The existence of uniform hypergraphs for which interpolation property of complete coloring fails

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Abstract

In 1967 Harary, Hedetniemi, and Prins showed that every graph G admits a complete t-coloring for every t with $\chi(G) \leq t \leq \psi(G)$, where $\chi(G)$ denotes the chromatic number of G and $\psi(G)$ denotes the achromatic number of G which is the maximum number r for which G admits a complete r-coloring. Recently, Edwards and Rzążewski (2020) showed that this result fails for hypergraphs by proving that for every integer k with $k \geq 9$, there exists a k-uniform hypergraph H with a complete $\chi(H)$ -coloring and a complete $\psi(H)$ -coloring, but no complete t-coloring for some t with $\chi(H) < t < \psi(H)$. They also asked whether there would exist such an example for 3-uniform hypergraphs and posed another problem to strengthen their result. In this paper, we generalize their result to all cases k with $k \geq 3$ and settle their problems by giving several kinds of 3-uniform hypergraphs. In particular, we disprove a recent conjecture due to Matsumoto and the third author (2020) who suggested a special family of 3-uniform hypergraph to satisfy the desired interpolation property.

Keywords: Hypergraph; complete coloring; triangulation; face hypergraph.

1 Introduction

In this paper, all hypergraphs are considered simple. Let H be a hypergraph. The vertex set and the hyperedge set of H are denoted by V(H) and E(H), respectively. A vertex subset of V(H) is said to be *independent*, if there is no hyperedge of H including two different vertices of it. The *incidence graph* of H refers to a bipartite graph G with $V(G) = V(H) \cup E(H)$ in which a vertex $v \in V(H)$ and a hyperedge $e \in E(H)$ are adjacent in G if and only if $v \in e$. A hypergraph is said to be k-uniform, if the size of all of hyperedges are the same number k. We say that a vertex set S covers a hyperedge e, if S includes at least one vertex of e. A face hypergraph refers a hypergraph obtained from a embedded graph G whose vertices are

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the same vertices of G and there is a one-to-one correspondence between the faces of G and hyperedges of Hsuch that each hyperedge of H consists all vertices of its corresponding face. This concept was introduced by Kündgen and Ramamurthi [5]. The minimum number of colors needed to color the vertices of H such that any two vertices lying in the same hyperedge have different colors (proper property) is denoted by $\chi(H)$. A complete t-coloring of a k-uniform hypergraph H is a coloring of whose vertices, using t colors, such that any two vertices lying in the same hyperedge have different colors, and also every arbitrary set of k different colors appears in at least one hyperedge. Note that an arbitrary uniform hypergraph may have not a complete coloring, see [2]. If H has a complete t-coloring, we denote by $\psi(H)$ the maximum number of such integers t; otherwise, we define $\psi(H) = 0$. The numbers $\chi(H)$ and $\psi(H)$ are called the chromatic number and the achromatic number of H, respectively. It was proved in [3, 6] that a given uniform hypergraph H may have not a complete $\chi(H)$ -coloring even if it admits a complete coloring. We say that a hypergraph H satisfies interpolation property, if it admits a complete t-coloring for every integer t with $\chi(H) \leq s < t \leq \psi(H)$, provided that H has a complete s-coloring.

In 1967 Harary, Hedetniemi, and Prins studied interpolation property for complete coloring of graphs and established the following result.

Theorem 1.1.([4]) Every graph G admits a complete t-coloring for every t with $\chi(G) \le t \le \psi(G)$.

Recently, Edwards and Rzążewski (2020) showed that Theorem 1.1 cannot be developed to k-uniform hypergraphs for all integers k with $k \ge 9$.

Theorem 1.2.([3]) Let k be a positive integer with $k \ge 9$. There exists a k-uniform hypergraph H which has a complete $\chi(H)$ -coloring, and a complete $\psi(H)$ -coloring, but no complete coloring for some t with $\chi(H) < t < \psi(H)$.

In addition, they asked the following two problems for generalizing Theorem 1.2 to 3-uniform hypergraphs, and for studying a weaker version of interpolation property of complete coloring of hypergraphs.

