The cone of quasi-semimetrics and exponent matrices of tiled orders

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Abstract

Finite quasi semimetrics on n can be thought of as nonnegative valuations on the edges of a complete directed graph on n vertices satisfying all possible triangle inequalities. They comprise a polyhedral cone whose symmetry groups were studied for small n by Deza, Dutour and Panteleeva. We show that the symmetry and combinatorial symmetry groups are as they conjectured.

Integral quasi semimetrics have a special place in the theory of tiled orders, being known as exponent matrices, and can be viewed as monoids under componentwise maximum; we provide a novel derivation of the automorphism group of that monoid. Some of these results follow from more general consideration of polyhedral cones that are closed under componentwise maximum.

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1. Introduction

Metric spaces are ubiquitous, making metrics a well known concept. Quasi-semimetrics (the term is not totally standard) are a weakened form of metrics: for a given space X, $d(x,y)$ is required to be a nonnegative real, and the triangle inequality $d(x,y) \leqslant d(x,z) + d(z,y)$ has to be satisfied. It relaxes two requirements on the definition of a metric, namely, it is not required to be symmetric, and distinct elements are allowed to be at “distance” zero. Usually, a space with a fixed metrics or quasi-semimetrics is studied, for the implied geometrical and topological structure. There is a plethora of such specific examples in [12].

Here we take a different viewpoint: we fix the space X and consider the set of all quasi-semimetrics on it as the main object. Quasi-semimetrics form a convex cone of real functions on $X \times X$; for finite X, that is a rational polyhedral cone, which we denote $\hat{\mathcal{E}}_X$ (also $\hat{\mathcal{E}}_n$ if $X = [n] = \{1, 2, \ldots, n\}$) and it has been the object of some study (Deza, Dutour, and Panteleeva [9], Deza, Deza, and Dutour Sikirić [10], Deza and Panteleeva [11] and Deza, Deza, and Dutour Sikirić [8], which denote it as $QMET_n$). It is convenient, for the discussion below, to have the explicit description of $\hat{\mathcal{E}}_n$ as the subset of $M_n(\mathbb{R})$ consisting of matrices $X = (x_{ij})$

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such that for all pairwise distinct i, j, k,

$$\begin{array}{lrl} T_{ijk}: & x_{ij} + x_{jk} & \geqslant x_{ik}\,, \\ N_{ij}\ : & x_{ij} & \geqslant 0\,, \\ & x_{ii} & = 0\,. \end{array} \tag{1}$$

It is also convenient to think of such X as an assignment of values to the edges of the complete directed graph with vertex set $[n]$. Following [10], the T_{ijk} are called *triangle inequalities*, while N_{ij} are *nonnegativity inequalities*. Also, to avoid special cases requiring definition acrobatics, we stipulate that $n \geqslant 3$, whenever $\hat{\mathcal{E}}_n$ is considered.

This family of cones is quite thoroughly discussed in [10], and one aspect will be relevant here: the determination of their (Euclidean) symmetry group and combinatorial symmetry group (those are defined in Section 2).

Given the description of $\hat{\mathcal{E}}_n$, there is a natural class of linear maps that preserve the cone: permutations of coordinates that leave the system describing $\hat{\mathcal{E}}_n$ invariant. Those are of two types:

a. Any permutation π on the indices induces the permutation $P_\pi : x_{ij} \mapsto x_{\pi(i)\pi(j)}$.
b. The transpose map (called reversal in [8]) τ: $x_{ij} \mapsto x_{ji}$.

We call those *system automorphisms*, and denote the group they form by $\mathcal{S}_n$. Since τ commutes with all permutations of the first type, it follows that $\mathcal{S}_n \cong S_n \times \mathbb{Z}_2$.

These maps are isometries, so they are symmetries of cone; they naturally induce a subgroup of the combinatorial automorphism group of $\hat{\mathcal{E}}_n$, which we also will denote as $\mathcal{S}_n$. Deza et al. [10] verified computationally that $\mathcal{S}_n$ is the whole symmetry and combinatorial symmetry group of $\hat{\mathcal{E}}_n$ for small n and conjectured (also in [8]) that this was the case in general. Our main result here settles those conjectures:

Theorem 3.7. *The combinatorial symmetry group of $\hat{\mathcal{E}}_n$ is $\mathcal{S}_n$.*

As every cone, $\hat{\mathcal{E}}_X$ is a semigroup under addition; it is, moreover, closed on an additional operation, which also turns it into a commutative semigroup: componentwise maximum. It is convenient to use the infix notation $a \oplus b = \max(a, b)$, for real a, b, and extend the notation componentwise to real vectors or matrices: if $u, v \in \mathbb{R}^I$, $u \oplus v$ is defined by $(u \oplus v)_i = u_i \oplus v_i$; we refer to this operation simply as max. These two operations (and the addition of a $-\infty$) turn $\mathbb{R}^X$ into a semiring, a coproduct of copies of the tropical semiring. In this context, $\hat{\mathcal{E}}_X$ is a subsemiring of $\mathbb{R}^X$, but not, however, a subsemimodule.

We will be especially interested in integer valued quasi-semimetrics, and denote $\mathcal{E}_n = \hat{\mathcal{E}}_n \cap \mathbb{Z}^n$. There are many reasons to concentrate on the integer points in a rational cone (see [6], [3]), and in this particular case they have appeared in a context far removed from the usual study of polyhedra, the theory of tiled orders in algebras. Those are described based on a discrete valuation ring and a matrix, which, by [16, Lemma 1.1], is an integer valued quasi-semimetrics; in that context, those have been called *exponent matrices*. We refer to [16], [1], [7], [14] for definitions, more details and some applications; we will not mention tiled orders any more here, but honor them in the notation $\hat{\mathcal{E}}_X$.

Clearly $\mathcal{S}_n$ respects integrality and the operations of addition and max, so it also restricts to $(\mathcal{E}_n, +)$ and $(\mathcal{E}_n, \oplus)$ automorphisms. A natural question is whether new symmetries or automorphisms can crop up if we consider each of those monoids.

That is not the case: it was proved in [14] that the automorphism group of $(\mathcal{E}_n, +)$ and $(\mathcal{E}_n, \oplus)$ is $\mathcal{S}_n$. The proofs were considerably ad-hoc and elaborate, and here we will present new, and somewhat more conceptual proofs.

Any automorphism of $(\mathcal{E}_n, +)$ naturally extend to a linear automorphisms of $\hat{\mathcal{E}}_n$, thus respecting its face-lattice, as well as any symmetry of $\hat{\mathcal{E}}_n$ does. So, our main theorem implies that both groups coincide with $\mathcal{S}_n$ (we elaborate on that in Section 2).

For some perspective on this result, we refer the reader to [5]; there, some elegant algorithms for computing the symmetry and combinatorial symmetry group of a given polyhedral cone are shown; both are reduced to finding the automorphism group of a colored graph. There is a difference, though. The graph for the symmetry group has as vertices the extreme rays or the facets of the cone, so it has polynomial size in the description of the cone. For the combinatorial symmetry group, the graph is the incidence graph of extreme rays and facets; this may have exponential size relative to a given description, and this is indeed the case with $\hat{\mathcal{E}}_n$. This method underlies our proof of the main theorem, but we were able to finesse the problem of describing the extreme rays by showing that just a small part of them suffices.

