# The Alon-Tarsi number of $K_{5}$-minor-free graphs 

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#### Abstract

In this paper, we show the following three theorems. Let $G$ be a $K_{5}$-minor-free graph. Then Alon-Tarsi number of $G$ is at most 5, there exists a matching $M$ of $G$ such that the Alon-Tarsi number of $G-M$ is at most 4 , and there exists a forest $F$ such that the Alon-Tarsi number of $G-E(F)$ is at most 3 .


Key words. planar graph, defective-coloring, list coloring; Combinatorial Nullstellensatz; Alon-Tarsi number

## 1 Introductions

In this paper, we only deal with finite and simple graphs. A d-defective coloring of $G$ is a coloring $c: V(G) \rightarrow \mathbb{N}$ such that each color class induces a subgraph of maximum degree at most $d$. Especially, a 0-defective coloring is also called a proper coloring of $G$.

A $k$-list assignment of a graph $G$ is a mapping $L$ which assigns to each vertex $v$ of $G$ a set $L(v)$ of $k$ permissible colors. Given a $k$-list assignment $L$ of $G$, a $d$-defective $L$-coloring of $G$ is a $d$-defective coloring $c$ such that $c(v) \in L(v)$ for every vertex $v$. We say that $G$ is $d$-defective $k$-choosable if $G$ has a $d$-defective $L$-coloring for every $k$-list assignment $L$. Especially, we say that $G$ is $k$-choosable if $G$ is 0 -defective $k$-choosable. The choice number $\operatorname{ch}(G)$ is defined as the smallest integer $k$ such that $G$ is $k$-choosable.

Let $G$ be a graph and let ' $<$ ' be an arbitrary fixed ordering of the vertices of $G$. The graph polynomial of $G$ is defined as

$$
P_{G}(\boldsymbol{x})=\prod_{u \sim v, u<v}\left(x_{u}-x_{v}\right),
$$

where $u \sim v$ means that $u$ and $v$ are adjacent, and $\boldsymbol{x}=\left(x_{v}\right)_{v \in V(G)}$ is a vector of $|V(G)|$ variables indexed by the vertices of $G$. It is easy to see that a mapping $c: V(G) \rightarrow \mathbb{N}$ is a proper coloring of $G$ if and only if $P_{G}(\boldsymbol{c}) \neq 0$, where $\boldsymbol{c}=(c(v))_{v \in V(G)}$. Therefore, to find a proper coloring of $G$ is equivalent to find an assignment of $\boldsymbol{x}$ so that $P_{G}(\boldsymbol{x}) \neq 0$. The following theorem, which was proved by Alon and Tarsi, gives sufficient conditions for the existence of such assignments as above.

Theorem 1.1 ([1]) (Combinatorial Nullstellensatz) Let $\mathbb{F}$ be an arbitrary field and let $f=f\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be a polynomial in $\mathbb{F}\left[x_{1}, x_{2}, \ldots, x_{n}\right]$. Suppose that the degree $\operatorname{deg}(f)$ of $f$ is $\sum_{i=1}^{n} t_{i}$ where each $t_{i}$ is a nonnegative integer, and suppose that the coefficient of $\prod_{i=1}^{n} x_{i}^{t_{i}}$ of $f$ is nonzero. Then if $S_{1}, S_{2}, \ldots, S_{n}$ are subsets of $\mathbb{F}$ with $\left|S_{i}\right| \geq t_{i}+1$, then there are $s_{1} \in S_{1}, s_{2} \in S_{2}, \ldots, s_{n} \in S_{n}$ so that $f\left(s_{1}, s_{2}, \ldots, s_{n}\right) \neq 0$.

In particular, a graph polynomial $P_{G}(\boldsymbol{x})$ is a homogeneous polynomial and $\operatorname{deg}\left(P_{G}\right)$ is equal to $|E(G)|$. Therefore, if there exists a monomial $c \prod_{v \in V(G)} x_{v}{ }^{t_{v}}$ in the expansion of $P_{G}$ so that $c \neq 0$ and $t_{v}<k$ for each $v \in V(G)$, then $G$ is $k$-choosable. Jensen and Toft [7] defined the Alon-Tarsi number of a graph as follows.