Problem A (Edwards and Rzążewski (2020) [3]) Does there exist a 3-uniform example of a hypergraph for which interpolation fails?

Problem B (Edwards and Rzążewski (2020) [3]) Does there exist a hypergraph H with a complete $\chi(H)$ coloring and a complete $\psi(H)$ -coloring, but no complete t-coloring for every t satisfying $\chi(H) < t < \psi(H)$ in which $\psi(H) \ge \chi(H) + 2$?

In this paper, we generalize Theorem 1.2 to all cases k with $k \ge 3$ by modifying some parts of their proof. In Section 3, we answer Problem B positively by giving several kinds of 3-uniform hypergraphs, which consequently shows that the answer of Problem A is positive. In particular, we form the following stronger assertion.

Theorem 1.3. There exists a 3-uniform hypergraph H with a complete $\chi(H)$ -coloring and a complete $\psi(H)$ -coloring, but no complete t-coloring for every t satisfying $\chi(H) < t < \psi(H)$ in which $\psi(H) \ge 2\chi(H)$.

Recently, Matsumoto and the third author (2020) investigated complete coloring for a special family of face hypergraphs using terms of facial complete coloring of planar triangulations. They put forward the following conjecture in their paper to suggest a family of hypergraphs satisfying interpolation property. In the rest of this paper, we disprove this conjecture by a particular hypergraph of order 12 which seems to be the unique exceptional example for this conjecture. It is known that a planar triangulation is 3-colorable if and only if whose degrees are even [7].

Conjecture 1.4. (Matsumoto and Ohno (2020) [6]) Let H be a 3-uniform face hypergraph obtained from a planar triangulation. If H is 3-colorable, then it admits a complete t-coloring for every t with $\chi(H) \leq t \leq \psi(H)$.

2 The existence of uniform hypergraphs for which interpolation property fails

The following theorem makes a stronger version for Theorem 1.2.

Theorem 2.1. Let k be a positive integer with $k \ge 3$. There exists a k-uniform hypergraph H which has a complete $\chi(H)$ -coloring and a complete $\psi(H)$ -coloring, but no complete t-coloring for some t with $\chi(H) < t < \psi(H)$.

Proof. We may assume that $k \ge 4$, as the assertion holds for k = 3 with respect to Theorem 3.1. Let r be a large enough integer number compared to k. Define H to be the k-uniform hypergraph with $V(H) = \{v_{i,j} : 1 \le i \le k, 1 \le j \le r\}$ and $E(H) = E_1 \cup E_2$ such that

$$E_1 = \{\{v_{i,p_i} : 1 \le i \le k\} : (p_1, \dots, p_k) \in \mathcal{A} \text{ and } f(p_1, \dots, p_k) \le 1\}, \text{ and}$$
$$E_2 = \{\{v_{i,p_i} : 1 \le i \le k\} : (p_1, \dots, p_k) \in \mathcal{A} \text{ and } p_1 < \dots < p_k\},$$

where \mathcal{A} denotes the set of all sequences (p_1, \ldots, p_k) such that all p_i are distinct and $1 \leq p_i \leq r$ and $f(p_1, \ldots, p_k) = |\{(i, j) : |p_i - p_j| = 1 \text{ and } 1 \leq i < j \leq k\}|$. We call the *i*-th part of H as the set of all vertices $v_{i,j}$ with $1 \leq j \leq r$, and call the *j*-th position of H as the set of all vertices $v_{i,j}$ with $1 \leq j \leq r$, and call the *j*-th position of H as the set of all vertices $v_{i,j}$ with $1 \leq i \leq k$. According to this construction, one can prove the following three assertions:

- (a1) There is no hyperedge including two vertices of the same position.
- (a2) There is no hyperedge including two vertices of the same part.
- (a3) For any two vertices in different parts and different positions, there is a hyperedge including them.

