THE SCATTERING MATRIX WITH RESPECT TO AN HERMITIAN MATRIX OF A GRAPH

Takashi KOMATSU Department of Bioengineering, School of Engineering, The University of Tokyo Bunkyo, Tokyo, 113-8656, JAPAN e-mail: komatsu@coi.t.u-tokyo.ac.jp Norio KONNO Department of Applied Mathematics, Faculty of Engineering, Yokohama National University Hodogaya, Yokohama 240-8501, JAPAN e-mail: konno-norio-bt@ynu.ac.jp Iwao SATO Oyama National College of Technology Oyama, Tochigi 323-0806, JAPAN e-mail: isato@oyama-ct.ac.jp

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Abstract

Recently, Gnutzmann and Smilansky [5] presented a formula for the bond scattering matrix of a graph with respect to a Hermitian matrix. We present another proof for this Gnutzmann and Smilansky's formula by a technique used in the zeta function of a graph. Furthermore, we generalize Gnutzmann and Smilansky's formula to a regular covering of a graph. Finally, we define an *L*-fuction of a graph, and present a determinant expression. As a corollary, we express the generalization of Gnutzmann and Smilansky's formula to a regular covering of a graph by using its *L*-functions.

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The contact author for correspondence: Iwao Sato Oyama National College of Technology, Oyama, Tochigi 323-0806, JAPAN Tel: +81-285-20-2176 Fax: +81-285-20-2880 E-mail: isato@oyama-ct.ac.jp

1 Introduction

Ihara zeta functions of graphs started from Ihara zeta functions of regular graphs by Ihara [9]. Originally, Ihara presented *p*-adic Selberg zeta functions of discrete groups, and showed that its reciprocal is a explicit polynomial. Serre [13] pointed out that the Ihara zeta function is the zeta function of the quotient T/Γ (a finite regular graph) of the one-dimensional Bruhat-Tits building T (an infinite regular tree) associated with $GL(2, k_p)$.

A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [16, 17]. Hashimoto [8] treated multivariable zeta functions of bipartite graphs. Bass [1] generalized Ihara's result on the Ihara zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial. Various proofs of Bass' Theorem were given by Stark and Terras [15], Foata and Zeilberger [3], Kotani and Sunada [10]. Sato [12] defined the second weighted zeta function of a graph by using not an infinite product but a determinant.

The spectral determinant of the Laplacian on a quantum graph is closely related to the Ihara zeta function of a graph(see [2, 4, 5, 14]). Smilansky [14] considered spectral zeta functions and trace formulas for (discrete) Laplacians on ordinary graphs, and expressed some determinant on the bond scattering matrix of a graph G by using the characteristic polynomial of its Laplacian. Recently, Gnutzmann and Smilansky [5] presented a formula for the bond scattering matrix of a graph with respect to a Hermitian matrix.

In this paper, we another proof for the Gnutzmann and Smilansky's formula on the bond scattering matrix of a graph with respect to a Hermitian matrix. by a technique used in the zeta function of a graph, and treat some related topics. In Section 2, we review the Ihara zeta function and the bond scattering matrix of a graph G. In Section 3, we present another proof for the Gnutzmann and Smilansky's formula by a technique used in the zeta function of a graph. In Section 4, we we express a new zeta function of G on the bond scattering matrix of G with respect to a Hermitian matrix by using the Euler product. In Section 5, we generalize the Gnutzmann and Smilansky's formula to a regular covering of G. In Section 6, we define an L-fuction of G, and present its determinant expression. As a corollary, we express the generalization of the Gnutzmann and Smilansky's formula to a regular covering of G by using its L-functions.

2 The zeta functions and the bond scattering matrix of a graph

Graphs treated here are finite. Let G = (V(G), E(G)) be a connected graph (possibly multiple edges and loops) with the set V(G) of vertices and the set E(G) of unoriented edges uv joining two vertices u and v. For $uv \in E(G)$, an arc (u, v) is the oriented edge from u to v. Set $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $b = (u, v) \in D(G)$, set u = o(b) and v = t(b). Furthermore, let $b^{-1} = (v, u)$ be the *inverse* of b = (u, v).

A path P of length n in G is a sequence $P = (b_1, \dots, b_n)$ of n arcs such that $b_i \in D(G)$, $t(b_i) = o(b_{i+1})(1 \le i \le n-1)$, where indices are treated mod n. Set |P| = n, $o(P) = o(b_1)$ and $t(P) = t(b_n)$. Also, P is called an (o(P), t(P))-path. We say that a path $P = (b_1, \dots, b_n)$ has a backtracking or back-scatter if $b_{i+1} = b_i^{-1}$ for some $i(1 \le i \le n-1)$. A (v, w)-path is called a v-cycle (or v-closed path) if v = w. The inverse cycle of a cycle $C = (b_1, \dots, b_n)$ is the cycle $C^{-1} = (\hat{b}_n, \dots, \hat{b}_1)$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, \dots, e_m)$ and $C_2 = (f_1, \dots, f_m)$ are called *equivalent* if there exists k such that $f_j = e_{j+k}$ for all j. The inverse cycle of C is in general not equivalent to C. Let [C] be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *power* of B. A cycle C is *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle C is *prime* if it is not a power of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, u)$ of G at a vertex u of G. Furthermore, an equivalence class of prime cycles of a graph G is called a *primitive periodic* orbit of G (see [14]).

The *Ihara zeta function* of a graph G is a function of a complex variable t with |t| sufficiently small, defined by

$$\mathbf{Z}(G,t) = \mathbf{Z}_G(t) = \prod_{[p]} (1 - t^{|p|})^{-1},$$

where [p] runs over all equivalence classes of prime, reduced cycles of G(see [9]).

Theorem 1 (Ihara; Bass) Let G be a connected graph. Then the reciprocal of the Ihara zeta function of G is given by

$$\mathbf{Z}(G,t)^{-1} = (1-t^2)^{r-1} \det(\mathbf{I} - t\mathbf{A}(G) + t^2(\mathbf{D} - \mathbf{I})),$$

where r and $\mathbf{A}(G)$ are the Betti number and the adjacency matrix of G, respectively, and $\mathbf{D} = (d_{ij})$ is the diagonal matrix with $d_{ii} = v_i = \deg u_i$ where $V(G) = \{u_1, \dots, u_n\}$.

Let G be a connected graph and $V(G) = \{u_1, \dots, u_n\}$. Then we consider an $n \times n$ matrix $\mathbf{W} = (w_{ij})_{1 \le i,j \le n}$ with ij entry the complex variable w_{ij} if $(u_i, u_j) \in D(G)$, and $w_{ij} = 0$ otherwise. The matrix $\mathbf{W} = \mathbf{W}(G)$ is called the *weighted matrix* of G. Furthermore, let $w(u_i, u_j) = w_{ij}, u_i, u_j \in V(G)$ and $w(b) = w_{ij}, b = (u_i, u_j) \in D(G)$. For each path $P = (e_{i_1}, \dots, e_{i_r})$ of G, the norm w(P) of P is defined as follows: $w(P) = w(e_{i_1})w(e_{i_2})\cdots w(e_{i_r})$.

Let G be a connected graph with n vertices and m unoriented edges, and $\mathbf{W} = \mathbf{W}(G)$ a weighted matrix of G. Two $2m \times 2m$ matrices $\mathbf{B} = \mathbf{B}(G) = (\mathbf{B}_{e,f})_{e,f \in R(G)}$ and $\mathbf{J}_0 = \mathbf{J}_0(G) = (\mathbf{J}_{e,f})_{e,f \in R(G)}$ are defined as follows:

$$\mathbf{B}_{e,f} = \begin{cases} w(f) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise,} \end{cases} \quad \mathbf{J}_{e,f} = \begin{cases} 1 & \text{if } f = \hat{e}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the second weighted zeta function of G is defined by

$$\mathbf{Z}_1(G, w, t) = \det(\mathbf{I}_n - t(\mathbf{B} - \mathbf{J}_0))^{-1}.$$

If w(e) = 1 for any $e \in D(G)$, then the zeta function of G is the Ihara zeta function of G.

Theorem 2 (Sato) Let G be a connected graph, and let $\mathbf{W} = \mathbf{W}(G)$ be a weighted matrix of G. Then the reciprocal of the second weighted zeta function of G is given by

$$\mathbf{Z}_1(G, w, t)^{-1} = (1 - t^2)^{m-n} \det(\mathbf{I}_n - t\mathbf{W}(G) + t^2(\tilde{\mathbf{D}} - \mathbf{I}_n))$$

where n = |V(G)|, m = |E(G)| and $\tilde{\mathbf{D}} = (d_{ij})$ is the diagonal matrix with $d_{ii} = \sum_{o(b)=u_i} w(e)$, $V(G) = \{u_1, \cdots, u_n\}$.

