

# Acyclic matchings in graphs of bounded maximum degree\*

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## Abstract

A matching  $M$  in a graph  $G$  is acyclic if the subgraph of  $G$  induced by the set of vertices that are incident to an edge in  $M$  is a forest. We prove that every graph with  $n$  vertices, maximum degree at most  $\Delta$ , and no isolated vertex, has an acyclic matching of size at least  $(1 - o(1))\frac{6n}{\Delta^2}$ , and we explain how to find such an acyclic matching in polynomial time.

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# 1 Introduction

We consider simple, finite, and undirected graphs, and use standard terminology. Let  $M$  be a matching in a graph  $G$ , and let  $H$  be the subgraph of  $G$  induced by the set of vertices that are incident to an edge in  $M$ . If  $H$  is a forest, then  $M$  is an *acyclic* matching in  $G$  [7], and, if  $H$  is 1-regular, then  $M$  is an *induced* matching in  $G$  [14]. If  $\nu(G)$ ,  $\nu_{ac}(G)$ , and  $\nu_s(G)$  denote the largest size of a matching, an acyclic matching, and an induced matching in  $G$ , respectively, then, since every induced matching is acyclic, we have

$$\nu(G) \geq \nu_{ac}(G) \geq \nu_s(G).$$

In contrast to the matching number  $\nu(G)$ , which is a well known classical tractable graph parameter, both, the acyclic matching number  $\nu_{ac}(G)$  as well as the induced matching number  $\nu_s(G)$  are computationally hard [3, 7, 13, 14]. While induced matchings have been studied in great detail, see, in particular, [8–11] for lower bounds on  $\nu_s(G)$  for graphs  $G$  of bounded maximum degree as well as the references therein, only few results are known on the acyclic matching number. While the equality  $\nu(G) = \nu_s(G)$  can be decided efficiently for a given graph  $G$  [2, 12], it is NP-complete to decide whether  $\nu(G) = \nu_{ac}(G)$  for a given bipartite graph  $G$  of maximum degree at most 4 [6], and efficient algorithms computing the acyclic matching number are known only for certain graph classes [1, 4, 6, 13]. It is known [1] that  $\nu_{ac}(G) \geq \frac{m}{\Delta^2}$  for a graph  $G$  with  $m$  edges and maximum degree  $\Delta$ , which was improved [5] to  $\frac{m}{6}$  for connected subcubic graphs  $G$  of order at least 7. Since, for every  $\Delta$ -regular graph  $G$  with  $m$  edges, a simple edge counting argument implies  $\nu_{ac}(G) \leq \frac{m-1}{2(\Delta-1)}$ , the constructive proofs of these bounds yield an efficient  $\frac{\Delta^2}{2(\Delta-1)}$ -factor approximation algorithm for  $\Delta$ -regular graphs, and an efficient  $\frac{3}{2}$ -factor approximation algorithm for cubic graphs for the maximum acyclic matching problem.

In the present paper we show a lower bound on the acyclic matching number of a graph  $G$  with  $n$  vertices, maximum degree  $\Delta$ , and no isolated vertex, which is inspired by a result of Joos [9] who proved

$$\nu_s(G) \geq \frac{n}{(\lfloor \frac{\Delta}{2} \rfloor + 1)(\lceil \frac{\Delta}{2} \rceil + 1)} \quad (1)$$

provided that  $\Delta \geq 1000$ . (1) is tight for the graph that arises by attaching  $\lfloor \frac{\Delta}{2} \rfloor$  new vertices of degree 1 to every vertex of a complete graph of order  $\lceil \frac{\Delta}{2} \rceil + 1$ . In view of these graphs, we conjectured [4, 5] that twice the right hand side of (1) should be the right lower bound on the acyclic matching number of the considered graphs for sufficiently large  $\Delta$ , that is, we believe that our following main result can be improved by a factor of roughly  $\frac{4}{3}$ .

