

(1,0,0)-colorability of planar graphs without cycles of length 4 or 6

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Abstract

A graph G is (d_1, d_2, d_3) -colorable if the vertex set $V(G)$ can be partitioned into three subsets V_1, V_2 and V_3 such that for $i \in \{1, 2, 3\}$, the induced graph $G[V_i]$ has maximum vertex-degree at most d_i . So, $(0, 0, 0)$ -colorability is exactly 3-colorability.

The well-known Steinberg's conjecture states that every planar graph without cycles of length 4 or 5 is 3-colorable. As this conjecture being disproved by Cohen-Addad etc. in 2017, a similar question, whether every planar graph without cycles of length 4 or i is 3-colorable for a given $i \in \{6, \dots, 9\}$, is gaining more and more interest. In this paper, we consider this question for the case $i = 6$ from the viewpoint of improper colorings. More precisely, we prove that every planar graph without cycles of length 4 or 6 is $(1, 0, 0)$ -colorable, which improves on earlier results that they are $(2, 0, 0)$ -colorable and also $(1, 1, 0)$ -colorable, and on the result that planar graphs without cycles of length from 4 to 6 are $(1, 0, 0)$ -colorable.

Keywords: planar graphs, $(1, 0, 0)$ -colorings, cycles, discharging, super-extension

1 Introduction

The graphs considered in this paper are finite and simple. A graph is planar if it is embeddable into the Euclidean plane. A plane graph (G, Σ) is a planar graph G together with an embedding Σ of G into the Euclidean plane, that is, (G, Σ) is a particular drawing of G in the Euclidean plane. In what follows, we will always say a plane graph G instead of (G, Σ) , which causes no confusion since in this paper no two embeddings of the same graph G will be involved in.

In the field of 3-colorings of planar graphs, one of the most active topics is about a conjecture proposed by Steinberg in 1976: every planar graph without cycles of length 4 or 5 is 3-colorable. There had been no progress on this conjecture for a long time, until Erdős [16] suggested a relaxation of it: does there exist a constant k such that every planar graph without cycles of length from 4 to k is 3-colorable? Abbott and Zhou [1] confirmed that such k exists and $k \leq 11$. This result was later on improved to $k \leq 9$ by Borodin [2] and, independently, by Sanders and Zhao [15], and to $k \leq 7$ by Borodin etc. [3]. Steinberg's conjecture was recently disproved by Cohen-Addad etc. [6]. Hence, associated to Erdős' relaxation, only one question remains unsettled.

Problem 1.1. *Is it true that planar graphs without cycles of length from 4 to 6 are 3-colorable?*

A more general problem than Steinberg's Conjecture was formulated in [14]:

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Problem 1.2. *What is the maximal subset \mathcal{A} of $\{5, 6, \dots, 9\}$ such that for $i \in \mathcal{A}$, every planar graph with cycles of length neither 4 nor i is 3-colorable?*

The refutation of Steinberg's Conjecture shows that $5 \notin \mathcal{A}$. For any other i , the question whether $i \in \mathcal{A}$ is still unsettled. In this paper, we consider such question for the case $i = 6$, i.e., the question whether every planar graph without cycles of length 4 or 6 is 3-colorable.

Let d_1, d_2 and d_3 be non-negative integers. A graph G is (d_1, d_2, d_3) -colorable if the vertex set $V(G)$ can be partitioned into three subsets V_1, V_2 and V_3 such that for $i \in \{1, 2, 3\}$, the induced graph $G[V_i]$ has maximum vertex-degree at most d_i . The associated coloring, assigning the vertices of V_i with the color i for $i \in \{1, 2, 3\}$, is an improper coloring, a concept which allows adjacent vertices to receive the same color. Clearly, $(0, 0, 0)$ -colorability is exactly 3-colorability. Improper coloring is a relaxation of proper coloring, providing us a way to approach the solution to some hard conjectures. It has been combined with many different kinds of colorings of graphs, such as improper k -colorings, improper list colorings, improper acyclic colorings and so on.

The coloring of planar graphs gain particular attention. There are a serial of known results on the (d_1, d_2, d_3) -colorability of planar graphs, motivated by Steinberg's conjecture. For example, Cowen etc. [7] proved that planar graphs are $(2, 2, 2)$ -colorable. Xu [19] showed that planar graphs with neither adjacent triangles nor cycles of length 5 are $(1, 1, 1)$ -colorable. So far, the best known results for planar graphs having no cycles of length 4 or 5 are that, they are $(1, 1, 0)$ -colorable [10, 21] and also $(2, 0, 0)$ -colorable [5], improving on some results in [9, 19]. Because of the refutation of Steinberg's conjecture, the following question is the only one in this direction that remains open.

Problem 1.3. *Is it true that planar graphs having no cycles of length 4 or 5 are $(1, 0, 0)$ -colorable?*

Analogously, for planar graphs having no cycles of length 4 or 6, it is known that they are $(1, 1, 0)$ -colorable [17, 20] and also $(2, 0, 0)$ -colorable [18]. In this paper, we prove that they are further $(1, 0, 0)$ -colorable, which improves on these two results.

Theorem 1.4. *Planar graphs with neither 4-cycles nor 6-cycles are $(1, 0, 0)$ -colorable.*

Towards Problem 1.1, Wang etc. [17] shown that planar graphs having no cycles of length from 4 to 6 are $(1, 0, 0)$ -colorable. Theorem 1.4 improves on this result as well. To our best knowledge, Theorem 1.4 is the first result on $(1, 0, 0)$ -colorability of planar graphs with neither 4-cycles nor i -cycles for $i \in \{5, 6, 7, 8, 9\}$, motivated by Problem 1.2.

The proof of this main result uses discharging method for improper colorings. In Section 2, we formulate a proposition that is stronger than Theorem 1.4, namely super-extended theorem. Section 3 addresses the proof of the super-extended theorem, which consists of two parts: reducible configurations and discharging procedure. For more information on discharging method, we refer to [8, 11, 12].

2 Super-extended theorem

Let G be a plane graph. For a set S such that $S \subseteq V(G)$ or $S \subseteq E(G)$, let $G[S]$ denote the subgraph of G induced by S . Let C be a cycle of G . Denote by $int(C)$ (resp. $ext(C)$) the set of vertices lying inside (resp. outside) C . Let H be a subgraph of G whose edges lie inside C (ends on C allowed) and let $H_0 = H - V(C)$, such that $d_H(v) = 3$ for each $v \in V(H_0)$. Call H a *claw* of C if H_0 is a vertex, an *edge-claw* if H_0 is an edge, a *path-claw* if H_0 is a path of length 2, and a *pentagon-claw* if H_0 is a pentagon.

Let \mathcal{G} denote all the connected plane graphs without cycles of length 4 or 6. For a cycle C , whose length is at most 11, of a graph from \mathcal{G} , C is good if it contains no claws, edge-claws, path-claws or pentagon-claws; bad otherwise.

Let G be a graph, H a subgraph of G , and ϕ a $(1, 0, 0)$ -coloring of H . We say that ϕ can be *super-extended* to G if G has a $(1, 0, 0)$ -coloring c such $c(u) = \phi(u)$ for each $u \in V(H)$ and that $c(v) \neq c(w)$ whenever $v \in V(H)$, $w \in V(G) \setminus V(H)$ and $vw \in E(G)$.

We shall prove the following theorem, called super-extended theorem, that is stronger than Theorem 1.4.

Theorem 2.1. (*Super-extended theorem*) *Let $G \in \mathcal{G}$. If the boundary D of the unbounded face of G is a good cycle, then every $(1, 0, 0)$ -coloring of $G[V(D)]$ can be super-extended to G .*

By assuming the truth of Theorem 2.1, we can easily derive Theorem 1.4 as follows. We may assume that G is connected since otherwise, we argue on each component. If G has no triangles, then by Three Color Theorem, G is 3-colorable. Hence, we may assume that G has a triangle, say T . By Theorem 2.1, we can super-extend any given $(1, 0, 0)$ -coloring of T respectively to its interior and exterior.

The rest of this section contributes to some necessary notations.

Let C be a cycle of a plane graph and T be a claw, or an edge-claw, or a path-claw, or a pentagon-claw of C . We call the graph H consisting of C and T a *bad partition* of C . Every facial cycle (except C) of H is called a *cell* of H .

The length of a path is the number of edges it contains. Denote by $|P|$ the length of a path P , by $|C|$ the length of a cycle C and by $d(f)$ the size of a face f . A k -*vertex* (resp. k^+ -vertex and k^- -vertex) is a vertex v with $d(v) = k$ (resp. $d(v) \geq k$ and $d(v) \leq k$). Similar notations are applied for paths, cycles and faces by constitute $d(v)$ for $|P|$, $|C|$ and $d(f)$, respectively.

Consider a plane graph. A vertex is *external* if it lies on the exterior face; *internal* otherwise. A 3^+ -vertex is *light* if it is internal and of degree 3; *heavy* otherwise. Let d_1, d_2, d_3 be three integers greater than 2. A (d_1, d_2, d_3) -*face* is a 3-face whose vertices are all internal and have degree d_1, d_2 and d_3 , respectively. A k -cycle with vertices v_1, \dots, v_k in cyclic order is denoted by $[v_1 \dots v_k]$. Let $f = [uxy]$ be a 3-face and v be a neighbor of u other than x and y . If u is an internal 3-vertex, then we call v an *outer neighbor* of u (or of f), u a *pendent vertex* of v , and f a *pendent 3-face* of v . A 3-face is *weak* if it has at least one outer neighbor that is light. A path is a *splitting path* of a cycle C if its two end-vertices lie on C and all other vertices lie inside C . A cycle C is *separating* if neither $int(C)$ nor $ext(C)$ is empty.

3 The proof of Theorem 2.1

Suppose to the contrary that Theorem 2.1 is false. From now on, let $G = (V, E)$ be a counterexample to Theorem 2.1 with the smallest $|V| + |E|$. Thus, we may assume that the boundary D of the exterior face of G is a good cycle, and that there exists a $(1, 0, 0)$ -coloring ϕ of $G[V(D)]$ which cannot be super-extended to G . By the minimality of G , we deduce that D has no chord.

Denote by $\{1, 2, 3\}$ the color set for ϕ where the color 1 might be assigned to two adjacent vertices. We define that, to *3-color* a vertex v means to assign v with a color from $\{1, 2, 3\}$ when this color has not been used by its neighbors yet; and to *$(1, 0, 0)$ -color* v means either to 3-color v or to assign v with the color 1 when precisely one neighbor of v is of color 1.

3.1 Structural properties of the minimal counterexample G

Lemma 3.1. *Every internal vertex of G has degree at least 3.*

Proof. Suppose to the contrary that G has an internal vertex v of degree at most 2. We can super-extend ϕ to $G - v$ by the minimality of G , and then to G by 3-coloring v . \square

Lemma 3.2. *G has no separating good cycle.*

Proof. Suppose to the contrary that G has a separating good cycle C . We super-extend ϕ to $G - \text{int}(C)$. Furthermore, since C is a good cycle, the restriction of ϕ on C can be super-extended to its interior, yielding a super-extension of ϕ to G . \square

Lemma 3.3. *G is 2-connected. Particularly, the boundary of each face of G is a cycle.*

Proof. Otherwise, let B a pendant block of G of minimum order, and let v be a cut vertex of G associated with B . By the minimality of G , we can super-extend ϕ to $G - (B - v)$. If we can 3-color B , then permute the color classes of B so that the colors assigned to v coincide, which completes a super-extension of ϕ to G . By the minimality of B , B is 2-connected. If B has no triangles, then Grötsch's Theorem yields that B is 3-colorable. So, let T be a triangle of B . By Lemma 3.2, T is a 3-face. Assign distinct colors to its three vertices, and by the minimality of G , we can super-extend the coloring of T , as an exterior face of B , to B . This gives a 3-coloring of B . \square

By the definition of a bad cycle, one can easily conclude the following lemma.