Next, we state the bond scattering matrix of a graph. Let G be a connected graph with n vertices and m edges, $V(G) = \{u_1, \ldots, u_n\}$ and $D(G) = \{b_1, \ldots, b_m, b_{m+1}, \ldots, b_{2m}\}$ such that $b_{m+j} = b_j^{-1} (1 \le j \le m)$. The Laplacian (matrix) $\mathbf{L} = \mathbf{L}(G)$ of G is defined by

$$\mathbf{L} = \mathbf{L}(G) = -\mathbf{A}(G) + \mathbf{D}$$

Let λ be a eigenvalue of **L** and $\psi = (\psi_1, \dots, \psi_n)$ the eigenvector corresponding to λ . For each arc $b = (u_j, u_l)$, one associates a *bond wave function*

$$\psi_b(x) = a_b \mathrm{e}^{i\pi x/4} + a_{b^{-1}} \mathrm{e}^{-i\pi x/4}, \ x = \pm 1$$

under the condition

$$\psi_b(1) = \psi_j, \psi_b(-1) = \psi_l.$$

We consider the following three conditions:

- 1. uniqueness: The value of the eigenvector at the vertex u_j , ψ_j , computed in the terms of the bond wave functions is the same for all the arcs emanating from u_j .
- 2. ψ is an eigenvector of **L**;
- 3. *consistency*: The linear relation between the incoming and the outgoing coefficients (1) must be satisfied simultaneously at all vertices.

By the uniqueness, we have

$$a_{b_1} \mathrm{e}^{i\pi/4} + a_{b_1^{-1}} \mathrm{e}^{-i\pi/4} = a_{b_2} \mathrm{e}^{i\pi/4} + a_{b_2^{-1}} \mathrm{e}^{-i\pi/4} = \dots = a_{b_{d_j}} \mathrm{e}^{i\pi/4} + a_{b_{d_j}^{-1}} \mathrm{e}^{-i\pi/4},$$

where $b_1, b_2, \ldots, b_{d_j}$ are arcs emanating from u_j , and $d_j = \deg u_j$, $i = \sqrt{-1}$. By the condition 2, we have

$$-\sum_{k=1}^{d_j} (a_{b_k} \mathrm{e}^{-i\pi/4} + a_{b_k^{-1}} \mathrm{e}^{i\pi/4}) = (\lambda - v_j) \frac{1}{v_j} \sum_{k=1}^{d_j} (a_{b_k} \mathrm{e}^{i\pi/4} + a_{b_k^{-1}} \mathrm{e}^{-i\pi/4}).$$

Thus, for each arc b with $o(b) = u_j$,

$$a_b = \sum_{t(c)=u_j} \sigma_{b,c}^{(u_j)}(\lambda) a_c, \tag{1}$$

where

$$\sigma_{b,c}^{(u_j)}(\lambda) = i(\delta_{b^{-1},c} - \frac{2}{d_j} \frac{1}{1 - i(1 - \lambda/d_j)}),$$

and $\delta_{b^{-1},c}$ is the Kronecker delta. The bond scattering matrix $\mathbf{U}(\lambda) = (U_{ef})_{e,f \in D(G)}$ of G is defined by

$$U_{ef} = \begin{cases} \sigma_{e,f}^{(t(f))} & \text{if } t(f) = o(e), \\ 0 & \text{otherwise} \end{cases}$$

By the consistency, we have

$$\mathbf{U}(\lambda)\mathbf{a} = \mathbf{a},$$

where $\mathbf{a} = {}^{t}(a_{b_1}, a_{b_2}, \dots, a_{b_{2m}})$. This holds if and only if

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = 0.$$

Theorem 3 (Smilansky) Let G be a connected graph with n vertices and m edges. Then the characteristic polynomial of the bond scattering matrix of G is given by

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = \frac{2^m i^n \det(\lambda \mathbf{I}_n + \mathbf{A}(G) - \mathbf{D})}{\prod_{j=1}^n (d_j - id_j + \lambda i)} = \prod_{[p]} (1 - a_p(\lambda)),$$

where [p] runs over all primitive periodic orbits of G, and

$$a_p(\lambda) = \sigma_{b_1, b_n}^{(t(b_n))} \sigma_{b_n, b_{n-1}}^{(t(b_{n-1}))} \cdots \sigma_{b_2, b_1}^{(t(b_1))}, \ p = (b_1, b_2, \dots, b_n)$$

Mizuno and Sato [11] presented another proof for this Smilansky's formula by using the determinant expression of the second weighted zeta function of a graph.

3 The scattering matrix of a graph with respect to a Hermitian matrix

Let G be a connected graph with n vertices and m edges, $V(G) = \{1, \ldots, n\}$ and $D(G) = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{2m}\}$ such that $e_{m+j} = e_j^{-1} (1 \le j \le m)$. Furthermore, let an Hermitian matrix $\mathbf{H} = \mathbf{H}(G) = (H_{uv})_{u,v \in V(G)}$ be given as follows:

$$H_{uv} = \begin{cases} h_f e^{2i\gamma_f} & \text{if } f = (u, v) \in D(G), \\ 0 & \text{otherwise,} \end{cases}$$

where, for each $f \in D(G)$,

$$h_f = h_{f^{-1}} \ge 0 \text{ and } \gamma_f = -\gamma_{f^{-1}} \in [-\pi/2, \pi/2].$$

If $H_{uv} = H_f$ is real and negative, then we choose $\gamma_f = \pi/2$ if $u \ge v$ and $\gamma_f = -\pi/2$ if u < v. Set

$$h(u,v) = h_{uv} = h_f \text{ and } \gamma(u,v) = \gamma_{uv} = \gamma_f \text{ for } f = (u,v) \in D(G).$$

Now, let λ be an eigenvalue of **H** and $\psi = (\psi_1, \dots, \psi_n)$ the eigenvector corresponding to λ . For each arc b = (u, v), one associates a bond wave function

$$\psi_b(x) = \frac{\mathrm{e}^{i\gamma_b}}{\sqrt{h_b}} (a_{b^{-1}} \mathrm{e}^{i\pi x/4} + a_b \mathrm{e}^{-i\pi x/4}), \ x = \pm 1$$

under the condition

$$\psi_b(1) = \psi_u, \psi_b(-1) = \psi_v.$$

We consider the following three conditions:

- 1. uniqueness: The value of the eigenvector at the vertex u, ψ_u , computed in the terms of the bond wave functions is the same for all the arcs emanating from u.
- 2. ψ is an eigenvector of **H**;
- 3. *consistency*: The linear relation between the incoming and the outgoing coefficients (1) must be satisfied simultaneously at all vertices.

By the uniqueness 1, we have

$$\frac{\mathrm{e}^{i\gamma_{b_1}}}{\sqrt{h_{b_1}}} (a_{b_1^{-1}} \mathrm{e}^{i\pi/4} + a_{b_1} \mathrm{e}^{-i\pi/4}) = \frac{\mathrm{e}^{i\gamma_{b_2}}}{\sqrt{h_{b_2}}} (a_{b_2^{-1}} \mathrm{e}^{i\pi/4} + a_{b_2} \mathrm{e}^{-i\pi/4}) = \cdots$$
$$= \frac{\mathrm{e}^{i\gamma_{b_d}}}{\sqrt{h_{b_d}}} (a_{b_d^{-1}} \mathrm{e}^{i\pi/4} + a_{b_d} \mathrm{e}^{-i\pi/4}) = \psi_u,$$

where b_1, b_2, \ldots, b_d are arcs emanating from u, and $d = \deg u$, $i = \sqrt{-1}$. By the condition 2, we have

$$(H_{uu} - \lambda)\psi_u + \sum_{v \in \mathcal{E}_u} H_{uv}\psi_v = 0,$$

and so,

$$(H_{uu} - \lambda) \frac{\mathrm{e}^{i\gamma_{b_1}}}{\sqrt{h_{b_1}}} (a_{b_1^{-1}} \mathrm{e}^{i\pi/4} + a_{b_1} \mathrm{e}^{-i\pi/4}) = -\frac{1}{d} \sum_{k=1}^d H_{b_j} \frac{\mathrm{e}^{i\gamma_{b_k}}}{\sqrt{h_{b_k}}} (a_{b_k^{-1}} \mathrm{e}^{i\pi/4} + a_{b_k} \mathrm{e}^{-i\pi/4})$$

where $\mathcal{E}_u = \{f \in D(G) \mid o(f) = u\}$. Thus, for each arc b with o(b) = u,

$$a_{b}^{-1} = ia_{b} - 2\sum_{k=1}^{d} \frac{\sqrt{h_{b}}\sqrt{h_{b_{k}}}}{H_{uu} - \lambda - i\Gamma_{u}} e^{i(\gamma_{b_{k}} + \gamma_{b^{-1}})} a_{b_{k}},$$

where

$$\Gamma_u = \sum_{k=1}^d h_{b_k}.$$

Let $e = b^{-1}$, $f = b_k$ and

$$\sigma_{ef}^{(u)}(\lambda) = i\delta_{e^{-1}f} - 2\frac{\sqrt{h_e}\sqrt{h_f}}{H_{uu} - \lambda - i\Gamma_u} e^{i(\gamma_f + \gamma_e)},$$

where $\delta_{e^{-1}f}$ is the Kronecker delta. Then we have

$$a_e = \sum_{o(f)=u} \sigma_{ef}^{(u)}(\lambda) a_f \tag{2}$$

for each arc e such that t(e) = u. The bond scattering matrix $\mathbf{U}(\lambda) = (U_{ef})_{e,f \in D(G)}$ of G is defined by

$$U_{ef} = \begin{cases} \sigma_{e,f}^{(t(e))} & \text{if } t(e) = o(f), \\ 0 & \text{otherwise} \end{cases}$$

By the consistency 3, we have

 $\mathbf{U}(\lambda)\mathbf{a}=\mathbf{a},$

where $\mathbf{a} = {}^{t}(a_{b_1}, a_{b_2}, \dots, a_{b_{2m}})$. This holds if and only if

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = 0.$$

We present another proof of Theorem 4 by using the technique on the Ihara zeta function, which is different from a proof in [5].