**Theorem 1.** *If  $G$  is a graph with  $n$  vertices, maximum degree at most  $\Delta$ , and no isolated vertex, then*

$$\nu_{ac}(G) \geq \frac{6n}{\Delta^2 + 12\Delta^{\frac{3}{2}}}.$$

Note that, for graphs that are close to  $\Delta$ -regular, the bound  $\nu_{ac}(G) \geq \frac{m}{\Delta^2}$  is stronger than Theorem 1. We prove Theorem 1 in the next section. In the conclusion we discuss algorithmic aspects of its proof and possible generalizations to so-called degenerate matchings [1].

## 2 Proof of Theorem 1

We prove the theorem by contradiction. Therefore, suppose that  $G$  is a counterexample of minimum order. Clearly,  $G$  is connected. If  $\Delta = 1$ , then  $G$  is  $K_2$ , and, hence,  $\nu_{ac}(G) = \frac{n}{2}$ . If  $\Delta = 2$ , then  $G$  is a path or a cycle, which implies  $\nu_{ac}(G) \geq \frac{n-2}{2}$ . These observations imply  $\Delta \geq 3$ . At several points within the proof we consider an acyclic matching  $M$  in  $G$ , and we consistently use

- $V_M$  to denote the set of vertices of  $G$  that are incident to an edge in  $M$ ,
- $N_M$  to denote the set of vertices in  $V(G) \setminus V_M$  that have a neighbor in  $V_M$ ,
- $G_M$  to denote the graph  $G - (V_M \cup N_M)$ ,
- $I_M$  to denote the set of isolated vertices of  $G_M$ , and
- $G'_M$  to denote the graph  $G_M - I_M$ .

Since  $G'_M$  is no counterexample, and the union of  $M$  with any acyclic matching in  $G'_M$  is an acyclic matching in  $G$ , we obtain

$$\frac{6n}{\Delta^2 + 12\Delta^{\frac{3}{2}}} > \nu_{ac}(G) \geq |M| + \frac{6(n - |V_M \cup N_M \cup I_M|)}{\Delta^2 + 12\Delta^{\frac{3}{2}}},$$

which implies

$$|V_M| + |N_M| + |I_M| > \left( \frac{\Delta^2}{6} + 2\Delta^{\frac{3}{2}} \right) |M|. \quad (2)$$

**Claim 1.** *For every edge  $uv$  in  $G$ , we have  $d_G(u) + d_G(v) > 2\sqrt{\Delta}$ .*

*Proof.* Suppose, for a contradiction, that  $d_G(u) + d_G(v) \leq 2\sqrt{\Delta}$  for some edge  $uv$  of  $G$ . For  $M = \{uv\}$ , we obtain  $|V_M| + |N_M| + |I_M| \leq 2 + (2\sqrt{\Delta} - 2) + (2\sqrt{\Delta} - 2)(\Delta - 1) \leq 2\Delta^{\frac{3}{2}}$ , contradicting (2).  $\square$

Let  $S$  be the set of vertices of degree at most  $\sqrt{\Delta}$ . By Claim 1, the set  $S$  is independent.

**Claim 2.**  *$S$  is not empty.*

*Proof.* Suppose, for a contradiction, that the minimum degree  $\delta$  of  $G$  is larger than  $\sqrt{\Delta}$ . Let  $uv$  be an edge of  $G$  such that  $u$  is of minimum degree. Let  $M = \{uv\}$ . Since every vertex in  $I_M$  has degree at least  $\delta$ , we have

$$|V_M| + |N_M| + |I_M| \leq 2 + (\Delta + \delta - 2) + \frac{(\Delta + \delta - 2)(\Delta - 1)}{\delta} \leq \frac{(\Delta + \delta)^2}{\delta}.$$

If  $\Delta = 3$ , then  $\delta$  is 2 or 3, and in both cases  $2 + (\Delta + \delta - 2) + \frac{(\Delta + \delta - 2)(\Delta - 1)}{\delta}$  is less than the right hand side of (2), contradicting (2). For  $\Delta \geq 4$ , we obtain that  $\frac{(\Delta + \delta)^2}{\delta} \leq \frac{(\Delta + \sqrt{\Delta})^2}{\sqrt{\Delta}}$  is less than the right hand side of (2). Hence, also in this case, we obtain a contradiction (2).  $\square$

Let  $N$  be the set of vertices that have a neighbor in  $S$ , and, for a vertex  $v$  in  $G$ , let  $d_S(v)$  be the number of neighbors of  $v$  in  $S$ . Since  $S$  is independent, the sets  $S$  and  $N$  are disjoint.