Lemma 3.4. *If C is a bad cycle of a plane graph of \mathcal{G} , then C has a bad partition isomorphic to one of the eight graphs shown in Figure 1. In particular, C has length 9 or 10 or 11. If $|C| = 9$ then C has a $(5,5,5)$ -claw; if $|C| = 10$ then C has a $(3,7,3,7)$ - or $(5,5,5,5)$ -edge-claw, or a $(5,5,5,5,5)$ -pentagon-claw; if $|C| = 11$ then C has a $(3,7,7)$ - or $(5,5,7)$ -claw, or a $(3,7,3,8)$ -edge-claw, or a $(5,5,5,5,5)$ -path-claw.*

From Lemmas 3.2 and 3.4, one can deduce the following remark.

Remark 3.5. *Let C be a bad cycle of G . The following statements hold true.*

- (1) *Every cell of C is facial except that an 8-cell may have a $(3,7)$ -chord connecting two vertices of C .*
- (2) *Every vertex inside C has degree 3 in G .*
- (3) *Every vertex on C has at most one neighbor inside C .*
- (4) *Every vertex on C is incident with at most two edges that locate inside C , where the exact case happens if and only if C has a $(3,7,3,8)$ -edge-claw.*
- (5) *For any set S of four consecutive vertices on C , G has at most two edges connecting a vertex from S to a vertex inside C .*

Lemma 3.6. *G has no light vertex with neighbors all light.*

Proof. Otherwise, let v be such a light vertex. Remove v and its three neighbors, obtaining a smaller graph G' . By the minimality of G , ϕ can be super-extended to G' . We further extend ϕ to being a $(1,0,0)$ -coloring of G in such way: 3-color all the neighbors of v and consequently, v can be $(1,0,0)$ -colored. \square

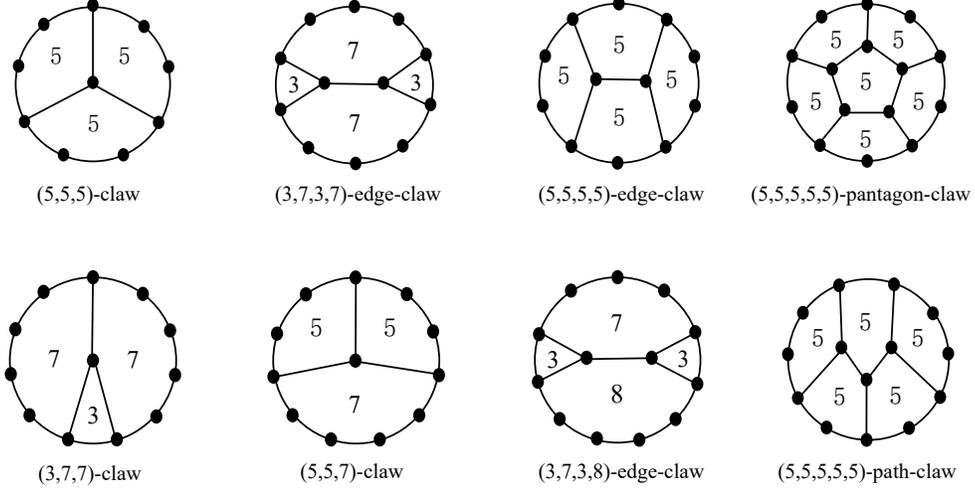


Figure 1: bad partitions of a cycle in a plane graph from \mathcal{G} , where the numbers indicate the length of each cell. A further name for the claw, edge-claw, path-claw or pentagon claw, which corresponds to each bad partition, is given below each drawing.

Lemma 3.7. *Every $(3, 3, 4)$ -face of G has no light outer neighbors.*

Proof. Suppose to the contrary that $f = [uvw]$ is a $(3, 3, 4)$ -face of G having a light outer neighbor x . W.l.o.g., Let u be adjacent to x and let $d(w) = 4$. Remove u, v, w and x from G , obtaining a smaller graph G' . By the minimality of G , ϕ can be super-extended to G' and further to G in such way: 3-color w, v and x in turn, and then $(1, 0, 0)$ -color u . \square

Lemma 3.8. *Let P be a splitting path of D which divides D into two cycles, say D' and D'' . The following four statements hold true.*

- (1) *If $|P| = 2$, then there is a triangle between D' and D'' .*
- (2) *If $|P| = 3$, then there is a 5-cycle between D' and D'' .*
- (3) *If $|P| = 4$, then there is a 5- or 7-cycle between D' and D'' .*
- (4) *If $|P| = 5$, then there is a 7- or 8- or 9-cycle between D' and D'' .*

Proof. Since D has length at most 11, we have $|D'| + |D''| = |D| + 2|P| \leq 11 + 2|P|$.

(1) Let $P = xyz$. Suppose to the contrary that $|D'|, |D''| \geq 5$. It follows that $|D'|, |D''| \leq 10$. By Lemma 3.1, y has a neighbor other than x and z , say y' . The vertex y' is internal since otherwise, D is a bad cycle with a claw. W.l.o.g., let y' lie inside D' . Now D' is a separating cycle. By Lemma 3.2, D' is not good. Recall that $|D'| \leq 10$. So D' is a bad 9- or 10-cycle and D'' is a 5-cycle. By Lemma 3.4, D' has a $(5, 5, 5)$ -claw or a $(5, 5, 5, 5)$ -edge-claw or a $(3, 7, 3, 7)$ -edge-claw or a $(5, 5, 5, 5, 5)$ -pentagon-claw, which would lead to a $(5, 5, 5, 5)$ -edge-claw or a $(5, 5, 5, 5, 5)$ -path-claw of D for the first two cases, to a 6-cycle for the third case, and to y' being a light vertex with three light neighbors for the last case, a contradiction.

(2) Let $P = wxyz$. Suppose to the contrary that $|D'|, |D''| \geq 7$. It follows that $|D'|, |D''| \leq 10$. Let x' and y' be neighbors of x and y not on P , respectively. If both x' and y' are external, then D has an edge-claw. Hence, we may assume that x' lies inside D' . By Lemmas 3.2 and 3.4, we deduce that D' is a bad 9- or 10-cycle.

So, D'' is a 7- or 8-cycle, which is good. Since every cell of D' is facial, y' must lie on D'' . The application of this lemma to the splitting 2-path $y'yz$ yields that yy' a (3,7)-chord of D'' . So, D' is a 9-cycle, which has a (5,5,5)-claw. Now the triangle $[yy'z]$ is adjacent to some 5-cell of D' , a contradiction.

(3) Let $P = vwxyz$. Suppose to the contrary that $|D'|, |D''| \geq 8$. It follows that $|D'|, |D''| \leq 11$. If $wy \in E(G)$, then by applying this lemma to the splitting 3-path $vwyz$ of D , either D' or D'' has length 6, a contradiction. Hence, $wy \notin E(G)$. Similarly, $vx, xz \notin E(G)$. Since G has no 4-cycles and D has no chord, we can further conclude that G has no edges connecting two nonconsecutive vertices on P , i.e., $G[V(P)]$ is P .

By Lemma 3.1, x has a neighbor x' besides w and y . We claim that x' lies inside D . Suppose to the contrary that $x' \in V(D')$. By applying this lemma to the splitting 3-paths $vwxx'$ and $x'xyz$, xx' is a (5,5)-chord of D' . Since $d(w) \geq 3$, let w' be a neighbor of w other than v and x . Clearly, w' lies either on D'' or inside it. Recall that w' is not on P . If w' lies on $D'' \setminus V(P)$, then $vw w'$ is splitting 2-path of D , which forms a triangle adjacent to a 5-cell of D' , a contradiction. Hence, w' lies inside D'' . Similarly, y' lies inside D'' as well. Clearly, w' and y' are distinct vertices. Notice that w and y have distance 2 along D'' . So, as a bad cycle, whose possible interior is given by Lemma 3.4, D'' has a (5,5,5,5)-edge-claw or a (5,5,5,5,5)-path-claw or a (5,5,5,5,5)-pentagon-claw, which implies a pentagon-claw of D for the first case, and w' being a light vertex with three light neighbors for the last two cases, a contradiction.

W.l.o.g., let x' lies inside D' . So D' is a bad cycle. By Remark 3.5(2), $d(x') = 3$. Denote by I the set of edges connecting a vertex from $\{w, x, y\}$ to a vertex not on P . Recall that $G[V(P)]$ is P . So, Lemma 3.1 implies that $|I| \geq 3$. By applying Lemma 3.6 to x , we further have $|I| \geq 4$.

Suppose that D'' is also a bad cycle, then one of D' and D'' has length 9 and the other has length 9 or 10, which implies that one contains at most one edge from I inside and the other contains at most two edges from I inside, contradicting the fact that $|I| \geq 4$. Hence, we may assume that D'' is a good cycle.

We conclude that $d(x) = 3$. This is because x has no neighbors on D by the same argument as for x' , no neighbors inside D'' since D'' is a good cycle, and no neighbors besides x' inside D' by Remark 3.5(4).

Recall that D'' is a good cycle, so w (as well as y) has no neighbors inside D'' . Moreover, since D has no claws, w (as well as y) has at most one neighbor on $D \setminus \{v, z\}$. It follows with $|I| \geq 4$ that, inside D' there exists a vertex t adjacent to w or y . By Remark 3.5(3) and (5), such t is unique. W.o.l.g, let $tw \in E(G)$. This implies that $|I| = 4$ and each of w and y have a neighbor on $D - V(P)$. If $t = x'$, then $[wx x']$ is a pendent (3,3,4)-face of y , contradicting Lemma 3.7. So, t and x' are distinct. Moreover, t and x' are not adjacent since otherwise G has a 4-cycle. Hence, we can conclude that D' has a path-claw or a pentagon-claw, making all cells of length 5. This yields that y must have no neighbors other than z on D , a contradiction.

(4) Let $P = uvwxyz$. Suppose to the contrary that $|D'|, |D''| \geq 10$. Since $|D'| + |D''| \leq 21$, we have $|D'|, |D''| \leq 11$. We claim that G has no edges connecting two nonconsecutive vertices on P , i.e., $G[V(P)]$ is P . Otherwise, let $e = t_1 t_2$ be such an edge. Let P' be obtained from P by constituting e for the subpath of P between t_1 and t_2 . Clearly, P' is a splitting 4⁻-path of D . Applying this lemma to P' yields that either D' or D'' has length at most 8, a contradiction. By this claim and Lemma 3.1, we may let v', w', x' and y' be a neighbor of v, w, x and y not on P , respectively.

We claim that both w and x have no neighbors on D . Otherwise, w.l.o.g., let w' be on D' . By applying this lemma to the splitting 3-path $uvw w'$ and the splitting 4-path $w' wxyz$ of D , we deduce that ww' is a (5,7)-chord of D' . Hence, the interior of D' contains no edges incident with v, x or y . If x' lies on D'' then similarly, xx' is a (5,7)-chord of D'' , resulting in no positions for u' and y' , a contradiction. Hence, x' must lie inside D'' . So, D'' is a bad cycle. Since a bad cycle has at most one chord, Remark 3.5(5) implies that the interior of D''

contains at most three edges incident with v, w, x or y . It follows that $d(v) = d(w) = d(x) = d(y) = 3$. By Remark 3.5(2), $d(x') = 3$. Now x is a light vertex with three light neighbors, contradicting Lemma 3.6.