Theorem 4 (Gnutzmann and Smilansky) Let G be a connected graph with n vertices $1, \ldots, n$ and m edges. Then, for the bond scattering matrix of G,

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = \frac{(-1)^n 2^m \det(\lambda \mathbf{I}_n - \mathbf{H})}{\prod_{j=1}^n (H_{jj} - \lambda - i\Gamma_j)},$$

Proof. The argument is an analogue of Watanabe and Fukumizu's method [18].

Let G be a connected graph with n vertices and m edges, $V(G) = \{1, \dots, n\}$ and $D(G) = \{b_1, \dots, b_m, b_1^{-1}, \dots, b_m^{-1}\}$. Set $d_j = \deg j$ and

$$x_j = \frac{2}{H_{jj} - \lambda - i\Gamma_j}$$

for each j = 1, ..., n. Furthermore, for $e \in D(G)$, let

$$w(e) = \sqrt{h_e} \mathrm{e}^{i\gamma_e}.$$

Them we have

$$\sigma_{ef}^{(t(e))}(\lambda) = i\delta_{e^{-1}f} - x_{t(e)}w(e)w(f).$$

Now, we consider a $2m \times 2m$ matrix $\mathbf{B} = (B_{ef})_{e,f \in D(G)}$ given by

$$B_{ef} = \begin{cases} x_{o(f)}w(e)w(f) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{K} = (\mathbf{K}_{i,j})_{1 \le i \le 2m; 1 \le j \le n}$ be the $2m \times n$ matrix defined as follows:

$$\mathbf{K}_{i,j} := \begin{cases} x_j w(b_i) & \text{if } o(b_i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we define two $2m \times n$ matrices $\mathbf{L} = (\mathbf{L}_{i,j})_{1 \le i \le 2m; 1 \le j \le n}$ and $\mathbf{M} = (\mathbf{M}_{i,j})_{1 \le i \le 2m; 1 \le j \le n}$ as follows:

$$\mathbf{L}_{i,j} := \begin{cases} w(b_i) & \text{if } t(b_i) = j, \\ 0 & \text{otherwise,} \end{cases} \quad \mathbf{M}_{i,j} := \begin{cases} w(b_i) & \text{if } o(b_i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\mathbf{K} = \mathbf{M} \begin{bmatrix} x_1 & 0 \\ & \ddots & \\ 0 & & x_n \end{bmatrix} = \mathbf{M} \mathbf{X}.$$
 (3)

Furthermore, we have

$$\mathbf{L}^t \mathbf{K} = \mathbf{B} \tag{4}$$

and

$${}^{t}\mathbf{ML} = \mathbf{H}.$$
 (5)

Note that

$$H_{uv} = w(u, v)^2 \ if \ (u, v) \in D(G)$$

But, since

$$U_{ef} = \begin{cases} -x_{t(e)}w(e)w(f) & \text{if } t(e) = o(f) \text{ and } f \neq e^{-1}, \\ i - x_{t(e)}w(e)w(f) & \text{if } f = e^{-1}, \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$\mathbf{U}(\lambda) = i\mathbf{J}_0 - \mathbf{B}$$

Furthermore, if **A** and **B** are an $r \times s$ and an $s \times r$ matrix, respectively, then we have

$$\det(\mathbf{I}_r - \mathbf{AB}) = \det(\mathbf{I}_s - \mathbf{BA}).$$

Thus,

$$det(\mathbf{I}_{2m} - u\mathbf{U}(\lambda)) = det(\mathbf{I}_{2m} - u(i\mathbf{J}_0 - \mathbf{B}))$$

= $det(\mathbf{I}_{2m} - iu\mathbf{J}_0 + u\mathbf{L} {}^t\mathbf{K})$
= $det(\mathbf{I}_{2m} + u\mathbf{L} {}^t\mathbf{K}(\mathbf{I}_{2m} - iu\mathbf{J}_0)^{-1}) det(\mathbf{I}_{2m} - iu\mathbf{J}_0)$
= $det(\mathbf{I}_n + u {}^t\mathbf{K}(\mathbf{I}_{2m} - iu\mathbf{J}_0)^{-1}\mathbf{L}) det(\mathbf{I}_{2m} - iu\mathbf{J}_0).$

Arrange arcs of D(G) as follows: $b_1, b_1^{-1}, \ldots, b_m, b_m^{-1}$. Then we have

$$\det(\mathbf{I}_{2m} - iu\mathbf{J}_0) = \det(\begin{bmatrix} 1 & -iu & \dots & 0\\ -iu & 1 & & \\ \vdots & & \ddots & \\ 0 & & & \end{bmatrix}) = (1+u^2)^m.$$

Furthermore,

$$(\mathbf{I}_{2m} - iu\mathbf{J}_0)^{-1} = \begin{bmatrix} 1 & -iu & \dots & 0\\ -iu & 1 & & \\ \vdots & & \ddots & \\ 0 & & & \end{bmatrix}^{-1}$$
$$= \frac{1}{1+u^2} \begin{bmatrix} 1 & iu & \dots & 0\\ iu & 1 & & \\ \vdots & & \ddots & \\ 0 & & & \end{bmatrix}$$
$$= \frac{1}{1+u^2} (\mathbf{I}_{2m} + iu\mathbf{J}_0).$$

Therefore, it follows that

$$\det(\mathbf{I}_{2m} - u\mathbf{U}(\lambda))$$

$$= \det(\mathbf{I}_n + \frac{u}{1+u^2} {}^t\mathbf{K}(\mathbf{I}_{2m} + iu\mathbf{J}_0)\mathbf{L})(1+u^2)^m$$

$$= (1+u^2)^{m-n}\det((1+u^2)\mathbf{I}_n + u {}^t\mathbf{K}\mathbf{L} + iu^2 {}^t\mathbf{K}\mathbf{J}_0\mathbf{L}).$$

But, we have

$$^{t}\mathbf{KL} = \mathbf{X}^{t}\mathbf{ML} = \mathbf{XH}.$$

Furthermore, we have

$${}^{t}\mathbf{K}\mathbf{J}_{0}\mathbf{L}=\mathbf{X}^{t}\mathbf{M}\mathbf{J}_{0}\mathbf{L}.$$

Then, for $u, v \in V(G)$, we have

$$({}^{t}\mathbf{M}\mathbf{J}_{0}\mathbf{L})_{uv}$$

$$= \delta_{uv}\sum_{o(e)=u}({}^{t}\mathbf{M})_{ue}(\mathbf{J}_{0})_{ee^{-1}}(\mathbf{L})_{e^{-1}v}$$

$$= \delta_{uv}\sum_{o(e)=u}w(e)\cdot 1\cdot w(e^{-1})$$

$$= \delta_{uv}\sum_{o(e)=u}\sqrt{h_{e}}e^{i\gamma_{e}}\sqrt{h_{e}}e^{-i\gamma_{e}}$$

$$= \delta_{uv}\sum_{o(e)=u}h_{e} = \delta_{uv}\Gamma_{u}.$$

Now, let

$$\mathbf{D}_L = \begin{bmatrix} \Gamma_1 & 0 \\ & \ddots & \\ 0 & & \Gamma_n \end{bmatrix}.$$

Then

$${}^{t}\mathbf{K}\mathbf{J}_{0}\mathbf{L}=\mathbf{X}\mathbf{D}_{L}.$$

Thus,

$$\det(\mathbf{I}_{2m} - u\mathbf{U}(\lambda)) = (1+u^2)^{m-n} \det((1+u^2)\mathbf{I}_n + u\mathbf{X}\mathbf{H} + iu^2\mathbf{X}\mathbf{D}_L).$$

Substituting u = 1, we obtain

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda))$$

$$= 2^{m-n} \det(2\mathbf{I}_n + \mathbf{X}\mathbf{H} + i\mathbf{X}\mathbf{D}_L)$$

$$= 2^{m-n} \det(\begin{bmatrix} \dots & 2 + i\Gamma_u \frac{2}{H_{uu} - \lambda - i\Gamma_u} & \dots & \frac{2}{H_{uu} - \lambda - i\Gamma_u} h_{uv} e^{2i\gamma_{uv}} & \dots \end{bmatrix})$$

$$= \frac{2^m}{\prod_{u=1}^n (H_{uu} - \lambda - i\Gamma_u)} \det(-\lambda \mathbf{I}_n + \mathbf{H})$$

$$= \frac{(-1)^n 2^m}{\prod_{u=1}^n (H_{uu} - \lambda - i\Gamma_u)} \det(\lambda \mathbf{I}_n - \mathbf{H}).$$

4 The Euler product with respect to the scattering matrix

We present the Euler product for the determinant formula of the scattering matrix $\mathbf{U}(\lambda)$ of a graph.

