The Alon-Tarsi number of K_5 -minor-free graphs

Toshiki Abe^a, Seog-Jin Kim^b, Kenta Ozeki^a

^aGraduate school of Environmental and Information Science, Yokohama National University

^bDepartment of Mathematics Education, Konkuk university

November 12, 2019

Abstract

In this paper, we show the following three theorems. Let G be a K_5 -minor-free graph. Then Alon-Tarsi number of G is at most 5, there exists a matching M of G such that the Alon-Tarsi number of G-M is at most 4, and there exists a forest F such that the Alon-Tarsi number of G - E(F) is at most 3.

Key words. planar graph, defective-coloring, list coloring; Combinatorial Nullstellensatz; Alon-Tarsi number

1 Introductions

In this paper, we only deal with finite and simple graphs. A *d*-defective coloring of G is a coloring $c: V(G) \to \mathbb{N}$ such that each color class induces a subgraph of maximum degree at most d. Especially, a 0-defective coloring is also called a *proper coloring* of G.

A k-list assignment of a graph G is a mapping L which assigns to each vertex v of G a set L(v) of k permissible colors. Given a k-list assignment L of G, a d-defective L-coloring of G is a d-defective coloring c such that $c(v) \in L(v)$ for every vertex v. We say that G is d-defective k-choosable if G has a d-defective L-coloring for every k-list assignment L. Especially, we say that G is k-choosable if G is 0-defective k-choosable. The choice number ch(G) is defined as the smallest integer k such that G is k-choosable. Let G be a graph and let '<' be an arbitrary fixed ordering of the vertices of G. The graph polynomial of G is defined as

$$P_G(\boldsymbol{x}) = \prod_{u \sim v, u < v} (x_u - x_v),$$

where $u \sim v$ means that u and v are adjacent, and $\boldsymbol{x} = (x_v)_{v \in V(G)}$ is a vector of |V(G)| variables indexed by the vertices of G. It is easy to see that a mapping $c : V(G) \to \mathbb{N}$ is a proper coloring of G if and only if $P_G(\boldsymbol{c}) \neq 0$, where $\boldsymbol{c} = (c(v))_{v \in V(G)}$. Therefore, to find a proper coloring of G is equivalent to find an assignment of \boldsymbol{x} so that $P_G(\boldsymbol{x}) \neq 0$. The following theorem, which was proved by Alon and Tarsi, gives sufficient conditions for the existence of such assignments as above.

Theorem 1.1 ([1]) (Combinatorial Nullstellensatz) Let \mathbb{F} be an arbitrary field and let $f = f(x_1, x_2, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, x_2, \ldots, x_n]$. Suppose that the degree deg(f) of f is $\sum_{i=1}^{n} t_i$ where each t_i is a nonnegative integer, and suppose that the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ of f is nonzero. Then if S_1, S_2, \ldots, S_n are subsets of \mathbb{F} with $|S_i| \ge t_i + 1$, then there are $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ so that $f(s_1, s_2, \ldots, s_n) \neq 0$.

In particular, a graph polynomial $P_G(\boldsymbol{x})$ is a homogeneous polynomial and deg (P_G) is equal to |E(G)|. Therefore, if there exists a monomial $c \prod_{v \in V(G)} x_v^{t_v}$ in the expansion of P_G so that $c \neq 0$ and $t_v < k$ for each $v \in V(G)$, then G is k-choosable. Jensen and Toft [7] defined the Alon-Tarsi number of a graph as follows.

Definition 1.2 The Alon-Tarsi number of a graph G, denoted by AT(G), is the minimum k for which there exists a monomial $c \prod_{v \in V(G)} x_v^{t_v}$ in the expansion of $P_G(\mathbf{x})$ such that $c \neq 0$ and $t_v < k$ for all $v \in V(G)$.

As explained above, $ch(G) \leq AT(G)$ for every graph G. Moreover, it is known that the gap between ch(G) and AT(G) can be arbitrary large. Nevertheless, it is also known that the upper bounds of ch(G) and AT(G)are the same for several graph classes. For example, Thomassen [11] proved that every planar graph is 5-choosable. Later, Zhu proved the following.