Suppose that one of D' and D'' , say D' , is a good cycle. In this case, both w' and x' lie inside D' . Remark 3.5(3) implies that such w' and x' are unique. So, $d(w) = d(x) = 3$. By Remark 3.5(5), both v' and y' are on D . Clearly, such v' and y' are also unique since otherwise, D has a claw. So, $d(v) = d(y) = 3$. By Remark 3.5(2), $d(w') = d(x') = 3$. Now x is a light vertex having three light neighbors, contradicting Lemma 3.6. Therefore, both D' and D'' are bad.

Denote by I the set of edges not on P and incident with a vertex from $\{v, w, x, y\}$. Notice that a bad cycle has a chord only if it is of length 11, but not both D' and D'' have length 11. So, I has at most one edge taking a vertex on D as an end. Moreover, Remark 3.5(5) implies that I has at most four edges taking a vertex inside D' or D'' as an end. Therefore, $|I| \leq 5$. This leads to the only case that $d(v) = d(y) = 3$ and between w and x , one has degree 3 and the other 4 since otherwise, at least one of w and x would be a light vertex with three light neighbors. W.l.o.g., let $d(x) = 4$. Since Remark 3.5(3), we may assume that w' and x' lie inside D' . Lemma 3.7 implies that w' and x' can not coincide. Notice that w and x are consecutive on D' . By the specific interior of a bad cycle, we can deduce that D' is a 11-cycle having a (5,5,5,5)-path-claw. This implies that both D' and D'' have no chords, a contradiction. \square

Loops and multiple edges are regarded as 1-cycles and 2-cycles, respectively.

Lemma 3.9. *Let G' be a connected plane graph obtained from G by deleting vertices, inserting edges, identifying vertices, or any combination of them. If G' is smaller than G and the following holds:*

- (i) *identify no pair of vertices of D and insert no edges connecting two vertices of D , and*
- (ii) *create no k -cycles for any $k \in \{1, 2, 4, 6\}$, and*
- (iii) *D is good in G' ,*

then ϕ can be super-extended to G' .

Proof. By Term (ii), the graph G' is simple and $G' \in \mathcal{G}$. The term (i) guarantees that the new graph G' has the same D as the boundary of its exterior face, and that ϕ is a (1,0,0)-coloring of $G'[V(D)]$. Since D is good in G' and G' is smaller than G , the lemma holds true by the minimality of G . \square

Lemma 3.10. *Let G' be a connected plane graph obtained from G by deleting a set of internal vertices together with either identifying two vertices or inserting an edge between two vertices. If the following holds true for this graph operation:*

- (a) *identify no pair of vertices of D , insert no edges connecting two vertices of D , and*
- (b) *create no 6^- -cycles or triangular 7 -cycles,*

then ϕ can be super-extended to G' .

Proof. Lemma 3.9 shows that, to complete the proof, it suffices to showing that D is a good cycle of G' . Suppose to the contrary that D has a bad partition H in G' . We distinguish two cases on the graph operation.

Case 1: assume that the graph operation includes identifying two vertices. Denote by v_1 and v_2 the two vertices we identify and by v the resulting vertex. Lemma 3.4 lists all the possible structure for H . Recall that D stays the same during the operation. If either $v \notin V(H)$ or $v \in V(H)$ such that $d_H(v) = 2$, then H stays

the same during the operation, contradicting the fact that D is a good cycle in G . Hence, v lies on H and $d_H(v) = 3$. If all the three neighbors of v in H are adjacent in G to a common vertex from $\{v_1, v_2\}$, then again H stays the same during the operation, a contradiction. Hence, one neighbor is adjacent to v_1 and the other two adjacent to v_2 . This implies that there are two cells around v that are created by our graph operation. It follows by the possible structure of H that, we create either a 6^- -cycle or a triangular 7^- -cycle, contradicting the assumption (b).

Case 2: assume that the graph operation includes inserting an edge, say e . Recall that D stays the same during the operation. If $e \notin E(H) \setminus E(D)$, then H is a bad partition of D also in G , a contradiction; otherwise, the two cells of H containing e are created by our operation, contradicting the assumption (b). \square

Lemma 3.11. *G contains no internal 4-vertices having a pendent $(3, 3, 3)$ -face and another pendent $(3, 3, 4^-)$ -face.*

Proof. Suppose to the contrary that G has such a vertex x . Denote by $[u_1u_2u_3]$ a $(3, 3, 3)$ -face and by $[v_1v_2v_3]$ a $(3, 3, 4^-)$ -face, with u_1 and v_1 as pendent vertices of x and with v_3 as the 4^- -vertex. Denote by x_1 and x_2 the remaining neighbors of x . We distinguish two cases.

Case 1: assume that x_1 and x_2 lie on different sides of the path u_1xv_1 , i.e., x_1 and x_2 are not consecutive in the cyclic order around x . Remove $x, u_1, u_2, u_3, v_1, v_2, v_3$ from G and identify x_1 with x_2 , obtaining a smaller graph G' than G . If this operation satisfies both terms in Lemma 3.10, then the pre-coloring ϕ of D can be super-extended to G' by the minimality of G , and further to G in such way: 3-color $v_3, v_2, v_1, x, u_2, u_3$ in turn and consequently, we can $(1, 0, 0)$ -color u_1 .

(Term *a*) If our operation identifies two vertices of D , or creates an edge that connects two vertices of D , then the path x_1xx_2 is contained in a splitting 2- or 3-path of D . By Lemma 3.8, this splitting path divides D into two parts, one of which is a 3- or 5-cycle. So this cycle is a good cycle but now it separates v_1 from u_1 , contradicting Lemma 3.2.

(Term *b*) If our operation creates a new 7^- -cycle, then this cycle corresponds to a 7^- -path of G between x_1 and x_2 , which together with the path x_1xx_2 forms a 9^- -cycle of G , say C . Clearly, C separates u_1 from v_1 . So, C is a bad 9-cycle having a $(5, 5, 5)$ -claw. But now C contains a 3-face inside, either $[u_1u_2u_3]$ or $[v_1v_2v_3]$, a contradiction.

Case 2: assume that x_1 and x_2 lie on the same side of the path u_1xv_1 . W.l.o.g., let u_1, x_1, x_2, v_1 locate in clockwise order around x and so do u_1, u_2, u_3 along the cycle $[u_1u_2u_3]$. Denote by y the remaining neighbor of u_2 . Delete $x, u_1, u_2, u_3, v_1, v_2, v_3$ and identify x_2 with y , obtaining a smaller graph G' than G . If our graph operation satisfies both terms of Lemma 3.10, then ϕ can be super-extended to G' by the minimality of G and further to G in such way: 3-color x and u_3 ; since x and y receive different colors, we can 3-color u_1 and u_2 ; 3-color v_3 and v_2 in turn and finally, we can $(1, 0, 0)$ -color v_1 .

Let us show that both terms of Lemma 3.10 do hold:

(Term *a*) Otherwise, the path $yu_2u_1xx_2$ is contained in a splitting 4- or 5-path of D . By Lemma 3.8, this splitting path divides D into two parts, one of which is a 9^- -cycle, say C . Now C separates v_1 from u_3 . Hence, C is a bad 9-cycle with a $(5, 5, 5)$ -claw. But C has to contain a 3-face inside, either $[u_1u_2u_3]$ or $[v_1v_2v_3]$, a contradiction.

(Term *b*) Suppose our operation creates a new 7^- -cycle, then it corresponds to a 7^- -path of G between y and x_2 , which together with the path $yu_2u_1xx_2$ forms a 11^- -cycle of G , say C . Clearly, C separates v_3 from u_3 . So C is a bad cycle containing either u_3 or v_3 inside. For the former case, because of the existence of $[u_1u_2u_3]$

and xx_1 , Remark 3.5(4) implies that xx_1 is a chord of C , which thereby has a $(3, 7, 3, 8)$ -edge-claw. Now u_3 is a light vertex with three light neighbors, a contradiction to Lemma 3.6. For the latter case, the interior of C , as a bad cycle, contains the triangle $[v_1v_2v_3]$, which is impossible. \square

Lemma 3.12. *G contains no internal 4-vertices incident with a $(3, 4^-, 4)$ -face and having a pendent $(3, 3, 4^-)$ -face.*

Proof. Suppose to the contrary that such vertex exists, say u . Denote by u_1, \dots, u_4 the neighbors of u locating in clockwise order around u . W.l.o.g., let $[uu_1u_2]$ be a $(3, 4^-, 4)$ -face and $[u_3u'_3u''_3]$ be a pendent $(3, 3, 4^-)$ -face of u . Delete u_1, u, u_3, u'_3, u''_3 from G and identify u_2 with u_4 , obtaining a new smaller graph G' . Similarly, to complete the proof, it suffices to doing two things.

Firstly, we shall show that both terms in Lemma 3.10 hold.

(Term *a*) If our operation identifies two vertices of D , or creates an edge that connects two vertices of D , then the path u_2uu_4 is contained in a splitting 2- or 3-path of D . By Lemma 3.8, this splitting path divides D into two parts, one of which is a 3- or 5-cycle, say C . Now C separates u_1 from u_3 , a contradiction.

(Term *b*) If our operation creates a new 7^- -cycle, then G has a 9^- -cycle C that contains the path u_2uu_4 . Since C separates u_1 from u_3 , C is a bad 9-cycle with a $(5, 5, 5)$ -claw, contradicting that C contains a triangle either $[uu_1u_2]$ or $[u_3u'_3u''_3]$ inside.

Secondly, we shall show that any $(1, 0, 0)$ -coloring of G' can be super-extended to G . This can be done in the following way. Since one of u'_3 and u''_3 has degree 3 and the other degree at most 4, we can 3-color them. Notice that u_1 has degree either 3 or 4. Since u_2 and u_4 receive the same color, if we can 3-color u_1 , then consequently we can 3-color u and $(1, 0, 0)$ -color u_3 in turn, we are done. Hence, we may assume that u_1 has degree 4 and its neighbors except u are colored pairwise distinct. In this case, give the color of u_2 to u_1 . Since u_2 has degree 3, we can recolor it properly. Since u_1 and u_4 are colored the same, we can 3-color u and then $(1, 0, 0)$ -color u_3 . \square

Lemma 3.13. *G has no 4-vertices incident with two $(3, 4^-, 4)$ -faces.*

Proof. Suppose to the contrary that G has such a 4-vertex v , incident with two $(3, 4^-, 4)$ -faces $T_1 = [vv_1v_2]$ and $T_2 = [vv_3v_4]$. W.l.o.g., let v_1, v_2, v_3, v_4 locate in clockwise order around v .