Theorem 5 Let G be a connected graph with m edges, and $\mathbf{H} = \mathbf{H}(G) = (H_{uv})_{u,v \in V(G)}$ an Hermitian matrix defined in Section 2. Then the characteristic polynomial of the bond scattering matrix of G induced from **H** is given by

$$\det(\mathbf{I}_{2m} - u\mathbf{U}(\lambda)) = \prod_{[C]} (1 - w_C u^{|C|}),$$

let c runs over all equivalence classes of prime cycles in G, and

$$w_C = \sigma_{e_1e_2}^{(t(e_1))} \sigma_{e_2e_3}^{(t(e_2))} \cdots \sigma_{e_ne_1}^{(t(e_n))}, \ C = (b_1, b_2, \dots, b_n)$$

Proof. Let $D(G) = \{b_1, \dots, b_{2m}\}$ such that $b_{m+j} = b_j^{-1} (1 \le j \le m)$. Set $\mathbf{U} = \mathbf{U}(\lambda)$. Since

$$\log \det(\mathbf{I} - u\mathbf{F}) = \operatorname{Tr} \log(\mathbf{I} - u\mathbf{F}),$$

for a square matrix \mathbf{F} , we have

$$\log \det(\mathbf{I} - u\mathbf{U}) = \operatorname{Tr} \log(\mathbf{I} - u\mathbf{U}) = -\sum_{k=1}^{\infty} \frac{\operatorname{Tr}(\mathbf{U}^k)}{k} u^k.$$

Here,

$$\operatorname{Tr}(\mathbf{U}^k) = \sum_C w_C,$$

where C runs over all cycles of length k in G, and

$$w_C = \sigma_{e_1 e_2}^{(t(e_1))} \sigma_{e_2 e_3}^{(t(e_2))} \cdots \sigma_{e_k e_1}^{(t(e_k))}, \ C = (b_1, b_2, \dots, b_k)$$

Thus,

$$u \frac{d}{du} \log \det(\mathbf{I}_{2m} - u\mathbf{U}) = \sum_{k=1}^{\infty} \operatorname{Tr}(\mathbf{U}^k) u^k$$
$$= \sum_C w_C u^{|C|},$$

where C runs over all cycles in G.

Now, let C be any cycle in G. Then there exists exactly one prime cycle D such that

$$C = D^l$$

Thus, we have

$$u\frac{d}{du}\log\det(\mathbf{I}_{2m}-u\mathbf{U})=-\sum_{D}\sum_{k=1}^{\infty}w_{D}^{k}u^{k|D|},$$

and so,

$$\frac{d}{du}\log\det(\mathbf{I}_{2m}-u\mathbf{U}) = -\sum_{D}\sum_{k=1}^{\infty} w_D^k u^{k|D|-1},$$

where D runs over all prime cycles in G. Therefore, it follows that

$$\log \det(\mathbf{I}_{2m} - u\mathbf{U}) = -\sum_{D} \sum_{k=1}^{\infty} \frac{w_{D}^{k}}{k|D|} u^{k|D|}$$
$$= -\sum_{[D]} \sum_{k=1}^{\infty} \frac{|D|}{k|D|} w_{D}^{k} u^{k|D|}$$
$$= -\sum_{[D]} \sum_{k=1}^{\infty} \frac{1}{k} w_{D}^{k} u^{k|D|}$$
$$= \sum_{[D]} \log(1 - w_{D} u^{|D|}).$$

Hence,

$$\det(\mathbf{I}_{2m} - u\mathbf{U}(\lambda)) = \prod_{[C]} (1 - w_C u^{|C|}),$$

5 Scattering matrix of a regular covering of a graph

Let G be a connected graph, and let $N(v) = \{w \in V(G) \mid (v, w) \in D(G)\}$ denote the neighbourhood of a vertex v in G. A graph H is a covering of G with projection $\pi : H \longrightarrow G$ if there is a surjection $\pi : V(H) \longrightarrow V(G)$ such that $\pi|_{N(v')} : N(v') \longrightarrow N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. When a finite group Π acts on a graph G, the quotient graph G/Π is a graph whose vertices are the Π -orbits on V(G), with two vertices adjacent in G/Π if and only if some two of their representatives are adjacent in G. A covering $\pi : H \longrightarrow G$ is regular if there is a subgroup B of the automorphism group Aut H of H acting freely on H such that the quotient graph H/B is isomorphic to G.

Let G be a graph and Γ a finite group. Then a mapping $\alpha : D(G) \longrightarrow \Gamma$ is an ordinary voltage assignment if $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in D(G)$. The pair (G, α) is an ordinary voltage graph. The derived graph G^{α} of the ordinary voltage graph (G, α) is defined as follows: $V(G^{\alpha}) = V(G) \times \Gamma$ and $((u, h), (v, k)) \in D(G^{\alpha})$ if and only if $(u, v) \in D(G)$ and $k = h\alpha(u, v)$. The natural projection $\pi : G^{\alpha} \longrightarrow G$ is defined by $\pi(u, h) = u$. The graph G^{α} is a derived graph covering of G with voltages in Γ or a Γ -covering of G. Note that $|\mathcal{E}_{(u,h)}| = |\mathcal{E}_u|$ for each $(u, h) \in V(G^{\alpha})$. The natural projection π commutes with the right multiplication action of the $\alpha(e), e \in D(G)$ and the left action of Γ on the fibers: $g(u, h) = (u, gh), g \in \Gamma$, which is free and transitive. Thus, the Γ -covering G^{α} is a $|\Gamma|$ -fold regular covering of G with covering transformation group Γ . Furthermore, every regular covering of a graph G is a Γ -covering of G for some group Γ (see [6]).

Let G be a connected graph, Γ be a finite group and $\alpha : D(G) \longrightarrow \Gamma$ be an ordinary voltage assignment. In the Γ -covering G^{α} , set $v_g = (v,g)$ and $e_g = (e,g)$, where $v \in V(G), e \in D(G), g \in \Gamma$. For $e = (u, v) \in D(G)$, the arc e_g emanates from u_g and terminates at $v_{g\alpha(e)}$. Note that $e_q^{-1} = (e^{-1})_{g\alpha(e)}$. Let G be a connected graph, Γ be a finite group and $\alpha : D(G) \longrightarrow \Gamma$ be an ordinary voltage assignment. Furthermore, let $\mathbf{H} = \mathbf{H}(G) = (H_{uv})_{u,v \in V(G)}$ be an Hermitian matrix such that

$$H_{uv} = \begin{cases} h_f e^{2i\gamma_f} & \text{if } f = (u, v) \in D(G), \\ 0 & \text{otherwise,} \end{cases}$$

where, for each $f \in D(G)$,

$$h_f = h_{f^{-1}} \ge 0 \text{ and } \gamma_f = -\gamma_{f^{-1}} \in [-\pi/2, \pi/2].$$

We give the function $\tilde{h}: D(G^{\alpha}) \longrightarrow \mathbb{R}$ and $\tilde{\gamma}: D(G^{\alpha}) \longrightarrow [-\pi/2, \pi/2]$ induced from h and γ , respectively, as follows:

$$\tilde{h}(u_g, v_k) = h_{uv} \text{ and } \tilde{\gamma}(u_g, v_k) = \gamma_{uv} \text{ if } (u, v) \in D(G) \text{ and } k = g\alpha(u, v)$$

Furthermore, we consider the Hermitian matrix $\tilde{\mathbf{H}} = \mathbf{H}(G^{\alpha}) = (H_{u_g v_k})_{u_g v_k \in V(G^{\alpha})}$ of G^{α} induced from **H**. At first, let

$$H_{u_q u_q} = H_{uu} \text{ for each } g \in \Gamma.$$

For $(u_g, v_k) \in D(G^{\alpha})$, we have

$$H_{u_g v_k} = \tilde{h}(u_g, v_k) e^{2i\tilde{\gamma}(u_g, v_k)} = h_{uv} e^{2i\gamma_{uv}}.$$

Thus,

$$H_{u_gv_k} = \begin{cases} h_{uv}e^{2i\gamma_{u_gv_k}} & \text{if } (u,v) \in D(G) \text{ and } k = g\alpha(u,v), \\ H_{uu} & \text{if } u = v \text{ and } k = g, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we consider the bond wave function of the regular covering G^{α} of G. Let $V(G) = \{v_1, \ldots, v_n\}$, $D(G) = \{e_1, \ldots, e_m, e_1^{-1}, \ldots, e_m^{-1}\}$ and $\Gamma = \{g_1 = 1, g_2, \ldots, g_p\}$. Let λ be a eigenvalue of $\tilde{\mathbf{H}} = \mathbf{H}(G^{\alpha})$, and let $\tilde{\phi} = (\phi_{v_1,g_1}, \ldots, \phi_{v_1,g_p}, \ldots, \phi_{v_n,g_1}, \ldots, \phi_{v_n,g_p})$ be the eigenvector corresponding to λ , where ϕ_{v_i,g_j} corresponds to the vertex (v_i,g_j) $(1 \le i \le n; 1 \le j \le p)$ of G^{α} . Furthermore let $b_g = (v_g, z_{g\alpha(b)})$ be any arc of G^{α} , where $b = (v, z) \in D(G)$, $g \in \Gamma$. Then the bond wave function of G^{α} is

$$\phi_{b_g}(x) = \frac{e^{i\gamma_b}}{\sqrt{h_b}} (a_{b_g^{-1}} e^{i\pi x/4} + a_{b_g} e^{-i\pi x/4}), \ x = \pm 1, \ i = \sqrt{-1}$$

under the condition

$$\phi_{b_g}(1) = \phi_{v_g} \text{ and } \phi_{b_g}(-1) = \phi_{z_{g\alpha(b)}}$$