Theorem 1.3 ([15]) Let G be a plane graph. Then $AT(G) \leq 5$.

Moreover, it was shown in [3] that every planar graph is 1-defective 4choosable. Recently, Grytczuk and Zhu have proved the following theorem.

Theorem 1.4 ([5]) Let G be a plane graph. Then there exists a matching M of G such that $AT(G - M) \leq 4$.

This result implies that every planar graph is 1-defective 4-choosable. Furthermore, it was shown independently in [4] and [9] that every planar graph is 2-defective 3-choosable. In this context, it seems natural to ask whether there exists a subgraph H of G such that $AT(G - E(H)) \leq 3$ and $d_H(v) \leq 2$ for every $v \in V(G)$. Since if it was true, this implies that every planar graph is 2-defective 3-choosable. However, this is not true and it was shown in [8] that there exists a planar graph G such that for any subgraph of H of Gwith maximum degree at most 3, G - E(H) is not 3-choosable. On the other hand, the following was also proved in the same paper.

Theorem 1.5 ([8]) Let G be a plane graph. Then there exists a forest F in G such that $AT(G - E(F)) \leq 3$.

A graph H is a *minor* of a connected graph G if we obtain H from G by deleting or contracting some edges recursively. A graph G is H-minor-free if H is not a minor of G. If multiple edges appear by a contraction, we replace them with simple edge.

As another extension of Thomassen's result, it was shown in [6] and [10] that every K_5 -minor-free graph is 5-choosable. Moreover, it is also shown in [14] that every K_5 -minor-free graph is 1-defective 4-choosable. In this paper, we extend these results from choice number to Alon-Tarsi number.

Theorem 1.6 Let G be a K_5 -minor-free graph. Then all of the following hold.

(i) $AT(G) \leq 5$.

(ii) There exists a matching M of G such that $AT(G - M) \leq 4$.

Theorem 1.7 For every K_5 -minor-free graph G, there exists a forest F such that G - E(F) is 2-degenerate.

Thus we have the following corollary.

Corollary 1.8 For every K_5 -minor-free graph G, there exists a forest F such that $AT(G - E(F)) \leq 3$.

This paper is organized as follows. In Section 2, we prepare some lemmas in order to show the main theorems. And in Section 3, we prove Theorem 1.6 and Theorem 1.7. In Section 4, we have some remarks that Theorem 1.6 and Corollary 1.8 can be extended to singed graphs.

2 Orientations and Alon-Tarsi number

2.1 An alternative definition of the Alon-Tarsi number.

Indeed Alon-Tarsi number is already defined algebraically in Section 1, Alon and Tarsi [2] found a combinatorial interpretation of the coefficient for each monomial in the graph polynomials in terms of orientations and Eulerian subgraphs. For an orientation D of G, $d_D^+(v)$ (resp. $d_D^-(v)$) denotes outdegree (resp. in-degree) of a vertex v in D. The maximum out-degree of D is denoted by $\Delta^+(D)$. A subgraph H of D is called *Eulerian* if V(H) = V(G)and $d_H^-(v) = d_H^+(v)$ for every $v \in V(H)$ with respect to D. Note that Hmight not be connected. Let EE(D) (resp. OE(D)) denote the set of all Eulerian subgraphs of D with even (resp. odd) number of edges. Especially, we say that an orientation D is *acyclic* if D does not contain any directed cycles.

Theorem 2.1 ([2]) Let G be a graph, let P_G be the graph polynomial of G and let D be an orientation of G with out-degree sequence $\mathbf{d} = (d_v)_{v \in V(G)}$. Then the coefficient of $\prod_{v \in V(G)} x_v^{d_v}$ in the expansion of P_G is equal to $\pm (|EE(D)| - |OE(D)|)$

We say that orientation D of G is an AT-orientation if D satisfies $|EE(D)| - |OE(D)| \neq 0$.

2.2 Orientations of planar graphs

Now let us focus on planar graphs. We say a plane graph G is a *near triangulation* if each internal face in G is triangular. In the papers [5], [8] and [15], the following are shown respectively.