**Claim 3.**  $\max\{d_S(v) : v \in V(G)\} = \alpha\Delta$  for some  $\alpha$  with  $0.2 \leq \alpha \leq 0.8$ .

*In other words, we have  $d_S(v) \leq 0.8\Delta$  for every vertex  $v$  of  $G$ , and  $d_S(v) \geq 0.2\Delta$  for some vertex  $v$  of  $G$ .*

*Proof.* Let the vertex  $v$  maximize  $d_S(v)$ . Suppose, for a contradiction, that  $d_S(v) = \alpha\Delta$  for some  $\alpha$  with either  $\alpha < 0.2$  or  $\alpha > 0.8$ . Let  $u$  be a neighbor of  $v$  of minimum degree. By Claim 2, we have  $d_S(v) \geq 1$ , which implies  $d_G(u) \leq \sqrt{\Delta}$ . Let  $M = \{uv\}$ . Clearly,

$$|V_M| + |N_M| \leq \sqrt{\Delta} + \Delta.$$

Let  $I_1$  be the set of vertices in  $I_M$  that have a neighbor in  $N_G(u) \cup (N_G(v) \cap S)$ , let  $I_2 = (I_M \setminus I_1) \cap S$ , and let  $I_3 = I_M \setminus (I_1 \cup I_2)$ .

We obtain

$$\begin{aligned} |I_1| &\leq (\Delta - 1)(d_G(u) - 1) + (\sqrt{\Delta} - 1) |N_G(v) \cap S| \\ &\leq (\Delta - 1) (\sqrt{\Delta} - 1) + (\sqrt{\Delta} - 1) \alpha\Delta \\ &\leq (1 + \alpha)\Delta^{\frac{3}{2}} - (\sqrt{\Delta} + \Delta). \end{aligned}$$

Let  $N' = N_G(v) \setminus (N_G(u) \cup S)$ . Note that  $|N'| \leq (1 - \alpha)\Delta$ , and that the vertices in  $I_2 \cup I_3$  have all their neighbors in  $N'$ . By the choice of  $v$ , every vertex in  $N'$  has at most  $\alpha\Delta$  neighbors in  $S$ , which implies

$$|I_2| \leq \alpha\Delta|N'| \leq \alpha(1 - \alpha)\Delta^2.$$

Since there are at most  $\Delta|N'|$  edges between  $N'$  and  $I_3$ , and every vertex in  $I_3$  has degree more than  $\sqrt{\Delta}$ , we obtain

$$|I_3| < \frac{\Delta|N'|}{\sqrt{\Delta}} \leq (1 - \alpha)\Delta^{\frac{3}{2}}.$$

Altogether, we obtain

$$\begin{aligned} |V_M| + |N_M| + |I_M| &\leq \sqrt{\Delta} + \Delta + (1 + \alpha)\Delta^{\frac{3}{2}} - (\sqrt{\Delta} + \Delta) + \alpha(1 - \alpha)\Delta^2 + (1 - \alpha)\Delta^{\frac{3}{2}} \\ &= \alpha(1 - \alpha)\Delta^2 + 2\Delta^{\frac{3}{2}} \\ &\leq 0.16\Delta^2 + 2\Delta^{\frac{3}{2}}, \end{aligned}$$

contradicting (2). □

Note that, so far in the proof of each claim, we had  $|M| = 1$ , and iteratively applying the corresponding reductions would eventually lead to an induced matching in  $G$  similarly as in [9]. In order to improve (1), we now choose  $M$  non-locally in some sense: Let  $M$  be an acyclic matching in  $G$  such that

- (i)  $M$  only contains edges incident to a vertex in  $S$ ,
- (ii) every vertex in  $V_M \cap S$  has degree one in the subgraph of  $G$  induced by  $V_M$ ,
- (iii) every vertex  $v$  in  $V_M \cap N$  satisfies  $d_S(v) \geq 0.2\Delta$ , and

$M$  maximizes

$$\sum_{v \in V_M \cap N} d_S(v). \tag{3}$$

among all acyclic matchings satisfying (i), (ii), and (iii). By Claim 3, the matching  $M$  is non-empty.

