

DISSECTING THE SQUARE INTO SEVEN OR NINE CONGRUENT PARTS

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ABSTRACT. We give a computer-based proof of the following fact: If a square is divided into seven or nine convex polygons, congruent among themselves, then the tiles are rectangles. This confirms a new case of a conjecture posed first by Yuan, Zamfirescu and Zamfirescu and later by Rao, Ren and Wang. Our method allows us to explore other variants of this question, for example, we also prove that no rectangle can be tiled by five or seven congruent non-rectangular polygons.

1. INTRODUCTION

Let P and T be convex polygons. We say that P can be tiled by n copies of T if there are convex polygons $T_1 \dots, T_n$, all congruent to T such that $P = \bigcup T_i$ and the T_i have disjoint interiors.

When can a polygon P be tiled by n copies of T ? In this paper we are mainly interested in the case when P is either a square or a rectangle.

It is easy to see that a square can always be tiled by n congruent rectangles. This can be done by dividing the square by $n - 1$ vertical lines. When n is not prime, there are many other ways in which this can be done and tiles need not be constructed with vertical lines. However, it is not known if there is an odd number n and a non-rectangular tile T for which a square can be tiled by n copies of T . The standing conjecture, as stated in [RRW20], is as follows.

Conjecture 1. *If n is an odd positive integer, then a square can be tiled by n congruent copies of a convex polygon T only if T is a rectangle.*

This conjecture, for $n = 3$, was posed as a problem by Rabinowitz in the journal *Cruz Mathematicorum* and was answered by Maltby [Mal91]. Maltby later generalized his result by showing that it is impossible to tile a rectangle by 3 copies of T unless T is also a rectangle [Mal94].

For $n = 5$, Conjecture 1 was verified by Yuan *et al.* [YZZ16]. They attribute a similar problem to Danzer, who conjectured that a square may not be tiled by 5

Key words and phrases. Tiling; Congruent; Equiangular; Computational geometry.

congruent polygons (convex or not), except when T is a rectangle. Danzer's conjecture remains open. They also posed Conjecture 1 for prime numbers n .

Apart from these two special cases, it is known that if a square can be tiled by an odd number of copies of T , then T cannot be a triangle. This follows from the work of Thomas and Monsky [Tho68, Mon70] who, independently, proved that a square cannot be tiled by an odd number of triangles with the same area. Recently Rao *et al.* showed that T may not have more than 6 sides and that T may not be a right-angle trapezoid [RRW20].

In this paper we give a computer-based proof of the validity of Conjecture 1 for $n = 7$ and $n = 9$.

Theorem 2. *Let $n = 7$ or $n = 9$, then a square cannot be tiled by n copies of a convex polygon T unless it is a rectangle.*

Using the same techniques, we are able to prove the following result for rectangles.

Theorem 3. *Let $n = 5$ or $n = 7$, then no rectangle can be tiled by n copies of a convex polygon T unless it is also a rectangle.*

Using slight modifications to our method, we are able to classify all tilings of the square using non-rectangular equiangular convex polygons. Here, two convex polygons are *equiangular* if there is a bijection between their vertices, respecting the order of the vertices, such that the angles at corresponding vertices be equal.

Theorem 4. *There are 31 ways (in the sense described in Section 3) in which a square can be tiled by 5 non-rectangular equiangular convex polygons.*

Having this list, we are able to answer the first three open problems stated at the end of [YZZ16].

The rest of the paper is devoted to proving these theorems. The main layout is as follows. We assume that a square or rectangle P can be tiled by n copies of a convex polygon T which is not a rectangle. In Section 2 we deduce properties that the tiles should have. In Section 3 we construct a polyhedral graph associated to a tiling of P . In Section 4 we describe the algorithm we used to see which of these graphs could be obtained by a tiling of P by congruent tiles. If there are no such graphs then the tiling does not exist. In Section 5 we mention the modifications needed to prove Theorem 4 and give several examples of such tilings. We give some final remarks in Section 6.

The code we used is available at <https://github.com/XGEu2X/TilingSquare/>. The `Readme.md` file in this repository contains a small explanation of how the code works. It is possible to print out parts of the proof to see the steps taken to reach our conclusions.