Lemma 2.2 Let G be a plane graph with simple boundary cycle $C = v_1 v_2 \dots v_m$. Then all of the following hold.

- (i) ([15]) G has an AT-orientation D such that $d_D^+(v_1) = 0$, $d_D^+(v_2) = 1$, $d_D^+(v_i) \le 2$ for each $i \in \{3, ..., m\}$ and $d_D^+(u) \le 4$ for each interior vertex u.
- (ii) ([5]) There exists a matching M and an AT-orientation D of G Msuch that $d_D^+(v_1) = d_D^+(v_2) = 0$, $d_D^+(v_i) \le 2 - d_M(v_i)$ for each $i \in \{3, ..., m\}$ and $d_D^+(u) \le 3$ for each interior vertex u.

(iii) ([8]) There exists a forest F in G and an acylic orientation D of G - E(F) such that $d_D^+(v_1) = d_D^+(v_2) = 0$, $d_D^+(v_i) = 1$ for each $i \in \{3, ..., m\}$ and $d_D^+(u) \leq 2$ for each interior vertex u.

In order to show the main theorem, we need an orientation which has some stronger properties.

Lemma 2.3 Let G be a plane graph with a boundary cycle $v_1v_2v_3$. Then all of the following hold.

- (i) There exists a matching M of G and an AT-orientation D of G Msuch that M does not cover v_3 , $d_D^+(v_1) = d_D^+(v_2) = 0$, $d_D^+(v_3) = 2$ and $d_D^+(y) \leq 3$ for $y \in V(G) - \{v_1, v_2, v_3\}$.
- (ii) There exists a forest F of G and an acyclic orientation D of G E(F)such that $v_1v_3 \notin E(F)$, $d_D^+(v_1) = d_D^+(v_2) = 0$, $d_D^+(v_3) = 1$ and $d_D^+(y) \le 2$ for $y \in V(G) - \{v_1, v_2, v_3\}$.

Proof. Let $G' = G - v_3$ and let $N(v_3) = \{v_1, u_1, ..., u_k, v_2\}$ be the neighborhood of v_3 as this rotation. Since G' is a plane graph, we have a matching M and an AT-orientation D' of G' - M such that $d_{D'}^+(v_1) = d_{D'}^+(v_2) = 0$, $d_{D'}^+(u_i) \leq 2$ for $i \in \{1, 2, ..., k\}$ and $d_{D'}^+(u) \leq 3$ for each interior vertex u by Lemma 2.2 (See Figure 1). Let D be the orientation of G - M obtained from D' by adding the vertex v_3 and k+2 oriented edges (u_i, v_3) for $i \in \{1, 2, ..., k\}$, (v_3, v_1) and (v_3, v_2) .

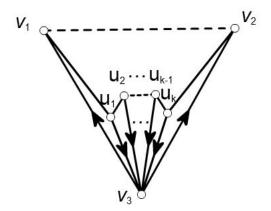


Figure 1: The orientation D of G - M

It is easy to see that D also satisfies the out-degree conditions and that M does not cover v_3 . Moreover, since the vertices v_1 and v_2 have out-degree 0, D is also an AT-orientation. \Box

Let G and H be a graph which contain a clique of the same size. The *clique-sum* of G and H is a operation that forms a new graph obtained from their disjoint union by identifying a clique of G and one of H with the same size and possibly deleting some edges in the clique. A *k*-clique-sum is a clique-sum in which both cliques have at most *k*-vertices.

Lemma 2.4 Let G be a graph which can be obtained by the 3-clique-sum of G_1 and G_2 and let $T = \{x_1, x_2, x_3\}$ be its clique. Moreover, let $G'_i = G \cap G_i$. Suppose that G'_1 has an AT-orientation D'_1 with $\Delta(D'_1) \leq k$ and that G_2 has an AT-orientation D_2 such that $d^+_{D_2}(x_1) = 0$, $d^+_{D_2}(x_2) \leq 1$, $d^+_{D_2}(x_3) \leq 2$ and $\Delta(D_2) \leq k$ and x_i is directed only to $x_{i'}$, where $x_i, x_{i'} \in V(T)$. Then G has an AT-orientation D such that $d^+_D(v) = d^+_{D'_1}(v)$ for each $v \in V(G_1)$ and maximum out-degree of D is at most k.