We now define certain relevant sets, see Figure 1 for an illustration.

- Let  $X$  be the set of vertices in  $N_M$  that are not adjacent to a vertex in  $V_M \cap S$  and that have at least one neighbor in  $S$  that is not adjacent to a vertex in  $V_M$ .  
(Note that  $X \subseteq N$ , and that the edges between vertices in  $X$  and suitable neighbors in  $S$  are possible candidates for modifying  $M$ .)
- Let  $Y$  be the set of vertices in  $N_M \setminus X$  that are not adjacent to a vertex in  $V_M \cap S$ .  
(Note that  $Y$  contains  $N_M \setminus N = (N_M \cap S) \cup (N_M \setminus (S \cup N))$ .)
- Let  $Z = (N \cap N_M) \setminus (X \cup Y)$ .  
(Note that  $Z$  consists of the vertices in  $N_M$  that have a neighbor in  $V_M \cap S$ .)
- Let  $I_1$  be the set of vertices in  $I_M \cap S$  that have a neighbor in  $N_M \setminus X$ .  
(Note that, by the definition of  $X$ , no vertex in  $I_1$  can have a neighbor in  $Y \cap N$ , which implies that every vertex in  $I_1$  has a neighbor in  $Z$ .)
- Let  $I_2$  be the set of vertices in  $I_M \setminus S$  that have a neighbor in  $Z$ .
- Let  $I_3$  be the set of vertices in  $I_M \cap S$  that only have neighbors in  $X$ .  
(Note that  $I_1 \cup I_3 = I_M \cap S$ .)
- Finally, let  $I_4 = I_M \setminus (I_1 \cup I_2 \cup I_3)$ .

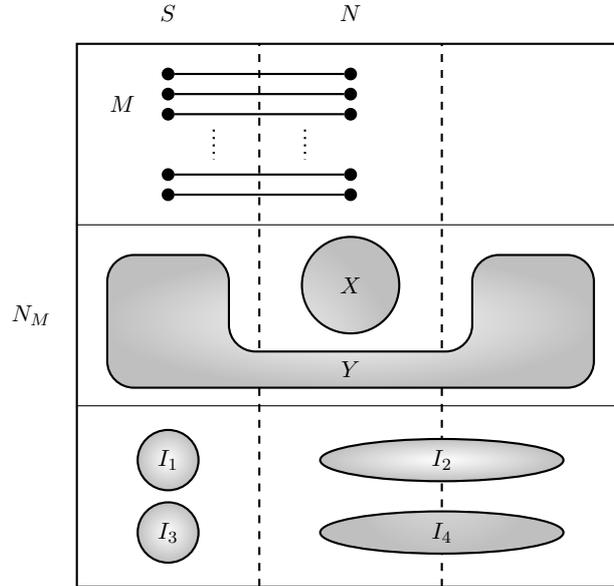


Figure 1: An illustration of the different relevant sets.