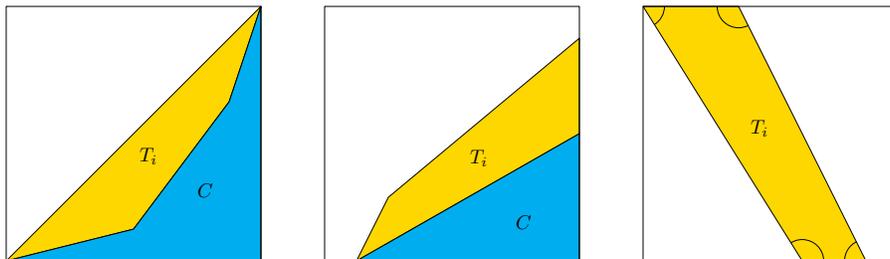


FIGURE 1. Three types of impossible tiles corresponding to parts (2) and (3) of Lemma 5. In the first two pictures, the diameter of any convex body contained in C is smaller than the diameter of T_i . In the last picture, the interior angles of T_i are all greater than $\pi/4$.

2. PROPERTIES OF TILINGS

We start by listing four properties that the tiles should have if they are to successfully tile a unit square. The first three properties are combinatorial and help to reduce the number of cases that must be analyzed later. The last property is geometric and is used later in the algorithm.

Lemma 5. *Let $n \geq 3$ be an odd integer and let P be a rectangle. Let s_0, s_1, s_2, s_3 be the (closed) sides of P ordered cyclically, where the indices are taken mod 4. If P can be tiled by n copies T_1, \dots, T_n of a convex body T , then the following hold:*

- (1) T has at least 4 sides.
- (2) If a tile T_i intersects two consecutive sides s_k and s_{k+1} of P , then T_i contains the vertex of P common to s_k and s_{k+1} .

Furthermore; if P is a square, T has exactly 4 sides and is not a rectangle then:

- (3) A tile T_i cannot have two sides a, b such that $a \subset s_k$ and $b \subset s_{k+2}$ for some k .
- (4) T does not have two consecutive right angles.

Proof. As mentioned before, if P is a square then (1) follows immediately from the main results in [Tho68, Mon70]. If P is a rectangle then we can apply a linear transformation which sends it to a square. Since linear transformations preserve areas, the same result holds.

Part (4) is Theorem 1.2 in [RRW20].

To prove (2), assume that T_i intersects sides s_k and s_{k+1} but does not intersect the corner contained in $s_k \cap s_{k+1}$. There are two cases to consider here. If T_i touches two opposite corners of P , as in Figure 1 (left), then the diameter of T_i equals the diameter of P . Consider the connected components of $P \setminus T_i$: there are at least two

of them or T_i would cover half of P . One of these components must be properly contained in one of the right triangles obtained by splitting P by its diagonal, but any tile T_j contained in this component must have diameter smaller than P . This leads to a contradiction. The argument when T_i does not touch two opposite corners of P , as in Figure 1 (middle), is similar. Let T_j be a tile containing the corner of P common to s_k and s_{k+1} . Then the diameter of T_j must be smaller than the diameter of T_i , leading to a contradiction.

Finally, assume that (3) does not hold. We consider two cases, the first is when some corner of P is incident with exactly one tile, then T must have a right angle. Since T_i has two parallel sides, it has two consecutive right angles which contradicts (4). The second case is when every corner of P is incident with at least two tiles. Then T must have an interior angle of at most $\pi/4$. But since T_i has two vertices on s_k and its other two vertices on s_{k+2} , all of its interior angles are strictly greater than $\pi/4$ which is a contradiction. This is exemplified in Figure 1 (right). \square

3. THE GRAPH ASSOCIATED TO A TILING

In this section we construct a polyhedral graph that contains the combinatorial structure of a tiling of the square.

Let T_1, \dots, T_n be convex polygons which tile a square or rectangle P with closed sides S_1, S_2, S_3, S_4 in cyclic order.

Construct a graph $G = (V, E)$, where $V = \{S_1, \dots, S_4, T_1, \dots, T_n\}$. The edges of this graph are defined by the following rules:

- $\{A, B\} \in E$ if $A, B \in V$ and $A \cap B$ is a segment of positive length.
- $\{S_i, S_j\} \in E$ if $i - j \equiv 1 \pmod{4}$.

G contains all the combinatorial properties of the tiling.

We can think of G as the dual of a pyramid with a square or rectangular base in which the base has been tiled in the same way as P . Furthermore, the following proposition gives us one more useful property.