The automorphism group of $(\mathcal{E}_n, \oplus)$ are derived from a more general view of cones closed under max, and their integer point submonoids. In this case, in general, a $\oplus$-automorphism does not need to respect the face-lattice (Example 5.1), and may even not be extendable to the whole cone. We present some conditions which imply that the automorphisms of the integer submonoid of a max-closed cone are induced by permutations of the coordinates. This is enough to show that $\mathrm{Aut}(\mathcal{E}_n, \oplus) = \mathrm{Aut}(\mathcal{E}_n, +)$

The article proceeds as follows: We start by recalling some basics on polyhedral cones and proving some initial facts in Section 2. Section 3 proves the main theorem. Section 4 presents some basic facts about max-closed cones. This is followed by Section 5 where we prove the aforementioned result on $\oplus$-automorphism of the integer submonoid of a max-closed cone, entailing, in particular, that $\mathrm{Aut}(\mathcal{E}_n, \oplus) = \mathcal{S}_n$.

2. Preliminaries on polyhedral cones

We present here a summary of facts and terminology about polyhedral cones; some of these have been appropriately streamlined for our needs. For more detailed information and proofs the reader is referred to [6, 8, 18]. Besides the definitions, we make several assertions about cones without further ado; they are well-known facts that can be found in the references, and are easy exercises. As the cones we are interested in are the $\hat{\mathcal{E}}_n$, we illustrate the concepts as they directly apply to them.

In what follows, I, J, N will denote finite sets; $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{R}_+$, $\mathbb{N}$ will stand for the sets of real numbers, integers, non-negative reals and non-negative integers, respectively. In the vector space $\mathbb{R}^N$ we single out the *canonical basis vectors* e_i and the *one vector* $\mathbf{1} = \sum_i e_i$; on $\mathbb{R}^N$, $x \leqslant y$ means $x_i \leqslant y_i$ for all $i \in N$, and $x \geqslant y$ means $y \leqslant x$. The *support* of $v \in \mathbb{R}^N$ is $\mathrm{supp}(v) = \{i \in N \mid v_i \neq 0\}$ and its cardinality will be denoted $s(v)$. A subset of $\mathbb{R}^N$ is *full-dimensional* if it linearly spans the whole space. We will consider subsets S, S' of $\mathbb{R}^N$ to be equivalent if there is a bijection between S and S' such that the image of each vector is a positive scalar multiple of it. A *ray* is an equivalence class of a nonzero singleton, and we will say that S is *clean* if its elements belong to different rays. A point (or ray) x that satisfies a linear inequality $ax \geqslant 0$, does it *exactly* if $ax = 0$ and *strictly* if $ax > 0$. A ray is *rational* if it contains a vector with rational coordinates.

Example 2.1: *As defined, the cone $\hat{\mathcal{E}}_n$ lies in the subspace of $n \times n$ real matrices with null diagonal; it is convenient to consider this subspace to be the whole ambient space. So, for a fixed n, let $N_n = \{(i,j) \mid 1 \leqslant i, j \leqslant n, i \neq j\}$, and we take the space $\mathbb{R}^{N_n}$ to be the one where $\hat{\mathcal{E}}_n$ is defined. The vectors in this space are still better visualized (and referred to) as matrices with a blotted diagonal, rather than a linear list of coordinates. Written in the format $ax \geqslant 0$, the defining inequalities T_{ijk} take the form $x_{ij} + x_{jk} - x_{ik} \geqslant 0$, whose coefficient vector a has support of size 3.*

A finite set of non-zero vectors S is said to be a $\mathscr{V}$*-description* of the set $\{\sum_{v\in S} \lambda_v v \mid \lambda_v \in \mathbb{R}_+ \text{ for all } v \in S\}$, and S is also called an $\mathscr{H}$*-description* of $\{x \in \mathbb{R}^N \mid v^t x \geqslant 0 \text{ for all } v \in S\}$. We may think of S as the set of rows of a matrix A; then S is an $\mathscr{H}$-description of $\{x \in \mathbb{R}^N \mid Ax \geqslant 0\}$. The *Weyl-Minkowski Theorem* (see [18] and [6]) states that a set has a $\mathscr{V}$-description if and only if it has an $\mathscr{H}$-description; further, it has a $\mathscr{V}$-description with rational rays if and only if it has an $\mathscr{H}$-description in which the matrix has only rational entries. A set with either description is called a *polyhedral cone*, and it is a *rational polyhedral cone* if it has either description using only rational data. Clearly, equivalent sets describe the same cones; either way, just clean descriptions suffice.

A cone $\mathcal{C}$ is *pointed* if the only linear subspace it contains is (0). A cone $\mathcal{C}$ is full-dimensional if there is a point that satisfies all inequalities of an $\mathscr{H}$-description strictly.

Example 2.2: *$\hat{\mathcal{E}}_n$ was defined by inequalities; that is, we have an explicit $\mathscr{H}$-description of $\hat{\mathcal{E}}_n$, the corresponding set S consisting of coefficient vectors (matrices, actually) of those inequalities. The coefficients are just $0, 1, -1$, which shows that $\hat{\mathcal{E}}_n$ is a rational polyhedral cone, and also that the description is clean. Moreover, each inequality induces a facet (a concept defined below and a fact proved in Section 3). The matrix $\mathbf{1}$ satisfies all inequalities strictly, showing that $\hat{\mathcal{E}}_n$ is full-dimensional; the nonnegativity inequalities easily imply that $\hat{\mathcal{E}}_n$ is pointed.*

The Weyl-Minkowski Theorem implies that $\hat{\mathcal{E}}_n$ also has a $\mathscr{V}$-description. Describing it explicitly is a possibly impossible task. In [9] and [8] there are explicit descriptions of the rays for $n \leqslant 4$. After that, there are descriptions of some families and some computational results. The number of rays grows exponentially with n^2, so computations quickly stop short. However, a small family of rays described in Section 3 will be crucial in the proof of the main theorem.

A linear inequality $ax \geqslant 0$, with $a \neq 0$ that holds for every $x \in \mathcal{C}$ is a *valid inequality* for $\mathcal{C}$; the *face* of $\mathcal{C}$ it *induces* is the set $\{x \in \mathcal{C} \mid ax = 0\}$. We also consider $\mathcal{C}$ a (improper) face. The faces of a cone, ordered by inclusion, comprise a lattice, the *face-lattice* of the cone, with intersection as the meet operation. The face lattice is finite and graded. A *facet* is a maximal proper face. If $\mathcal{C}$ is full-dimensional, every facet is induced by a unique (up to equivalence) inequality, and the collection of such *facet-inequalities* comprises the unique minimal $\mathscr{H}$-description of $\mathcal{C}$. The face-lattice is *coatomistic*, that is, every proper face is an intersection of facets; equivalently, in any $\mathscr{H}$-description, a face is a subset of $\mathcal{C}$ that satisfies some fixed subset of the inequalities exactly. A point is *interior* to a face if the valid inequalities it satisfies exactly are precisely those that induce the face; every face has an interior point. Equivalently, a point p is interior to a face F if and only if the facets containing p are those that contain F; in particular, in a clean $\mathscr{H}$-description, a facet-inequality is one such that there is a point satisfying that one exactly, and all other inequalities strictly. It also follows that $\mathcal{C}$ is full-dimensional if and only if it has an interior point. If the cone is pointed, the minimal non-zero faces are rays, so called *extreme rays*, and these comprise the unique minimal $\mathscr{V}$-description of the cone. The face-lattice is also *atomistic*: every face is a join of extreme rays.

