, c_2, c_3\}]$ has degree at least $n - 6$. By Lemma 6.2.5, for each $i \in \{1, \ldots, 10\}$, there are $a_{i1}, a_{i2}, a_{i3} \in \{a_1, a_2, \ldots, a_{n-5}\}$ such that $G[\{a_{i1}, a_{i2}, a_{i3}, c_1, c_2, c_3\}]$ contains a subgraph isomorphic to Z_i. Furthermore, the isomorphism can be chosen so that a_{i1} is mapped to v_1 in Z_i. Thus, $G[V(S_{n-2}) \cup \{w_1\}]$ contains $T_{n-1,|V(Z_i)|-1}(v_1; Z_i)$. If a_0 is adjacent to a vertex in $U_2 \setminus \{w_1\}$, then $G[V(S_{n-2}) \cup \{w_1\}]$ contains $T_{n,|V(Z_i)|-1}(v_1; Z_i)$.

Next, by Lemma 6.2.6, for each integer $i = 11, \ldots, 14$, there are elements $a_{i1}, a_{i2}, a_{i3} \in \{a_1, a_2, \ldots, a_{n-5}\}$ such that $G[\{a_{i1}, a_{i2}, a_{i3}, c_1, c_2, c_3\}]$ contains a subgraph isomorphic to Z_i. Furthermore, the isomorphism can be chosen so that a_{i1} is mapped to v_1 and a_{i2} is mapped to v_2 in Z_i. Thus, $G[V(S_{n-2}) \cup \{w_1\}]$ contains $T_{n-1,4}(v_1, v_2; X_{i1}, X_{i2})$. If a_0 is adjacent to a vertex in $U_2 \setminus \{w_1\}$, then $G[V(S_{n-2}) \cup \{w_1\}]$ contains $T_{n,4}(v_1, v_2; X_{i1}, X_{i2})$.

By Lemma 6.2.7, there are elements $a_{j_1}, a_{j_2}, a_{j_3} \in \{a_1, \ldots, a_{n-5}\}$ such that $G[\{a_{j_1}, a_{j_2}, a_{j_3}, c_1, c_2, c_3\}]$ contains a subgraph U isomorphic to Z_{15}. Furthermore, the isomorphism can be chosen so that a_{j_1} is mapped to v_1, a_{j_2} is mapped to v_2 and a_{j_3} is mapped to v_3 in Z_{15}. Therefore, $G[V(S_{n-2}) \cup \{w_1\}]$ contains the subgraph $T_{n-1,3}(v_1, v_2, v_3; X_1, X_2, X_3)$. If a_0 is adjacent to a vertex in $U_2 \setminus \{w_1\}$, then $G[V(S_{n-2}) \cup \{w_1\}]$ contains $T_{n,3}(v_1, v_2, v_3; X_1, X_2, X_3)$.

Hence, we may assume that a_0 is not adjacent to any vertex in $U_2 \setminus \{w_1\}$. Since a_0 was chosen arbitrarily, no vertex in $V(S_{n-2}) \cup \{w_1\}$ is adjacent to any vertex in $U_2 \setminus \{w_1\}$. Choose any vertices $d_1, \ldots, d_4 \in V(S_{n-2}) \cup \{w_1\}$. Now, $|U_2 \setminus \{w_1\}| = n$ and none of $d_1, \ldots, d_4$ is adjacent to any vertex in $U_2 \setminus \{w_1\}$. Thus by Lemma 6.2.11, $G[U_2 \setminus \{w_1\}]$ contains $T_{n,|V(Z_i)|-1}(v_1; Z_i)$ for $1 \leq i \leq 10$, G contains $T_{n,4}(v_1, v_2; X_{i1}, X_{i2})$ for $11 \leq i \leq 14$ and G contains $T_{n,3}(v_1, v_2, v_3; X_1, X_2, X_3)$.

This completes the proof of the lemma. □

6.3 Ramsey numbers for large tree graphs with maximum degree of at most $n-6$ versus the wheel graph of order 9

Now, we present the Ramsey number $R(T_n, W_8)$ for large tree with $\Delta(T_n) \leq n-6$.

Theorem 6.3.1. *Let $k \geq 5$ be a positive integer and $T = T_{n,k}(v_1, \ldots, v_t; Q_1, \ldots, Q_t)$ be the tree defined in Definition 6.1.1. Then there is a positive integer $n_0(k)$ such that, for each integer $n \geq n_0(k)$, $R(T, W_8) = 2n - 1$.*

Proof. Clearly, $G' = K_{n-1} \cup K_{n-1}$ does not contain T and $\overline{G'}$ does not contain W_8. So, $R(T, W_8) \geq 2n - 1$.