Case 1: assume that at least one of T_1 and T_2 is a $(3, 3, 4)$ -face, w.l.o.g, say T_1 . Delete v, v_1, \dots, v_4 , obtaining a smaller graph G' than G . Since we only remove vertices, both terms in Lemma 3.10 hold. Hence, ϕ can be super-extended to G' by the minimality of G , and further to G in such way: 3-color the vertices of T_2 . Denote by v'_1 and v'_2 the remaining neighbors of v_1 and v_2 , respectively. We can always 3-color v_1 and v_2 except the case $\phi(v'_1) = \phi(v'_2) \neq \phi(v)$, for which we distinguish three subcases: if $1 \notin \{\phi(v'_1), \phi(v)\}$, then give the color 1 to both v_1 and v_2 , completing the super-extension; if $\phi(v) = 1$, then assign v_1 with the color 1 and consequently, we can 3-color v_2 ; if $\phi(v'_1) = 1$, then recolor v by the color 1, and then 3-color both v_1 and v_2 .

Case 2: assume that both T_1 and T_2 are $(3, 4, 4)$ -faces. W.l.o.g., let $d(v_1) = 4$. We distinguish two cases.

Case 2.1: assume that $d(v_3) = 4$. Denote by v'_2 and v'_4 the outer neighbors of v_2 and v_4 , respectively. We delete all vertices of T_1 and T_2 , and identify v'_2 with v'_4 , obtaining a new graph G' . We will show that both terms in Lemma 3.10 do hold:

(Term *a*) If our operation identifies two vertices of D , or creates an edge that connects two vertices of D , then the path $v'_2v_2vv_4v'_4$ is contained in a splitting 4- or 5-path of D . By Lemma 3.8, this splitting path divides

D into two parts, one of which is a 9^- -cycle, say C . Now C separates v_1 from v_3 and contains a triangle inside, a contradiction.

(Term b) If our operation creates a new 7^- -cycle, then G has a 11^- -cycle C that contains the path $v'_2v_2vv_4v'_4$. Now C separates v_1 from v_3 , both has degree 4, contradicting Remark 3.5(2).

We will show that any $(1,0,0)$ -coloring of G' can be super-extended to G : 3-color v_1 and v_3 . Denote by α the color v'_2 and v'_4 received. If α has not been used by both v_1 and v_3 , then give α to v and consequently, we can 3-color v_2 and v_4 . W.l.o.g., we may next assume that v_3 has color α . 3-color v_2 and then $(1,0,0)$ -color v . Since v_3 and v'_4 received the same color, we can 3-color v_4 .

Case 2.2: assume that $d(v_4) = 4$. Denote by v'_i the neighbor of v_i for $i \in \{2,3\}$, and by v'_i and v''_i the remaining neighbors of v_i locating in clockwise order around v_i for $i \in \{1,4\}$. Delete all vertices of T_1 and T_2 and identify v'_1 with v'_3 . Denote by z the resulting vertex and G' the resulting graph. Notice that our operation may create some new 7^+ -cycles.

Firstly, by the same argument as in Case 2.1, Term (a) does hold.

Secondly, we claim that the operation creates no 6^- -cycles. Otherwise, G has a 10^- -cycle C that contains the path $v'_1v_1vv_3v'_3$. So, C is a bad cycle containing either v_2 or v_4 inside. For the former case, since a bad 10^- -cycle has no chords, v_1 has two neighbors inside C , contradicting Remark 3.5(3). For the latter case, $d(v_4) = 4$ contradicts Remark 3.5(2).

Finally, we do not make D bad. Otherwise, since we create no 6^- -cycles, by the argument for the proof of Lemma 3.10, we can deduce that the new vertex z is incident with two cells of D in G' that are created by our operation, where one cell has length 7 and the other length 7 or 8. These two cells correspond to two cycles of G containing the path $v'_1v_1vv_3v'_3$, one cycle (say C') contains v_2 inside and the other (say C'') contains v_4 inside. Clearly, one of C' and C'' has length 11 and the other length 11 or 12. Since $d(v_4) = 4$, we can deduce that $|C''| = 12$ by Remark 3.5 (2). So, $|C'| = 11$. Hence, the way we make D bad is that our operation make D have a $(3,7,3,8)$ -edge-claw in G' where the 7-cell and 8-cell are created. Let e denote the common edge of these two cells. Since v_1 is incident with two edges $v_1v'_1$ and v_1v_2 inside C' , we can deduce that $v_1v'_1$ is a chord of C' , which has a $(3,7,3,8)$ -edge-claw in G by Remark 3.5 (3). Let $C' = [v'_3v_3vv_1v'_1v''_1y_1 \cdots y_5]$. Recall that v'_1 and v'_3 are the two vertices we identified. So, e corresponds to either v'_3y_5 or $v'_1v''_1$. For the former case, the vertices v''_1, y_1, \cdots, y_4 lie on D . A contradiction follows by applying Lemma 3.8 to the splitting 4-path $v''_1v_1v_2v''_2y_4$ of D in G . For the latter case, by substituting $v_1v''_1$ for $v_1v'_1v''_1$ from C'' , we obtain a 11-cycle of G that contains v_4 inside, a contradiction.

Because of the conclusions in the previous three paragraphs, by the minimality of G , we can super-extend ϕ from D to G' . We complete a $(1,0,0)$ -coloring of G as follows: 3-color v_4 and v_1 . Since v_1 and v'_3 receive different colors, we can 3-color v_3 and v . Finally, we can $(1,0,0)$ -color v_2 except the case that v'_2 has the color 1 and between v and v_1 , one has the color 2 and the other 3. Notice that the colors of v_4 , v_3 and v are pairwise distinct. Recolor v by 1 and finally, we can 3-color v_2 . \square

Lemma 3.14. G has no internal 5-vertices incident with two faces, one is a weak $(3,3,5)$ -face and the other is a $(3,4^-,5)$ -face.

Proof. Suppose to the contrary that G has such a vertex v . Denote by v_1, \dots, v_5 the neighbors of v locating in clockwise order around v with $[vv_1v_2]$ being a weak $(3,3,5)$ -face and $[vv_3v_4]$ being a $(3,4^-,5)$ -face. Let x' be a light outer neighbor of $[vv_1v_2]$. Between v_1 and v_2 , denote by x the one adjacent to x' and by y the other. Clearly, v_4 is of degree 3 or 4. We distinguish two cases.

Case 1: assume $d(v_4) = 3$. Delete v, v_1, v_2, x', v_4 and identify v_3 with v_5 , obtaining a smaller graph G' than G . We shall show that both terms in Lemma 3.10 hold.

(Term *a*) If our operation identifies two vertices of D , or creates an edge that connects two vertices of D , then the path v_3vv_5 is contained in a splitting 2- or 3-path of D . By Lemma 3.8, this splitting path divides D into two parts, one of which is a 3- or 5-cycle, say C . Now C separates v_2 from v_4 , a contradiction.

(Term *b*) If our operation creates a new 7^- -cycle, then G has a 9^- -cycle C that contains the path v_3vv_5 . Since C separates v_2 from v_4 , C is a bad 9-cycle with a (5,5,5)-claw, contradicting that C contains a triangle either $[vv_1v_2]$ or $[vv_3v_4]$ inside.

Hence, the coloring ϕ of D can be super-extended to G' by Lemma 3.10 and further to G as follows: 3-color v_4, v, x', y in turn and consequently, we can (1,0,0)-color x . This is a contradiction.

Case 2: assume $d(v_4) = 4$. It follows that $d(v_3) = 3$. Let v'_3 be the remaining neighbor of v_3 . Delete $v, v_1, v_2, v_3, v_4, x'$ and insert an edge between v'_3 and v_5 , obtaining a smaller graph G' than G .

(Term *a*) Notice that our operation identifies no vertices. Suppose to the contrary that it creates an edge that connects two vertices of D , then the path $v'_3v_3vv_5$ is contained in a splitting 3-path of D . By Lemma 3.8, this splitting path divides D into two parts, one of which is a 5-cycle. Now this cycle separates v_2 from v_4 , a contradiction.

(Term *b*) If our operation creates a new 7^- -cycle, then G has a 9^- -cycle C containing path $v'_3v_3vv_5$. Clearly, C separates v_2 from v_4 . Hence, C is a bad 9-cycle that contains a triangle either $[vv_1v_2]$ or $[vv_3v_4]$ inside, a contradiction.

Hence, ϕ can be super-extended to G' by Lemma 3.10 and further to G as follows: 3-color v_4 . If $\phi(v'_3) \neq \phi(v_5)$ or $\phi(v'_3) = \phi(v_5) = \phi(v_4)$, then we can first 3-color v and v_3 , next 3-color x' and y in turn and consequently, we can (1,0,0)-color x , we are done. Hence, we may assume that $\phi(v'_3) = \phi(v_5) \neq \phi(v_4)$. Since v'_3 and v_5 are adjacent in G' , both v'_3 and v_5 have color 1 and have no other neighbors colored 1. So we can give the color 1 to v_3 and then 3-color v . By the same way as above, we color v_2, v_1 and v , we are done as well. \square

Lemma 3.15. *If v is an internal 5-vertex of G incident with two 3-faces, one is a weak (3,3,5)-face and the other is a weak (3,5,5⁺)-face, then v has no pendent (3,3,3)-faces.*

Proof. Denote by v_1, \dots, v_5 the neighbors of v , whose order around v has not been given yet. Suppose to the contrary that v has a pendent (3,3,3)-face, say $[v_1w_1w_2]$. Let $[vv_2v_3]$ be a weak (3,3,5)-face with v'_3 being a light outer neighbor of v_3 . Let $[vv_4v_5]$ be a weak (3,5,5⁺)-face with v'_4 being a light outer neighbor of v_4 . Delete $v, v_1, \dots, v_4, w_1, w_2, v'_3, v'_4$ from G , obtaining a graph G' . By the minimality of G , the pre-coloring ϕ of D can be super-extended to G' , and further to G in such way: 3-color v'_4, v_4 and v in turn. If v has color 1, then exchange the colors of v and v_4 . Hence, w.l.o.g., we may assume that v has color 2. 3-color v'_3, v_2, w_1, w_2 in turn. Consequently, we can (1,0,0)-color v_3 and v_1 . \square

Lemma 3.16. *If v is an internal 6-vertex of G incident with two weak (3,3,6)-faces, then v is incident with no other (3,4⁻,6)-faces,*

Proof. Denote by v_1, \dots, v_6 the neighbors of v locating around v in clockwise order. Let $[vv_3v_4]$ and $[vv_5v_6]$ be two weak (3,3,6)-faces. Suppose to the contrary that $[vv_1v_2]$ is a (3,4⁻,6)-face. W.l.o.g., let $d(v_2) = 3$. Denote by v'_i the remaining neighbor of v_i for $i \in \{2, \dots, 6\}$. Since $[vv_3v_4]$ is weak, denote by x' a light outer neighbor of $[vv_3v_4]$. Between v_3 and v_4 , denote by x the one adjacent to x' and by y the other. Delete vertices

v, v_1, \dots, v_6, x' from G and identify v'_2 with v'_5 , obtaining a new graph G' . We will show that both terms in Lemma 3.10 do hold:

(Term *a*) Otherwise, the path $v'_2v_2vv_5v'_5$ is contained in a splitting 4- or 5-path of D . By Lemma 3.8, this splitting path divides D into two parts, one of which is a 9^- -cycle, say C . Now C separates v_4 from v_6 and contains a triangle either $[vv_3v_4]$ or $[vv_5v_6]$ inside, a contradiction.

(Term *b*) If our operation creates a new 7^- -cycle, then G has a 11^- -cycle C that contains the path $v'_2v_2vv_5v'_5$. Since C separates v_4 from v_5 , C is a bad cycle. Now v is a vertex on C which has two neighbors either v_3, v_4 or v_1, v_6 inside C , contradicting Remark 3.5(3).