Clearly,

$$|V_M| + |N_M| \leq (\sqrt{\Delta} + \Delta) |M|. \quad (4)$$

Since every vertex in  $I_1 \cup I_2$  has a neighbor in  $Z$ , and every vertex in  $Z$  has a neighbor in  $V_M \cap S$ , we have

$$|I_1 \cup I_2| \leq (\Delta - 1) |Z| \leq (\Delta - 1) (\sqrt{\Delta} - 1) |M| = \left( \Delta^{\frac{3}{2}} - \Delta - \sqrt{\Delta} + 1 \right) |M|. \quad (5)$$

Since every vertex in  $I_4$  has degree more than  $\sqrt{\Delta}$  and has all its neighbors in  $X \cup Y$ , and every vertex in  $X \cup Y$  has a neighbor in  $V_M \cap N$ , we have

$$|I_4| \leq \frac{(\Delta - 1)|X \cup Y|}{\sqrt{\Delta}} \leq \frac{(\Delta - 1)^2|M|}{\sqrt{\Delta}} = \left( \Delta^{\frac{3}{2}} - 2\sqrt{\Delta} + \frac{1}{\sqrt{\Delta}} \right) |M|. \quad (6)$$

Combining (4), (5), and (6), we obtain

$$|V_M| + |N_M| + |I_M| - |I_3| \leq 2\Delta^{\frac{3}{2}}. \quad (7)$$

In order to estimate  $|I_3|$ , we partition the set  $X$  as follows:

- Let  $X_1$  be the set of vertices  $v$  in  $X$  with  $d_S(v) < 0.2\Delta$ ,
- let  $X_2$  be the set of vertices in  $X \setminus X_1$  with at least four neighbors in  $V_M$ , and
- let  $X_3 = X \setminus (X_1 \cup X_2)$ .

For a vertex  $v$  in  $V_M \cap N$ , let  $d_3(v)$  be the number of neighbors of  $v$  in  $X_3$ .

**Claim 4.**  $|I_3| \leq 0.2\Delta|X_1| + 0.8\Delta|X_2| + \frac{2}{3} \sum_{v \in V_M \cap N} d_S(v)d_3(v)$ .

*Proof.* By Claim 3, we obtain that

$$|I_3| \leq \sum_{w \in X} d_S(w) = \sum_{w \in X_1 \cup X_2 \cup X_3} d_S(w) \leq 0.2\Delta|X_1| + 0.8\Delta|X_2| + \sum_{w \in X_3} d_S(w).$$

Let  $w$  be a vertex in  $X_3$ . By the definition of  $X$ , the vertex  $w$  has a neighbor  $u$  in  $S$  that is not adjacent to a vertex in  $V_M$ . If  $w$  has only one neighbor in  $V_M$ , then  $M \cup \{wu\}$  is an acyclic matching satisfying (i), (ii), and (iii) that has a larger value in (3), contradicting the choice of  $M$ . Hence, we may assume that  $w$  has either  $k = 2$  or  $k = 3$  neighbors  $v_1, \dots, v_k$  in  $V_M$ . Let  $u_1v_1, \dots, u_kv_k$  be edges in  $M$ , and suppose that  $d_S(v_1) \leq \dots \leq d_S(v_k)$ . Since

$$M' = (M \cup \{wu\}) \setminus \{u_1v_1, \dots, u_{k-1}v_{k-1}\}$$

is an acyclic matching satisfying (i), (ii), and (iii), the choice of  $M$  implies that the value of  $M'$  in (3) is at most the one of  $M$ , which implies

$$d_S(w) \leq \sum_{i=1}^{k-1} d_S(v_i) \leq \frac{k-1}{k} \sum_{i=1}^k d_S(v_i) \leq \frac{2}{3} \sum_{i=1}^k d_S(v_i).$$

Now, we obtain

$$\sum_{w \in X_3} d_S(w) \leq \frac{2}{3} \sum_{w \in X_3} \sum_{v \in V_M \cap N \cap N_G(w)} d_S(v) = \frac{2}{3} \sum_{v \in V_M \cap N} d_3(v)d_S(v),$$

which completes the proof.  $\square$

For a vertex  $v$  in  $V_M \cap N$ , let  $d_1(v)$  be the number of neighbors of  $v$  in  $X_1 \cup X_2$ . By property (iii), we have  $d_S(v) \geq 0.2\Delta$ , which implies that  $d_1(v) \leq 0.8\Delta$ . Using Claim 4,  $xy \leq \frac{(x+y)^2}{4}$  for  $x, y \geq 0$ , and