By (1), we have

$$\begin{aligned} a_{b_g^{-1}} &= i\delta_{b_g^{-1}e_g} - 2\sum_{o(e_g)=v_g} \frac{\sqrt{\tilde{h}_{e_g}}\sqrt{\tilde{h}_{b_g}}}{H_{v_gv_g} - \lambda - i\Gamma_{v_g}} e^{i(\tilde{\gamma}_{b_g} + \tilde{\gamma}_{e_g})} a_{e_g} \\ &= \sum_{o(e_g)=v_g} \sigma_{b_ge_g}^{(v_g)} a_{e_g} \end{aligned}$$

for each arc b_g with $o(b_g) = v_g$, where

$$\sigma_{b_g e_g}^{(v_g)} = i\delta_{b_g^{-1}e_g} - 2\frac{\sqrt{\tilde{h}_{e_g}}\sqrt{\tilde{h}_{b_g}}}{H_{v_g v_g} - \lambda - i\Gamma_{v_g}}e^{i(\tilde{\gamma}_{b_g} + \tilde{\gamma}_{e_g})}a_{e_g}$$

and

$$\tilde{h}_{e_g} = \tilde{h}(e_g), \ \tilde{\gamma}_{e_g} = \tilde{\gamma}(e_g).$$

By the definitions of \tilde{h} , $\tilde{\gamma}$ and $\tilde{\mathbf{H}}$, we have

$$\sigma_{b_g e_g}^{(v_g)} = i\delta_{b^{-1}e} - 2\frac{\sqrt{h_e}\sqrt{h_b}}{H_{vv} - \lambda - i\Gamma_v}e^{i(\gamma_b + \gamma_e)} = \sigma_{be}^{(v)} = \sigma_{be}^{(t(b))}$$

Note that $\mathcal{E}_{(v,g)} = \mathcal{E}_v$. Thus,

$$a_{b_g^{-1}} = \sum_{o(e_g)=v_g} \sigma_{be}^{(t(b))} a_{e_g}$$

Therefore, the bond scattering matrix $\tilde{\mathbf{U}}(\lambda) = (U(e_g, f_h))_{e_g, f_h \in D(G^{\alpha})}$ of G^{α} is given by

$$U(e_g, f_h) = \begin{cases} \sigma_{ef}^{(t(e))} & \text{if } t(f_h) = o(e_g), \\ 0 & \text{otherwise.} \end{cases}$$

But, we have

$$x_{v_g} = \frac{2}{H_{vv} - \lambda - i\Gamma_v} = x_v$$

for $v_g \in V(G^{\alpha})$. Furthermore, let $\tilde{w} : D(G^{\alpha}) \longrightarrow \mathbb{C}$ be given as follows:

$$\tilde{w}(e_g) = \sqrt{\tilde{h}_{e_g}} e^{i\tilde{\gamma}_{e_g}} \text{ for each } e_g \in D(G^{\alpha}).$$

Then we have

$$\tilde{w}(e_g) = \sqrt{h_e} e^{i\gamma_e} = w(e), \ e_g \in D(G^{\alpha}).$$

For $g \in \Gamma$, let the matrix $\mathbf{H}_g = (H_{uv}^{(g)})$ be defined by

$$H_{uv}^{(g)} = \begin{cases} h_{uv}e^{2i\gamma_{uv}} & \text{if } \alpha(u,v) = g \text{ and } (u,v) \in D(G), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, let $\mathbf{U}_g = (U^{(g)}(e, f))$ be given by

$$U^{(g)}(e,f) = \begin{cases} \sigma_{ef}^{(t(e))} & \text{if } t(e) = o(f) \text{ and } \alpha(e) = g, \\ 0 & \text{otherwise,} \end{cases}$$

Let $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$ be the block diagonal sum of square matrices $\mathbf{M}_1, \ldots, \mathbf{M}_s$. If $\mathbf{M}_1 = \mathbf{M}_2 = \cdots = \mathbf{M}_s = \mathbf{M}$, then we write $s \circ \mathbf{M} = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$. The Kronecker product $\mathbf{A} \bigotimes \mathbf{B}$ of matrices \mathbf{A} and \mathbf{B} is considered as the matrix \mathbf{A} having the element a_{ij} replaced by the matrix $a_{ij}\mathbf{B}$.

Theorem 6 Let G be a connected graph with n vertices $v_1, \ldots v_n$ and m unoriented edges, Γ be a finite group and $\alpha : D(G) \longrightarrow \Gamma$ be an ordinary voltage assignment. Set $|\Gamma| = p$. Furthermore, let $\rho_1 = 1, \rho_2, \cdots, \rho_k$ be the irreducible representations of Γ , and f_i be the degree of ρ_i for each i, where $f_1 = 1$.

If the Γ -covering G^{α} of G is connected, then, for the bond scattering matrix of G^{α} ,

$$\det(\mathbf{I}_{2mp} - \tilde{\mathbf{U}}(\lambda)) = \det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) \prod_{i=2}^{k} \det(\mathbf{I}_{2mf_i} - \sum_{h} \rho_i(h) \bigotimes \mathbf{U}_h)^{f_i}$$

$$=\frac{2^{mp}(-1)^{np}\det(\lambda\mathbf{I}_n-\mathbf{H})}{\prod_{u\in V(G)}(H_{uu}-\lambda-i\Gamma_u)}\prod_{i=2}^k\det(\lambda\mathbf{I}_{nf_i}-\sum_{h\in\Gamma}\rho_i(h)\bigotimes\mathbf{H}_h-\mathbf{I}_{f_i}\bigotimes\operatorname{diag}(\mathbf{H}))^{f_i},$$

where

$$\operatorname{diag}(\mathbf{H}) = \begin{bmatrix} H_{v_1v_1} & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & H_{v_nv_n} \end{bmatrix}.$$

Proof. Let $|\Gamma| = p$. By Theorem 4, for the bond scattering matrix of G^{α} , we have

$$\det(\mathbf{I}_{2mp} - \tilde{\mathbf{U}}(\lambda)) = \frac{2^{mp}(-1)^{np}\det(\lambda\mathbf{I}_{np} - \mathbf{H}(G^{\alpha}))}{\prod_{u \in V(G)}(H_{uu} - \lambda - i\Gamma_u)^p}$$

Let $D(G) = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{2m}\}$ such that $e_{m+j} = e_j^{-1} (1 \le j \le m)$ and $\Gamma = \{1 = g_1, g_2, \ldots, g_p\}$. Arrange arcs of G^{α} in p blocks: $(e_1, 1), \ldots, (e_{2m}, 1); (e_1, g_2), \ldots, (e_{2m}, g_2); \ldots;$ $(e_1, g_p), \ldots, (e_{2m}, g_p)$. We consider the matrix $\tilde{\mathbf{U}}(\lambda)$ under this order. For $h \in \Gamma$, the matrix $\mathbf{P}_h = (p_{ij}^{(h)})$ is defined as follows:

$$p_{ij}^{(h)} = \begin{cases} 1 & \text{if } g_i h = g_j, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $p_{ij}^{(h)} = 1$, i.e., $g_j = g_i h$. Then $U(e_{g_i}, f_{g_j}) \neq 0$ if and only if $t(e, g_j) = o(f, g_i)$. Furthermore, $t(e, g_j) = o(f, g_i)$ if and only if $(o(f), g_j) = o(f, g_j) = t(e, g_i) = (t(e), g_i \alpha(e))$. Thus, t(e) = o(f) and $\alpha(e) = g_i^{-1}g_j = g_i^{-1}g_i h = h$. Thus, we have

$$\widetilde{\mathbf{U}}(\lambda) = \sum_{h \in \Gamma} \mathbf{P}_h \bigotimes \mathbf{U}_h.$$

Furthermore, we have

$$\operatorname{diag}(\mathbf{H}(G^{\alpha})) = \mathbf{I}_p \bigotimes \operatorname{diag}(\mathbf{H}).$$

Let ρ be the right regular representation of Γ . Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_k$ be all inequivalent irreducible representations of Γ , and f_i the degree of ρ_i for each i, where $f_1 = 1$. Then we have $\rho(g) = \mathbf{P}_g$ for $g \in \Gamma$. Furthermore, there exists a nonsingular matrix \mathbf{P} such that $\mathbf{P}^{-1}\rho(g)\mathbf{P} = (1) \oplus f_2 \circ \rho_2(g) \oplus \cdots \oplus f_k \circ \rho_k(g)$ for each $g \in \Gamma$ (see [12]). Thus, we have

$$\mathbf{P}^{-1}\mathbf{P}_{g}\mathbf{P} = (1) \oplus f_{2} \circ \rho_{2}(g) \oplus \cdots \oplus f_{k} \circ \rho_{k}(g).$$

Putting $\mathbf{F} = (\mathbf{P}^{-1} \bigotimes \mathbf{I}_{2m}) \tilde{\mathbf{U}}(\lambda) (\mathbf{P} \bigotimes \mathbf{I}_{2m})$, we have

$$\mathbf{F} = \sum_{g \in \Gamma} \{(1) \oplus f_2 \circ \rho_2(g) \oplus \cdots \oplus f_k \circ \rho_k(g)\} \bigotimes \mathbf{U}_g.$$

Note that $\mathbf{U}(\lambda) = \sum_{g \in \Gamma} \mathbf{U}_g$ and $1 + f_2^2 + \cdots + f_k^2 = p$. Therefore it follows that

$$\det(\mathbf{I}_{2mp} - \tilde{\mathbf{U}}(\lambda)) = \det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) \prod_{i=2}^{k} \det(\mathbf{I}_{2mf_i} - \sum_{g} \rho_i(g) \bigotimes \mathbf{U}_g)^{f_i}.$$

Next, let $V(G) = \{v_1, \ldots, v_n\}$. Arrange vertices of G^{α} in p blocks: $(v_1, 1), \ldots, (v_n, 1)$; $(v_1, g_2), \ldots, (v_n, g_2); \ldots; (v_1, g_p), \ldots, (v_n, g_p)$. We consider the matrix $\mathbf{H}(G^{\alpha})$ under this order.