By Lemma 3.10, ϕ can be super-extended to G' . We will further super-extend ϕ to G in the following way. Let α be the color v'_2 and v'_5 receive. 3-color v_1, v_2 and v in turn. If v has color α , then we can 3-color v_6 and v_5 in turn and separately, 3-color x' and y in turn and then $(1,0,0)$ -color x , we are done. Hence, we may assume that the color of v is not α . Since the colors of v, v_1 and v_2 are pairwise distinct, v_1 has color α . We may assume that the color of v is not 1 since otherwise, we exchange the colors of v and v_2 . 3-color x' and y in turn and consequently, we can $(1,0,0)$ -color x . Remove the color of an outer neighbor (say z) of $[vv_5v_6]$ and in the same way, we color z, v_5, v_6 , as desired. \square

Let W be a subgraph of G consisting of a $(4, 4, 4)$ -face $[uvw]$ and three 3-faces $[uu_1u_2], [vv_1v_2]$ and $[ww_1w_2]$ of G that share precisely one vertex (respectively, u, v and w) with $[uvw]$. Let u, v, w as well as $u_1, u_2, v_1, v_2, w_1, w_2$ be in clockwise order around $[uvw]$. Call W a *wheel*, written as $(uvw, u_1u_2v_1v_2w_1w_2)^W$, if $d(u_1) = d(v_1) = d(w_1) = 3$ and $d(u_2) = d(v_2) = d(w_2) = 4$. Call W an *antiwheel*, written as $(uvw, u_1u_2v_1v_2w_1w_2)^{AW}$, if $d(u_1) = d(v_1) = d(w_2) = 3$ and $d(u_2) = d(v_2) = d(w_1) = 4$.

Lemma 3.17. G has no wheels.

Proof. Suppose to the contrary that G has a wheel, say $W = (uvw, u_1u_2v_1v_2w_1w_2)^W$. Let u'_1, v'_1 and w'_1 be the remaining neighbors of u_1, v_1 and w_1 , respectively. Delete all vertices of W and insert three edges making $[u'_1v'_1w'_1]$ a triangle. We thereby obtain a graph G' smaller than G . We shall use Lemma 3.9.

Suppose that our operation connects two vertices of D . W.l.o.g., let u'_1 and v'_1 locate on D . Then as a splitting 5-path of D , $u'_1u_1uvv_1v'_1$ divides D into two parts, one of which is a 9^- -cycle. Now this cycle separates u_2 from w and contains a triangle either $[uu_1u_2]$ or $[uvw]$ inside, a contradiction. Hence, Term (i) holds true.

Suppose that our operation creates a new 7^- -cycle C' other than $[u'_1v'_1w'_1]$. Since C' is new, C' must share edges with $[u'_1v'_1w'_1]$. If they have precisely two common edges (w.l.o.g., say $u'_1v'_1$ and $v'_1w'_1$), then the cycle obtained from C' by constituting the edge $u'_1w'_1$ for the path $u'_1v'_1w'_1$ is also created and has smaller length than C' . Take this cycle as the choice for C' . Hence, we may assume that C' and $[u'_1v'_1w'_1]$ have one edge in common, say $u'_1v'_1$. So, C' corresponds to a 11^- -cycle C of G that contains the path $u'_1u_1uvv_1v'_1$. Since C separates u_2 from w , C is a bad cycle containing either u_2 or w inside, both of which have degree 4. This contradicts Remark 3.5 (2). Therefore, our operation creates no 7^- -cycles C' other than $[u'_1v'_1w'_1]$. In particular, Term (ii) holds true.

Suppose that our operation makes D bad. So, D has a bad partition H in G' . If H and $[u'_1v'_1w'_1]$ have no edges in common, then H is a bad partition of D in G as well, a contradiction. Hence, let e be a common edge of H and $[u'_1v'_1w'_1]$. Recall that among the vertices of $[u'_1v'_1w'_1]$, at most one lies on D . So, e is not an edge of D . This implies that e is incident with two cells of H , both of which are new. That is to say, we created a 7^- -cycle other than $[u'_1v'_1w'_1]$, a contradiction. Therefore, Term (iii) holds true.

By Lemma 3.9, ϕ can be super-extended to G' . We will further super-extend ϕ to G . Since $[u'_1v'_1w'_1]$ is a triangle of G' , we distinguish two cases as follows.

Case 1: assume that the colors of u'_1, v'_1 and w'_1 are pairwise distinct. W.l.o.g., let $\phi(u'_1) = 3, \phi(v'_1) = 2$ and $\phi(w'_1) = 1$. 3-color u_2, v_2 and w_2 . If $\phi(u_2) \neq 3$ and $\phi(v_2) \neq 2$, then assign u, v, w with colors 3, 2, 1, respectively. Consequently, we can 3-color u_1, v_1 and w_1 , we are done. W.l.o.g., we may next assume that $\phi(u_2) = 3$. Assign u_1 with color 2 and u with color 1. Since u and v'_1 have different colors, we can 3-color v and v_1 . If w_2 has color 1, then we can 3-color w and w_1 in turn; otherwise, assign w with the color 1 and then 3-color w_1 .

Case 2: assume that the colors of u'_1, v'_1 and w'_1 are not pairwise distinct. Since the extension of ϕ in G' is a $(1, 0, 0)$ -coloring, precisely two of u'_1, v'_1 and w'_1 have the color 1, say u'_1 and v'_1 . 3-color u_2, v_2, w_2, w_1, w in turn. We may assume that the color of w is not 1 since otherwise, we can exchange the colors of w and w_1 . W.l.o.g., let w be of color 3. Since both u'_1 and v'_1 have color 1 that is different from the color of w , regardless of the edge uv , we can 3-color u, u_1 and v, v_1 . The resulting coloring gives a $(1, 0, 0)$ -coloring of G unless both u and v have color 2. For this remaining case, we can deduce that u_1 has color 3 and u_2 has color 1. Reassign u with the color 1, we are done. \square

Lemma 3.18. *G has no antiwheel whose outer neighbors are all light.*

Proof. Suppose to the contrary that G has such an antiwheel, say $W = (uvw, u_1u_2v_1v_2w_1w_2)^{AW}$. Denote by u'_1, v'_1 and w'_2 outer neighbors of u_1, v_1 and w_2 , respectively. Delete all the vertices of W except v_2 , identify v_2 with w'_2 , and insert an edge between u'_1 and v'_1 , obtaining a new graph G' from G . We shall use Lemma 3.9.

Suppose that our operation identifies two vertices of D , or inserts an edge that connects two vertices of D . So, D has a splitting 4- or 5-path in G containing either $v_2vww_2w'_2$ or $u'_1u_1uvv_1v'_1$. By Lemma 3.8, this splitting path divides D into two parts, one of which is a 9^- -cycle, say C . Now C separates u_2 from w_1 and contains a triangle either $[uu_1u_2]$ or $[ww_1w_2]$ inside, a contradiction. Hence, Term (i) holds true.

Suppose that our operation creates a new 7^- -cycle, say C' . C' corresponds to a subgraph (say P) of G that can be distinguished in four cases: (1) a 6^- -path between u'_1 and v'_1 ; (2) a 7^- -path between w'_2 and v_2 ; (3) the union of two vertex-disjoint paths, one between u'_1 and w'_2 and the other between v'_1 and v_2 ; (4) the union of two vertex-disjoint paths, one between u'_1 and v_2 and the other between v'_1 and w'_2 . For the first case, P and the path $u'_1u_1uvv_1v'_1$ together form a 11^- -cycle which contains a 4-vertex either u_2 or w_1 inside, a contradiction to Remark 3.5 (2). For the case (2), P and the path $w'_2w_2wvv_2$ together form a 11^- -cycle which contains a 4-vertex either u_2 or w_1 inside, again a contradiction to Remark 3.5 (2). For the case (3), since G has no 6^- -cycles adjacent to a triangle, we can deduce that G has no 4^- -paths between v'_1 and v_2 by the existence of $[vv_1v_2]$ and no edges between u'_1 and w'_2 by the existence of $[uvw]$. It follows that P has length at least 8, a contradiction. Case (4) is impossible by the planarity of G . Therefore, our operation creates no 7^- -cycles. In particular, Term (ii) holds true.

Suppose that our operation makes D bad. Let H be a bad partition of D in G' . Since both terms of Lemma 3.10 holds, if $u'_1v'_1 \notin E(H)$, then the proof of Lemma 3.10 shows that identifying w'_2 with v_2 can not make D bad. So, $u'_1v'_1$ belongs to H . Since Term (i) holds true, $u'_1v'_1$ is incident with two cells of H . Clearly, these two cells are created and at least one of them is a 7^- -cycle, contradicting the conclusion above that our operation creates no 7^- -cycles. Therefore, Term (iii) holds.

By Lemma 3.9, ϕ can be super-extended to G' . Denote by α the color v_2 and w'_2 receive and by β the color u'_1 receives. 3-color u_2 and w_1 . We distinguish two cases according to the colors of u_2 and w_1 .

Case 1: suppose that not both u_2 and w_1 have color α . So, we can 3-color u, v and w . Since both u'_1 and w'_2 have degree 3, we can 3-recolor them. Consequently, we can $(1, 0, 0)$ -color u_1 and w_2 . If not all the colors occur on the neighbors of v_2 , then we can 3-recolor v_2 and eventually, 3-recolor v'_1 and $(1, 0, 0)$ -color v_1 in turn, we are done. So, we may next assume that v_2 has all the colors around. It follows that v_2 is of color 1 and v not. W.l.o.g., Let v be of color 3. We may assume that v'_1 is of color 2 since otherwise, we can 3-color v_1 . Since G' has an edge between u'_1 and v'_1 , $\beta \neq 2$. Now we recolor some vertices as follows. Assign v_1 with 1, reassign v_2 with 3 and v with 2, remove the colors of u_1, u, w, w_2 , and give the color 1 back to w'_2 and β back to u'_1 . Since now u'_1 and v have different colors, we can 3-color u and u_1 . Clearly, w'_2 has no neighbors of color 2 since v_2 already has one. If w_1 has color 2, then we can 3-color w and $(1, 0, 0)$ -color w_2 in turn; otherwise, assign w_2 with 2 and we can $(1, 0, 0)$ -color w .

Case 2: suppose that both u_2 and w_1 have color α . If $\alpha = 1$, then assign u with α and we can 3-color u_1, v_1, v, w, w_2 in turn, we are done. W.l.o.g., we may next assume that $\alpha = 2$. If $\beta \neq 3$, then we can 3-color v_1, v, w, w_2 in turn, assign u with the color 1, and 3-color u_1 at last; otherwise, since v'_1 is of color different from β , we assign u, w_2 and v_1 with 3, and u_1, w and v with 1. We are done in both situations. \square

Lemma 3.19. *G has no 5-faces whose vertices are all light.*

Proof. Suppose G has such a 5-face, say $f = [u_1 u_2 \dots u_5]$. For $i \in \{1, 2, \dots, 5\}$, let u'_i denote the remaining neighbor of u_i . If both u'_1 and u'_3 belong to D , then as being a splitting 4-path of D , $u'_1 u_1 u_2 u_3 u'_3$ divides D into two parts, one of which is a 5- or 7-cycle. This cycle is actually a face but now contains an edge either $u_2 u'_2$ or $u_3 u_4$ inside, a contradiction. Therefore, at least one of u'_1 and u'_3 is internal. For the same reason, this is even true for u_i and u_{i+2} for each $i \in \{1, 2, \dots, 5\}$, where the index is added in modulo 5. Hence, we can always get three internal vertices u'_i, u'_{i+1} and u'_{i+2} for some $i \in \{1, 2, \dots, 5\}$. W.l.o.g., let u'_5, u'_1 and u'_2 be internal. Remove all the vertices of f from G and insert an edge between u'_2 and u'_5 , obtaining a new graph G' . We shall use Lemma 3.9. Clearly, Term (i) holds true.