We prove only the last assertion as the other ones are obvious. Let $v_{i,j}$ and $v_{i',j'}$ be two arbitrary vertices of H in different parts and different positions so that $i \neq i'$ and $j \neq j'$. Since r is large enough, there is an integer s with $1 \leq s \leq r$ such that $\{j, j'\} \cap \{s, \ldots, s + 2k + 2\} = \emptyset$. Consider the sequence (p_1, \ldots, p_k) satisfying $p_i = j$, $p_{i'} = j'$, and $p_t = s + 2t$ for every $t \in \{1, \ldots, k\} \setminus \{i, i'\}$. Obviously, this sequence is in \mathcal{A} and $f(p_1, \ldots, p_k) \leq 1$. Thus the hyperedge corresponding to this sequence must be in E_1 . Note that this hyperedge includes both of $v_{i,j}$ and $v_{i',j'}$. Hence the claim holds.

To show that this hypergraph has a complete k-coloring, we take color set $\{c_1, c_2, \ldots, c_k\}$ and for each $1 \leq i \leq k$, we color all vertices in the *i*-th part with the color c_i . By (a_2) this is a proper coloring and each hyperedge contains all k colors. For complete r-coloring, we take a color set $\{c_1, c_2, \ldots, c_r\}$ and for each $1 \leq j \leq r$, we color all vertices in the *j*-th position with the color c_j . According to (a_1) , it is a proper coloring. In addition, if $\{c_{p_1}, c_{p_2}, \ldots, c_{p_k}\}$ is a k-subset of $\{c_1, c_2, \ldots, c_r\}$ with $p_1 < \cdots < p_k$, then the hyperedge $\{v_{1,p_1}, v_{2,p_2}, \ldots, v_{k,p_k}\}$ of E_2 contains this color set. Therefore, $\chi(H) = k$ and $\psi(H) \geq r$.

Now, we show that H has no complete t-coloring for every integer t with $\frac{k-2}{k-1}r + k + 1 \le t < r$. Suppose, to the contrary, that H has a complete t-coloring using colors c_1, \ldots, c_t . Define X to be the set of colors appearing in at least two parts and define Y to be the set of colors appearing in only one part. We are going to prove the following two assertions:

- (b1) Each color of X appears in only one position and all vertices of this position colored only by this color.
- (b2) Each part has only one color from Y so that |Y| = k and |X| = t k.

Consider a color $x \in X$. If $x \in X$ occurred in more than one position, then by the definition of X, there must be two vertices having the same color x with different parts and different positions. Thus by (a_3) there is a hyperedge including both of them. This shows that the coloring is not proper, a contradiction. Thus all occurrences of x are in the same position. Now, since |X| < r, there is one position whose colors are not in X. In other words, there are k vertices with different parts whose colors are in Y. On the other hand, each part contains at most one color of Y; otherwise, if two colors of Y are in the same part, then by (a2) there is no hyperedge including them which is impossible. Therefore, |Y| = k and |X| = t - k. Consequently, we can define y_i to be the unique color in Y appearing in the *i*-th part, where $1 \le i \le k$. Assume that the color $x \in X$ appears in the *j*-th position. We are going to show that all vertices of this position are colored by this color. If we consider a given arbitrary vertex $v_{i,j}$ of this position, then there is one hyperedge of H containing all colors of the set $\{y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_k\}$. Let $(p_1, \ldots, p_k) \in \mathcal{A}$ be the sequence corresponding to this hyperedge. Obviously, the color of v_{t,p_t} must be y_t for every $t \in \{1, \ldots, k\}$ with $t \neq i$. Thus the color x must be appeared on the *i*-th part, and so the vertex $v_{i,j}$ must be colored with x. Therefore, all of vertices of the *j*-th position are colored with the color x. Hence the assertions hold.

