On the construction of cospectral nonisomorphic bipartite graphs^{*}

M. Rajesh Kannan[†] Shivaramakrishna Pragada[‡]

Hitesh Wankhede §

April 12, 2022

Abstract

In this article, we construct bipartite graphs which are cospectral for both the adjacency and normalized Laplacian matrices using the notion of partitioned tensor products. This extends the construction of Ji, Gong, and Wang [9]. Our proof of the cospectrality of adjacency matrices simplifies the proof of the bipartite case of Godsil and McKay's construction [4], and shows that the corresponding normalized Laplacian matrices are also cospectral. We partially characterize the isomorphism in Godsil and McKay's construction, and generalize Ji et al.'s characterization of the isomorphism to biregular bipartite graphs. The essential idea in characterizing the isomorphism uses Hammack's cancellation law as opposed to Hall's marriage theorem used by Ji et al.

AMS Subject Classification(2010): 05C50.

Keywords. Adjacency matrix, Normalized Laplacian matrix, Cospectral bipartite graphs, Hammack's cancellation law, Partitioned tensor product.

1 Introduction

We consider simple and undirected graphs. Let G = (V(G), E(G)) be a graph with the vertex set $V(G) = \{1, 2, ..., n\}$ and the edge set E(G). If two vertices *i* and *j* of *G* are adjacent, we denote it by $i \sim j$. For a graph *G* on *n* vertices, the *adjacency matrix* $A(G) = [a_{ij}]$ is the $n \times n$ matrix defined by

$$a_{ij} = \begin{cases} 1, & \text{if } i \sim j, \\ 0, & \text{otherwise.} \end{cases}$$

^{*}The contents of this article are included in the last author's MS thesis [14].

[†]Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur 721 302, India. Email: rajeshkannan@maths.iitkgp.ac.in, rajeshkannan1.m@gmail.com

[‡]Department of Aerospace Engineering, Indian Institute of Technology Kharagpur, Kharagpur 721 302, India. Email: shivaram@iitkgp.ac.in, shivaramkratos@gmail.com

[§]Department of Mathematics, Indian Institute of Science Education and Research Pune, Pune 411 008, India. Email: hitesh.wankhede@students.iiserpune.ac.in, hiteshwankhede9@gmail.com

The spectrum of a graph G is the set of all eigenvalues of A(G), with corresponding multiplicities. Two graphs are *cospectral for the adjacency matrices* if they have the same adjacency spectrum. It has been a longstanding problem to characterize graphs that are determined by their spectrum [12, 13]. If any graph which is cospectral with G is also isomorphic to it, then G is said to be determined by its spectrum (DS graph for short), otherwise we say that the graph G has a cospectral mate or we say that G is not determined by its spectrum (NDS for short). To show that a graph is NDS, we provide a construction of a cospectral mate. In [11], Schwenk proved that almost all trees are NDS. In [5], Godsil and McKay provided a method for constructing cospectral nonisomorphic graphs. In [12], van Dam and Haemers mentioned that: If we were to bet, it would be for: 'almost all graphs are DS'. Later Haemers conjectured the same in [6]. For each vertex i of a graph G, let d_i denote the degree of the vertex i. Let D(G) denote the diagonal degree matrix whose (i, i)-th entry is d_i . Then the matrix L(G) = D(G) - A(G) is the Laplacian matrix of the graph G, and if G has no isolated vertices, then the matrix $\mathcal{L}(G) = I - D(G)^{-\frac{1}{2}}A(G)D(G)^{-\frac{1}{2}}$ is the normalized Laplacian matrix of G. The normalized Laplacian spectrum of a graph G is the spectrum of $\mathcal{L}(G)$. Two graphs are cospectral for the normalized Laplacian matrices if they have the same normalized Laplacian spectrum. For more details, we refer to [1, 3, 5, 12, 13].