Proof. Let $D'_2 \,\subset D_2$ be the orientation of G'_2 and let $D = D'_1 \cup (D'_2 - E(T))$. Then it is easy to see that $d^+_D(v) = d^+_{D'_1}(v)$ for each $v \in V(G'_1)$, $d^+_D(v) = d^+_{D'_2}(v)$ for each $v \in V(G'_2) - V(T)$ and hence maximum out-degree of D is at most k. For the orientation D_2 , the vertices in T has a direction only to other vertices of T and no Eulerian subgraphs in D_2 contain the edge in T by the out-degree conditions of D_2 . Thus D'_2 is also an AT-orientation of G'_2 and any spanning Eulerian sub-digraphs H of D has an edge-disjoint decomposition $H = H_1 \cup H_2$ where H_1 and H_2 are Eulerian sub-digraphs in D'_1 and D'_2 , respectively. Therefore, we have the bijection τ so that

- $\tau(EE(D)) = (EE(D'_1) \times EE(D'_2)) \cup (OE(D'_1) \times OE(D'_2))$ and
- $\tau(OE(D)) = (OE(D'_1) \times EE(D'_2)) \cup (EE(D'_1) \times OE(D'_2)).$

Hence

$$|EE(D)| - |OE(D)|$$

$$= (|EE(D'_1)| \times |EE(D'_2)| + |OE(D'_1)| \times |OE(D'_2)|)$$

$$-(|EE(D'_1)| \times |OE(D'_2)| + |OE(D'_1)| \times |EE(D'_2)|)$$

$$= (|EE(D'_1)| - |OE(D'_1)|) \cdot (|EE(D'_2)| - |OE(D'_2)|)$$

$$\neq 0.$$

These imply that the orientation D is an AT-orientation of G with desired properties. \Box

2.3 Characterizations of K_5 -minor-free graphs.

Now, let us focus on K_5 -minor-free graphs. In order to show the main theorem, we use the following results.

Lemma 2.5 ([12]) A graph G is K_5 -minor-free if and only if G can be formed from some 3-clique-sums of graphs, each of which is either planar or the Wagner graph W as shown in Figure 2.

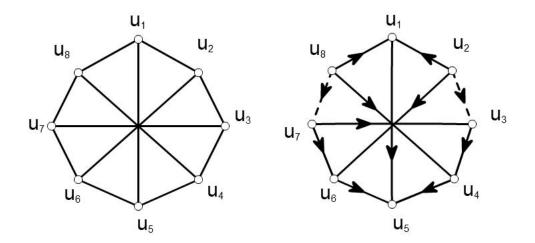


Figure 2: The left is the Wagner graph W and the right is an acyclic orientation with maximum out-degree 3. Doted edges denote elements of a matching or forest. If we delete the two doted edges, the orientation has maximum out-degree 2.

3 Proof of main Theorem.

Theorem 1.6 and Theorem 1.7 follow from the lemma below.

Lemma 3.1 Let G be a K_5 -minor-free graph and let H_i be a subgraph of G which is isomorphic to $uv \in E(G)$ or $\{uv, vw, wu\} \subset E(G)$ for each $i \in \{1, 2, 3\}$. Then all of the following hold.

(i) There exists an AT-orientation D such that $d_D^+(u) = 0$, $d_D^+(v) = 1$, $(d_D^+(w) = 2 \text{ if } H_1 \text{ is isomorphic to } K_3) \text{ and } d_D^+(y) \leq 4 \text{ for } y \in V(G) - \{u, v, w\}.$

- (ii) There exists a matching M of G and an AT-orientation D of G Msuch that $d_D^+(u) = d_D^+(v) = 0$, $(d_D^+(w) = 2$ and M does not cover w if H_2 is isomorphic to K_3) and $d_D^+(y) \leq 3$ for $y \in V(G) - \{u, v, w\}$.
- (iii) There exists a forest F of G and an acyclic orientation D of G E(F)such that $uw \notin E(F)$, $d_D^+(u) = d_D^+(v) = 0$, $(d_D^+(w) = 1$ if H_3 is isomorphic to K_3) and $d_D^+(y) \leq 2$ for $y \in V(G) - \{u, v, w\}$.