Suppose the graph operation creates a k -cycle with $k \in \{1, 2, 4, 6\}$. So, G has a k -path between u'_2 and u'_5 . This path together with $u'_5 u_5 u_1 u_2 u'_2$ form a $(k + 3)$ -cycle, say C . By Lemma 3.6, $d(u'_1) \geq 4$. So, C can not contain u'_1 inside since otherwise, a contradiction to Remark 3.5 (2). Moreover, as a 9^- -cycle, C can not contain both u_3 and u_4 inside. Therefore, by planarity of G , u'_1 must locate on C . Now the cycle, obtained from C by constituting $u_2 u_3 u_4 u_5$ for $u_2 u_1 u_5$, is a bad 10-cycle but it has a claw and a 5-cell, which is impossible. Therefore, Term (ii) holds true.

Suppose that our operation makes D bad. Let H be a bad partition of D in G' . So, $u'_2 u'_5$ belongs to $H - E(D)$ since otherwise, H is a bad partition of D in G . Now, $u'_2 u'_5$ is incident with two cells of H , say h' and h'' . Denote by C' and C'' cycles obtained from h' and h'' by constituting the edge $u'_2 u'_5$ for the path $u'_2 u_2 u_1 u_5 u'_5$. Clearly, one of C' and C'' (w.l.o.g., say C') contains u'_1 inside or on C , and the other contains u'_3 and u'_4 inside. Since a cell has length at most 8, both C' and C'' have length at most 11. So, C'' is a bad cycle. Lemma 3.6 implies that both u'_3 and u'_4 are not light. So, C'' can not contain them inside by Remark 3.5(2). Instead, u'_3 and u'_4 are on C'' . Now C'' has an edge-claw, more precisely, an $(5, 5, 5, 5)$ -edge-claw. So, h'' is a non-triangular 7-cell of H , which implies that H must have a $(5, 5, 7)$ -claw in G' . This gives a contradiction since both u'_2 and u'_5 are internal vertices on H . Therefore, Term (iii) holds true.

By Lemma 3.9, ϕ can be super-extended to G' and further to G as follows. If there is a vertex from $\{u'_2, u'_5\}$ of color different from 1, w.l.o.g., say u'_5 , then 3-color u_1, \dots, u_4 in turn and finally, we can $(1, 0, 0)$ -color u_5 . So

we may assume that both u'_2 and u'_5 are of color 1. Again, 3-color u_1, \dots, u_4 in turn. Since u'_2 has no neighbors of color 1 in G , we can $(1, 0, 0)$ -color u_5 . \square

Lemma 3.20. *G has no 5-faces, four of whose vertices are light and the remaining one is an internal 4-vertex.*

Proof. Suppose to the contrary the G has such a 5-face $[u_1 \dots u_5]$. W.l.o.g., let u_1 be of degree 4. Denote by u'_1 and u''_1 the remaining neighbor of u_1 and for $i \in \{2, \dots, 5\}$, denote by u'_i the remaining neighbor of u_i . Remove all the vertices of $[u_1 \dots u_5]$ and insert an edge between u'_2 and u'_5 , obtaining a new graph G' . We will show that both terms in Lemma 3.10 do hold:

(Term *a*) Otherwise, both u'_2 and u'_5 belong to D . So, $u'_2 u_2 u_1 u_5 u'_5$ is a splitting 4-path of D , which divides D into two parts so that one part is a 5- or 7-cycle C , by Lemma 3.8. Notice that C is actually a face but now has to contain an edge either $u_1 u'_1$ or $u_2 u_3$ inside, a contradiction.

(Term *b*) Otherwise, G has a 9^- -cycle or a triangular 10-cycle C containing the path $u'_2 u_2 u_1 u_5 u'_5$. By the planarity of G , either C contains the edges $u_1 u'_1$ and $u_1 u''_1$ inside or C contains the vertices u_3 and u_4 inside. For the former case, since C has length at most 10, Remark 3.5(4) implies that C is not a bad cycle. So, u'_1 and u''_1 locate on C , yields the length of C at least 11, a contradiction. For the latter case, by Lemma 3.6, neither u'_3 nor u'_4 is light. So, they both locate on C'' , implied by Remark 3.5(2). Now C has a $(5, 5, 5, 5)$ -edge-claw, which gives a new triangular 7-cycle in G' , a contradiction.

By Lemma 3.10, ϕ can be super-extended to G' and further to G in the same way as in the proof of Lemma 3.19. \square

Lemma 3.21. *G has no two 5-faces f and g sharing precisely one edge, say uv , such that u is an internal 5-vertex and all other vertices on f or g are light.*

Proof. Suppose to the contrary that such f and g exist. By the minimality of G , we can super-extend ϕ to $G - V(f) \cup V(g)$ and further to G as follows: 3-color the vertices of f and g except v beginning with u along separately the boundary of f and one of g . Eventually, we can $(1, 0, 0)$ -color v . \square

3.2 Discharging in G

Let u be a vertex of a $(4, 4, 4)$ -face. u is *abnormal* if it is incident with a $(3, 4, 4)$ -face; otherwise, u is *normal*. A 5-face is *small* if it contains precisely four light vertices. Let P be the common part of D and a face f . f is *sticking* if P is a vertex, *i -ceiling* if P is a path of length i for $i \geq 1$.

Let V, E and F be the set of vertices, edges and faces of G , respectively. Denote by f_0 the exterior face of G . Give *initial charge* $ch(x)$ to each element x of $V \cup F$ defined as $ch(f_0) = d(f_0) + 24$, $ch(x) = 5d(x) - 14$ for $x \in V$, and $ch(x) = 2d(x) - 14$ for $x \in F \setminus \{f_0\}$. Move charges among elements of $V \cup F$ based on the following rules (called discharging rules):

- R1. Every internal 3-vertex sends to each incident face f charge 1 if $d(f) = 3$, and charge $\frac{1}{3}$ otherwise.
- R2. Every internal 4-vertex sends to each incident 3-face f charge $\frac{7}{2}$ if f is a $(3, 4, 4)$ -face, charge 3 if f is a $(3, 3, 4)$ -face, charge $\frac{8}{3}$ if f is a $(4, 4, 4)$ -face, charge $\frac{5}{2}$ otherwise.
- R3. Every internal 5-vertex sends to each incident 3-face f charge 6 if f is weak $(3, 3, 5)$ -face, charge $\frac{9}{2}$ if f is $(3, 4, 5)$ -face, charge $\frac{7}{2}$ if f is either a weak $(3, 5, 5)$ -face or a strong $(3, 3, 5)$ -face, charge 3 otherwise.

- R4. Every internal 6-vertex sends to each incident 3-face f charge 6 if f is weak $(3, 3, 6)$ -face, charge 5 if f is $(3, 4, 6)$ -face, charge 4 otherwise.
- R5. Every internal 7^+ -vertex sends to each incident 3-face charge 6.
- R6. Every internal 4^+ -vertex sends to each pendent 3-face f charge $\frac{5}{3}$ if f is $(3, 3, 3)$ -face, charge $\frac{3}{2}$ if f is a $(3, 3, 4)$ -face, and charge $\frac{5}{4}$ otherwise.
- R7. Every internal 4^+ -vertex u sends to each incident 5-face f charge $\frac{8}{3}$ if $d(u) \geq 5$ and f is small, and charge $\frac{3}{2}$ otherwise.
- R8. Within a $(4, 4, 4)$ -face, every normal vertex send to each abnormal vertex charge $\frac{1}{6}$.
- R9. Within an antiwheel, every strong $(3, 4, 4)$ -face sends to each vertex of the $(4, 4, 4)$ -face charge $\frac{1}{6}$.
- R10. The exterior face f_0 sends charge 3 to each incident vertex.
- R11. Every 2-vertex receives charge 1 from its incident face other than f_0 .
- R12. Every exterior 3^+ -vertex sends to each sticking 3-face charge 6, to each ceiling 3-face charge $\frac{7}{2}$, to each sticking 5-face charge $\frac{8}{3}$, to each 2-ceiling 5-face charge $\frac{13}{6}$, to each pendent 3-faces charge $\frac{5}{3}$, to each 1-ceiling 5-face charge $\frac{3}{2}$, to each 3-ceiling 7-face charge 1, to each 2-ceiling 7-face charge $\frac{1}{2}$.

Let $ch^*(x)$ denote the *final charge* of an element x of $V \cup F$ after discharging. On one hand, from Euler's formula $|V| + |E| - |F| = 2$, we deduce $\sum_{x \in V \cup F} ch(x) = 0$. Since the sum of charges over all elements of $V \cup F$ is unchanged during the discharging procedure, it follows that $\sum_{x \in V \cup F} ch^*(x) = 0$. On the other hand, we will show that $ch^*(x) \geq 0$ for $x \in V \cup F \setminus \{f_0\}$ and $ch^*(f_0) > 0$. So, this obvious contradiction completes the proof of Theorem 2.1.

Claim 3.21.1. $ch^*(f_0) > 0$.

Proof. Notice that R10 is the only rule making f_0 move charges out, charge 3 to each incident vertex. Recall that $ch(f_0) = d(f_0) + 24$ and $d(f_0) \leq 11$. So, $ch^*(f_0) \geq ch(f_0) - 3d(f_0) = 24 - 2d(f_0) > 0$. \square

Claim 3.21.2. $ch^*(v) \geq 0$ for $v \in V$.

Proof. Denote by $m_3(v)$ the number of pendent 3-faces of v , and by $n_i(v)$ the number of i -faces containing v for $i \in \{3, 5\}$, where these countings excludes f_0 . Since G has no cycles of length 4 or 6, we have

$$2n_3(v) + n_5(v) + m_3(v) \leq d(v). \quad (1)$$

Furthermore, if $n_5(v) \notin \{0, d(v)\}$, then

$$2n_3(v) + n_5(v) + m_3(v) \leq d(v) - 1. \quad (2)$$

Case 1: first assume that v is external. By R10, v always receives charge 3 from f_0 . Since D is a cycle, $d(v) \geq 2$. If $d(v) = 2$, then v receives charge 1 from the other incident face by R11, giving $ch^*(v) = ch(v) + 3 + 1 = 0$. Hence, we may next assume that $d(v) \geq 3$. Denote by f_1 and f_2 the two ceiling faces containing v . W.l.o.g., let $d(f_1) \leq d(f_2)$.

Case 1.1: suppose $d(v) = 3$. In this case, $ch(v) = 1$, and v sends charge to f_1 and f_2 when R12 is applicable to v . If $d(f_1) = 3$, then on one hand, $d(f_2) \geq 7$, since G has neither 4-cycles nor 6-cycles; on the other hand, f_2 is

not a 3-ceiling 7-face by using Lemma 3.8. So v sends to f_2 charge at most $\frac{1}{2}$, giving $ch^*(v) \geq ch(v) + 3 - \frac{7}{2} - \frac{1}{2} = 0$. We may next assume that $d(f_1) \geq 5$. Lemma 3.8 also implies that not both f_1 and f_2 are 2-ceiling 5-faces. So, v sends to f_1 and f_2 a total charge at most $\frac{13}{6} + \frac{3}{2}$, giving $ch^*(v) \geq ch(v) + 3 - \frac{13}{6} - \frac{3}{2} = \frac{1}{3} > 0$.