In [4], Godsil and McKay constructed cospectral graphs for the adjacency matrices using the notion of partitioned tensor products of matrices (See Section 2 for the definition). Recently, Ji, Gong, and Wang proposed a construction for cospectral bipartite graphs for the adjacency and normalized Laplacian matrices using the unfolding technique [9]. This construction is a generalization of the unfoldings of a bipartite graph considered by Butler[2]. In this paper, first, we note that the proof of the construction of cospectral bipartite graphs [9, Theorem 2.1] can be done by expressing the matrices involved as the partitioned tensor products, and our proof works for larger classes of graphs. This is done in Theorem 3.1 (for adjacency matrices) and Theorem 3.3 (for normalized Laplacian matrices). Also, the proof of Theorem 3.1 provides an alternate proof of Godsil and McKay's result for the bipartite graphs.

Weichsel proved that if G_1 and G_2 are two connected bipartite graphs, then their direct product $G_1 \times G_2$ has exactly two connected bipartite components [15]. Jha, Klavžar and Zmazek [8] showed that if either G_1 or G_2 admits an automorphism that interchanges its partite sets, then the components of $G_1 \times G_2$ are isomorphic. Hammack [7] proved that the converse is also true. Hammack's proof uses a cancellation property (see Theorem 2.2), which we call Hammack's cancellation law. Surprisingly, we are able to use this result to prove the isomorphism of the cospectral graphs that we construct in Section 3. The characterization theorem for isomorphism of Ji et al. [9, Theorem 3.1], is a particular case of our result. Also, their proof uses Hall's Marriage theorem, whereas we do not.

The outline of this paper is as follows: In Section 2, we include some needed known results for graphs and matrices. In Section 3, the main results about the construction of cospectral bipartite graphs for the adjacency and normalized Laplacian matrices are stated and proved. Section 4 is devoted to the study of the existence of isomorphisms between the cospectral pairs constructed in Section 3.

2 Preliminaries

The notion of partitioned tensor products of matrices is used extensively in this article. This is closely related to the well known Kronecker product of matrices. The *Kronecker product* of matrices $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ of size $m \times n$ and B of size $p \times q$, denoted by $A \otimes B$, is the $mp \times nq$ block matrix $\begin{bmatrix} a_{ij}B \end{bmatrix}$. The partitioned tensor product of two partitioned matrices $M = \begin{bmatrix} U & V \\ W & X \end{bmatrix}$ and $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, denoted by $M \underline{\otimes} H$, is defined as $\begin{bmatrix} U \otimes A & V \otimes B \\ W \otimes C & X \otimes D \end{bmatrix}$. Given the matrices U, V, W and X, define $\mathcal{I}(U, X) = \begin{bmatrix} U & 0 \\ 0 & X \end{bmatrix}$ and $\mathcal{P}(V, W) = \begin{bmatrix} 0 & V \\ W & 0 \end{bmatrix}$ where 0 is the zero matrix of appropriate order. A 2×2 block matrix is diagonal (resp., an anti diagonal) block matrix if it is of the form $\mathcal{I}(U, X)$ (resp., $\mathcal{P}(V, W)$). The above notions were introduced by Godsil and McKay [4]. The following proposition is easy to verify.

Proposition 2.1. Let Q and R be the matrices of the form $\mathcal{I}(Q_1, Q_2)$ and $\mathcal{I}(R_1, R_2)$, respectively. If $M = \begin{bmatrix} U & V \\ W & X \end{bmatrix}$ and $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ are 2×2 block matrices, then $(Q \otimes R)(M \otimes H) = (QM) \otimes (RH).$

The same holds true when the matrices Q and R are both of the form $\mathcal{P}(Q_1, Q_2)$ and $\mathcal{P}(R_1, R_2)$, respectively.

Two matrices A and B are said to be *equivalent*, if there exists invertible matrices P and Q such that $Q^{-1}AP = B$. If the matrices P and Q are orthogonal, then matrices A and B are said to be *orthogonally equivalent*. If the matrices P and Q are permutation matrices, then matrices A and B are said to be *permutationally equivalent*. Using the singular value decomposition, it is easy to see that any square matrix is orthogonally equivalent to its transpose.