Let G be a graph with $2n - 1$ vertices such that $\overline{G}$ does not contain W_8. Let $Z_1, \ldots, Z_{15}$ be defined as in Lemmas 6.2.1-6.2.3. For $11 \leq i \leq 14$, let $Z_i = X_{i1} \cup X_{i2}$ where X_{i1} is a tree and X_{i2} is an edge disjoint from X_{i2}. Let $Z_{15} = X_1 \cup X_2 \cup X_3$ where X_1, X_2, X_3 are disjoint edges. By Lemma 6.2.12, G contains $T_{n,|V(Z_i)|-1}(v_1; Z_i)$ for $1 \leq i \leq 10$, $T_{n,4}(v_1, v_2; X_{i1}, X_{i2})$ for $11 \leq i \leq 14$ and $T_{n,3}(v_1, v_2, v_3; X_1, X_2, X_3)$.

Without loss of generality, assume that $|V(Q_1)| \geq |V(Q_2)| \geq \cdots \geq |V(Q_t)| \geq 2$.

Suppose that $t = 1$. By Lemma 6.2.1, the subtree Q in $T = T_{n,k}(v_1; Q)$ contains Z_i for some $i \in \{1, \ldots, 10\}$. By Lemma 6.2.12(a), G contains $T_{n,|V(Z_i)|-1}(v_1; Z_i)$. By Corollary 6.2.10, G contains T.

Suppose that $t = 2$. By Lemma 6.2.2, the subforest $Q_1 \cup Q_2$ in the graph $T = T_{n,k}(v_1, v_2; Q_1, Q_2)$ contains Z_i for some $i \in \{11, \ldots, 14\}$. By Lemma 6.2.12(b), G contains $T_{n,4}(v_1, v_2; X_{i1}, X_{i2})$. By Corollary 6.2.10, G contains T.

Suppose that $t \geq 3$. By Lemma 6.2.3, the subforest $Q_1 \cup Q_2 \cup Q_3$ in T contains Z_{15}. By Lemma 6.2.12(b), G contains $T_{n,3}(v_1, v_2, v_3; X_1, X_2, X_3)$. By Corollary 6.2.10, G contains T.

This completes the proof of the theorem. □

Corollary 6.3.2. *Let $k \geq 5$ be a positive integer and T be a tree with n vertices and $\Delta(T) = n - k - 1$. Then there is a positive integer $n_0(k)$ such that, for each integer $n \geq n_0(k)$, $R(T, W_8) = 2n - 1$.*

Proof. Note that $T = T_{n,k}(v_1, \ldots, v_t; Q_1, \ldots, Q_t)$ for some disjoint trees $Q_1, \ldots, Q_t$. The corollary then follows from Theorem 6.3.1. $\square$

Note that if T is one of the graphs $S_n(\ell, k)$, $S_n(k)$ or $S_n[k]$, and $\Delta(T) = n-k-1$, then the following corollary follows from Corollary 6.3.2.

Corollary 6.3.3. *Let $k \geq 5$ be a fixed positive integer. For sufficiently large n, $R(T, W_8) = 2n - 1$ for each $T = S_n(\ell, k)$, $S_n(k)$, $S_n[k]$.*

Chapter 7

Conclusion and possible future work

7.1 Conclusion

Chen, Zhang and Zhang [18] conjectured that $R(T_n, W_m) = 2n-1$ for all tree graphs T_n of order $n \geq m-1$ when m is even and the maximum degree $\Delta(T_n)$ "is not too large". This conjecture was further refined by Hafidh and Baskoro [33] who specified the bound $\Delta(T_n) \leq n-m+2$. When n is large compared to m, $\Delta(T_n)$ is not required to be small: the refined conjecture then implies that, for each fixed even integer m, all but a vanishing proportion of the tree graphs T_n with $n \geq m-1$ satisfy $R(T_n, W_m) = 2n-1$.

Throughout this thesis, the aim has been to explore and partially verify this conjecture. We determined the Ramsey numbers $R(T_n, W_8)$ for all tree graphs T_n of order $n \geq 5$ with maximal degree $\Delta(T_n) \geq n-5$; see Chapters 4 and 5.

These Ramsey numbers show that the proportion of tree graphs T_n satisfying the equality $R(T_n, W_8) = 2n-1$ quickly grows as the maximal degree $\Delta(T_n)$ decreases. When $\Delta(T_n) \geq n-2$, no tree graph T_n satisfies the equality. In contrast when $\Delta(T_n) = n-3$, roughly one third of all tree graphs T_n satisfy the equality. When $\Delta(T_n) = n-4$, more than 85% of all tree graphs T_n satisfy the equality. And when $\Delta(T_n) = n-5$, roughly 94.7% of all tree graphs T_n satisfy the equality. Moreover, in Chapter 6, we proved that the Ramsey number $R(T_n, W_8)$ equals $2n-1$ for all tree graphs of sufficiently large order n. These results lend strong support for the conjecture described above by Chen et al. and Hafidh and Baskoro.