Proposition 6. *The graph G is 3-connected.*

Proof. Let $v_1, v_2 \in V$, each v_i is either a tile or a side of P . We must prove that for every selection of these vertices, $G' = G - v_1 - v_2$ is connected. It is enough to prove that, for each $T_i \in V(G')$ and each $S_j \in V(G')$, there is a path between T_i and S_j . In this way, any two vertices of G' may be connected by a path through some S_j or some T_i . Let p be a point in the interior of a $T_i \in V(G')$.

If v_1 and v_2 are both sides of P , then choose a segment l from p to any point in the relative interior of $S_j \neq v_1, v_2$. The segment l is contained in $P \setminus (v_1 \cup v_2)$. By taking the graph induced by tiles that intersect l (including S_j) we obtain a connected subgraph of G' which contains T_i and S_j . Therefore G' is connected.

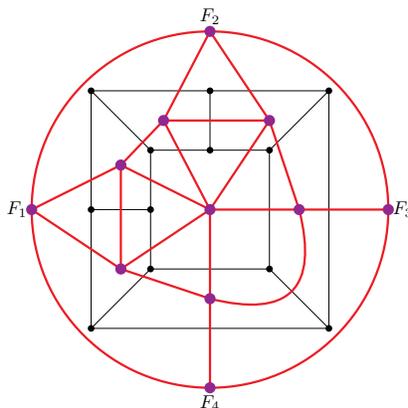


FIGURE 2. A tiling (black) and its associated graph (red).

Otherwise, consider a line l separating the relative interiors of v_1 and v_2 . Let l' be the line parallel to l that contains p . Since v_1 and v_2 are convex, the intersection between l' and $(v_1 \cup v_2)$ is a segment, a point or the empty set. In any case, there is a segment contained in l' connecting p and a side $S_j \neq v_1, v_2$ which does not intersect $v_1 \cup v_2$. As above, this implies that G' is connected. \square

Steinitz's theorem [Grü07] states that the family of polyhedral graphs (corresponding to convex polytopes) is precisely the family of 3-connected planar graphs. Therefore, our graph G is a polyhedral graph.

Note that the degree of a tile T in G may not correspond to the number of sides of T as a polygon. However, the number of sides of T is a lower bound for the degree of T in G .

In this way we have associated a graph to each convex tiling of the square by tiles of at least 4 sides. This graph has a distinguished set of vertices S which form a cycle. Figure 2 shows an example of a tiling with its associated graph G .

In what follows, we study pairs (G, S) where $G = (V, E)$ is a 3-connected planar graph with $n + 4$ vertices, and $S \subset V$ induces a 4-cycle in G . Since these graphs come from tilings of the square, we simply refer to the vertices of G in S as *sides* and the vertices in $V \setminus S$ as *tiles*. We say that two pairs (G, S) and (G', S') are isomorphic if there is an isomorphism ϕ between G and G' such that $\phi(S) = S'$.

To avoid confusion in the future, the union of the sets of vertices of the polygons T_i are called *tiling-vertices*. The set of tiling-vertices in the boundary of a tile T_i are called the *tiling-vertices of T_i* ; the tile T_i has internal angles at each of these, although some may be equal to π .

Remark. In practice, we need to generate all such graphs using `plantri` [BM07]. To facilitate this, we add an extra vertex S_0 to G which is adjacent to the vertices in S . Since we deal mostly with quadrilateral tiles, we can specify that this new graph has minimum degree 4. It also simplifies checking isomorphisms, since we only need this new vertex to be fixed.

4. EXPLORING THE TILINGS

Let P be either a rectangle with sides of length 1 and r , or the unit square (in which case we take $r = 1$ as a constant). Let S be the set of sides of P . Assume that P can be tiled by copies T_1, \dots, T_n of a convex body T which is not a rectangle. By part (1) of Lemma 5, T has at least 4 sides. Therefore, as described in Section 3, there is a pair (G, S) associated to this tiling where G is a 3-connected planar graph on $n + 4$ vertices such that its tiles have minimum degree 4.

Using `plantri` it is possible to generate all graphs G with these properties with $n \leq 9$. Once this is done, we search each graph G to find the possible distinguished 4-cycles. A single graph may have several possible distinguished 4-cycles, so we use the isomorphism algorithm in `NetworkX` [HSS08] to avoid including two pairs (G, S) which are isomorphic.

Now we filter these lists by using part (2) of Lemma 5. Since this is a purely combinatorial statement, it is easy to check it directly on (G, S) .