Definition 1.2 The Alon-Tarsi number of a graph $G$, denoted by $A T(G)$, is the minimum $k$ for which there exists a monomial $c \prod_{v \in V(G)} x_{v}^{t_{v}}$ in the expansion of $P_{G}(\boldsymbol{x})$ such that $c \neq 0$ and $t_{v}<k$ for all $v \in V(G)$.

As explained above, $\operatorname{ch}(G) \leq A T(G)$ for every graph $G$. Moreover, it is known that the gap between $\operatorname{ch}(G)$ and $A T(G)$ can be arbitrary large. Nevertheless, it is also known that the upper bounds of $\operatorname{ch}(G)$ and $A T(G)$ are the same for several graph classes. For example, Thomassen [11 proved that every planar graph is 5 -choosable. Later, Zhu proved the following.

Theorem 1.3 ([15]) Let $G$ be a plane graph. Then $A T(G) \leq 5$.
Moreover, it was shown in [3] that every planar graph is 1-defective 4choosable. Recently, Grytczuk and Zhu have proved the following theorem.

Theorem 1.4 ([5]) Let $G$ be a plane graph. Then there exists a matching $M$ of $G$ such that $A T(G-M) \leq 4$.

This result implies that every planar graph is 1-defective 4-choosable. Furthermore, it was shown independently in [4] and [9] that every planar graph is 2-defective 3-choosable. In this context, it seems natural to ask whether there exists a subgraph $H$ of $G$ such that $A T(G-E(H)) \leq 3$ and $d_{H}(v) \leq 2$ for every $v \in V(G)$. Since if it was true, this implies that every planar graph is 2 -defective 3 -choosable. However, this is not true and it was shown in [8] that there exists a planar graph $G$ such that for any subgraph of $H$ of $G$ with maximum degree at most $3, G-E(H)$ is not 3 -choosable. On the other hand, the following was also proved in the same paper.

Theorem 1.5 ([8]) Let $G$ be a plane graph. Then there exists a forest $F$ in $G$ such that $A T(G-E(F)) \leq 3$.

A graph $H$ is a minor of a connected graph $G$ if we obtain $H$ from $G$ by deleting or contracting some edges recursively. A graph $G$ is $H$-minor-free if $H$ is not a minor of $G$. If multiple edges appear by a contraction, we replace them with simple edge.

As another extension of Thomassen's result, it was shown in [6] and [10] that every $K_{5}$-minor-free graph is 5 -choosable. Moreover, it is also shown in [14] that every $K_{5}$-minor-free graph is 1 -defective 4 -choosable. In this paper, we extend these results from choice number to Alon-Tarsi number.

Theorem 1.6 Let $G$ be a $K_{5}$-minor-free graph. Then all of the following hold.
(i) $A T(G) \leq 5$.
(ii) There exists a matching $M$ of $G$ such that $A T(G-M) \leq 4$.

Theorem 1.7 For every $K_{5}$-minor-free graph $G$, there exists a forest $F$ such that $G-E(F)$ is 2-degenerate.

Thus we have the following corollary.
Corollary 1.8 For every $K_{5}$-minor-free graph $G$, there exists a forest $F$ such that $A T(G-E(F)) \leq 3$.

This paper is organized as follows. In Section 2, we prepare some lemmas in order to show the main theorems. And in Section 3, we prove Theorem 1.6 and Theorem 1.7. In Section 4, we have some remarks that Theorem 1.6 and Corollary 1.8 can be extended to singed graphs.