Case 1.2: suppose $d(v) \geq 4$. v sends charge out, only by $R12$, possibly to ceiling 3- or 5- or 7-faces, sticking 3- or 5-faces and pendent 3-faces. So,

$$ch^*(v) \geq \begin{cases} ch(v) + 3 - \frac{7}{2} - \frac{7}{2} - 6(n_3(v) - 2) - \frac{8}{3}n_5(v) - \frac{5}{3}m_3(v) = ch(v) - \eta(v) + 8, & \text{when } d(f_1) = d(f_2) = 3; \\ ch(v) + 3 - \frac{7}{2} - \frac{13}{6} - 6(n_3(v) - 1) - \frac{8}{3}(n_5(v) - 1) - \frac{5}{3}m_3(v) = ch(v) - \eta(v) + 7, & \text{when } d(f_1) = 3 \text{ and } d(f_2) = 5; \\ ch(v) + 3 - \frac{7}{2} - 1 - 6(n_3(v) - 1) - \frac{8}{3}n_5(v) - \frac{5}{3}m_3(v) = ch(v) - \eta(v) + \frac{9}{2}, & \text{when } d(f_1) = 3 \text{ and } d(f_2) \geq 7; \\ ch(v) + 3 - \frac{13}{6} - \frac{13}{6} - 6n_3(v) - \frac{8}{3}(n_5(v) - 2) - \frac{5}{3}m_3(v) = ch(v) - \eta(v) + 4, & \text{when } d(f_1) = d(f_2) = 5; \\ ch(v) + 3 - \frac{13}{6} - 1 - 6n_3(v) - \frac{8}{3}(n_5(v) - 1) - \frac{5}{3}m_3(v) = ch(v) - \eta(v) + \frac{5}{2}, & \text{when } d(f_1) = 5 \text{ and } d(f_2) \geq 7; \\ ch(v) + 3 - 1 - 1 - 6n_3(v) - \frac{8}{3}n_5(v) - \frac{5}{3}m_3(v) = ch(v) - \eta(v) + 1, & \text{when } d(f_1) \geq 7, \end{cases} \quad (3)$$

where $\eta(v) = 6n_3(v) + \frac{8}{3}n_5(v) + \frac{5}{3}m_3(v)$. Moreover, since f_0 is a face containing v , Equation (1) can be strengthened as:

$$\zeta(v) = 2n_3(v) + n_5(v) + m_3(v) \leq \begin{cases} d(v), & \text{when } d(f_1) = d(f_2) = 3; \\ d(v) - 1, & \text{when either } d(f_1) = 3 \text{ and } d(f_2) \geq 5 \text{ or } d(f_1) = d(f_2) = 5; \\ d(v) - 2, & \text{when } d(f_1) \geq 5 \text{ and } d(f_2) \geq 7. \end{cases} \quad (4)$$

Since $\eta(v) \leq 3\zeta(v)$, combining Equations (3) and (4) gives $ch^*(v) \geq ch(v) - 3d(v) + 7 = 2d(v) - 7 > 0$.

Case 2: it remains to assume that v is internal. By Lemma 3.1, $d(v) \geq 3$.

Case 2.1: suppose that $d(v) = 3$. In this case, $ch(v) = 1$ and $n_3(v) \leq 1$. Notice that only the rule $R1$ makes v send charge out. So, if v is triangular, $ch^*(v) = ch(v) - 1 = 0$; otherwise, $ch^*(v) = ch(v) - \frac{1}{3} \times 3 = 0$.

Case 2.2: suppose that $d(v) = 4$. In this case, $ch(v) = 6$. Notice that, if v is incident with no $(4, 4, 4)$ -faces, then exactly three rules $R2$, $R6$ and $R7$ make v send charge out, to incident 3-faces, pendent 3-faces and incident 5-faces, respectively; otherwise, an additional rule $R8$ is applied to v . Clearly, $n_3(v) \leq 2$. We distinguish three cases.

Case 2.2.1: assume that $n_3(v) = 0$. So, $m_3(v) + n_5(v) \leq 4$. If v has no pendent $(3, 3, 3)$ -faces, then v sends to each pendent 3-face or incident 5-face charge at most $\frac{3}{2}$, giving $ch^*(v) \geq ch(v) - \frac{3}{2}(m_3(v) + n_5(v)) \geq 0$. So, we may assume that v has a pendent $(3, 3, 3)$ -face. It follows that $n_5(v) \leq 2$. By Lemma 3.11, v has no other pendent $(3, 3, 3)$ - or $(3, 3, 4)$ -faces, which implies that v sends to any other pendent 3-face charge at most $\frac{5}{4}$. So, $ch^*(v) \geq ch(v) - \frac{5}{3} - \frac{3}{2} \times 2 - \frac{5}{4} = \frac{1}{12} > 0$.

Case 2.2.2: assume that $n_3(v) = 1$. In this case, either $n_5(v) = 1$ and $m_3(v) = 0$, or $n_5(v) = 0$ and $m_3(v) \leq 2$. For the former case, we have $ch^*(v) \geq ch(v) - \frac{7}{2} - \frac{3}{2} = 1 > 0$. For the latter case, we argue as follows. Denote by f the 3-face containing v . If f is a $(3, 4^-, 4)$ -face, then v has no pendent $(3, 3, 4^-)$ -faces by Lemma 3.12, giving $ch^*(v) \geq ch(v) - \frac{7}{2} - \frac{5}{4} \times 2 = 0$. So, let us assume f is not a $(3, 4^-, 4)$ -face. By $R2$, v sends to f charge at most $\frac{8}{3}$, and to abnormal vertices on f a total charge at most $\frac{1}{6} \times 2$ when $R8$ is applicable for v . Moreover, Combining Lemma 3.11 and the rule $R6$ yields that v sends to possible pendent 3-faces a total charge at most $\max\{\frac{5}{3} + \frac{5}{4}, \frac{3}{2} \times 2\}$, equal to 3. Therefore, $ch^*(v) \geq ch(v) - \frac{8}{3} - \frac{1}{6} \times 2 - 3 = 0$.

Case 2.2.3: assume that $n_3(v) = 2$. So, $m_3(v) = n_5(v) = 0$. Denote by f_1 and f_2 two 3-faces incident with v . If both f_1 and f_2 are not $(3, 4, 4)$ -faces, then no matter f_i has abnormal vertices or not, v sends to f_i and

possibly abnormal vertices on f_i a total charge at most 3, giving $ch^*(v) \geq ch(v) - 3 \times 2 = 0$. So, we may next assume that f_1 is a $(3, 4, 4)$ -face. By $R2$, v sends charge $\frac{8}{3}$ to f_1 . By Lemma 3.13, f_2 is not a $(3, 4^-, 4)$ -face. If f_2 is further not a $(4, 4, 4)$ -face, then v sends to f_2 charge at most $\frac{5}{2}$, giving $ch^*(v) \geq ch(v) - \frac{7}{2} - \frac{5}{2} = 0$. So, we may further assume that f_2 is a $(4, 4, 4)$ -face, that is, v is abnormal. If f_2 contains a normal vertex, then from it v receives charge $\frac{1}{6}$ by $R8$, giving $ch^*(v) \geq ch(v) - \frac{7}{2} - \frac{8}{3} + \frac{1}{6} = 0$. So, we may assume that all the vertices on f are abnormal. That is to say, f_2 together with three 3-faces intersecting with f_2 forms a wheel or an antiwheel, say W . Since G has no wheels by Lemma 3.17, W is an antiwheel. By Lemma 3.18, W has a heavy outer neighbor, that is, W has a strong $(3, 4, 4)$ -face. By the rule $R9$, v receives charge $\frac{1}{6}$ from this face, giving $ch^*(v) = ch(v) - \frac{7}{2} - \frac{8}{3} + \frac{1}{6} = 0$.

Case 2.3: suppose that $d(v) = 5$. In this case, $ch(v) = 11$ and $n_3(v) \leq 2$. Notice that only rules $R3$, $R6$ and $R7$ make v send charge out, to incident 3-faces, pendent 3-faces and incident 5-faces, respectively. We distinguish three cases.

Case 2.3.1: assume that $n_3(v) = 2$. So, $n_5(v) = 0$ and $m_3(v) \leq 1$. Denote by f_1 and f_2 the two 3-faces containing v and by f the pendent 3-face of v if it exists. If both f_1 and f_2 are not weak $(3, 3, 5)$ -faces, then v sends to each of them charge at most $\frac{9}{2}$, giving $ch^*(v) = ch(v) - \frac{9}{2} \times 2 - \frac{5}{3} = \frac{1}{3} > 0$. So, we may assume that v is incident with a weak $(3, 3, 5)$ -face, say f_1 . By Lemma 3.14, f_2 is neither a $(3, 3, 5)$ -face nor a $(3, 4, 5)$ -face. If f_2 is further not a weak $(3, 5, 5)$ -face, then v sends to f_2 charge $\frac{10}{3}$, giving $ch^*(v) = ch(v) - 6 - \frac{10}{3} - \frac{5}{3} = 0$. So, let f_2 be a weak $(3, 5, 5)$ -face. By Lemma 3.15, f is neither a $(3, 3, 3)$ -face nor a $(3, 3, 4)$ -face. So, v sends to f charge $\frac{5}{4}$, giving $ch^*(v) = ch(v) - 6 - \frac{7}{2} - \frac{5}{4} = \frac{1}{4} > 0$.

Case 2.3.2: assume that $n_3(v) = 1$. We can deduce that, $n_5(v) = 2$ and $m_3(v) = 0$, or $n_5(v) = 1$ and $m_3(v) \leq 1$, or $n_5(v) = 0$ and $m_3(v) \leq 3$. For the first case, Lemma 3.21 implies that not both 5-faces incident with v are small. So v sends to at least one of them charge $\frac{3}{2}$, giving $ch^*(v) \geq ch(v) - 6 - \frac{8}{3} - \frac{3}{2} = \frac{5}{6} > 0$. For the latter two cases, a direct calculation gives $ch^*(v) \geq ch(v) - 6 - \frac{8}{3} - \frac{5}{3} = \frac{2}{3} > 0$ and $ch^*(v) \geq ch(v) - 6 - \frac{5}{3} \times 3 = 0$, respectively.

Case 2.3.3: assume that $n_3(v) = 0$. Lemma 3.21 implies that v has at most two small 5-faces around. For any other incident 5-face or any pendent 3-face, v sends to it charge no greater than $\frac{5}{3}$, giving $ch^*(v) \geq ch(v) - \frac{8}{3} \times 2 - \frac{5}{3}(n_5(v) + m_3(v) - 2) \geq \frac{2}{3} > 0$, where Equation (1) has been used for the second inequality.