Proof. Suppose that the Lemma is false and let G_i be a counterexample for each $i \in \{1, 2, 3\}$ respectively with $|V(G_i)|$ as small as possible. By the minimality of G_i , G_i is connected. Moreover, it is easy to check that G_i does not have a cut vertex. Thus we may assume G_i is 2-connected.

First we suppose that G_i is a plane graph. If H_i is isomorphic to K_2 or K_3 which bounds a face, without loss of generality, H_i lies on the boundary of G. In this case, a desired AT-orientation exists by Lemma 2.2. Thus H_i consists a separating 3-cycle in G_i . We let $G_{i,1}$ and $G_{i,2}$ be subgraphs of G_i so that $G_{i,1} \cup G_{i,2} = G_i$ and $G_{i,1} \cap G_{i,2} = H_i$. By Lemma 2.2 and Lemma 2.3, for $j \in \{1,2\}$ we have an AT-orientation $D_{1,j}$ of $G_{1,j}$ which satisfies the conditions. Similarly, we have that there exists a matching M_j and an AT-orientation of $G_{2,j} - M_j$ and that there exists a forest F_j and acyclic orientation of $G_{3,j} - E(F_j)$, which satisfy the conditions. It is easy to see that $M = M_1 \cup M_2$ is also a matching of G_2 and $F = F_1 \cup F_2$ is a forest of G_3 . Therefore, we get a desired AT-orientation respectively by Lemma 2.4. Thus G_i is not planar.

Next, suppose that G_i is the Wagner graph W. Since W does not contain a triangle, H_i must be isomorphic to K_2 . By the symmetry of W, we may assume that $H_i = u_1u_5$ or u_5u_6 . We let M be a matching of W and let F be a forest of W such that $M = E(F) = E(H) \cup \{u_2u_3, u_7u_8\}$. In this case, the orientation in Figure 2 is a desired AT-orientation respectively.

Thus we assume that G_i is neither planar graph nor the graph W. By Lemma 2.5, there exists K_5 -minor-free graphs $G_{i,1}$ and $G_{i,2}$ such that G_i can be obtained by a 3-clique-sum of $G_{i,1}$ and $G_{i,2}$. Let T be its clique and let $G'_{i,j} = G_{i,j} \cap G_i$ for $j \in \{1, 2\}$. It is easy to see that $H_i \subset G'_{i,1}$ or $G'_{i,2}$. Without loss of generality, we may assume that $H_i \subset G_{i,1}$. By the minimality of G_i , we have the following.

- (i) There exists an AT-orientation $D_{1,1}$ of $G'_{1,1}$ which satisfies the assumption (i) of Lemma 3.1.
- (ii) There exists a matching M_1 and an AT-orientation $D_{2,1}$ of $G'_{2,1} M$ which satisfies the assumption (ii) of Lemma 3.1.

(iii) There exists a forest F_1 and an acyclic orientation $D_{3,1}$ of $G'_{3,1} - E(F)$ which satisfies the assumption (iii) of Lemma 3.1.

First, we consider the case when i = 1. By the minimality of G_1 , we get an AT-orientation $D_{1,2}$ of $G_{1,2}$ with $d_{D_{1,2}}^+(x_1) = 0$, $d_{D_{1,2}}^+(x_2) = 1$, $d_{D_{1,2}}^+(x_3) = 2$ and the maximum degree of $D_{1,2}$ is at most 4. By Lemma 2.4, we get a desired AT-orientation D in G_1 .