## 2 Orientations and Alon-Tarsi number

### 2.1 An alternative definition of the Alon-Tarsi number.

Indeed Alon-Tarsi number is already defined algebraically in Section 1, Alon and Tarsi [2] found a combinatorial interpretation of the coefficient for each monomial in the graph polynomials in terms of orientations and Eulerian subgraphs. For an orientation $D$ of $G, d_{D}^{+}(v)$ (resp. $d_{D}^{-}(v)$ ) denotes outdegree (resp. in-degree) of a vertex $v$ in $D$. The maximum out-degree of $D$ is denoted by $\Delta^{+}(D)$. A subgraph $H$ of $D$ is called Eulerian if $V(H)=V(G)$ and $d_{H}^{-}(v)=d_{H}^{+}(v)$ for every $v \in V(H)$ with respect to $D$. Note that $H$ might not be connected. Let $E E(D)$ (resp. $O E(D)$ ) denote the set of all Eulerian subgraphs of $D$ with even (resp. odd) number of edges. Especially, we say that an orientation $D$ is acyclic if $D$ does not contain any directed cycles.

Theorem 2.1 ([2]) Let $G$ be a graph, let $P_{G}$ be the graph polynomial of $G$ and let $D$ be an orientation of $G$ with out-degree sequence $\boldsymbol{d}=\left(d_{v}\right)_{v \in V(G)}$. Then the coefficient of $\prod_{v \in V(G)} x_{v}^{d_{v}}$ in the expansion of $P_{G}$ is equal to $\pm(|E E(D)|-$ $|O E(D)|)$

We say that orientation $D$ of $G$ is an $A T$-orientation if $D$ satisfies $|E E(D)|-$ $|O E(D)| \neq 0$.

### 2.2 Orientations of planar graphs

Now let us focus on planar graphs. We say a plane graph $G$ is a near triangulation if each internal face in $G$ is triangular. In the papers [5], 8] and [15], the following are shown respectively.

Lemma 2.2 Let $G$ be a plane graph with simple boundary cycle $C=v_{1} v_{2} \ldots v_{m}$. Then all of the following hold.
(i) ([15]) $G$ has an AT-orientation $D$ such that $d_{D}^{+}\left(v_{1}\right)=0, d_{D}^{+}\left(v_{2}\right)=1$, $d_{D}^{+}\left(v_{i}\right) \leq 2$ for each $i \in\{3, \ldots, m\}$ and $d_{D}^{+}(u) \leq 4$ for each interior vertex u.
(ii) ([5]) There exists a matching $M$ and an AT-orientation $D$ of $G-M$ such that $d_{D}^{+}\left(v_{1}\right)=d_{D}^{+}\left(v_{2}\right)=0, d_{D}^{+}\left(v_{i}\right) \leq 2-d_{M}\left(v_{i}\right)$ for each $i \in$ $\{3, \ldots, m\}$ and $d_{D}^{+}(u) \leq 3$ for each interior vertex $u$.
(iii) ([8]) There exists a forest $F$ in $G$ and an acylic orientation $D$ of $G$ $E(F)$ such that $d_{D}^{+}\left(v_{1}\right)=d_{D}^{+}\left(v_{2}\right)=0, d_{D}^{+}\left(v_{i}\right)=1$ for each $i \in\{3, \ldots, m\}$ and $d_{D}^{+}(u) \leq 2$ for each interior vertex $u$.
In order to show the main theorem, we need an orientation which has some stronger properties.

Lemma 2.3 Let $G$ be a plane graph with a boundary cycle $v_{1} v_{2} v_{3}$. Then all of the following hold.
(i) There exists a matching $M$ of $G$ and an AT-orientation $D$ of $G-M$ such that $M$ does not cover $v_{3}, d_{D}^{+}\left(v_{1}\right)=d_{D}^{+}\left(v_{2}\right)=0, d_{D}^{+}\left(v_{3}\right)=2$ and $d_{D}^{+}(y) \leq 3$ for $y \in V(G)-\left\{v_{1}, v_{2}, v_{3}\right\}$.
(ii) There exists a forest $F$ of $G$ and an acyclic orientation $D$ of $G-E(F)$ such that $v_{1} v_{3} \notin E(F), d_{D}^{+}\left(v_{1}\right)=d_{D}^{+}\left(v_{2}\right)=0, d_{D}^{+}\left(v_{3}\right)=1$ and $d_{D}^{+}(y) \leq 2$ for $y \in V(G)-\left\{v_{1}, v_{2}, v_{3}\right\}$.