Case 2.4: suppose that $d(v) = 6$. In this case, $ch(v) = 16$ and only rules $R4$, $R6$ and $R7$ make v send charge out, to incident 3-faces, pendent 3-faces and incident 5-faces, respectively. If $n_5(v) = 6$, then $ch^*(v) \geq ch(v) - \frac{8}{3} \times 6 = 0$, we are done. Moreover, if $n_5(v) \in \{1, 2, \dots, 5\}$, then we have $ch^*(v) \geq ch(v) - 6n_3(v) - \frac{8}{3}n_5(v) - \frac{5}{3}m_3(v) \geq ch(v) - \frac{2}{3}n_3(v) - \frac{8}{3}(2n_3(v) + n_5(v) + m_3(v)) \geq 16 - \frac{2}{3}n_3(v) - \frac{8}{3}(d(v) - 1) = \frac{8}{3} - \frac{2}{3}n_3(v) > 0$, where the third inequality follows from Equation (2). Hence, we may next assume that $n_5(v) = 0$. Analogously, by using Equation (1) instead of Equation (2), we can deduce that $ch^*(v) \geq ch(v) - 6n_3(v) - \frac{5}{3}m_3(v) \geq ch(v) - \frac{8}{3}n_3(v) - \frac{5}{3}(2n_3(v) + m_3(v)) \geq 16 - \frac{8}{3}n_3(v) - \frac{5}{3}d(v) = 6 - \frac{8}{3}n_3(v) > 0$, provided by $n_3(v) \leq 2$. Hence, we may next assume that $n_3(v) = 3$. If v is incident with at most one weak $(3, 3, 6)$ -face, then $ch^*(v) \geq ch(v) - 6 - 5 \times 2 = 0$; otherwise, Lemma 3.16 implies that v is incident with a 3-face f that is neither $(3, 3, 6)$ -face nor $(3, 4, 6)$ -face. So, v sends to f charge 4, giving $ch^*(v) \geq ch(v) - 6 \times 2 - 4 = 0$.

Case 2.5: suppose that $d(v) \geq 7$. In this case, v sends to any incident 3-face charge 6 by $R5$, to any incident 5-face charge at most $\frac{8}{3}$ by $R7$, and to any pendent 3-face charge at most $\frac{5}{3}$ by $R6$. So, $ch^*(v) \geq ch(v) - 6n_3(v) - \frac{8}{3}n_5(v) - \frac{5}{3}m_3(v) \geq ch(v) - 3(2n_3(v) + n_5(v) + m_3(v)) \geq (5d(v) - 14) - 3d(v) \geq 0$, where the last two inequalities follow from Equation (1) and the assumption $d(v) \geq 7$, respectively. \square

Claim 3.21.3. $ch^*(f) \geq 0$ for $f \in F \setminus \{f_0\}$.

Proof. Since G has neither 4-cycles nor 6-cycles, $d(f) \notin \{4, 6\}$.

Case 1: assume that f contains vertices of D . Denote by $n_2(f)$ the number of 2-vertices on f . Lemma 3.8 implies that, if $d(f) \in \{3, 5, 7\}$ then the common part of f and D must be a path of length at most $\frac{d(f)-1}{2}$, say the path P . Here, a path of length 0 or 1 means a vertex or an edge, respectively. So, $n_2(f) \leq \frac{d(f)-1}{2} - 1$. We distinguish four cases.

Case 1.1: let $d(f) = 3$. In this case, $ch(f) = -8$ and P is either a vertex or an edge. Notice that f receives charge at least 1 from each incident internal vertex by rules from $R1$ to $R5$. If P is a vertex, then f is a sticking 3-face, which receives charge 6 from P by $R12$, giving $ch^*(f) = ch(f) + 6 + 1 \times 2 = 0$, we are done. If P is an edge, then f is a 1-ceiling 3-face, which receives charge $\frac{7}{2}$ from both vertices of P by $R12$, giving $ch^*(f) = ch(f) + \frac{7}{2} \times 2 + 1 = 0$, we are done as well.

Case 1.2: let $d(f) = 5$. By $R1$ and $R7$, f receives from each exterior vertex of f charge at least $\frac{1}{3}$. Clearly, $ch(f) = -4$ and P is a vertex or an edge or a 2-path. If P is a vertex, then f receives charge $\frac{8}{3}$ from this vertex by $R12$, giving $ch^*(f) = ch(f) + \frac{1}{3} \times 4 + \frac{8}{3} = 0$. If P is an edge, then f receives charge $\frac{3}{2}$ from both vertices of P by $R12$, giving $ch^*(f) = ch(f) + \frac{1}{3} \times 3 + \frac{3}{2} \times 2 = 0$. If P is a 2-path, then f receives charge $\frac{13}{6}$ from each end vertex of P by $R12$ and sends charge 1 to the unique 2-vertex of P , giving $ch^*(f) = ch(f) + \frac{1}{3} \times 2 + \frac{13}{6} \times 2 - 1 = 0$. We are done in all the three situations above.

Case 1.3: let $d(f) = 7$. In this case, f sends charge to incident 2-vertices by $R11$ and receives charge from incident exterior 3^+ -vertices by $R12$, no other charges moving about f . Recall that $ch(f) = 2d(f) - 14 = 0$ and $n_2(f) \leq \frac{d(f)-1}{2} - 1 = 2$. If $n_2(f) = 2$, i.e., f is a 3-ceiling face, then f receives charge 1 from each end vertex of P , giving $ch^*(f) = ch(f) + 1 \times 2 - 1 \times n_2(f) = 0$. If $n_2(f) = 1$, i.e., f is a 2-ceiling face, then f receives charge $\frac{1}{2}$ from each end vertex of P , giving $ch^*(f) = ch(f) + \frac{1}{2} \times 2 - 1 \times n_2(f) = 0$. If $n_2(f) = 0$, then f has no charges moving in or out, giving $ch^*(f) = ch(f) = 0$. We are done in all the three situations above.

Case 1.4: let $d(f) \geq 8$. Since f is not f_0 , f contains an internal vertex. That is to say, f contains a splitting 2^+ -path of D , say Q . By Lemma 3.8, if $|Q| \leq 4$, then Q divides D into two parts, one of which together with Q forms a face. Now Q contains internal 2-vertices, contradicting Lemma 3.1. So, $|Q| \geq 5$. It follows that $n_2(f) \leq d(f) - 6$. By our discharging rules, 8^+ -faces send charge only to incident 2-vertices, charge 1 to each by $R11$. So, $ch^*(f) = ch(f) - 1 \times n_2(f) \geq (2d(f) - 14) - (d(f) - 6) = d(f) - 8 \geq 0$.

Case 2: assume that f is vertex-disjoint with D . We distinguish three cases.

Case 2.1: let $d(f) \geq 7$. By our discharging rules, f has no charges moved in or out in this case. So, $ch^*(f) = ch(f) = 2d(f) - 14 \geq 0$.

Case 2.2: let $d(f) = 5$. In this case, $ch(f) = -4$. By our discharging rules, f sends no charges out and receives from each incident 4^+ -vertex charge at least $\frac{1}{3}$ by $R1$ or $R7$. By Lemma 3.19, f contains a 4^+ -vertex, say u . If u is the only 4^+ -vertex on f , i.e., f is small, then Lemma 3.20 implies that u is further a 5^+ -vertex, which sends to f charge $\frac{8}{3}$ by $R7$, giving $ch^*(f) \geq ch(f) + \frac{8}{3} + \frac{1}{3} \times 4 = 0$; otherwise, f has at least two 4^+ -vertices, from each f receives charge $\frac{3}{2}$, giving $ch^*(f) \geq ch(f) + \frac{3}{2} \times 2 + \frac{1}{3} \times 3 = 0$.

Case 2.3: let $d(f) = 3$. In this case, $ch(f) = -8$ and f receives charge from all the incident vertices and from all heavy outer neighbors, and sends charge out only when $R9$ applied. In particular, f receives charge 1 from each incident 3-vertex by $R1$.

If f is a $(3, 3, 3)$ -face, then Lemma 3.7 implies that f has three heavy outer neighbors, each sends charge $\frac{5}{3}$ to f by $R6$ or $R12$. So, $ch^*(f) = ch(f) + \frac{5}{3} \times 3 + 1 \times 3 = 0$.

If f is a $(3, 3, 4)$ -face, then f has precisely two heavy outer neighbors by Lemma 3.7, each sends charge at least $\frac{3}{2}$ to f by $R6$ or $R12$. Moreover, f receives charge 3 from the 4-vertex of f by $R2$. So, $ch^*(f) = ch(f) + \frac{3}{2} \times 2 + 3 + 1 \times 2 = 0$.

If f is a weak $(3, 3, 5)$ -face or a weak $(3, 3, 6)$ -face or a $(3, 3, 7^+)$ -face, then f receives charge 6 from the 5^+ -vertex of f by $R3$ or $R4$ or $R5$, respectively. So, $ch^*(f) = ch(f) + 6 + 1 \times 2 = 0$.

If f is a strong $(3, 3, 5)$ -face or a strong $(3, 3, 6)$ -face, then f receives charge at least $\frac{7}{2}$ from the 5^+ -vertex of f by $R3$ or $R4$, respectively. Moreover, f receives charge at least $\frac{5}{4}$ from both heavy outer neighbors of f by $R6$ or $R12$. So, $ch^*(f) \geq ch(f) + \frac{7}{2} + \frac{5}{4} \times 2 + 1 \times 2 = 0$.

If f is a weak $(3, 4, 4)$ -face or a weak $(3, 5, 5)$ -face, then f receives charge $\frac{7}{2}$ from both 4-vertices or 5-vertices of f by $R2$ or $R3$, respectively. So, $ch^*(f) = ch(f) + \frac{7}{2} \times 2 + 1 = 0$.

If f is a strong $(3, 4, 4)$ -face, then f might send charge out by $R9$. Notice that f is contained in at most two antiwheels, that is, f sends charge to at most six abnormal vertices, charge $\frac{1}{6}$ to each. Moreover, since f is strong, f has a heavy outer neighbor, from which f receives charge at least $\frac{5}{4}$ by $R6$ or $R12$. So, $ch^*(f) \geq ch(f) - \frac{1}{6} \times 6 + \frac{5}{4} + \frac{7}{2} \times 2 + 1 = \frac{1}{4} > 0$.

If f is a $(3, 4, 5^+)$ -face, then f receives charge $\frac{5}{2}$ from the 4-vertex of f by $R2$ and charge at least $\frac{9}{2}$ from the 5^+ -vertex of f by $R3$ or $R4$ or $R5$. So, $ch^*(f) \geq ch(f) + \frac{5}{2} + \frac{9}{2} + 1 = 0$.

If f is a strong $(3, 5, 5)$ -face, then f receives charge at least $\frac{5}{4}$ from the heavy outer neighbor by $R6$ or $R12$ and charge $\frac{7}{2}$ from both 5-vertices of f by $R3$. So, $ch^*(f) \geq ch(f) + \frac{5}{4} + \frac{7}{2} \times 2 + 1 = \frac{1}{4} > 0$.

If f is a $(3, 5^+, 6^+)$ -face, then f receives charge 3 and charge at least 4 from the 5^+ -vertex and the 6^+ -vertex on f , respectively. So, $ch^*(f) \geq ch(f) + 3 + 4 + 1 = 0$.

If f is a $(4, 4, 4)$ -face, then f receives charge $\frac{8}{3}$ from each incident vertex by $R2$, giving $ch^*(f) = ch(f) + \frac{8}{3} \times 3 = 0$.

If f is a $(4, 4^+, 5^+)$ -face, then f receives charge $\frac{5}{2}$, charge at least $\frac{5}{2}$ and charge at least 3 from the 4-vertex, the 4^+ -vertex and the 5^+ -vertex, respectively. So, $ch^*(f) \geq ch(f) + \frac{5}{2} + \frac{5}{2} + 3 = 0$. \square

By the previous three claims, the proof of Theorem 2.1 is completed.

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