At this point, the number of sides of T becomes relevant. We split the analysis into cases. If T is a quadrilateral then we use part (3) of Lemma 5 to filter the list once more. If not, then the tiles in (G, S) necessarily have degree at least 5, so we may discard the graphs that do not satisfy this.

In order to filter this list further, we must use geometrical properties of T such as the values of its angles and side-lengths. Let p_1, p_2, \dots, p_k be the vertices of T , α_i the internal angle of T at p_i and t_i be the length of the segment $p_i p_{i+1}$. The angles must satisfy the equation $\alpha_1 + \alpha_2 + \dots + \alpha_k = (k - 2)\pi$.

Recall that if v is a tiling-vertex of a tile T_i , then the internal angle of T_i at v is either π or one of the α_j . For every tiling-vertex v , the sum of the internal angles at v of the tiles that contain v must be either 2π , π or $\pi/2$ depending on whether v is in the interior, on a side or on a vertex of P . This is shown in Figure 3. Therefore, the angles $\alpha_1, \alpha_2, \dots, \alpha_k$ must satisfy a system of non-homogeneous linear equations.

Something similar happens for the side-lengths of T . For every side s of P , take the tiles T_i adjacent to s . The sum of the sides of these tiles contained in s must add up to the length of s which is either 1 or r . We do not fix the value of r but treat it as a variable. So this gives us four non-homogeneous linear equations for the t_i and, if applicable, r .

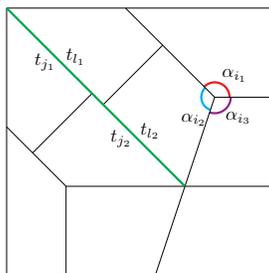


FIGURE 3. In this tiling we see that $\alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3} = 2\pi$, and $t_{j_1} + t_{j_2} = t_{l_1} + t_{l_2}$.

Furthermore, if two tiles T_i and T_j intersect in a segment v_1v_2 such that v_1 and v_2 are consecutive corners of both T_i and T_j , then we can deduce an equation of the form $t_k = t_l$. More generally, if there is a line ℓ which contains a side of the tiles T_{i_1}, \dots, T_{i_l} and $T_{j_1}, \dots, T_{j_{l'}}$ such that the T_{i_m} and $T_{j_{m'}}$ are on different sides of ℓ and the T_{i_m} cover the same segment of ℓ as the $T_{j_{m'}}$, then we may deduce another linear equation for the t_i (see Figure 3).

In the quadrilateral case we may also check non-linear equations which involve both sides and angles of T . The area of T is r/n , so

$$t_1t_2 \sin(\alpha_2) + t_3t_4 \sin(\alpha_4) = t_2t_3 \sin(\alpha_3) + t_4t_1 \sin(\alpha_1) = 2r/n$$

and by computing the length of the diagonals of T we obtain

$$\begin{aligned} t_1^2 + t_2^2 - 2t_1t_2 \cos(\alpha_2) &= t_3^2 + t_4^2 - 2t_3t_4 \cos(\alpha_4) \\ t_2^2 + t_3^2 - 2t_2t_3 \cos(\alpha_3) &= t_4^2 + t_1^2 - 2t_4t_1 \cos(\alpha_1). \end{aligned}$$

All of these equations must be satisfied. The problem is that, from the graph G , we do not know which tiling-vertices of a tile T_i correspond to which vertices of T . If the degree of T_i in G is larger than 4 then we must also decide which tiling-vertices of T_i have internal angles equal to π . We could try computing all possibilities but the number of cases is too large even for a single graph, so instead we do something slightly more efficient which allows us to discard most of the remaining graphs.

Each angle of T is labeled by **a**, **r** or **o** depending on whether the angle is acute, right or obtuse. We also label the angles of each tile T_i at each tiling-vertex of T_i with **a**, **r**, **o**, **p** depending on whether the angle is acute, right, obtuse or plain (meaning that the internal angle at this point is π). We call the label of an angle its *angle-type*. We use the fact that the angles of each T_i that are not labeled with **p** must appear in the same cyclic order as the angles of T .