Proof. Let $G^{\prime}=G-v_{3}$ and let $N\left(v_{3}\right)=\left\{v_{1}, u_{1}, \ldots, u_{k}, v_{2}\right\}$ be the neighborhood of $v_{3}$ as this rotation. Since $G^{\prime}$ is a plane graph, we have a matching $M$ and an AT-orientation $D^{\prime}$ of $G^{\prime}-M$ such that $d_{D^{\prime}}^{+}\left(v_{1}\right)=d_{D^{\prime}}^{+}\left(v_{2}\right)=0$, $d_{D^{\prime}}^{+}\left(u_{i}\right) \leq 2$ for $i \in\{1,2, \ldots, k\}$ and $d_{D^{\prime}}^{+}(u) \leq 3$ for each interior vertex $u$ by Lemmar 2.2 (See Figure 1). Let $D$ be the orientation of $G-M$ obtained from $D^{\prime}$ by adding the vertex $v_{3}$ and $k+2$ oriented edges $\left(u_{i}, v_{3}\right)$ for $i \in\{1,2, \ldots, k\}$, $\left(v_{3}, v_{1}\right)$ and $\left(v_{3}, v_{2}\right)$.


Figure 1: The orientation $D$ of $G-M$

It is easy to see that $D$ also satisfies the out-degree conditions and that $M$ does not cover $v_{3}$. Moreover, since the vertices $v_{1}$ and $v_{2}$ have out-degree $0, D$ is also an AT-orientation.

Let $G$ and $H$ be a graph which contain a clique of the same size. The clique-sum of $G$ and $H$ is a operation that forms a new graph obtained from their disjoint union by identifying a clique of $G$ and one of $H$ with the same size and possibly deleting some edges in the clique. A $k$-clique-sum is a clique-sum in which both cliques have at most $k$-vertices.

Lemma 2.4 Let $G$ be a graph which can be obtained by the 3 -clique-sum of $G_{1}$ and $G_{2}$ and let $T=\left\{x_{1}, x_{2}, x_{3}\right\}$ be its clique. Moreover, let $G_{i}^{\prime}=G \cap G_{i}$. Suppose that $G_{1}^{\prime}$ has an AT-orientation $D_{1}^{\prime}$ with $\Delta\left(D_{1}^{\prime}\right) \leq k$ and that $G_{2}$ has an AT-orientation $D_{2}$ such that $d_{D_{2}}^{+}\left(x_{1}\right)=0, d_{D_{2}}^{+}\left(x_{2}\right) \leq 1, d_{D_{2}}^{+}\left(x_{3}\right) \leq 2$ and $\Delta\left(D_{2}\right) \leq k$ and $x_{i}$ is directed only to $x_{i^{\prime}}$, where $x_{i}, x_{i^{\prime}} \in V(T)$. Then $G$ has an AT-orientation $D$ such that $d_{D}^{+}(v)=d_{D_{1}^{\prime}}^{+}(v)$ for each $v \in V\left(G_{1}\right)$ and maximum out-degree of $D$ is at most $k$.

Proof. Let $D_{2}^{\prime} \subset D_{2}$ be the orientation of $G_{2}^{\prime}$ and let $D=D_{1}^{\prime} \cup\left(D_{2}^{\prime}-E(T)\right)$. Then it is easy to see that $d_{D}^{+}(v)=d_{D_{1}^{\prime}}^{+}(v)$ for each $v \in V\left(G_{1}^{\prime}\right), d_{D}^{+}(v)=d_{D_{2}^{\prime}}^{+}(v)$ for each $v \in V\left(G_{2}^{\prime}\right)-V(T)$ and hence maximum out-degree of $D$ is at most $k$. For the orientation $D_{2}$, the vertices in $T$ has a direction only to other vertices of $T$ and no Eulerian subgraphs in $D_{2}$ contain the edge in $T$ by the out-degree conditions of $D_{2}$. Thus $D_{2}^{\prime}$ is also an AT-orientation of $G_{2}^{\prime}$ and any spanning Eulerian sub-digraphs $H$ of $D$ has an edge-disjoint decomposition $H=H_{1} \cup H_{2}$ where $H_{1}$ and $H_{2}$ are Eulerian sub-digraphs in $D_{1}^{\prime}$ and $D_{2}^{\prime}$, respectively. Therefore, we have the bijection $\tau$ so that