$d_S(v) + d_1(v) + d_3(v) \leq \Delta$  and  $d_1(v)^2 \leq 0.8\Delta d_1(v)$  for  $v \in V_M \cap N$ , we obtain

$$\begin{aligned}
|I_3| &\leq 0.2\Delta|X_1| + 0.8\Delta|X_2| + \frac{2}{3} \sum_{v \in V_M \cap N} d_S(v)d_3(v) \\
&\leq 0.2\Delta(|X_1| + 4|X_2|) + \frac{1}{6} \sum_{v \in V_M \cap N} (d_S(v) + d_3(v))^2 \\
&\leq 0.2\Delta \sum_{v \in V_M \cap N} d_1(v) + \frac{1}{6} \sum_{v \in V_M \cap N} (\Delta - d_1(v))^2 \\
&= \frac{\Delta^2}{6}|M| + \Delta \left( \frac{1}{5} - \frac{1}{3} \right) \sum_{v \in V_M \cap N} d_1(v) + \frac{1}{6} \sum_{v \in V_M \cap N} d_1(v)^2 \\
&\leq \frac{\Delta^2}{6}|M| + \Delta \left( \frac{2}{15} - \frac{2}{15} \right) \sum_{v \in V_M \cap N} d_1(v) \\
&= \frac{\Delta^2}{6}|M|,
\end{aligned}$$

and together with (7), we obtain a final contradiction to (2) completing the proof.  $\square$

### 3 Conclusion

While the choice of  $M$  after Claim 3 in the proof is non-constructive, the proof of Theorem 1 easily yields an efficient algorithm that returns an acyclic matching in a given input graph  $G$  as considered in Theorem 1 with size at least  $\frac{6n}{\Delta^2 + 12\Delta^{\frac{3}{2}}}$ . If the statements of Claims 1, 2, or 3 fail, then their proofs contain simple reduction rules, each fixing one edge in the final acyclic matching and producing a strictly smaller instance  $G'_M$ . Adding that fixed edge to the output on the instance  $G'_M$  yields the desired acyclic matching. The matching  $M$  chosen after Claim 3 can be initialized as any acyclic matching satisfying (i), (ii), and (iii). If Claim 4 fails, then its proof contains simple update procedures that increase the value in (3). Since this value is integral and polynomially bounded, after polynomially many updates the statement of Claim 4 holds, and adding  $M$  to the output on the instance  $G'_M$  yields the desired acyclic matching.

The acyclic matchings  $M$  produced by the proof of Theorem 1 actually have a special structure because the subgraph  $H$  of  $G$  induced by the set of vertices that are incident to an edge in  $M$  is not just any forest but a so-called *corona* of a forest, that is, every vertex  $v$  of  $H$  of degree at least 2 in  $H$  has a unique neighbor  $u$  of degree 1 in  $H$ , and all the edges  $uv$  form  $M$ .

As a generalization of acyclic matchings, [1] introduced the notion of a *k-degenerate matching* as a matching  $M$  in a graph  $G$  such that the subgraph  $H$  of  $G$  defined as above is  $k$ -degenerate. If the *k-degenerate matching number*  $\nu_k(G)$  of  $G$  denotes the largest size of a  $k$ -degenerate matching in  $G$ , then  $\nu_1(G)$  coincides with the acyclic matching number. We conjecture that

$$\nu_k(G) \geq \frac{(k+1)n}{(\lfloor \frac{\Delta}{2} \rfloor + 1)(\lceil \frac{\Delta}{2} \rceil + 1)}$$

for every graph  $G$  with  $n$  vertices, sufficiently large maximum degree  $\Delta$ , and no isolated vertex. A

straightforward adaptation of the proof of Theorem 1 yields

$$\frac{\nu_k(G)}{n} \geq \begin{cases} (1 - o(1)) \frac{4(k+3)}{3\Delta^2} & \text{for } k \in \{2, 3, 4, 5, 6\} \text{ and} \\ (1 - o(1)) \frac{k+4}{\Delta^2} & \text{for } k \geq 7. \end{cases}$$

for these graphs  $G$ .

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