Next, we consider the case when i = 2. By the minimality of G_2 , we get a matching M_2 of $G_{2,2}$ and an AT-orientation $D_{2,2}$ of $G_{2,2} - M_2$ such that $d_{D_{2,2}}^+(x_1) = 0$, $d_{D_{2,2}}^+(x_2) = 0$, $d_{D_{2,2}}^+(x_3) = 2$, maximum out-degree of $D_{2,2}$ is at most 3 and M_2 does not cover x_3 . Let $M = M_1 \cup (M_2 - \{x_1x_2\})$. It is easy to see that M is a matching of G. By Lemma 2.4, we get a desired AT-orientation D in $G_2 - M$.

Finally, we consider the case when i = 3. By the minimality of G_3 , we get a forest F_2 of $G_{3,2}$ and an acyclic orientation $D_{3,2}$ of $G_{3,2} - E(F_2)$ with $d_{D_{3,2}}^+(x_1) = d_{D_{3,2}}^+(x_2) = 0$, $d_{D_{3,2}}^+(x_3) = 1$ and maximum out-degree of $D_{3,2}$ is at most 2. Let $F = F_1 \cup (F_2 - E(T))$. Similarly, we can show that F is a forest and D_3 is an acyclic orientation of $G_3 - E(F)$ with desired properties. This is a contradiction and we completes the proof. \Box

[Proof of Theorem 1.6 and Theorem 1.7]

Theorem 1.6 follows immediately from (i) and (ii) in Lemma 3.1. For Theorem 1.7, each K_5 -minor-free graph G has a forest F and and an acyclic orientation D of G - E(F) with maximum out-degree at most 2 by Lemma 3.1. Since G - E(F) is finite and D is acyclic, there exists a vertex v with $d_D^-(v) = 0$ and hence the vertex v has degree at most 2 in G - E(F). \Box

4 Some remarks

A signed graph is a pair (G, σ) , where G is a graph and σ is a signature of G which assigns to each edge e = uv of G a sign $\sigma_{uv} \in \{1, -1\}$. Let

$$N_k = \begin{cases} \{0, \pm 1, \dots, \pm q\} & \text{if } k = 2q + 1 \text{ is an odd integer,} \\ \{\pm 1, \dots, \pm q\} & \text{if } k = 2q \text{ is an even integer.} \end{cases}$$

Note that $|N_k| = k$ for each integer k. A proper coloring of (G, σ) is a mapping $c : V(G) \to N_k$ such that $c(x) \neq \sigma_{xy}c(y)$ for each edge xy. The chromatic number $\chi(G, \sigma)$ of (G, σ) is minimum integer t such that there exists a proper coloring $c : V(G) \to N_t$. The choice number $ch(G, \sigma)$ of (G, σ) is minimum integer k such that for every k-list assignment L, there exists a proper coloring c of (G, σ) so that $c(v) \in L(v)$ for every $v \in V(G)$.

Let (G, σ) be a signed graph and let '<' be an arbitrary fixed ordering of the vertices of (G, σ) . The singed graph polynomial of (G, σ) is defined as

$$P_{G,\sigma}(\boldsymbol{x}) = \prod_{u \sim v, u < v} (x_u - \sigma_{uv} x_v),$$

where $u \sim v$ means that u and v are adjacent, and $\boldsymbol{x} = (x_v)_{v \in V(G)}$ is a vector of |V(G)| variables indexed by the vertices of G. It is easy to see that a mapping $c : V(G) \to \mathbb{Z}$ is a proper coloring of (G, σ) if and only if $P_{G,\sigma}(\boldsymbol{c}) \neq 0$, where $\boldsymbol{c} = (c(v))_{v \in V(G)}$. The Alon-Tarsi number of (G, σ) is defined similarly and we have $\chi(G, \sigma) \leq ch(G, \sigma) \leq AT(G, \sigma)$.

Let (G, σ) be a singled graph and let D be an orientation of (G, σ) . Let $\sigma EE(D)$ (resp. $\sigma OE(D)$) denote the set of all spanning Eulerian subdigraphs of D with even (resp. odd) number of positive edges on σ . It was shown in [13] that Theorem 2.1 can be extended to signed one as follows.