- $\tau(E E(D))=\left(E E\left(D_{1}^{\prime}\right) \times E E\left(D_{2}^{\prime}\right)\right) \cup\left(O E\left(D_{1}^{\prime}\right) \times O E\left(D_{2}^{\prime}\right)\right)$ and
- $\tau(O E(D))=\left(O E\left(D_{1}^{\prime}\right) \times E E\left(D_{2}^{\prime}\right)\right) \cup\left(E E\left(D_{1}^{\prime}\right) \times O E\left(D_{2}^{\prime}\right)\right)$.

Hence

$$
\begin{aligned}
& |E E(D)|-|O E(D)| \\
= & \left(\left|E E\left(D_{1}^{\prime}\right)\right| \times\left|E E\left(D_{2}^{\prime}\right)\right|+\left|O E\left(D_{1}^{\prime}\right)\right| \times\left|O E\left(D_{2}^{\prime}\right)\right|\right) \\
& -\left(\left|E E\left(D_{1}^{\prime}\right)\right| \times\left|O E\left(D_{2}^{\prime}\right)\right|+\left|O E\left(D_{1}^{\prime}\right)\right| \times\left|E E\left(D_{2}^{\prime}\right)\right|\right) \\
= & \left(\left|E E\left(D_{1}^{\prime}\right)\right|-\left|O E\left(D_{1}^{\prime}\right)\right|\right) \cdot\left(\left|E E\left(D_{2}^{\prime}\right)\right|-\left|O E\left(D_{2}^{\prime}\right)\right|\right) \\
\neq & 0 .
\end{aligned}
$$

These imply that the orientation $D$ is an AT-orientation of $G$ with desired properties.

### 2.3 Characterizations of $K_{5}$-minor-free graphs.

Now, let us focus on $K_{5}$-minor-free graphs. In order to show the main theorem, we use the following results.

Lemma 2.5 ([12]) $A$ graph $G$ is $K_{5}$-minor-free if and only if $G$ can be formed from some 3-clique-sums of graphs, each of which is either planar or the Wagner graph $W$ as shown in Figure 2.



Figure 2: The left is the Wagner graph $W$ and the right is an acyclic orientation with maximum out-degree 3. Doted edges denote elements of a matching or forest. If we delete the two doted edges, the orientation has maximum out-degree 2 .

## 3 Proof of main Theorem.

Theorem 1.6 and Theorem 1.7 follow from the lemma below.
Lemma 3.1 Let $G$ be a $K_{5}$-minor-free graph and let $H_{i}$ be a subgraph of $G$ which is isomorphic to $u v \in E(G)$ or $\{u v, v w, w u\} \subset E(G)$ for each $i \in\{1,2,3\}$. Then all of the following hold.
(i) There exists an AT-orientation $D$ such that $d_{D}^{+}(u)=0, d_{D}^{+}(v)=1$, $\left(d_{D}^{+}(w)=2\right.$ if $H_{1}$ is isomorphic to $\left.K_{3}\right)$ and $d_{D}^{+}(y) \leq 4$ for $y \in V(G)-$ $\{u, v, w\}$.
(ii) There exists a matching $M$ of $G$ and an AT-orientation $D$ of $G-M$ such that $d_{D}^{+}(u)=d_{D}^{+}(v)=0,\left(d_{D}^{+}(w)=2\right.$ and $M$ does not cover $w$ if $H_{2}$ is isomorphic to $K_{3}$ ) and $d_{D}^{+}(y) \leq 3$ for $y \in V(G)-\{u, v, w\}$.
(iii) There exists a forest $F$ of $G$ and an acyclic orientation $D$ of $G-E(F)$ such that uw $\notin E(F), d_{D}^{+}(u)=d_{D}^{+}(v)=0,\left(d_{D}^{+}(w)=1\right.$ if $H_{3}$ is isomorphic to $K_{3}$ ) and $d_{D}^{+}(y) \leq 2$ for $y \in V(G)-\{u, v, w\}$.