An *integer cone* is the intersection $\mathcal{C}_\mathbb{Z}$ of a rational cone $\mathcal{C}$ with $\mathbb{Z}^N$. Such a cone is an additive submonoid of $\mathbb{Z}^N$, and it is finitely generated. If $\mathcal{C}$ is pointed, there exists a unique minimal set of generators, called a *Hilbert basis*, and it is finite (see [18, Theorem 16.4], [6, Chapter 2]); it contains one point in each extreme ray, and usually some more points.

The presentation above relies on a fixed system of coordinates, given by the basis of elementary vectors. A more elegant, coordinate free approach is used in [6], and it gives an account of all relevant concepts related to the face-lattice. However, working with a fixed basis comes naturally when handling systems of linear inequalities; moreover, the max operation is naturally and traditionally defind based on coordinates.

A *linear automorphism* of a cone $\mathcal{C} \subseteq \mathbb{R}^N$ is a linear automorphism φ of $\mathbb{R}^N$ such that $\varphi(\mathcal{C}) = \mathcal{C}$. If φ is an isometry preserving Euclidean distance, it is said to be a *isometry* of $\mathcal{C}$ ([8] calls it a *symmetry* of $\mathcal{C}$). It is clear from the definition that any linear automorphism of

$\mathcal{C}$ maps faces to faces, and induces an automorphism of the face lattice of $\mathcal{C}$; in particular, the families of extreme rays and of facets are invariant.

We single out four symmetry groups associated with a given cone $\mathcal{C}$ (we combine the notation of [5] and [8], with occasional slight change of meaning):

- $\mathrm{Comb}(\mathcal{C})$ - the *combinatorial symmetry group*, consisting of all automorphisms of the face-lattice of $\mathcal{C}$.
- $\mathrm{Lin}(\mathcal{C})$ - the *linear symmetry group*, consisting of all linear automorphisms of $\mathcal{C}$.
- $\mathrm{Symm}(\mathcal{C})$ - the *symmetry group*, consisting of all isometries of $\mathcal{C}$ (named as in [8]).
- $\mathrm{Lin}_{\mathbb{Z}}(\mathcal{C})$ - the *integral symmetry group*, consisting of all linear automorphisms leaving $\mathcal{C} \cap \mathbb{Z}^N$ invariant.

So, any linear automorphism of $\mathcal{C}$ induces an automorphism of the face lattice of $\mathcal{C}$, and this induction is indeed a group homomorphism $\mathrm{ind}\colon \mathrm{Lin}(\mathcal{C}) \to \mathrm{Comb}(\mathcal{C})$. As the face-lattice is both atomistic and coatomistic, both the set of facets and the set of extreme rays are bases for the permutation group $\mathrm{Comb}(\mathcal{C})$; that is, any element of this group is fully determined by its action on either set. The approach favored in [5] is to represent automorphisms by their action on extreme rays, while in Section 3 we find it convenient to represent them by their action on the facets. After all, convenience depends on the available description of the cone.

Proposition 2.1. *If $\mathcal{C}$ is a full dimensional pointed rational cone, the restrictions of* ind *to* $\mathrm{Symm}(\mathcal{C})$ *and* $\mathrm{Lin}_{\mathbb{Z}}(\mathcal{C})$ *are injective.*

Proof. Consider first the restriction to $\mathrm{Lin}_{\mathbb{Z}}(\mathcal{C})$. If φ is in the kernel of this map, it leaves each ray invariant. But the set of integral vectors in the ray is also invariant, and that implies that φ is the identity on that ray. So it is the identity automorphism. For $\mathrm{Symm}(\mathcal{C})$, we apply the same argument to the unit vector in each ray. □

EXAMPLE 2.3: *Notice that, in spite of the similarity exposed in the proof of Proposition 2.1,* $\mathrm{Symm}(\mathcal{C})$ *and* $\mathrm{Lin}_{\mathbb{Z}}(\mathcal{C})$ *can be quite different. Consider the cone* $\mathcal{C}_1 = \{x \in \mathbb{R}^2 \mid x_2 \geqslant 0, x_1 - x_2 \geqslant 0\}$*; the map given by* $\left(\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\right)$ *is in* $\mathrm{Lin}_{\mathbb{Z}}(\mathcal{C}_1)$ *but not in* $\mathrm{Symm}(\mathcal{C}_1)$*. On the other hand, for* $\mathcal{C}_2 = \{x \in \mathbb{R}^2 \mid x_2 \geqslant 0, 3x_1 - 4x_2 \geqslant 0\}$*, the map given by* $\frac{1}{5}\left(\begin{smallmatrix} 4 & 3 \\ 3 & -4 \end{smallmatrix}\right)$ *is in* $\mathrm{Symm}(\mathcal{C}_2)$ *but not in* $\mathrm{Lin}_{\mathbb{Z}}(\mathcal{C}_2)$*.*

The restrictions of ind above (actually, ind itself) can be far from surjective. To see this, take your favorite highly symmetric cone and apply to it a linear transformation that is neither orthogonal nor integral. The poor image's symmetry and integral symmetry groups becomes severely handicapped, while the combinatorial symmetry group gets away scot-free.

The astute reader may complain that this is a trick; a deeper construction of Bokowski, Ewald, and Kleinschmidt [4] presents a polytope lattice and a combinatorial symmetry that cannot be realized linearly for any polytope with that face-lattice. A standard construction turns this into a result about cones.

Going back to the cone $\hat{\mathcal{E}}_n$, we notice that $\mathcal{S}_n$ consists of maps that are both isometries and integral, that is, we have the following diagram of monomorphisms:

As it turns out in Theorem 3.8, the composition $\mathcal{S}_n \to \mathrm{Comb}(\hat{\mathcal{E}}_n)$ is surjective, hence all inclusions are equalities.

3. Symmetries and combinatorial automorphisms of $\hat{\mathcal{E}}_n$

The main result in this section is Theorem 3.7, which describes the combinatorial automorphism group of $\hat{\mathcal{E}}_n$. For $n \leqslant 5$ this has been done in [9], computationally. As noted before, while some shortcuts exist, the general method for computing the combinatorial automorphism

$$
\begin{tikzcd}
 & & \mathrm{Symm}(\hat{\mathscr{E}}_n) \arrow[dr, "\mathrm{ind}"] & \\
\mathcal{S}_n \arrow[r, "\subseteq"] & \mathrm{Symm}(\hat{\mathscr{E}}_n) \cap \mathrm{Lin}_{\mathbb{Z}}(\hat{\mathscr{E}}_n) \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"] & & \mathrm{Comb}(\hat{\mathscr{E}}_n) \\
 & & \mathrm{Lin}_{\mathbb{Z}}(\hat{\mathscr{E}}_n) \arrow[ur, "\mathrm{ind}"] &
\end{tikzcd}
$$

Figure 1:

group of a cone is to determine the bipartite incidence graph of extreme rays and facets, and then computing the automorphism group of the graph. Although no polynomial algorithm is yet known for such computation, there are very good programs [17] that can handle graphs of fairly large size.