Obviously, there are r - |X| positions are not colored by colors of X. Since $r - |X| \le r/(k-1) - 1$, we can conclude that there are k - 1 consecutive positions $\{s, s + 1, \ldots, s + k - 2\}$ of H colored only with colors of X. Define Z to be the set of all those k - 1 colors along with the color y_2 . By the assumption, there is a hyperedge $e \in E(H)$ including all colors of Z. Let $(p_1, \ldots, p_k) \in \mathcal{A}$ be the sequence corresponding to this hyperedge. Obviously, by (b2), the vertex v_{2,p_2} must be colored by y_2 . We know that $\{p_1, \ldots, p_k\} \setminus \{p_2\} = \{s, s+1, \ldots, s+k-2\}$. Since $k \ge 4$, there must be three integers $a, b, c \in \{1, \ldots, k\}$ such that $\{p_a, p_b, p_c\} = \{s, s+1, s+2\}$. Thus $f(p_1, \ldots, p_k) \ge 2$ and so $e \notin E_1$. Moreover, according to the situation of the position containing the color y_2 , we have either $\max\{p_1, p_3\} < p_2$ or $p_2 < \min\{p_1, p_3\}$ and so $e \notin E_2$. This is a contradiction. Hence the theorem is proved.

3 Answering to Problem **B** by 3-uniform hypergraphs

In this section, we are going to answer to Problem 4 in [3] by giving several kinds of 3-uniform hypergraphs.

3.1 A hypergraph of order 9

A positive answer to Problem B is given in the following theorem.

Theorem 3.1. There exists a 3-uniform hypergraph H of order 9 with a complete $\chi(H)$ -coloring and a complete $\psi(H)$ -coloring, but no complete t-coloring for every t satisfying $\chi(H) < t < \psi(H)$ in which $\psi(H) \ge \chi(H) + 2$.

Proof. Let *H* be the 3-uniform hypergraph of order 9 whose incidence graph is shown in Figure 1. If *H* has a complete *k*-coloring for $k \ge 6$, then it has at least twenty hyperedges. However, *H* has exactly ten hyperedges and hence $\psi(H) \le 5$. In fact, *H* has a complete 3-coloring and a complete 5-coloring (see Figures 1 and 2, respectively). Therefore, $\chi(H) = 3$ and $\psi(H) = 5$.



Figure 1: A complete 3-coloring of H

Figure 2: A complete 5-coloring of H

Next, we show that H has no complete 4-coloring. Suppose, to the contrary, that H has a complete 4-coloring using colors c_1, \ldots, c_4 . Since H has nine vertices, there exists at least one color appearing on at least three vertices of H, say color c_1 . Note that those vertices with the same color form an independent

set. It is easy to check that there are exactly three independent sets of H with size three (which shown as vertices numbered by 1, 2 and 3 in Figure 1). Since the vertices of every such vertex set cover all hyperedges of H, the triad $\{c_2, c_3, c_4\}$ does not appear on any hyperedge of H. Hence H has no complete 4-coloring and so it is a desired hypergraph.

3.2 A 3-regular 3-uniform hypergraph of order 15

Another positive answer to Problem B is given in the next theorem.

Theorem 3.2. There exists a 3-uniform 3-regular hypergraph of order 15 with a complete $\chi(H)$ -coloring and a complete $\psi(H)$ -coloring, but no complete t-coloring for every t satisfying $\chi(H) < t < \psi(H)$ in which $\psi(H) \ge \chi(H) + 2$.

Proof. Let *H* be the hypergraph with the vertex set $\{v_{i,j} : 1 \le i \le 3, 1 \le j \le 5\}$ consisting of those hyperedges e_{ij} with $1 \le i \le 3$ and $1 \le j \le 5$ in which

$$e_{ij} = \{v_{i,j+1}\} \cup \{v_{t,j} : 1 \le t \le 3, t \ne i\},\$$

where $v_{i,6} = v_{i,1}$. The incidence graph of this hypergraph is shown in Figure 3. Obviously, H is 3-uniform and 3-regular. If H has a complete k-coloring for $k \ge 6$, then H must have at least twenty hyperedges. However, H has exactly fifteen hyperedges and hence $\psi(H) \le 5$. In fact, H has a complete 3-coloring and a complete 5-coloring (see Figures 3 and 4, respectively). Therefore, $\chi(H) = 3$ and $\psi(H) = 5$.