In Chapter 3, we used Theorem 2.2.2 to find the Ramsey number $R(T_n, W_{s,6})$ by applying Lemma 3.1.1 repeatedly. We can apply Lemma 3.1.1 similarly for $R(T_n, W_{s,8})$, especially for those tree graphs with $R(T_n, W_8) = 2n-1$.

Definition 7.1.1. *Let $\mathcal{T}$ be the family consisting of the following tree graphs:*

1. $S_n(2,1)$ *for odd* $n \geq 7$*;*
2. $S_n(3)$ *for odd* $n \geq 9$*;*
3. $S_n(1,3)$, $T_A(n)$ *or* $T_B(n)$ *for* $n \geq 7$ *and* $n \not\equiv 0 \pmod 4$*;*
4. $S_n[4]$, $S_n(1,4)$, $S_n(2,2)$, $T_D(n)$ *or* $T_N(n)$ *for* $n \geq 9$ *and* $n \not\equiv 0 \pmod 4$*;*
5. $T_C(n)$, $S_n(3,1)$, $S_n(5)$, $S_n[5]$, $S_n(4,1)$, $T_G(n)$, $T_H(n)$, $T_J(n)$, $T_K(n)$, $T_L(n)$, $T_M(n)$, $T_P(n)$, $T_Q(n)$, $T_R(n)$ *or* $T_S(n)$ *for all* $n \geq 8$*;*
6. $S_n(4)$, $T_E(n)$ *or* $T_F(n)$ *for all* $n \geq 9$*;*
7. T_n *with* $\Delta(T_n) \leq n-6$ *and sufficiently large* n*.*

Theorem 7.1.2. *Let $n \geq 7$ and $s \geq 2$. For all $T \in \mathcal{T}$,*

$$R(T, W_{s,8}) = (s+1)(n-1)+1\,.$$

Proof. By the various theorems in Chapters 4, 5 and 6, $R(T, W_{1,8}) = 2n - 1$. By applying Lemma 3.1.1 repeatedly, we conclude that $R(T, W_{s,8}) \leq (s+1)(n-1)+1$. Furthermore, since $\chi(W_{s,8}) = s+2$ and $t(W_{s,8}) = 1$, Theorem 2.2.7 implies that $R(T, W_{s,8}) \geq (s+1)(n-1)+1$. Hence, $R(T, W_{s,8}) = (s+1)(n-1)+1$. □

Similarly, we have the following result for $W_{s,9}$.

Theorem 7.1.3. *Let $n \geq 7$ and $s \geq 1$. For all $T \in \mathcal{T}$,*

$$R(T, W_{s,9}) = (s+2)(n-1)+1\,.$$

Proof. By Theorem 2.2.7, $\chi(W_{s,9}) = s+3$ and $t(W_{s,9}) = 1$. Therefore, for any tree graph T of order n, $R(T, W_{s,9}) \geq (s+2)(n-1)+1$. Since $W_{s,9}$ is a subgraph of $W_{s+1,8}$, Theorem 3.3.1 implies that $R(T, W_{s,9}) \leq R(T, W_{s+1,8}) = (s+2)(n-1)+1$. Hence, $R(T, W_{s,9}) = (s+2)(n-1)+1$. □

7.2 Possible future work

As described in Section 3.4, we propose Conjecture 3.4.1, here restated as follows.

Conjecture. *Suppose that $m \geq 3$ and $s \geq 2$. Then for sufficiently large n,*

$$R(T_n, W_{s,m}) = \begin{cases} (s+1)(n-1)+1, & \text{if } m \text{ is even;} \\ (s+2)(n-1)+1, & \text{if } m \text{ is odd.} \end{cases}$$

For $m = 8$ and $m = 9$, we have proved that this conjecture is true for all tree graphs $T \in \mathcal{T}$. To complete all of the cases, we need to find the analogous results for all other trees separately.

Furthermore, in Chapters 4 and 5, we have determined the Ramsey numbers $R(T_n, W_8)$ for all tree graphs T_n with maximum degree of at least $n-5$ versus the wheel graph W_8. In Chapter 6, we have determined the Ramsey numbers $R(T_n, W_8)$ for all tree graphs T_n with maximum degree of at most $n-6$ where n is sufficiently large versus W_8. To determine the remaining Ramsey numbers $R(T_n, W_8)$, the next step would be to focus on the smaller tree graphs with maximum degree of at most $n-6$.

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