As per Corollary 3.3, $\hat{\mathscr{E}}_n$ has $n^2(n-1)/2$ facets. However, [9] tells us that for $n \geqslant 6$, the number of extreme rays is already too big for polite computational society.

As it turns out, there is an orbit of $\mathcal{S}_n$, denoted $\mathscr{L}_n$, consisting of $2n$ extreme rays, such that it is enough to consider the incidence graph of facets and $\mathscr{L}_n$ to clinch $\mathrm{Comb}(\hat{\mathscr{E}}_n)$; the following nice properties hold:

a. $\mathscr{L}_n$ is an orbit of $\mathrm{Comb}(\hat{\mathscr{E}}_n)$ (Lemma 3.10).
b. The action of $\mathrm{Comb}(\hat{\mathscr{E}}_n)$ on $\mathscr{L}_n$ is the same as the action of $\mathcal{S}_n$ (Lemma 3.12).
c. The action above is faithful.

The last item is what will finally establish the main theorem.

Recall that T_{ijk} and N_{ij} are the defining inequalities for $\hat{\mathscr{E}}_n$; in what follows, the same labels in boldface ($\mathbf{T}_{ijk}$, etc.) will denote the corresponding faces of $\hat{\mathscr{E}}_n$, which turn out to be facets of $\hat{\mathscr{E}}_n$.

The following facts about some special members of $\hat{\mathscr{E}}_n$ appear in part in [9, Theorem 5], and in [14, Theorem 1.1]. We present them here with proofs, for completeness. For each proper subset I of $\{1, \ldots, n\}$, the associated *oriented cut quasi-semimetrics* ([9, 11, 12]) is the binary exponent matrix $\delta(I)$ such that $\delta(I)_{ij} = 1$ if and only if $i \in I, j \notin I$.

Proposition 3.1. *Considering* $\hat{\mathscr{E}}_n$*:*

a. *The oriented cut quasi-semimetrics are those with minimal nonempty supports.*
b. *If* $A \in \hat{\mathscr{E}}_n$ *and* $\mathrm{supp}(A) = \mathrm{supp}(\delta(I))$*, then* A *is a scalar multiple of* $\delta(I)$*.*
c. *All oriented cut quasi-semimetrics are extreme rays of* $\hat{\mathscr{E}}_n$*.*

Proof. For (a), let $A \in \hat{\mathscr{E}}_n$, and suppose $A_{rs} > 0$. Let $I = \{k \mid A_{rk} = 0\}$; this is a proper subset of indices, as $r \in I$, $s \notin I$. Then, if $i \in I, j \notin I$, $A_{ri} = 0 \neq A_{rj}$, and T_{rij} implies $A_{ij} > 0$. It follows that $\mathrm{supp}(\delta(I)) \subseteq \mathrm{supp}(A)$. For (b), let $i \in I$ and suppose there exist distinct $j, k \notin I$. Applying T_{ijk} and T_{ikj} we conclude that $A_{ij} = A_{jk}$; that is, all nonzero terms on each row of A are equal; the same argument applies to columns. So, all nonzero elements of A are equal, and the result follows. Finally, for (c), let A be an interior point of the minimal face containing $\delta(I)$. It must satisfy exactly the same inequalities as $\delta(I)$; in particular, the same N_{ij}, hence $\mathrm{supp}(A) = \mathrm{supp}(\delta(I))$. From part (b), it follows that the face has dimension 1. $\square$

This is a technical workhorse for what follows:

Lemma 3.2. *For all three distinct indices* i, j, k *the only defining inequalities of* $\hat{\mathscr{E}}_n$ *exactly satisfied by* $\mathbf{N}_{ij} \cap \mathbf{N}_{jk}$ *are* N_{ij}, N_{jk}, N_{ik} *and* T_{ijk}*.*

Proof. If $x \in \mathbf{N}_{ij} \cap \mathbf{N}_{jk}$, then $x_{ij} = x_{jk} = 0$, and T_{ijk} implies that $x_{jk} \leqslant 0$. This implies that x satisfies both N_{ik} and T_{ijk} exactly.

In order to show that no other inequality is satisfied exactly, we construct an exponent matrix $H = H(i,j,k)$ as follows. For distinct r, s,

$$H_{rs} = \begin{cases} 0 & \text{if } rs = ij, jk, ik, \\ 3 & \text{if } rs = ji, kj \quad \text{or} \quad r = i, s \neq j, k \quad \text{or} \quad s = k, r \neq i, j, \\ 4 & \text{if } rs = ki \quad \text{or} \quad r = j, s \neq i, k \quad \text{or} \quad s = j, r \neq i, k, \\ 5 & \text{otherwise.} \end{cases}$$

See Fig. 2 for an illustration.

$$\begin{pmatrix} 0 & 0 & 0 & 3 & 3 & 3 & \to \\ 3 & 0 & 0 & 4 & 4 & 4 & \to \\ 4 & 3 & 0 & 5 & 5 & 5 & \to \\ 5 & 4 & 3 & 0 & 5 & 5 & \to \\ 5 & 4 & 3 & 5 & 0 & 5 & \to \\ 5 & 4 & 3 & 5 & 5 & 0 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \end{pmatrix}$$

Figure 2: $H(1,2,3)$. Arrows mean repeat the term in that direction.

It is quite clear that the only nonnegativity inequalities satisfied by H are N_{ij}, N_{jk}, N_{ik}, and it also satisfies T_{ijk} exactly. To see that $H \in \hat{\mathcal{E}}_n$, as well as that it does not satisfy any other triangle inequality exactly can be done by case analysis. Separating the remaining T_{rst} according with $r = i, r = j, r = k$, $s = i, s = j, s = k$ (some of these cases are not mutually exclusive), and all remaining cases, leads to an easy verification that H satisfies all these T_{rst}, no one exactly. $\square$

The fact below is proved in [8] as a consequence of a method of lifting facets from $\hat{\mathcal{E}}_n$ to $\hat{\mathcal{E}}_{n+1}$; that kind of obscures its simplicity.

Corollary 3.3. *All nonnegativity and triangle inequalities are facet defining for $\hat{\mathcal{E}}_n$.*

Proof. Lemma 3.2 implies that no face induced by each nonnegativity inequality is contained in any other face, so they all induce facets. But clearly $\hat{\mathcal{E}}_n \neq \mathbb{R}^n$, so, at least one triangle inequality is facet-inducing. As the group $\mathcal{S}_n$ acts transitively on the set of triangular inequalities, they all induce facets as well. $\square$

Here is a more direct proof:

> The matrix whose ij-entry is 0 if $ij = rs$, 2 if $i = s$ or $j = r$ and 1 otherwise is an interior point of $\mathbf{N}_{rs}$. The matrix whose ij-entry is 1 if $ij = rs$ or st and is 2 otherwise is an interior point of $\mathbf{T}_{rst}$. Each of these facts can be verified by inspection.