Figure 4: A complete 5-coloring of H

Suppose, to the contrary, that H has a complete 4-coloring using colors c_1, \ldots, c_4 . Since |V(H)| = 15, there exists a color appearing on at least four vertices, say c_1 . Define $X_i = \{v_{i,j} : 1 \le j \le 5\}$ for each iwith $1 \le i \le 3$. According to the construction of H, it is not difficult to check that every independent set of size four must be a subset of X_1, X_2 , or X_3 . Hence the color c_1 only appears on vertices of a set X_t where $1 \le t \le 3$. If c_1 appears on five vertices, then it must appear on all vertices of X_t . In this case, the triad $\{c_2, c_3, c_4\}$ does not appear, because all hyperedges of H are covered by the vertices of X_t . Therefore, each color appears on at most four vertices. Since H has 15 vertices, every color must appear on four vertices, except one color which appears on three vertices. We may assume that for each $i \in \{1, 2, 3\}$, the color c_i appears on exactly four vertices of X_i . Then the remaining three vertices are colored by c_4 so that each X_i includes exactly one of them. Let us define $Y_j = \{v_{i,j} : 1 \le i \le 3\}$ for each j with $1 \le j \le 5$. It is easy to check that if a vertex in X_i and a vertex in $X_{i'}$ are colored by the same color provided that $i \neq i'$, both of them cannot be in the set $Y_j \cup Y_{j+1}$ for all $j \in \{1, \ldots, 5\}$; where $Y_6 = Y_1$. Now, since three vertices are colored by c_4 and each X_i includes exactly one of them, we derive a contradiction. Therefore, H has no complete 4-coloring and it is a desired hypergraph.

3.3 Answering to a stronger version of Problem B

Our aim in this subsection is to present a 3-uniform 3-colorable hypergraph having a complete 6-coloring but no complete t-coloring for each $t \in \{4, 5\}$. To find such a hypergraph, we first made a complete 3-uniform hypergraph H of order 6 with size $\binom{6}{3}$ so that for any triad of vertices, there is a hyperedge including all of them. Next, we tried to generate new hypergraphs by splitting every vertex into two vertices and examine the other necessary properties using a special computer search. By this way, we succeeded to prove the following assertion. This method was already used to make the hypergraph stated in the proof of Theorem 3.1.

Theorem 3.3. There exists a 3-uniform hypergraph H with a complete $\chi(H)$ -coloring and a complete $\psi(H)$ -coloring, but no complete t-coloring for every t satisfying $\chi(H) < t < \psi(H)$ in which $\psi(H) \ge 2\chi(H)$.

Proof. Let *H* be the 3-uniform hypergraph whose incidence graph is shown in Figure 5. If *H* has a complete *k*-coloring for $k \ge 7$, then it has at least thirty-five hyperedges. However, *H* has exactly twenty hyperedges and hence $\psi(H) \le 6$. In fact, *H* has a complete 3-coloring and a complete 6-coloring (see Figures 5 and 6, respectively). Therefore, $\chi(H) = 3$ and $\psi(H) = 6$.



Figure 5: A complete 3-coloring of H.

Figure 6: A complete 6-coloring of H.

Next, we show that H has neither a complete 4-coloring nor a complete 5-coloring. According to the construction of H, it is not hard to check that there are exactly three independent sets X_1, X_2 and X_3

of size four (which shown as vertices numbered by 1, 2 and 3 in Figure 5, respectively). Moreover, every independent set of size three must be a subset of X_1 , X_2 , or X_3 . Suppose, to the contrary, that H has a complete 4-coloring using colors c_1, \ldots, c_4 . First, we assume that there exists a color appearing on at least four vertices of H, say color c_1 . Since H has no independent sets of size five, the color c_1 must appear on all four vertices of a set X_i , where $i \in \{1, 2, 3\}$. Since these four vertices cover all hyperedges of H, the triad $\{c_2, c_3, c_4\}$ does not appear on any hyperedge of H, a contradiction. Now, since H has 12 vertices, we may assume that every color appears on exactly three vertices of H. On the other hand, H has at most three disjoint independent sets of size three, a contradiction. Therefore, H has no complete 4-coloring.