There are several ways in which the internal angles of T can be labeled. This depends on whether P is a square or a rectangle, but they are easy to list. For

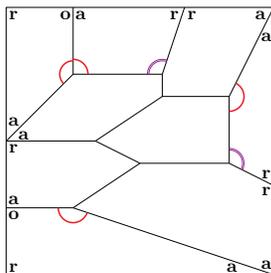


FIGURE 4. This is a representation of a tiling where the angles of T are labeled as **aror**. At this point we are only working with the graph G , so the angles in this representation may not correspond to the actual angles of the tiles and some angles may even be plain. In this case there is only one possibility for the angle-types of the angles that lie in the boundary of P . From there it follows that the red angles must be labeled with **r** and the purple angles with **o**.

example, if T is a quadrilateral, the internal angles of T , ordered cyclically and without taking orientation into account, can be labeled in exactly 9 different ways: **aaao**, **aaro**, **aaoo**, **arao**, **aror**, **aroo**, **aoao**, **aoro** and **aooo**. In this case we are using property (4) of Lemma 5 to discard labels with two consecutive right angles. Since T is not a rectangle, we do not include **rrrr**. If T has 5 sides, the number of ways to label the angles is greater.

There are several conditions which the angle-types of the tiles must satisfy. For example, assume that v_k is a tiling-vertex of T_i . If v_k is also a corner of P , then the angle-type of T_i at v_k is either **a** or **r**, depending on whether there are other tiles containing v_k or not. To state another example, assume that T_i and T_j are the only two tiles containing a tiling-vertex v_k which lies on a side of P . Then either the angle-types of T_i and T_j at v_k are both **r** or one is **a** and the other is **o**. See Figure 4 for an example of a tiling with some angle-types.

In order to avoid listing these properties individually, we introduce the following definitions. Let $\varepsilon > 0$ be a small real number. Define $\text{minang}(T_i, v_k)$ as ε , $\pi/2$, $\pi/2 + \varepsilon$ or π depending on whether the angle-type of T_i at v_k is **a**, **r**, **o**, or **p**, respectively. Likewise, define $\text{maxang}(T_i, v_k)$ as $\pi/2 - \varepsilon$, $\pi/2$, $\pi - \varepsilon$ or π depending on whether the angle-type of T_i at v_k is **a**, **r**, **o**, or **p**. If ε is small enough, then values of the angles of T_i at v_k are in the interval $[\text{minang}(T_i, v_k), \text{maxang}(T_i, v_k)]$. Thus, for every

tiling-vertex v_k we have that

$$\sum_{T_i \ni v_k} \max\text{ang}(T_i, v_k) \geq \alpha(v_k),$$

$$\sum_{T_i \ni v_k} \min\text{ang}(T_i, v_k) \leq \alpha(v_k),$$

where $\alpha(v_k)$ is $\pi/2$, π or 2π if v_k is in a corner, side or interior of P , respectively. In practice, this allows us to decide the angle-types of the tiles fairly quickly, even if we do not know the values of the angles of T . Another useful observation comes from noting that each of the previous sums has at most n terms; since we only work with $n \leq 9$, any value of ε smaller than $\pi/18$ allows us to discriminate between valid and invalid assignments of angle-types.

Now we are ready to filter the list of pairs (G, S) . For each of these pairs, select one of the possible angle-type assignments for the internal angles of T . For each tile T_i , we wish to decide which tiling-vertices of T_i correspond to which vertices of T . So we assign to (T_i, v_k) , where v_k is a tiling-vertex of T_i , a vertex of T or a mark indicating that the angle-type of T_i at v_k is \mathbf{p} . In order to do this, we use a deep search algorithm to explore the tree of possibilities. The process is divided into three main steps:

Selection: Select a tile T_i and one of its tiling-vertices v_k such that (T_i, v_k) is unassigned.

Assignment: Makes a list of valid assignments for (T_i, v_k) taking into account the conditions that the angle-types must satisfy, described above.

Equation verification: For each assignment, check if a new equation for the angles or the sides of T is generated. If so, verify that the equations still have solutions. This is done with `Sympy` [MSP⁺17] which works symbolically instead of numerically, this guarantees that we do not discard equations when solutions actually do exist. If possible, check that the non-linear equations are not violated. This final step is done numerically, so a small tolerance is allowed.

The surviving assignments are added to the stack for further exploration.

The order in which the tiles and tiling-vertices are explored is important. We always start with the corners of the rectangle and the tiles that contain them. We continue with tiles adjacent to a side of P and tiling-vertices in this side in a cyclic order. In many cases, exploring these pairs of tiles and tiling-vertices is enough to discard (G, S) for a given angle-type labeling of T .