The oriented cut quasi-semimetrics associated to sets of size 1 and $n-1$ play a very special role, already detected in [14]. Denote $R^{(r)} = \delta(\{r\})$, $C^{(r)} = \delta(\{1, \ldots, n\} \backslash \{r\}$. In matrix form, $R^{(r)}$ has row r with all ones off diagonal, and is zero elsewhere; ditto for $C^{(r)}$ and column r, so that $C^{(r)} = \tau(R^{(r)})$. We denote $\mathcal{L}_n = \{R^{(r)}, C^{(r)} \mid 1 \leqslant r \leqslant n\}$ and refer to its members as *lines*. As noted before, all rays in $\mathcal{L}_n$ are extreme in $\hat{\mathcal{E}}_n$; an alternative proof is in [14, Lemma 3.2].

Recall that $s(A)$ is the size of the support of A, that is the number of nonzero entries in A.

Proposition 3.4. *If $0 \neq A \in \hat{\mathcal{E}}_n$, then $s(A) \geqslant n-1$. If $s(A) = n-1$, then it is a multiple of some line.*

Proof. This can be read directly from Proposition 3.1, as $s(\delta(I)) = |I|(n-|I|)$. $\square$

Direct inspection shows that:

Proposition 3.5. *Recall that when mentioning T_{ijk} and N_{ij} all indices are distinct.*

a. *$R^{(r)}$ satisfies exactly only $\{T_{ijk} \mid j \neq r\}$ and $\{N_{ij} \mid i \neq r\}$.*
b. *$C^{(s)}$ satisfies exactly only $\{T_{ijk} \mid j \neq s\}$ and $\{N_{ij} \mid j \neq s\}$.*
c. *N_{ij} is satisfied exactly on $\mathscr{L}_n$ only by those $R^{(r)}$ such that $r \neq i$ and those $C^{(s)}$ such that $s \neq j$.*
d. *T_{ijk} is satisfied exactly on $\mathscr{L}_n$ only by those $R^{(r)}$ such that $r \neq j$ and those $C^{(s)}$ such that $s \neq j$.*

For each facet F, let $e(F) = \{R \in \mathscr{L}_n \mid R \text{ satisfies the } F \text{ inequality exactly}\}$. If we think of the members of $\mathscr{L}_n$ as rays, $e(F) = \{R \in \mathscr{L}_n \mid R \subseteq F\}$.

Proposition 3.6. *If for some $\mathbf{N}_{ij}$, $e(F) = e(\mathbf{N}_{ij})$, then $F = \mathbf{N}_{ij}$.*

Proof. From Proposition 3.5, $e(\mathbf{N}_{ij}) = \mathscr{L}_n \backslash \{R^{(i)}, C^{(j)}\}$, while $e(\mathbf{T}_{ijk}) = \mathscr{L}_n \backslash \{R^{(j)}, C^{(j)}\}$, and the result follows. $\square$

We now state the main result in this section.

Theorem 3.7. *The combinatorial automorphism group of $\hat{\mathcal{E}}_n$ consists precisely of those permutations of the face-lattice induced by $\mathcal{S}_n$, that is, the restriction of* ind *to $\mathcal{S}_n$ is an isomorphism.*

In view of Proposition 2.1, we have

Theorem 3.8. $\mathrm{Symm}(\hat{\mathcal{E}}_n) = \mathrm{Lin}_{\mathbb{Z}}(\hat{\mathcal{E}}_n) = \mathcal{S}_n$.

Some of these results are proved in [8] for small values of n and conjectured to hold in general.

Proof of Theorem 3.7. By (Fig. 1) and Proposition 2.1, it is enough to show that $\mathrm{ind}(\mathcal{S}_n) = \mathrm{Comb}(\hat{\mathcal{E}}_n)$. The set $\mathscr{F}$ of facets is invariant under $\mathrm{Comb}(\hat{\mathcal{E}}_n)$; since every face of $\hat{\mathcal{E}}_n$ is a meet of facets, it is enough to show that for every $\varphi \in \mathrm{Comb}(\hat{\mathcal{E}}_n)$ there exists a $\psi \in \mathcal{S}_n$ whose action on $\mathscr{F}$ coincides with that of φ.

Some Lemmas below pave the way to Lemma 3.12, which shows $\mathscr{L}_n$ is invariant under $\mathrm{Comb}(\hat{\mathcal{E}}_n)$, and each combinatorial automorphism acts on $\mathscr{L}_n$ in the same way as some automorphism induced from $\mathcal{S}_n$.

To finish the proof, let $\varphi \in \mathrm{Comb}(\hat{\mathcal{E}}_n)$, and ψ be given by Lemma 3.12; then $\gamma = (\mathrm{ind}\,\psi)^{-1}\varphi \in \mathrm{Comb}(\hat{\mathcal{E}}_n)$ is the identity on $\mathscr{L}_n$.

As γ is a combinatorial automorphism, it commutes with e. Hence, given a nonnegativity facet N_{ij}, $e(\gamma(N_{ij})) = \{\gamma(R) \mid R \in e(N_{ij})\} = e(N_{ij})$, and it follows from Proposition 3.6 that $\gamma(N_{ij}) = N_{ij}$; Lemma 3.2 implies that γ fixes the triangular facets as well. Hence γ is the identity, and $\varphi = \mathrm{ind}\,\psi$. $\square$

Those were the facts used in the proof of Theorem 3.7:

Lemma 3.9. *Any $A \in \hat{\mathcal{E}}_n$ strictly satisfies at least $(n-2)p(A) + s(A)$ defining inequalities, where $p(A)$ is the number of pairs $\{i, j\}$ such that at least one of a_{ij}, a_{ji} is positive.*

Proof. If A satisfies both T_{ijk} and T_{jik} exactly, then $a_{ij} + a_{ji} = 0$, hence these entries are 0. So, if at least one of a_{ij}, a_{ji} is positive, at least one of T_{ijk} and T_{jik} is strict for A, for each k. That gives $(n-2)p(A)$ strict inequalities. The number of strict nonnegativity inequalities is $s(A)$. $\square$

Lemma 3.10. *The set $\mathscr{L}_n$ is invariant under any combinatorial automorphism of $\hat{\mathcal{E}}_n$.*

Proof. Proposition 3.5 implies that the ray spanned by each $R^{(r)}$ and $C^{(s)}$ is contained in $(n-1)^2(n-2)$ triangular facets and $(n-1)^2$ nonnegativity ones, so it is contained in precisely $(n-1)^3$ facets. We will show that any other nonzero face is contained in fewer facets. This immediately implies the result.

Let F be a nonzero face of $\hat{\mathcal{E}}_n$ and let A be an interior point of F. As there exist $n(n-1)(n-2)$ triangular facets and $n(n-1)$ nonnegativity facets, for a total of $n(n-1)^2$, we want to show that if $F \notin \mathscr{L}_n$ then A strictly satisfies more than $(n-1)^2 = n(n-1)^2 - (n-1)^3$ inequalities.