Proof. Suppose that the Lemma is false and let $G_{i}$ be a counterexample for each $i \in\{1,2,3\}$ respectively with $\left|V\left(G_{i}\right)\right|$ as small as possible. By the minimality of $G_{i}, G_{i}$ is connected. Moreover, it is easy to check that $G_{i}$ does not have a cut vertex. Thus we may assume $G_{i}$ is 2 -connected.

First we suppose that $G_{i}$ is a plane graph. If $H_{i}$ is isomorphic to $K_{2}$ or $K_{3}$ which bounds a face, without loss of generality, $H_{i}$ lies on the boundary of $G$. In this case, a desired AT-orientation exists by Lemma 2.2. Thus $H_{i}$ consists a separating 3 -cycle in $G_{i}$. We let $G_{i, 1}$ and $G_{i, 2}$ be subgraphs of $G_{i}$ so that $G_{i, 1} \cup G_{i, 2}=G_{i}$ and $G_{i, 1} \cap G_{i, 2}=H_{i}$. By Lemma 2.2 and Lemma 2.3, for $j \in\{1,2\}$ we have an AT-orientation $D_{1, j}$ of $G_{1, j}$ which satisfies the conditions. Similarly, we have that there exists a matching $M_{j}$ and an AT-orientation of $G_{2, j}-M_{j}$ and that there exists a forest $F_{j}$ and acyclic orientation of $G_{3, j}-E\left(F_{j}\right)$, which satisfy the conditions. It is easy to see that $M=M_{1} \cup M_{2}$ is also a matching of $G_{2}$ and $F=F_{1} \cup F_{2}$ is a forest of $G_{3}$. Therefore, we get a desired AT-orientation respectively by Lemma 2.4 . Thus $G_{i}$ is not planar.

Next, suppose that $G_{i}$ is the Wagner graph $W$. Since $W$ does not contain a triangle, $H_{i}$ must be isomorphic to $K_{2}$. By the symmetry of $W$, we may assume that $H_{i}=u_{1} u_{5}$ or $u_{5} u_{6}$. We let $M$ be a matching of $W$ and let $F$ be a forest of $W$ such that $M=E(F)=E(H) \cup\left\{u_{2} u_{3}, u_{7} u_{8}\right\}$. In this case, the orientation in Figure 2 is a desired AT-orientation respectively.

Thus we assume that $G_{i}$ is neither planar graph nor the graph $W$. By Lemma 2.5, there exists $K_{5}$-minor-free graphs $G_{i, 1}$ and $G_{i, 2}$ such that $G_{i}$ can be obtained by a 3-clique-sum of $G_{i, 1}$ and $G_{i, 2}$. Let $T$ be its clique and let $G_{i, j}^{\prime}=G_{i, j} \cap G_{i}$ for $j \in\{1,2\}$. It is easy to see that $H_{i} \subset G_{i, 1}^{\prime}$ or $G_{i, 2}^{\prime}$. Without loss of generality, we may assume that $H_{i} \subset G_{i, 1}$. By the minimality of $G_{i}$, we have the following.
(i) There exists an AT-orientation $D_{1,1}$ of $G_{1,1}^{\prime}$ which satisfies the assumption (i) of Lemma 3.1.
(ii) There exists a matching $M_{1}$ and an AT-orientation $D_{2,1}$ of $G_{2,1}^{\prime}-M$ which satisfies the assumption (ii) of Lemma 3.1.
(iii) There exists a forest $F_{1}$ and an acyclic orientation $D_{3,1}$ of $G_{3,1}^{\prime}-E(F)$ which satisfies the assumption (iii) of Lemma 3.1.