This does not eliminate all possible pairs (G, S) , so we also check the additional geometric conditions. These are described in the following statements.

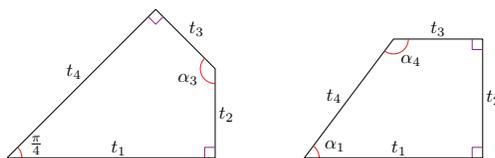


FIGURE 5. The quadrilaterals in Lemma 7

Lemma 7. *Let T be a quadrilateral with area A and let α_i and t_i be as above. If T is labeled with angle-types **aror** and $\alpha_1 = \pi/4$, as in the left part of Figure 5, then*

$$\sqrt{2A} < t_1, t_4 \leq 2\sqrt{A}.$$

*If instead T is labeled with angle-types **arro**, as in the right part of Figure 5, then*

$$\begin{aligned} t_1 &= t_3 + t_4 \cos(\alpha_1), \\ t_2 &= t_4 \sin(\alpha_1). \end{aligned}$$

Proof. It is easy to see that $t_4 = (t_1 + t_2)/\sqrt{2}$ and $t_3 = (t_1 - t_2)/\sqrt{2}$, therefore

$$t_1 t_2 + \frac{t_1^2 - t_2^2}{2} = 2A.$$

Thinking of t_1 as a function of t_2 and taking the derivative with respect to t_2 yields

$$t_1' = \frac{t_2 - t_1}{t_2 + t_1}.$$

Since $\alpha_1 = \pi/4$, we have $0 < t_2 < t_1$, so t_1' is negative and the extremal values for t_1 occur when $t_2 = 0$ and $t_2 = t_1$ which gives the desired result.

The second part is straightforward. \square

Lemma 8. *Let T be a quadrilateral which tiles P and has two angles labeled as **r**. Assume some tile T_i has a vertex on corner of P with angle-type **a**, then the acute angle of T must be $\pi/3$ or $\pi/4$.*

Proof. Let α_a and α_o be the values of the acute and obtuse angles of T . In the tiling, the number of obtuse angles must be equal to the number of acute angles. Note that no obtuse angle coincides with a corner of P and any obtuse angle which coincides with a side of P shares a vertex with exactly one acute angle of a different tile. Since there is an acute angle which coincides with a corner, there exists a tiling-vertex v in the interior of P incident to more obtuse angles than acute angles. Assume that the number of acute, obtuse and right angles incident to v are N_a , N_o and N_r , respectively. The sum of angles in each interior tiling-vertex is 2π , so $N_o\alpha_o + N_a\alpha_a + N_r\pi/2 = (N_o - N_a)\alpha_o + N_a\pi + N_r\pi/2 = 2\pi$. Since $\alpha_o > \pi/2$, we

have that $N_o - N_a \leq 3$, so α_o is of the form $p\pi/(2q)$ with $1 \leq p \leq 4$ and $1 \leq q \leq 3$. From here we may conclude that the only possible values for α_o are $2\pi/3$ and $3\pi/4$ and therefore α_a is $\pi/3$ or $\pi/4$. \square

When P is a square and $n \leq 9$ is an odd integer or when P is a rectangle and $n \leq 7$ is an odd integer, every pair (G, S) is discarded with this method. This proves Theorems 2 and 3.

5. EQUIANGULAR PIECES

Using the same method, we explore another variant of the conjecture. At the end of [YZZ16] there are four open problems:

- (1) Does every dissection of the square into five similar convex tiles use right isosceles triangles or rectangles as tiles?
- (2) Does every dissection of the square into five equiangular convex polygons use only angles measuring $\pi/4$, $\pi/2$, $3\pi/4$?
- (3) Find all dissections of the square into five equiangular non-rectangular convex polygons.
- (4) Is every dissection of the square into n congruent convex tiles necessarily the “standard” one (i.e. dividing it by $n - 1$ vertical or horizontal lines) if $n \geq 3$ is a prime number?

Our only contribution to the fourth problem is that of Theorem 2. However, we are able to solve the other three. The first two questions have a negative answer, this can be seen immediately after solving the third problem.

By slightly modifying the algorithm described in Section 4, we are able to list all the possible equiangular tilings (in the sense described in Section 3). These are shown in Figures 6 and 7.

The equivalence relation we use might not be the one one might expect. For example, in Figure 7 there are four pairs of tilings enclosed in dashed rectangles. The two tilings in each rectangle have the same pair (G, S) , but one might argue that they are geometrically distinct.