By Proposition 3.1, we need to consider only two cases

a. There exists a subset I of $\mathbb{N}^N$ such that $2 \leqslant |I| \leqslant n-2$ and for every $i \in I, j \notin I$, $a_{ij} > 0$. Referring to Lemma 3.9, both $p(A)$ and $s(A)$ are at least $|I|(n-|I|)$; so A strictly satisfies at least $(n-1)|I|(n-|I|)$ inequalities. As $2 \leqslant |I| \leqslant n-2$, $|I|(n-|I|) > n-1$, and the result follows.
b. There exist two members of $\mathscr{L}_n$ such that the support of A contains the union of their supports, hence $s(A) \geqslant 2n-3$. Trivially, $p(A) \geqslant n-1$, so A strictly satisfies at least $(n-2)(n-1) + 2n - 3 > (n-1)^2$ inequalities.

$\square$

Denote $\mathscr{R} = \{R^{(i)} \mid i = 1, \ldots, n\}$, $\mathscr{C} = \{C^{(i)} \mid i = 1, \ldots, n\}$.

Lemma 3.11. *The partitions $(\mathscr{R}, \mathscr{C})$ and $\left(\{R^{(i)}, C^{(i)}\}_{i=1,\ldots,n}\right)$ of $\mathscr{L}_n$ are preserved by any combinatorial automorphism of $\hat{\mathcal{E}}_n$.*

Proof. We denote by $c(A,B)$ the number of facets containing lines A, B; this is clearly a combinatorial invariant. Let us compute these numbers using Proposition 3.5 (a) and (b). There are three cases to consider:

a. $A = R^{(r)}, B = R^{(s)}$ or $A = C^{(r)}, B = C^{(s)}$, $r \neq s$. We count the T_{ijk} with $j \neq r, s$ and N_{ij} with $i \neq r, s$ (case R) or $j \neq r, s$ (case C). Hence $c(A,B) = (n-1)(n-2)^2 + (n-1)(n-2) = (n-1)^2(n-2)$.
b. $A = R^{(r)}, B = C^{(s)}$, $r \neq s$. We count the T_{ijk} with $j \neq r, s$ and N_{ij} with $i \neq r$ and $j \neq s$. Hence $c(A,B) = (n-1)(n-2)^2 + (n-1)(n-2) + 1 = (n-1)^2(n-2) + 1$.
c. $A = R^{(r)}, B = C^{(r)}$. We count the T_{ijk} with $j \neq r$ and N_{ij} with $i \neq r$ and $j \neq r$. Hence $c(A,B) = (n-1)^2(n-2) + (n-1)(n-2) = n(n-1)(n-2)$.

Consider now the complete graph with vertex set $\mathscr{L}_n$ and edges colored by the values of $c(A,B)$ just computed. By Lemma 3.10, any combinatorial automorphism of $\hat{\mathcal{E}}_n$ permutes the vertices of the graph, and, since the colors are combinatorial invariants, such permutation is an automorphism of the colored graph. $\square$

Lemma 3.12. *If φ is a combinatorial automorphism of $\hat{\mathcal{E}}_n$, then there exists $\psi \in \mathcal{S}_n$ such that for every $R \in \mathscr{L}_n$, $\varphi(R) = (\operatorname{ind}\psi)(R)$.*

Proof. Since $\mathscr{L}_n$ is invariant, $\varphi\left(R^{(1)}\right) = R^{(j)}$ or $C^{(j)}$, for some j. Consider first the case $\varphi\left(R^{(1)}\right) = R^{(j)}$. It follows from Lemma 3.11, $\varphi(\mathscr{R}) = \mathscr{R}$, hence there exists $\pi \in S_n$ such that for every i, $\varphi\left(R^{(i)}\right) = R^{(\pi(i))}$. Also, for every i, $\varphi\left(\{R^{(i)}, C^{(i)}\}\right) = \{R^{(\pi(i))}, \varphi\left(C^{(i)}\right)\}$, and again by Lemma 3.11, we have that $\varphi\left(C^{(i)}\right) = C^{(\pi(i))}$, and the result follows, with $\psi = P_\pi$. In the case $\varphi\left(R^{(1)}\right) = C^{(j)}$, we argue as above for $(\operatorname{ind}\tau) \circ \varphi$, and conclude the result with $\psi = \tau P_\pi$. $\square$

4. Max closed cones

Here we have a glimpse on cones that are also $\oplus$-monoids, i.e. max-*closed cones*, restricted to cones contained in $\mathbb{Q}_+^N$. As we have two additive monoids on $\mathbb{Q}_+^N$, we label by the $\oplus$ symbol all notions like submonoid and homomorphism to make it clear which structure we are referring to. While the restriction to rational instead of real cones appears out of the blue here, Example 5.4 gives a rationale for that.

Proposition 4.1. *A rational cone $\mathcal{C}$ is a $\oplus$-submonoid of $\mathbb{Q}_+^N$ if and only if its integer cone $\mathcal{C}_\mathbb{Z} = \mathcal{C} \cap \mathbb{Z}^N$ is a $\oplus$-submonoid of $\mathbb{N}^N$.*

Proof. Only the 'if' part requires a proof. Let $u, v \in \mathcal{C}$. Choose a positive integer r such that $ru, rv \in \mathbb{N}^N$. Clearly those vectors are in $\mathcal{C}$, hence so is $ru \oplus rv = r(u \oplus v)$. The last equality shows that $u \oplus v \in \mathcal{C}$, as required. $\square$

Proposition 4.2. *Let H be a half-space given by a linear inequality $ax \geqslant 0$. Then,*

a. *If a has at most one negative component, H is* max-*closed.*
b. *If a has at least two negative components, then any small neighborhood in the bounding hyperplane of H contains points whose* max *is outside H.*

Proof. If a is nonnegative, it is clear that H is max-closed. Suppose it has a single negative component; we can rewrite the inequality as $cx - bx_i \geqslant 0$, where $c \geqslant 0, c_i = 0, b > 0$. Let $u, v \in H$; Without loss of generality, $\max(u_i, v_i) = v_i$. Then, $c(u \oplus v) \geqslant cv \geqslant bv_i = b(u \oplus v)_i$, showing that $u \oplus v \in H$. This shows part (a).

Suppose that there are distinct $r, s \in N$ such that $a_r, a_s < 0$. Let u be a point on the hyperplane $ax = 0$ and choose any $\varepsilon > 0$. Let $z \in \mathbb{N}^N$ have components $z_r = -a_s$, $z_s = a_r$, all other components being 0; then, $az = 0$, and both $u + \varepsilon z$ and $u - \varepsilon z$ lie in the hyperplane. Let $v = (u + \varepsilon z) \oplus (u - \varepsilon z)$; then, $v_s = u_s - \varepsilon a_r$, $v_r = u_r - \varepsilon a_s$, and $v_i = u_i$ otherwise. But $av = au - 2\varepsilon a_r a_s < 0$, so $v \notin H$. $\square$

Theorem 4.3. *A full dimensional cone is* max-*closed if only if for every facet inequality $ax \geqslant 0$, a has at most one negative component.*

Proof. One one hand, if the inequalities are of the form given, the cone is an intersection of max-closed half-spaces, hence max-closed.