Suppose that $p_{ij}^{(h)} = 1$, i.e., $g_j = g_i h$. Then $((u, g_i), (v, g_j)) \in D(G^{\alpha})$ if and only if $(u, v) \in D(G)$ and $g_j = g_i \alpha(u, v)$. If $g_j = g_i \alpha(u, v)$, then $\alpha(u, v) = g_i^{-1}g_j = g_i^{-1}g_i h = h$. Thus we have

$$\mathbf{H}(G^{\alpha}) = \sum_{h \in \Gamma} \mathbf{P}_h \bigotimes \mathbf{H}_h + \mathbf{I}_p \bigotimes \operatorname{diag}(\mathbf{H}).$$

Putting $\mathbf{E} = (\mathbf{P}^{-1} \bigotimes \mathbf{I}_n) \mathbf{H}(G^{\alpha}) (\mathbf{P} \bigotimes \mathbf{I}_n)$, we have

$$\mathbf{E} = \sum_{h \in \Gamma} \{(1) \oplus f_2 \circ \rho_2(h) \oplus \cdots \oplus f_k \circ \rho_k(h)\} \bigotimes \mathbf{H}_h + \mathbf{I}_p \bigotimes \operatorname{diag}(\mathbf{H}).$$

Note that $\mathbf{H}(G) = \sum_{h \in \Gamma} \mathbf{H}_h + \text{diag}(\mathbf{H})$. Therefore it follows that

$$\det(\lambda \mathbf{I}_{np} - \mathbf{H}(G^{\alpha})) = \det(\lambda \mathbf{I}_{n} - \mathbf{H}(G))$$
$$\times \prod_{i=2}^{k} \det(\lambda \mathbf{I}_{nf_{i}} - \sum_{h \in \Gamma} \rho_{i}(h) \bigotimes \mathbf{H}_{h} - \mathbf{I}_{f_{i}} \bigotimes \operatorname{diag} \mathbf{H})^{f_{i}}.$$

Hence,

$$\det(\mathbf{I}_{2mp} - \tilde{\mathbf{U}}(\lambda)) = \det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) \prod_{i=2}^{k} \det(\mathbf{I}_{2mf_{i}} - \sum_{h} \rho_{i}(h) \bigotimes \mathbf{U}_{h})^{f_{i}}$$
$$= \frac{2^{mp}(-1)^{np} \det(\lambda \mathbf{I}_{n} - \mathbf{H}(G))}{\prod_{u \in V(G)} (H_{uu} - \lambda - i\Gamma_{u})^{p}} \prod_{i=2}^{k} \det(\lambda \mathbf{I}_{nf_{i}} - \sum_{h \in \Gamma} \rho_{i}(h) \bigotimes \mathbf{H}_{h} - \mathbf{I}_{f_{i}} \bigotimes \operatorname{diag} \mathbf{H})^{f_{i}}$$

6 L-functions of graphs

Let G be a connected graph with n vertices and m unoriented edges, Γ be a finite group and $\alpha : D(G) \longrightarrow \Gamma$ be an ordinary voltage assignment. Furthermore, let $\mathbf{H} = \mathbf{H}(G) = (H_{uv})_{u,v \in V(G)}$ be an Hermitian matrix such that

$$H_{uv} = \begin{cases} h_f e^{2i\gamma_f} & \text{if } f = (u, v) \in D(G), \\ 0 & \text{otherwise,} \end{cases}$$

where, for each $f \in D(G)$,

$$h_f = h_{f^{-1}} \ge 0 \text{ and } \gamma_f = -\gamma_{f^{-1}} \in [-\pi/2, \pi/2].$$

Let ρ be a unitary representation of Γ and d its degree. The *L*-function of G associated with ρ and α is defined by

$$\mathbf{Z}_{H}(G,\lambda,\rho,\alpha) = \det(\mathbf{I}_{2md} - \sum_{h\in\Gamma} \rho(h) \bigotimes \mathbf{U}_{h})^{-1}.$$

If $\rho = \mathbf{1}$ is the identity representation of Γ , then

$$\mathbf{Z}_H(G,\lambda,\mathbf{1},\alpha) = \det(\mathbf{I}_{2m} - \mathbf{U})^{-1}.$$

A determinant expression for the *L*-function of *G* associated with ρ and α is given as follows. For $1 \leq i, j \leq n$, the (i, j)-block $\mathbf{F}_{i,j}$ of a $dn \times dn$ matrix \mathbf{F} is the submatrix of \mathbf{K} consisting of $d(i-1) + 1, \ldots, di$ rows and $d(j-1) + 1, \ldots, dj$ columns.

Theorem 7 Let G be a connected graph with n vertices and m unoriented edges, Γ be a finite group and α : $D(G) \longrightarrow \Gamma$ be an ordinary voltage assignment. If ρ is a unitary representation of Γ and d is the degree of ρ , then the reciprocal of the L-function of G associated with ρ and α is

$$\mathbf{Z}_{H}(G,\lambda,\rho,\alpha)^{-1} = \frac{2^{md}(-1)^{nd}}{\prod_{u \in V(G)} (H_{uu} - \lambda - i\Gamma_{u})^{d}} \det(\lambda \mathbf{I}_{np} - \sum_{g \in \Gamma} \rho(g) \bigotimes \mathbf{H}_{g} - \mathbf{I}_{d} \bigotimes \operatorname{diag}(\mathbf{H})).$$

Proof. The argument is an analogue of Watanabe and Fukumizu's method [18].

Let $V(G) = \{v_1, \ldots, v_n\}$ and $D(G) = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{2m}\}$ such that $e_{m+i} = e_i^{-1}(1 \le i \le m)$. Note that the (e, f)-block $(\sum_{g \in \Gamma} \mathbf{U}_g \bigotimes \rho(g))_{ef}$ of $\sum_{g \in \Gamma} \mathbf{U}_g \bigotimes \rho(g)$ is given by

$$(\sum_{g\in\Gamma} \mathbf{U}_g \bigotimes \rho(g))_{ef} = \begin{cases} \rho(\alpha(e))\sigma_{ef}^{(t(e))} & \text{if } t(e) = o(f), \\ \mathbf{0}_d & \text{otherwise.} \end{cases}$$

For $g \in \Gamma$, two $2m \times 2m$ matrices $\mathbf{B}_g = (\mathbf{B}_{ef}^{(g)})_{e,f \in D(G)}$ and $\mathbf{J}_g = (\mathbf{J}_{ef}^{(g)})_{e,f \in D(G)}$ are defined as follows:

$$\mathbf{B}_{ef}^{(g)} = \begin{cases} x_{o(f)}w(e)w(f) & \text{if } t(e) = o(f) \text{ and } \alpha(e) = g, \\ 0 & \text{otherwise,} \end{cases} \quad \mathbf{J}_{ef}^{(g)} = \begin{cases} 1 & \text{if } f = e^{-1} \text{ and } \alpha(e) = g, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\mathbf{U}_g = i \mathbf{J}_g - \mathbf{B}_g \text{ for } g \in \Gamma.$$

Let $\mathbf{K} = (\mathbf{K}_{ij})_{1 \le i \le 2m; 1 \le j \le n}$ be the $2md \times nd$ matrix defined as follows:

$$\mathbf{K}_{ij} := \begin{cases} x_{v_j} w(e_i) \mathbf{I}_d & \text{if } o(e_i) = v_j, \\ \mathbf{0}_d & \text{otherwise.} \end{cases}$$

Furthermore, we define two $2md \times nd$ matrices $\mathbf{L} = (\mathbf{L}_{ij})_{1 \leq i \leq 2m; 1 \leq j \leq n}$ and $\mathbf{M} = (\mathbf{M}_{ij})_{1 \leq i \leq 2m; 1 \leq j \leq n}$ as follows:

$$\mathbf{L}_{ij} := \begin{cases} w(e_j)\rho(\alpha(e_i)) & \text{if } t(e_i) = v_j, \\ \mathbf{0}_d & \text{otherwise,} \end{cases} \quad \mathbf{M}_{ij} := \begin{cases} w(e_i)\mathbf{I}_d & \text{if } o(e_i) = v_j, \\ \mathbf{0}_d & \text{otherwise.} \end{cases}$$