The 31 ways referred to in Theorem 4 can be separated into two types of tiling; then ones in which the tiles are triangles and the ones in which they are quadrilaterals.

Figure 6 shows the 12 ways in which a square can be tiled using 5 similar triangles. The tiles in each case are right triangles, let α be the smallest angle of the triangles in each case. The top three tilings in Figure 6 (green) have $\tan(\alpha) = a \approx 0.56984$, where a is the real root of the polynomial $a^3 - a^2 + 2a - 1$. The tilings in the middle row (red) have $\tan(\alpha) = 1/2$. The bottom tilings (purple) have $\tan(\alpha) = 1$.

The quadrilateral case is shown in Figure 7. Here each pair (G, S) can be realized as a continuum of tilings as there are always several degrees of freedom. As mentioned

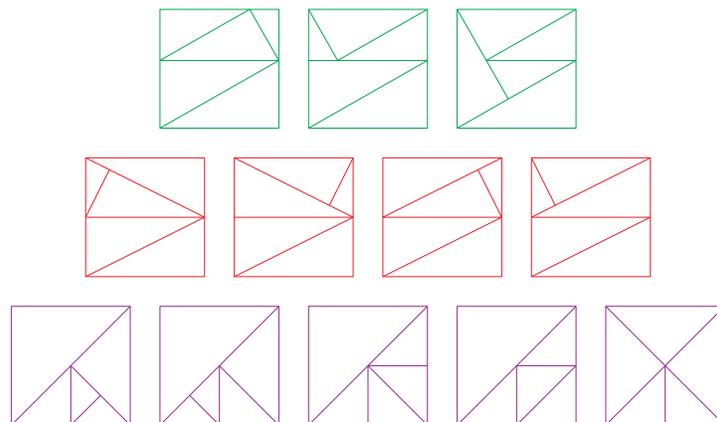
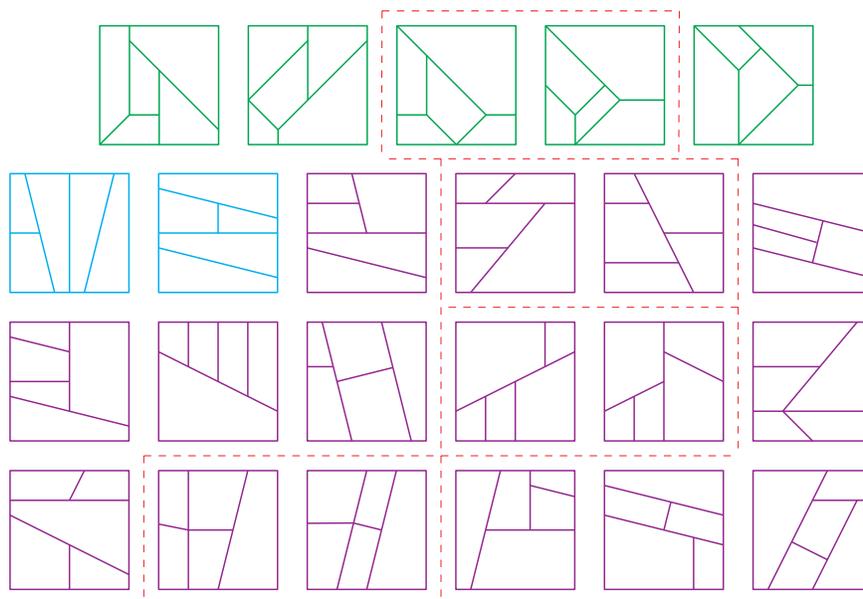


FIGURE 6. Equiangular tilings with five triangles.

FIGURE 7. Equiangular tilings using five quadrilaterals. The angles of the tiles are of the form $\alpha, \pi/2, \pi/2, \pi - \alpha$.

before, the tiles which are enclosed in dashed rectangles are equivalent and are included only to illustrate what the equivalence means. In total, there are 19 families of tilings of the square into 5 equiangular quadrilaterals such that the elements of each

class produce the same pair (G, S) . The families represented in the first row of Figure 7 (green) have angles $\pi/4, \pi/2, \pi/2, 3\pi/4$. The rest have angles $\alpha, \pi/2, \pi/2, \pi - \alpha$, where α can be chosen, in all but two tilings, in the interval $(0, \pi/4)$ and can sometimes take more values. The first two tilings (blue) are the exception; α can only be taken in the interval $(0, \arctan(1/2))$.