First, we consider the case when $i=1$. By the minimality of $G_{1}$, we get an AT-orientation $D_{1,2}$ of $G_{1,2}$ with $d_{D_{1,2}}^{+}\left(x_{1}\right)=0, d_{D_{1,2}}^{+}\left(x_{2}\right)=1, d_{D_{1,2}}^{+}\left(x_{3}\right)=2$ and the maximum degree of $D_{1,2}$ is at most 4. By Lemma 2.4, we get a desired AT-orientation $D$ in $G_{1}$.

Next, we consider the case when $i=2$. By the minimality of $G_{2}$, we get a matching $M_{2}$ of $G_{2,2}$ and an AT-orientation $D_{2,2}$ of $G_{2,2}-M_{2}$ such that $d_{D_{2,2}}^{+}\left(x_{1}\right)=0, d_{D_{2,2}}^{+}\left(x_{2}\right)=0, d_{D_{2,2}}^{+}\left(x_{3}\right)=2$, maximum out-degree of $D_{2,2}$ is at most 3 and $M_{2}$ does not cover $x_{3}$. Let $M=M_{1} \cup\left(M_{2}-\left\{x_{1} x_{2}\right\}\right)$. It is easy to see that $M$ is a matching of $G$. By Lemma [2.4, we get a desired AT-orientation $D$ in $G_{2}-M$.

Finally, we consider the case when $i=3$. By the minimality of $G_{3}$, we get a forest $F_{2}$ of $G_{3,2}$ and an acyclic orientation $D_{3,2}$ of $G_{3,2}-E\left(F_{2}\right)$ with $d_{D_{3,2}}^{+}\left(x_{1}\right)=d_{D_{3,2}}^{+}\left(x_{2}\right)=0, d_{D_{3,2}}^{+}\left(x_{3}\right)=1$ and maximum out-degree of $D_{3,2}$ is at most 2. Let $F=F_{1} \cup\left(F_{2}-E(T)\right)$. Similarly, we can show that $F$ is a forest and $D_{3}$ is an acyclic orientation of $G_{3}-E(F)$ with desired properties. This is a contradiction and we completes the proof.
[Proof of Theorem 1.6 and Theorem 1.7]
Theorem 1.6 follows immediately from (i) and (ii) in Lemma 3.1. For Theorem 1.7, each $K_{5}$-minor-free graph $G$ has a forest $F$ and and an acyclic orientation $D$ of $G-E(F)$ with maximum out-degree at most 2 by Lemma 3.1. Since $G-E(F)$ is finite and $D$ is acyclic, there exists a vertex $v$ with $d_{D}^{-}(v)=0$ and hence the vertex $v$ has degree at most 2 in $G-E(F)$.

## 4 Some remarks

A signed graph is a pair $(G, \sigma)$, where $G$ is a graph and $\sigma$ is a signature of $G$ which assigns to each edge $e=u v$ of $G$ a sign $\sigma_{u v} \in\{1,-1\}$. Let

$$
N_{k}= \begin{cases}\{0, \pm 1, \ldots, \pm q\} & \text { if } k=2 q+1 \text { is an odd integer } \\ \{ \pm 1, \ldots, \pm q\} & \text { if } k=2 q \text { is an even integer. }\end{cases}
$$

Note that $\left|N_{k}\right|=k$ for each integer $k$. A proper coloring of $(G, \sigma)$ is a mapping $c: V(G) \rightarrow N_{k}$ such that $c(x) \neq \sigma_{x y} c(y)$ for each edge $x y$. The chromatic number $\chi(G, \sigma)$ of $(G, \sigma)$ is minimum integer $t$ such that there exists a proper coloring $c: V(G) \rightarrow N_{t}$. The choice number $c h(G, \sigma)$ of $(G, \sigma)$ is minimum integer $k$ such that for every $k$-list assignment $L$, there exists a proper coloring $c$ of $(G, \sigma)$ so that $c(v) \in L(v)$ for every $v \in V(G)$.