Theorem 4.1 ([13]) Let (G, σ) be signed graph, let $P_{G,\sigma}$ be the signed graph polynomial of (G, σ) and let D be an orientation of (G, σ) with out-degree sequence $\mathbf{d} = (d_v)_{v \in V(G)}$. Then the coefficient of $\prod_{v \in V(G)} x_v^{d_v}$ in the expansion of $P_{G,\sigma}$ is equal to $\pm (|\sigma EE(D)| - |\sigma OE(D)|)$.

We say that an orientation D of (G, σ) is a σ AT-orientation if D satisfies $|\sigma EE(D)| - |\sigma OE(D)| \neq 0.$

Let us focus on planar graphs. In the papers [5] and [13], the following are shown respectively.

Lemma 4.2 Let (G, σ) be a signed near triangulation and let $C = v_1 v_2 ... v_m$ be the boundary cycle of G. Then all of the following hold.

- (i) ([13]) G has a σAT -orientation D such that $d_D^+(v_1) = 0$, $d_D^+(v_2) = 1$, $d_D^+(v_i) \leq 2$ for each $i \in \{3, ..., m\}$ and $d_D^+(u) \leq 4$ for each interior vertex u.
- (ii) ([5]) There exists a matching M and a σAT -orientation D of G Msuch that $d_D^+(v_1) = d_D^+(v_2) = 0$, $d_D^+(v_i) \leq 2 - d_M(v_i)$ for each $i \in \{3, ..., m\}$ and $d_D^+(u) \leq 3$ for each interior vertex u.

Although Lemma 4.2 only deals with near triangulations in the paper [13], it is not hard to extend the graph class from near triangulations to planar graphs. Moreover, since all the arguments of Lemmas in Section 2 and Lemma 3.1 work even if we replace AT-orientations into σ AT-orientations, we have the following results.

Theorem 4.3 Let (G, σ) be a signed K_5 -minor-free graph. Then all of the following hold.

- (i) $AT(G, \sigma) \leq 5$.
- (ii) There exists a matching M of G such that $AT(G M, \sigma) \leq 4$.
- (iii) There exists a forest F in G such that $AT(G E(F), \sigma) \leq 3$.

References

- N. Alon, Combinatorial Nullstellensatz, Combin. Probab. Comput. 8 (1999) 7–29.
- [2] N. Alon, M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (1992) 125–134.
- [3] W. Cushing, H.A. Kirestread, planar graphs are 1-relaxed, 4-choosable, European Journal of Combinatorics 31(2010) 1385–1397.
- [4] N. Eaton, T.Hull, Defective List Colorings of Planar Graphs, Bulletin of the Institute of Combinatorics and its applications, 25(1999) 79–87.
- [5] J. Grytczuk, X.Zhu, The Alon-Tarsi number of a planar graph minus a matching, arXiv1811.12012
- [6] W. He, W. Miao and Y. Shen, Another proof of the 5-choosability of K_5 -minor-free graphs, Discrete Math. 308 (2008) 4024–4026
- [7] T. Jensen, B.Toft, Graph Coloring Problems, Wiley, New York, 1995.
- [8] R. Kim, S. Kim and X. Zhu, The Alon-Tarsi number of subgraphs of a planar graph, arXiv:1906.01506
- [9] R. Skrekovski, List improper coloring s of planar graphs, Combinatorics, Probability and Computing, 8 (1994) 293–299
- [10] R. Skrekovski, Choosability of K_5 -minor-free graphs, Discrete Math. 190 (1998) 223–226.
- [11] C. Thomassen, Every planar graph is 5-choosable, J. Combin. Theory, Ser. B 62 (1994) 180–181.
- [12] K. Wagner. Uber eine Eigenschaft der ebenen Komplexe. Math. Ann, 114 (1937), 570–590

- [13] W. Wang, J. Qian and T. Abe, Alon-Tarsi Number and Modulo Alon-Tarsi Number of Signed Graphs, to appear in Graphs Combin.
- [14] D.R. Woodall, Defective choosability of graphs in surfaces, Discussiones Mathematicae Graph Theory 31 (2011) 441–459
- [15] X. Zhu, The Alon-Tarsi number of planar graphs, J. Combin. Theory, Ser. B.134 (2019) 354–358.