If a facet inequality of a cone is not of the specified form, then, applying Prop. 4.2 to a neighborhood of an interior point of that facet, we see that the cone is not max-closed. $\square$

Corollary 4.4. *A full dimensional nonnegative cone is* max-*closed if only if every facet inequality is either of form $x_j \geqslant 0$ or $ax \geqslant 0$, where a has exactly one negative component and at least one positive component.*

Proof. Since each $x_j \geqslant 0$ is a valid inequality for $\mathcal{C}$, all facet inequalities $ax \geqslant 0$ not of this type must have at least one negative coefficient, say, $a_i < 0$. By Theorem 4.3, it is exactly one and there must be a j such that $a_j > 0$, otherwise, any $x \in \mathcal{C}$ would satisfy $x_i \leqslant 0$ and be nonnegative, whence, $x_i = 0$, contradicting full dimension. $\square$

5. $\oplus$-automorphisms

In $\mathbb{R}^N$, $x \leqslant y$ if and only if $x \oplus y = y$, so any $\oplus$-automorphism also preserves $\leqslant$. We recall that in a partial order, y *covers* x if $x \lneq y$ and there is no third element z such that $x \lneq z \lneq y$. In $\mathbb{Z}^N$, y covers x if and only if $y = x + e_i$, for some i. When dealing with integer vectors, we will use interval notation to refer implicitly to $\mathbb{Z}^N$, that is $[x, y] = \{z \in \mathbb{Z}^N \mid x \leqslant z \leqslant y\}$.

We will be concerned here with $\oplus$-automorphisms of integer max-closed cones only. On one hand, this is motivated by our interest in exponential matrices; on the other hand, Example 5.4 presents a brief discussion on real and rational cones in this context, and their difficulties. Still, some facts that help tame additive automorphisms of integer max-closed cones do not hold for $\oplus$-automorphisms, and we will need a few additional hypotheses on the cone.

Example 5.1: *Here we show a family of* max-closed *cones, and a $\oplus$-automorphism of the corresponding integer cones which cannot be extended to an additive automorphism. An intuitive geometric explanation for that is that the cones are "too thin". For any positive integer k, let $\mathcal{C}_k$ be the 2-dimensional cone given by:*

$$\begin{cases} \quad -k\,x \;+\; (k+1)y \geqslant 0 \\ (k+1)x \;-\; k\,y \geqslant 0. \end{cases}$$

One readily verifies that $\mathcal{C}_k$ is symmetric about the line $x-y=0$.

The points $p=(k+1,k)$ and $q=(k,k+1)$ are special here: in the lattice $\mathcal{C}_k\cap\mathbb{Z}^2$ each of them covers only (k,k) and is covered only by $(k+1,k+1)$. To see this, suppose $(x,y)\in\mathcal{C}_k\cap\mathbb{Z}^2\backslash\{p,q\}$ satisfies $(x,y)\geqslant p$. We want to show that $(x,y)\geqslant(k+1,k+1)$; if $y>k$, then $y\geqslant k+1$ and we are done, and the case $y=k$ would require $x>k+1$, which is ruled out by the first defining inequality. All remaining verifications are similar.

It follows that the involution on $\mathcal{C}_k\cap\mathbb{Z}^2$ that interchanges p and q and fixes all other points is order preserving - hence a $\oplus$-automorphism of the integer cone.

Since this map moves p and fixes $2p$, it cannot be extended to an additive map.

We say that an arbitrary $S\subseteq\mathbb{R}^N$ is *very full* if, for every $i\in N$, $\mathbf{1}\pm e_i\in S$ (recall that $\mathbf{1}$ denotes the vector of all 1's).

Proposition 5.1. *A cone is very full if and only if for every facet inequality $ax\geqslant 0$, one has that $a\mathbf{1}\geqslant\max_{i\in N}|a_i|$.*

Proof. Consider a vector $w=\mathbf{1}+\alpha e_i$, where $\alpha=\pm1$. Then, $aw\geqslant 0$ if and only if $a\mathbf{1}\geqslant-\alpha a_i$, and that happens if and only if $a\mathbf{1}\geqslant|a_i|$. □

If $\mathcal{C}$ is a subset of $\mathbb{R}^N$, and φ is a bijective map from $\mathcal{C}$ to itself, we will say that φ is *permutational* if there exists a permutation π of N such that for every $a=(a_i)\in\mathcal{C}$, $\varphi(a)_{\pi(i)}=a_i$ for all $i\in N$. That means that φ is the restriction to $\mathcal{C}$ of the linear map whose action on the canonical basis $(e_i)_{i\in N}$ is given by $e_i\mapsto e_{\pi(i)}$.

Note that any permutational map is an additive homomorphism, a $\oplus$-homomorphism and an isometry.

Theorem 5.2. *Let $\mathcal{C}$ be a very full subset of $\mathbb{N}^N$ closed under $+$ and $\oplus$. Then, every $\oplus$-automorphism fixing a non-zero multiple of $\mathbf{1}$ is permutational.*

Proof. Notice that $\mathbf{1}=(\mathbf{1}-e_1)\oplus(\mathbf{1}-e_2)\in\mathcal{C}$, since $\mathbf{1}\pm e_i\in\mathcal{C}$. Furthermore, for every integer $k>0$, $i\in N$, $k\mathbf{1}\pm e_i=(k-1)\mathbf{1}+\mathbf{1}\pm e_i\in\mathcal{C}$. It will be convenient to denote $b_i^k=k\mathbf{1}-e_i$, and $B^k=\{b_i^k\mid i\in N\}$.

Let φ be a $\oplus$-automorphism fixing $r\mathbf{1}$.

We will proceed through a series of claims.

Claim 1 For every integer $k\geqslant 0$, φ fixes $k\mathbf{1}$.

It is enough to prove that, for $k>0$, if φ fixes $k\mathbf{1}$, then it fixes both $(k+1)\mathbf{1}$ and $(k-1)\mathbf{1}$, and then proceed by induction up and down, starting from $r\mathbf{1}$.

The set of vectors covering $k\mathbf{1}$ is invariant under φ. Those are $\{k\mathbf{1}+e_i\}_{i\in N}$, and so $(k+1)\mathbf{1}=\oplus_{i\in N}(k\mathbf{1}+e_i)$ is fixed by φ. Also, the set B^k of vectors covered by $k\mathbf{1}$ is invariant, so $\oplus\{x\in\mathcal{C}\mid x\leqslant b_i^k, \text{ for all } i\in N\}=(k-1)\mathbf{1}$ is also fixed, and the claim is proved.

Notice that for every integer $k \geqslant 1$, the set B^k consists of the coatoms of the interval $[0, k\mathbf{1}]$ in $\mathbb{Z}^N$, and, *a fortiori*, in $\mathcal{C}$, so, by Claim 1, B^k is invariant under φ. Let $\pi \in S_N$ be the permutation defined by $\varphi(b_i^1) = b_{\pi(i)}^1$.

Claim 2 For every $k \geqslant 1$, $i \in N$, $\varphi(kb_i^1) = kb_{\pi(i)}^1$.

Fix a $k > 1$. We know already from the proof of Claim 1, that φ permutes the vectors from B^k. Let us show φ also permutes $(kb_i^1)_{i \in N}$. This set is precisely

$$\{w \in \mathcal{C} \mid |[0, w] \cap B^1| = 1 = |[w, k\mathbf{1}] \cap B^k|\},$$

which shows our set is invariant under φ. Since one must have $\varphi(b_i^1) \leqslant \varphi(kb_i^1)$, the claim follows.