Suppose, to the contrary, that H has a complete 5-coloring using colors c_1, \ldots, c_5 . As we have observed above, no color can appear on at least four vertices. Since H has 12 vertices, there must be a color appearing on exactly three vertices of H, say c_1 . Call the set of all vertices having the color c_1 by S. Since the size of S is three, it must be a subset of X_1, X_2 , or X_3 , say X_1 . We may assume that the unique vertex in $X_1 \setminus S$ is colored by c_2 . Since X_1 covers all hyperedges of H, the triad $\{c_3, c_4, c_5\}$ does not appear on any hyperedge of H, a contradiction. Therefore, H has no complete 5-coloring and it is a desired hypergraph.

4 An exceptional example for Conjecture 1.4

A counterexample of Conjecture 1.4 is given in the following theorem which answers Problem A as well. This hypergraph was first found by writing a C++ code for checking complete coloring of hypergraphs and by applying it on the specified outputs of *plantri* program due to Brinkmann and McKay [1]. Note that this face hypergraph is unique by searching among all 3-colorable planar triangulations on up to 23 vertices.

Theorem 4.1. There is a 3-uniform 3-colorable face hypergraph of order 12, obtained from a planar triangulation, having a complete 6-coloring but with no complete 5-coloring.

Proof. Let *H* be the 3-uniform face hypergraph obtained from the planar triangulation shown in Figure 7. If *H* has a complete *k*-coloring for $k \ge 7$, then *H* has at least thirty-five hyperedges. However, *H* has exactly twenty hyperedges and hence $\psi(H) \le 6$. In fact, *H* has a complete 3-coloring and a complete 6-coloring (see Figures 7 and 8, respectively). Therefore, $\chi(H) = 3$ and $\psi(H) = 6$.

Suppose, to the contrary, that H has a complete 5-coloring using colors c_1, \ldots, c_5 . For every i with $1 \leq i \leq 6$, we call those two vertices of H specifying by the number i in Figure 8 by v_i and w_i such that w_i is the inner one. We may assume that w_1, w_2 , and w_3 are colored by c_1, c_2 , and c_3 , respectively. We may also assume that each of the colors c_4 and c_5 appears on at least one of w_4, w_5 , and w_6 ; otherwise, it is enough to change the colors of them to make this property along with maintaining the property of complete 5-coloring. According to the features of the hypergraph H, we can also assume that w_4, w_5 , and w_6 are colored by c_4, c_5 , and c_2 , respectively. It is not difficult to check that for a given arbitrary proper coloring of the octahedron, every pair of colors is contained in at most two kinds of triads appeared on faces of the octahedron. Thus the octahedron $v_1v_2 \cdots v_6$ has at most two kinds of colored faces including both of c_3 and



Figure 7: A complete 3-coloring of H



Figure 8: A complete 6-coloring of H

 c_4 . Since there exist three remaining triads containing c_3 and c_4 , one can conclude that the color c_4 must appear on either v_4 or v_6 . Similarly, with respect to the colors c_1 and c_5 on this octahedron, one can also conclude that the color c_5 must appear on either v_4 or v_5 . To complete the proof, we shall consider three cases.

Case A: The vertex v_4 is colored by c_2 .

In this case, the vertices v_5 and v_6 must be colored by c_5 and c_4 , respectively. Since at least one face is colored by the triad $\{c_1, c_4, c_5\}$, the color c_1 must also appear on the vertex v_1 . Consequently, it is easy to see that the triad $\{c_3, c_4, c_5\}$ cannot appear, which is a contradiction.

Case B: The vertex v_4 is colored by c_4 .

In this case, the vertex v_5 must be colored by c_5 and so the vertex v_6 must be colored by c_1 . Since at least one face is colored by the triad $\{c_3, c_4, c_5\}$, the color c_3 must also appear on the vertex v_3 . Consequently, it is easy to see that the triad $\{c_1, c_3, c_5\}$ cannot appear which is again a contradiction.

Case C: The vertex v_4 is colored by c_5 . The proof of this case is similar to Case B (by exchanging the colors c_4 and c_5 and using the symmetry of H).

Hence the proof is completed.

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