Let $(G, \sigma)$ be a signed graph and let ' $<$ ' be an arbitrary fixed ordering of the vertices of $(G, \sigma)$. The singed graph polynomial of $(G, \sigma)$ is defined as

$$
P_{G, \sigma}(\boldsymbol{x})=\prod_{u \sim v, u<v}\left(x_{u}-\sigma_{u v} x_{v}\right)
$$

where $u \sim v$ means that $u$ and $v$ are adjacent, and $\boldsymbol{x}=\left(x_{v}\right)_{v \in V(G)}$ is a vector of $|V(G)|$ variables indexed by the vertices of $G$. It is easy to see that a mapping $c: V(G) \rightarrow \mathbb{Z}$ is a proper coloring of $(G, \sigma)$ if and only if $P_{G, \sigma}(\boldsymbol{c}) \neq 0$, where $\boldsymbol{c}=(c(v))_{v \in V(G)}$. The Alon-Tarsi number of $(G, \sigma)$ is defined similarly and we have $\chi(G, \sigma) \leq \operatorname{ch}(G, \sigma) \leq A T(G, \sigma)$.

Let $(G, \sigma)$ be a singed graph and let $D$ be an orientation of $(G, \sigma)$. Let $\sigma E E(D)$ (resp. $\sigma O E(D)$ ) denote the set of all spanning Eulerian subdigraphs of $D$ with even (resp. odd) number of positive edges on $\sigma$. It was shown in 13 that Theorem 2.1 can be extended to signed one as follows.

Theorem 4.1 ([13]) Let $(G, \sigma)$ be signed graph, let $P_{G, \sigma}$ be the signed graph polynomial of $(G, \sigma)$ and let $D$ be an orientation of $(G, \sigma)$ with out-degree sequence $\boldsymbol{d}=\left(d_{v}\right)_{v \in V(G)}$. Then the coefficient of $\prod_{v \in V(G)} x_{v}^{d_{v}}$ in the expansion of $P_{G, \sigma}$ is equal to $\pm(|\sigma E E(D)|-|\sigma O E(D)|)$.

We say that an orientation $D$ of $(G, \sigma)$ is a $\sigma$ AT-orientation if $D$ satisfies $|\sigma E E(D)|-|\sigma O E(D)| \neq 0$.

Let us focus on planar graphs. In the papers [5] and [13], the following are shown respectively.

Lemma 4.2 Let $(G, \sigma)$ be a signed near triangulation and let $C=v_{1} v_{2} \ldots v_{m}$ be the boundary cycle of $G$. Then all of the following hold.
(i) ([13]) $G$ has a $\sigma A T$-orientation $D$ such that $d_{D}^{+}\left(v_{1}\right)=0, d_{D}^{+}\left(v_{2}\right)=1$, $d_{D}^{+}\left(v_{i}\right) \leq 2$ for each $i \in\{3, \ldots, m\}$ and $d_{D}^{+}(u) \leq 4$ for each interior vertex $u$.
(ii) ([5]) There exists a matching $M$ and a $\sigma A T$-orientation $D$ of $G-M$ such that $d_{D}^{+}\left(v_{1}\right)=d_{D}^{+}\left(v_{2}\right)=0, d_{D}^{+}\left(v_{i}\right) \leq 2-d_{M}\left(v_{i}\right)$ for each $i \in$ $\{3, \ldots, m\}$ and $d_{D}^{+}(u) \leq 3$ for each interior vertex $u$.

Although Lemma 4.2 only deals with near triangulations in the paper [13], it is not hard to extend the graph class from near triangulations to planar graphs. Moreover, since all the arguments of Lemmas in Section 2 and Lemma 3.1 work even if we replace AT-orientations into $\sigma$ AT-orientations, we have the following results.

Theorem 4.3 Let $(G, \sigma)$ be a signed $K_{5}$-minor-free graph. Then all of the following hold.
(i) $A T(G, \sigma) \leq 5$.
(ii) There exists a matching $M$ of $G$ such that $A T(G-M, \sigma) \leq 4$.
(iii) There exsits a forest $F$ in $G$ such that $A T(G-E(F), \sigma) \leq 3$.

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