In order to prove Theorem 4, we slightly modify the algorithm described in Section 4.

The obvious change is to ignore the side-lengths, so the equations only involve angles. Another helpful change, which greatly improves the running time of our algorithm, is to use more angle-types when tiling with triangles. In this case, instead of using **a** to label an acute angle, we use the labels (sa) , (ma) and (la) which correspond to acute angles smaller than $\pi/4$, equal to $\pi/4$ and greater than $\pi/4$, respectively. Analogously, we replace **o** with (so) , (mo) and (lo) which correspond to obtuse angles smaller than $3\pi/4$, equal to $3\pi/4$ and greater than $3\pi/4$, respectively. Since Lemma 5 does not apply in this case, we use the following observations.

Lemma 9. *Let $n \geq 3$ be an odd integer and P be a square. Let s_1, s_2, s_3, s_4 be the (closed) sides of P ordered cyclically, where the indices are taken mod 4. If P can be tiled by convex polygons T_1, \dots, T_n which are equiangular and the T_i are triangles, then the following hold:*

- (1) *If a tile T_i intersects a side s_k in a segment of positive length and one of its vertices v is in s_{k+2} , then the angle α of T_i at v satisfies $\alpha < \pi/2$.*
- (2) *If a tile T_i contains a side s_k and intersects s_{k+1} (or s_{k-1}) in a segment of positive length, then the angle α at the vertex of T_i in s_{k-1} (or s_{k+1}) of T_i satisfies $\alpha \leq \pi/4$.*
- (3) *A tile T_i cannot have two sides a, b such that $a \subset s_k$ and $b \subset s_{k+2}$ for some k .*

If instead the T_i are quadrilaterals, then

- (4) *A tile intersecting three consecutive sides of P in segments of positive length, cannot have an angle $\alpha \leq \pi/4$.*

Proof. In (1), the two sides adjacent to v have length at least 1, so the side in s_k is the smallest side of T_i . Thus, the smallest angle of T_i is α and therefore $\alpha \leq \pi/3 < \pi/2$. Statement (2) is straightforward from the fact that T_i is a right triangle and α is its smallest angle. Statement (3) follows from the fact that, in a triangle, every side is adjacent to the other two. Finally, in (4), suppose T_i intersects s_k, s_{k+1} and s_{k+2} . Since T_i is a quadrilateral, s_{k+1} must be one of its sides and the vertices of T_i which are not in s_{k+1} are in s_k and s_{k+2} . Therefore, the two angles incident to s_{k+1} are right angles and each of the other two are greater than $\pi/4$. \square

After running the modified program, we are left with 15 graphs in the triangular case and 27 graphs in the quadrilateral case. Of these, 3 graphs for the triangular case and 8 graphs for the quadrilateral do not produce valid tilings. This can easily be checked by hand. All these graphs can be seen in our repository, it is also possible to examine the unrealizable and unfiltered graphs.

6. FINAL REMARKS

We decided to use Python in order to make the code more accessible. The execution time turned out not to be an issue. Since each graph can be explored separately, the code is easily parallelizable. Proving Theorem 2 for $n = 9$ takes a couple of days in a modern home computer, the rest of cases take at most a few of hours.

In Theorems 2 and 3, the number of tiles may not be increased using our method. The number of graphs to be analyzed is simply too large. As for the equiangular case with 7 tiles, our algorithm produces around 2000 possible valid graphs, however we were unable to find a systematic way of deciding whether a graph is valid or not.

Notice that this method can be used to find tilings (using congruent or equiangular pieces) of other polygons. We may use other observations in the way of Lemma 5 to optimize the process. It might be necessary to consider labellings using different angle-types. What is important is to check that the underlying graph is 3-connected or that there is some other condition which allows us to generate the list of possible graphs.

Returning to the general conjecture, we noticed that when the angles of T are (cyclically) labeled as **aror**, we usually had to go deeper in the tree of possibilities before we could completely discard a given pair (G, S) . We tried to rule out this case in general but failed. It seems that the corresponding tiles have more structure so it may be possible to discard them by other means.

7. ACKNOWLEDGMENTS

We are thankful to the anonymous referee for his careful reading and fruitful suggestions. During this research the first author was supported by a CONACyT scholarship and the second author was supported by CONACyT project 282280.

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