Then we have

$$\mathbf{K} = \mathbf{M}(\mathbf{X} \bigotimes \mathbf{I}_d) = \mathbf{M}\mathbf{X}_d,$$

where

$$\mathbf{X}_d = \mathbf{X} \bigotimes \mathbf{I}_d$$

Furthermore, we have

$$\mathbf{L}^{t}\mathbf{K} = \sum_{h\in\Gamma} \mathbf{B}_{h} \bigotimes \rho(h) = \mathbf{B}_{\rho}$$
(6)

and

$${}^{t}\mathbf{ML} = \sum_{g \in \Gamma} \mathbf{H}_{g} \bigotimes \rho(g), \tag{7}$$

where

$$\mathbf{B}_{\rho} = \sum_{g \in \Gamma} \mathbf{B}_g \bigotimes \rho(g).$$

Thus,

$$\det(\mathbf{I}_{2md} - u \sum_{g \in \Gamma} \rho(g) \bigotimes \mathbf{U}_g) = \det(\mathbf{I}_{2md} - u \sum_{g \in \Gamma} \mathbf{U}_g \bigotimes \rho(g))$$
$$= \det(\mathbf{I}_{2md} - u \sum_{g \in \Gamma} (i\mathbf{J}_g - \mathbf{B}_g) \bigotimes \rho(g))$$
$$= \det(\mathbf{I}_{2md} - iu \sum_{g \in \Gamma} \mathbf{J}_g \bigotimes \rho(g) + u \sum_{g \in \Gamma} \mathbf{B}_g \bigotimes \rho(g)).$$

Now, let

$$\mathbf{J}_{\rho} = \sum_{g \in \Gamma} \mathbf{J}_g \bigotimes \rho(g).$$

Note that

$$\mathbf{J}_{\rho}^2 = \mathbf{I}_{2md}$$

Then we have

$$\det(\mathbf{I}_{2md} - u \sum_{g \in \Gamma} \rho(g) \otimes \mathbf{U}_g)$$

$$= \det(\mathbf{I}_{2md} - iu\mathbf{J}_{\rho} + u\mathbf{B}_{\rho})$$

$$= \det(\mathbf{I}_{2md} + u\mathbf{B}_{\rho}(\mathbf{I}_{2md} - iu\mathbf{J}_{\rho})^{-1})\det(\mathbf{I}_{2md} - iu\mathbf{J}_{\rho})$$

$$= \det(\mathbf{I}_{2md} + u\mathbf{L} \ {}^{t}\mathbf{K}(\mathbf{I}_{2md} - iu\mathbf{J}_{\rho})^{-1})\det(\mathbf{I}_{2md} - iu\mathbf{J}_{\rho})$$

$$= \det(\mathbf{I}_{nd} + u \ {}^{t}\mathbf{K}(\mathbf{I}_{2md} - iu\mathbf{J}_{\rho})^{-1}\mathbf{L})\det(\mathbf{I}_{2md} - iu\mathbf{J}_{\rho}).$$

But, we have

$$\det(\mathbf{I}_{2md} - iu\mathbf{J}_{\rho}) = \det\begin{pmatrix} \mathbf{I}_{d} & -iu\rho(\alpha(e_{1})) & \mathbf{0} \\ -iu\rho(\alpha(e_{1}^{-1})) & \mathbf{I}_{d} & \\ & \ddots & \\ \mathbf{0} & & \ddots & \\ \mathbf{0} & & & \end{pmatrix} = (1+u^{2})^{md}.$$

Furthermore, we have

$$(\mathbf{I}_{2md} - iu\mathbf{J}_{\rho})^{-1}$$

$$= \begin{bmatrix} \mathbf{I}_{d} & -iu\rho(\alpha(e_{1})) & \mathbf{0} \\ -iu\rho(\alpha(e_{1}^{-1})) & \mathbf{I}_{d} & & \\ \mathbf{0} & & \ddots & \\ \end{bmatrix}^{-1}$$

$$= \frac{1}{1+u^{2}} \begin{bmatrix} \mathbf{I}_{d} & iu\rho(\alpha(e_{1})) & \mathbf{0} \\ iu\rho(\alpha(e_{1}^{-1})) & \mathbf{I}_{d} & & \\ & & \ddots & \\ \mathbf{0} & & & \ddots & \\ \end{bmatrix}$$

$$= \frac{1}{1+u^{2}} (\mathbf{I}_{2md} + iu\mathbf{J}_{\rho}).$$

Thus, we have

$$\det(\mathbf{I}_{2md} - u \sum_{g \in \Gamma} \rho(g) \bigotimes \mathbf{U}_g)$$

$$= (1 + u^2)^{md} \det(\mathbf{I}_{nd} + u/(1 + u^2) {}^t \mathbf{K} (\mathbf{I}_{2md} + iu \mathbf{J}_\rho) \mathbf{L})$$

$$= (1 + u^2)^{md - nd} \det((1 + u^2) \mathbf{I}_{nd} + u {}^t \mathbf{K} \mathbf{L} + iu^2 {}^t \mathbf{K} \mathbf{J}_\rho \mathbf{L}).$$

Now, we have

$${}^{t}\mathbf{KL} = \mathbf{X}_{d} {}^{t}\mathbf{ML} = \mathbf{X}_{d} \sum_{g \in \Gamma} \mathbf{H}_{g} \bigotimes \rho(g).$$

Furthermore,

$${}^{t}\mathbf{K}\mathbf{J}_{\rho}\mathbf{L}=\mathbf{X}_{d} {}^{t}\mathbf{M}\mathbf{J}_{\rho}\mathbf{L}.$$

Then we have

$$({}^{t}\mathbf{M}\mathbf{J}_{
ho}\mathbf{L})_{uv}$$

$$= \delta_{uv} \sum_{o(e)=u} ({}^{t}\mathbf{M})_{ue} (\mathbf{J}_{\rho})_{ee^{-1}} (\mathbf{L})_{e^{-1}v}$$

$$= \delta_{uv} \sum_{o(e)=u} w(e) \mathbf{I}_{d} \rho(\alpha(e)) w(e^{-1}) \rho(\alpha(e^{-1}))$$

$$= \delta_{uv} \sum_{o(e)=u} \sqrt{h_{e}} e^{i\gamma_{e}} \sqrt{h_{e}} e^{-i\gamma_{e}} \mathbf{I}_{d}$$

$$= \delta_{uv} \sum_{o(e)=u} h_{e} \mathbf{I}_{d} = \delta_{uv} \Gamma_{u} \mathbf{I}_{d}.$$

Thus,

$${}^{t}\mathbf{K}\mathbf{J}_{\rho}\mathbf{L}=\mathbf{X}(\mathbf{D}_{\Gamma}\bigotimes\mathbf{I}_{d}),$$

where

$$\mathbf{D}_{\Gamma} = \begin{bmatrix} \Gamma_{v_1} & 0 \\ & \ddots & \\ 0 & & \Gamma_{v_n} \end{bmatrix}.$$

Therefore, it follows that

$$\det(\mathbf{I}_{2md} - u \sum_{g \in \Gamma} \rho(g) \otimes \mathbf{U}_g)$$

= $(1 + u^2)^{(m-n)d} \det((1 + u^2)\mathbf{I}_{nd} + u\mathbf{X}_d \sum_{g \in \Gamma} \mathbf{H}_g \otimes \rho(g) + iu^2 \mathbf{X}_d(\mathbf{D}_{\Gamma} \otimes \mathbf{I}_d)).$

Substituting u = 1, we obtain

$$\det(\mathbf{I}_{2md} - \sum_{g \in \Gamma} \rho(g) \otimes \mathbf{U}_g)$$

$$= 2^{(m-n)d} \det(2\mathbf{I}_{nd} + \mathbf{X}_d \sum_{g \in \Gamma} \mathbf{H}_g \otimes \rho(g) + i\mathbf{X}_d(\mathbf{D}_{\Gamma} \otimes \mathbf{I}_d))$$

$$= 2^{(m-n)d} \det(\mathbf{X}_d) \det(2\mathbf{X}_d^{-1} + \sum_{g \in \Gamma} \mathbf{H}_g \otimes \rho(g) + i\mathbf{D}_{\Gamma} \otimes \mathbf{I}_d).$$

Then we have

$$\det(\mathbf{X}_d) = \det(\mathbf{X}\bigotimes\mathbf{I}_d) = (\det(\mathbf{X}))^d = \frac{2^{nd}}{\prod_{u \in V(G)} (H_{uu} - \lambda - i\Gamma_u)^d}$$

Furthermore, since

$$\mathbf{X}_d^{-1} = \mathbf{X}^{-1} \bigotimes \mathbf{I}_d,$$

we have

$$(2\mathbf{X}_{d}^{-1} + i\mathbf{D}_{\Gamma} \bigotimes \mathbf{I}_{d})_{uu} = (2\frac{H_{uu} - \lambda - i\Gamma_{u}}{2} + i\Gamma_{u}) \bigotimes \mathbf{I}_{d}$$
$$= (H_{uu} - \lambda) \bigotimes \mathbf{I}_{d}.$$

That is,

$$2\mathbf{X}_{d}^{-1} + i\mathbf{D}_{\Gamma} \bigotimes \mathbf{I}_{d} = -\lambda \mathbf{I}_{nd} + \operatorname{diag}(\mathbf{H}) \bigotimes \mathbf{I}_{d}.$$

Therefore, it follows that

$$\det(\mathbf{I}_{2md} - \sum_{g \in \Gamma} \rho(g) \bigotimes \mathbf{U}_g)$$

$$= \frac{2^{md}}{\prod_{u \in V(G)} (H_{uu} - \lambda - i\Gamma_u)^d} \det(-\lambda \mathbf{I}_{nd} + \sum_{g \in \Gamma} \mathbf{H}_g \bigotimes \rho(g) + \operatorname{diag}(\mathbf{H}) \bigotimes \mathbf{I}_d)$$

$$= \frac{(-1)^{nd} 2^{md}}{\prod_{u \in V(G)} (H_{uu} - \lambda - i\Gamma_u)^d} \det(\lambda \mathbf{I}_{nd} - \sum_{g \in \Gamma} \rho(g) \bigotimes \mathbf{H}_g - \mathbf{I}_d \bigotimes \operatorname{diag}(\mathbf{H})).$$

By Theorems 6 and 7 the following result holds.