Claim 3 For every $k \geqslant 1$,$i \in N$, the interval $[kb_i^1, k\mathbf{1}]$ in $\mathcal{C}$ is a chain of height k (i.e. length $k+1$).

Clearly this interval consists of the vectors $kb_i^1 \oplus t\mathbf{1}$, $0 \leqslant t \leqslant k$, which gives the claim.

Now we finish the proof. Given any $v \in \mathcal{C}$, we want to show that $\varphi(v)_{\pi(i)} = v_i$, for each i. Choose k bigger than any component of v, and of $\varphi(v)$. The vector $kb_i^1 \oplus v$ has all components k, except for the i^{th}, which equals v_i. So, the interval $[kb_i^1, kb_i^1 \oplus v]$ is a chain of height v_i. It is mapped bijectively by φ to $[kb_{\pi(i)}^1, kb_{\pi(i)}^1 \oplus \varphi(v)]$, which is a chain of height $\varphi(v)_{\pi(i)}$. It follows that $\varphi(v)_{\pi(i)} = v_i$, as claimed. □

The examples below illustrate the precision of Theorem 5.2.

Example 5.2: *The set $\mathcal{C}$ needs not be an integer polyhedral cone: fix a real $0 < \alpha < 1$, and for $n \geqslant 3$, let $\mathcal{C} = \{x \in \mathbb{N}^n \mid \sum_{i \neq j} x_i \geqslant x_j^\alpha, j = 1, \ldots, n\}$. Since $x^\alpha + y^\alpha \geqslant (x+y)^\alpha$ for all $x, y \in \mathbb{R}_+$, this is an additive submonoid of $\mathbb{N}^n$, and it is clearly very full. Following the proof of Proposition 4.2, we see that $\mathcal{C}$ is also closed under $\oplus$. The elements of $\mathcal{C}$ with minimal support are all vectors with exactly two ones and zeros elsewhere; any $\oplus$-automorphism keeps this set invariant, hence it fixes $\mathbf{1}$, its max. Theorem 5.2 implies that every $\oplus$-automorphism is permutational (by the description symmetry, any permutation of coordinates yields a $\oplus$-automorphism). On the other hand, it is an exercise to show that the smallest polyhedral cone containing $\mathcal{C}$ is the positive orthant, which is, of course, too large, as $e_1 \notin \mathcal{C}$.*

Example 5.3: *Here we show that the requirement of being an additive monoid cannot be simply dismissed. For each $k \in \mathbb{N}$, let $D_k = \{v \in \mathbb{N}^N \mid k\mathbf{1} \leqslant v \leqslant (k+1)\mathbf{1}\}$, and let $D = \cup_{k \in \mathbb{N}} D_k$. Then D is $\oplus$-submonoid of $\mathbb{N}^N$, and very full, but it is not closed under $+$. Choose, for each k, a permutation $\pi_k \in S_N$, and let T be the map that acts like π_k on the layer D_k. This is a $\oplus$-automorphism, but, unless all the π_k are equal, it is not permutational.*

Example 5.4: *If $\mathcal{C}$ is a real, full dimensional $\oplus$-closed cone, then every $\oplus$-automorphism is fully determined by its action on $\mathcal{C} \cap \mathbb{Q}^N$. That is because each $x \in \mathcal{C}$ is the greatest lower bound of the set $\{y \in \mathcal{C} \cap \mathbb{Q}^N \mid x \leqslant y\}$. On the other hand, the rational cone $\mathbb{Q}_+^n$ admits a quite complicated automorphism group: choose on each component an increasing function on $\mathbb{Q}_+$; the whole choice can even be done so as to fix $\mathbb{N}^N$. Actually, in the same vein as in [2], Corollary 5.4 is likely also true for rational and real cones,* except *for orthants, but we do not pursue this here.*

Lemma 5.3. *Let $\mathcal{C}$ be a $\oplus$-submonoid of $\mathbb{N}^N$. Assume also that $\mathcal{C}$ is fixed by a group of permutational maps that is transitive on the canonical basis. Then every $\oplus$-automorphism of $\mathcal{C}$ fixes a multiple of $\mathbf{1}$.*

Proof. By Dickson's Lemma [13], [15], $\mathcal{C}$ has a finite set of minimal non-zero vectors; this set is invariant under any $\oplus$-automorphism of $\mathcal{C}$, so its $\oplus$-sum is fixed by those automorphisms. That is a non-zero vector fixed by a transitive permutation group, so it is a positive multiple of $\mathbf{1}$. □

Combining Theorem 5.2 with Lemma 5.3, we obtain:

Corollary 5.4. *Let $\mathcal{C}$ be a very full subset of $\mathbb{N}^N$ closed under $+$ and $\oplus$. Assume also that $\mathcal{C}$ is fixed by a group of permutational maps that is transitive on the canonical basis. Then every $\oplus$-automorphism of $\mathcal{C}$ is permutational.*

Let us apply this now to $\mathcal{E}_n$. This is very full, closed under $+$ and $\oplus$, and $\mathcal{S}_n \subseteq \mathrm{Aut}(\mathcal{E}_n, \oplus)$ is a group as required by Corollary 5.4, so every $\oplus$-automorphism of $\mathcal{E}_n$ is permutational. As permutational maps are linear, we have that $\mathrm{Aut}(\mathcal{E}_n, \oplus) \subseteq \mathrm{Lin}_{\mathbb{Z}}(\mathcal{E}_n) = \mathcal{S}_n$, and we have proved:

Theorem 5.5. [14, Theorem 4.3] $\mathrm{Aut}(\mathcal{E}_n, \oplus) = \mathcal{S}_n$.

Notice that $\hat{\mathcal{E}}_n$, as a rational polyhedral cone, satisfies the following properties: it is very full, non-negative, pointed, max-closed and each non-negativity inequality determines a facet. The next example shows that even all of these properties of a polyhedral cone are not enough to guarantee that each additive automorphism is also a $\oplus$-automorphism.

Example 5.5: *Consider the cone $\mathcal{C} = \{x \in \mathbb{Q}^3 \mid x_1 + x_2 \geqslant x_3, x_1, x_2, x_3 \geqslant 0\}$. The following hold:*

a. *$\mathcal{C}$ is very full, pointed, non-negative,* max-*closed.*
b. *Each nonnegativity inequality of $\mathcal{C}$ determines a facet.*
c. *There is an additive automorphism of $\mathcal{C}_{\mathbb{Z}}$ which is not permutational and does not preserve $\oplus$.*

Clearly (a) is satisfied. Fact (b) is proved by the respective interior points $(0,2,1), (2,0,1), (1,1,0)$. *The extreme rays of $\mathcal{C}$ are*

$$v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (1,0,1), v_4 = (0,1,1).$$

Let $\psi \in GL(3, \mathbb{Z})$ be given by the matrix

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Then ψ fixes v_1 and v_4 and interchanges v_2 and v_3. As it permutes extreme rays, it leaves $\mathcal{C}$ invariant, and is an additive automorphism of $\mathcal{C}_{\mathbb{Z}}$. On the other hand, $\psi(v_1 \oplus v_3) \neq \psi(v_1) \oplus \psi(v_3)$, showing (c).

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