Corollary 1 Let G be a connected graph with m edges, Γ be a finite group and $\alpha : D(G) \longrightarrow \Gamma$ be an ordinary voltage assignment. Then

$$\det(\mathbf{I}_{2mp} - \tilde{\mathbf{U}}(\lambda)) = \prod_{\rho} \mathbf{Z}_H(G, \lambda, \rho, \alpha)^{-\deg \rho},$$

where ρ runs over all inequivalent irreducible representations of Γ and $p = |\Gamma|$.

7 Example

We give an example. Let $G = K_3$ be the complete graph with three vertices 1, 2, 3 and six arcs $e_1, e_2, e_3, e_1^{-1}, e_2^{-1}, e_3^{-1}$, where $e_1 = (v_1, v_2), e_2 = (v_2, v_3), e_3 = (v_3, v_1)$. Furthermore, let

$$\mathbf{H} = \left[\begin{array}{ccc} a & be^{2i\alpha} & be^{2i\alpha} \\ be^{-2i\alpha} & a & be^{2i\alpha} \\ be^{-2i\alpha} & be^{-2i\alpha} & a \end{array} \right],$$

where a > 0, b > 0 and $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then we have

$$x_1 = x_2 = x_3 = \frac{2}{a - \lambda - 2ib}$$

Set $x = \frac{2}{a - \lambda - 2ib}$. Considering $\mathbf{U}(\lambda)$ under the order $e_1, e_2, e_3, e_1^{-1}, e_2^{-1}, e_3^{-1}$, we have

$$\mathbf{U}(\lambda) = \begin{bmatrix} -xbe^{2i\alpha} & -xbe^{2i\alpha} & i-xb & -xb & -xb \\ -xbe^{2i\alpha} & -xbe^{2i\alpha} & -xbe^{2i\alpha} & -xb & i-xb & -xb \\ -xbe^{2i\alpha} & -xbe^{2i\alpha} & -xbe^{2i\alpha} & -xb & i-xb \\ i-xb & -xb & -xb & -xbe^{-2i\alpha} & -xbe^{-2i\alpha} \\ -xb & i-xb & -xb & -xbe^{-2i\alpha} & -xbe^{-2i\alpha} \\ -xb & -xb & i-xb & -xbe^{-2i\alpha} & -xbe^{-2i\alpha} \\ -xb & -xb & i-xb & -xbe^{-2i\alpha} & -xbe^{-2i\alpha} \end{bmatrix}.$$

By Theorem 4, we have

$$det(\mathbf{I}_{6} - \mathbf{U}(\lambda)) = \frac{2^{3}(-1)^{3}}{(a - \lambda - 2ib)^{3}} det(\lambda \mathbf{I}_{3} - \mathbf{H})$$

$$= \frac{-8}{(a - \lambda - 2ib)^{3}} \begin{bmatrix} \lambda - a & -be^{2i\alpha} & -be^{2i\alpha} \\ -be^{-2i\alpha} & \lambda - a & -be^{2i\alpha} \\ -be^{-2i\alpha} & -be^{-2i\alpha} & \lambda - a \end{bmatrix}$$

$$= \frac{-8}{(a - \lambda - 2ib)^{3}} \{(\lambda - a)^{3} - 3b^{2}(\lambda - a) - b^{3}(e^{2i\alpha} + e^{-2i\alpha})\}$$

$$= \frac{-8}{(a - \lambda - 2ib)^{3}} \{(\lambda - a)^{3} - 3b^{2}(\lambda - a) - 2b^{3}\cos 2\alpha\}.$$

Next. let $\Gamma = \mathbb{Z}_3 = \{1, \tau, \tau^2\}(\tau^3 = 1)$ be the cyclic group of order 3, and let $\alpha : D(K_3) \longrightarrow \mathbb{Z}_3$ be the ordinary voltage assignment such that $\alpha(e_1) = \tau$, $\alpha(e_1^{-1}) = \tau^2$ and $\alpha(e_2) = \alpha(e_2^{-1}) = \alpha(e_3) = \alpha(e_3^{-1}) = 1$. Then the \mathbb{Z}_3 -coverng K_3^{α} of K_3 is the cycle graph of length 9.

The characters of \mathbb{Z}_3 are given as follows: $\chi_i(\tau^j) = (\xi^i)^j, 0 \le i, j \le 2$, where $\xi = \frac{-1+\sqrt{-3}}{2}$. Then we have

$$\mathbf{H}_{1} = \begin{bmatrix} 0 & 0 & be^{2i\alpha} \\ 0 & be^{2i\alpha} \\ be^{-2i\alpha} & be^{-2i\alpha} & 0 \end{bmatrix}, \mathbf{H}_{\tau} = \begin{bmatrix} 0 & be^{2i\alpha} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{H}_{\tau^{2}} = \begin{bmatrix} 0 & 0 & 0 \\ be^{-2i\alpha} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, by Theorem 7,

$$\zeta_{H}(K_{3},\lambda,\chi_{1},\alpha)^{-1} = \frac{2^{3}(-1)^{3}}{(a-\lambda-2ib)^{3}} \det(\lambda \mathbf{I}_{3} - \sum_{j=0}^{2} \chi_{1}(\tau^{j}) \mathbf{H}_{\tau^{j}} - \operatorname{diag}(\mathbf{H}))$$

$$= \frac{-8}{(a-\lambda-2ib)^{3}} \begin{bmatrix} \lambda - a & -b\xi e^{2i\alpha} & -be^{2i\alpha} \\ -b\xi^{2}e^{-2i\alpha} & \lambda - a & -be^{2i\alpha} \\ -be^{-2i\alpha} & -be^{-2i\alpha} & \lambda - a \end{bmatrix}$$

$$= \frac{-8}{(a-\lambda-2ib)^{3}} \{(\lambda-a)^{3} - 3b^{2}(\lambda-a) - b^{3}(\xi e^{2i\alpha} + \xi^{2}e^{-2i\alpha})\}$$

$$= \frac{-8}{(a-\lambda-2ib)^{3}} \{(\lambda-a)^{3} - 3b^{2}(\lambda-a) - 2b^{3}\cos 2(\alpha + \pi/3)\}.$$

Similarly, we have

$$\zeta_{H}(K_{3},\lambda,\chi_{2},\alpha)^{-1} = \frac{2^{3}(-1)^{3}}{(a-\lambda-2ib)^{3}} \det(\lambda \mathbf{I}_{3} - \sum_{j=0}^{2} \chi_{2}(\tau^{j}) \mathbf{H}_{\tau^{j}} - \operatorname{diag}(\mathbf{H}))$$

$$= \frac{-8}{(a-\lambda-2ib)^{3}} \begin{bmatrix} \lambda - a & -b\xi^{2}e^{2i\alpha} & -be^{2i\alpha} \\ -b\xi e^{-2i\alpha} & \lambda - a & -be^{2i\alpha} \\ -be^{-2i\alpha} & -be^{-2i\alpha} & \lambda - a \end{bmatrix}$$

$$= \frac{-8}{(a-\lambda-2ib)^{3}} \{(\lambda-a)^{3} - 3b^{2}(\lambda-a) - b^{3}(\xi^{2}e^{2i\alpha} + \xi e^{-2i\alpha})\}$$

$$= \frac{-8}{(a-\lambda-2ib)^{3}} \{(\lambda-a)^{3} - 3b^{2}(\lambda-a) - 2b^{3}\cos 2(\alpha+2\pi/3)\}.$$

By Corollary 1, it follows that

$$\det(\mathbf{I}_{18} - \tilde{\mathbf{U}}(\lambda)) = \det(\mathbf{I}_6 - \mathbf{U}(\lambda))\zeta_H(K_3, \lambda, \chi_1, \alpha)^{-1}\zeta_S(K_H, \lambda, \chi_2, \alpha)^{-1}$$
$$= \frac{-512}{(a - \lambda - 2ib)^9} \{(\lambda - a)^3 - 3b^2(\lambda - a) - 2b^3\cos 2\alpha\}$$
$$\times \{(\lambda - a)^3 - 3b^2(\lambda - a) - 2b^3\cos 2(\alpha + \pi/3)\}\{(\lambda - a)^3 - 3b^2(\lambda - a) - 2b^3\cos 2(\alpha + 2\pi/3)\}\}$$

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