A square matrix A is said to be a *PET matrix* if it is permutationally equivalent to its transpose. If the set of row sums of an $n \times n$ matrix A is different from the set of columns sums of A, then A is non-PET.

We recall the cancellation law of matrices given by Hammack.

Theorem 2.2. [7, Lemma 3] Let A, B and C be (0, 1)-matrices. Let C be a non-zero matrix and A be a square matrix with no zero rows. Then, the matrices $C \otimes A$ and $C \otimes B$ are permutationally equivalent if and only if A and B are permutationally equivalent. Similarly, the matrices $A \otimes C$ and $B \otimes C$ are permutationally equivalent if and only if A and B are permutationally equivalent.

An isomorphism of two graphs G_1 and G_2 is a bijection $f: V(G_1) \longrightarrow V(G_2)$ such that any two vertices u and v are adjacent in G_1 if and only if f(u) and f(v) are adjacent in G_2 . Two graphs G_1 and G_2 are isomorphic if there exists an isomorphism between them. It is easy to see that G_1 and G_2 are isomorphic if and only if the corresponding adjacency matrices are permutationally similar. An *automorphism* of a graph G is an isomorphism from the graph G to itself and the set of automorphisms $\operatorname{Aut}(G)$ of a graph is a group with respect to the composition of functions. Every automorphism of a graph G on n vertices can be represented by an $n \times n$ permutation matrix. Thus $\operatorname{Aut}(G)$ can be identified with the set of permutation matrices P such that $P^T A(G)P = A(G)$.

A graph G is *bipartite* if its vertex set can be partitioned into two parts X and Y such that every edge has one end in X and the other end in Y. We refer to $V(G) = X \cup Y$ as a bipartition of G, and X and Y as the *partite sets* of G. A bipartite graph is *balanced* if its partite sets have the same number of elements. If G is a bipartite graph with the adjacency matrix $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$, then the matrix B is the *biadjacency matrix* of G. Let G_1 and G_2 be two isomorphic bipartite graphs, and let $V(G_i) = X_i \cup Y_i$ be the bipartition of G_i for $i \in \{1, 2\}$. An isomorphism f from G_1 to G_2 respects the partite sets if it satisfies either $f(X_1) = X_2$ and $f(Y_1) = Y_2$ or $f(X_1) = Y_2$ and $f(Y_1) = X_2$. If G_1 and G_2 are two connected isomorphic bipartite graphs, then any isomorphism between them respects the partite sets.

Let G be a bipartite graph whose partite sets are X and Y. An automorphism f of G fixes the partite sets if f(X) = X and f(Y) = Y, and interchanges the partite sets if f(X) = Y and f(Y) = X.

Definition 2.3. [7] A connected bipartite graph has property π if it admits an automorphism that interchanges its partite sets.

The next proposition connects PET matrices to automorphisms of bipartite graphs. We skip the proof.

Proposition 2.4. Let G be a bipartite graph with the adjacency matrix $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$. Then the biadjacency matrix B is PET if and only if there exists an automorphism $f \in \operatorname{Aut}(G)$ that interchanges its partite sets, where the partite sets are induced by the biadjacency matrix B.

The direct product or tensor product $G_1 \times G_2$ of graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ (the cartesian product $V(G_1)$ and $V(G_2)$) and two vertices (g, h)and (g', h') are adjacent in $G_1 \times G_2$ if and only if g is adjacent to g' in G_1 and h is adjacent to h' in G_2 . The adjacency matrix of the graph $G_1 \times G_2$ is given by $A(G_1) \otimes A(G_2)$. Here $A(G_i)$ denotes the adjacency matrix of the graph G_i (for i = 1, 2), and \otimes denotes the Kronecker product.

3 Construction of the cospectral pairs

In this section, we give a construction of cospectral bipartite graphs for both the adjacency and the normalized Laplacian matrices. Let I_m and 0_n denote the identity and zero matrices of orders m and n, respectively. Let V be an $m \times n$ matrix and B be a $p \times q$ matrix. Define the matrices $L = \begin{bmatrix} 0 & V \\ V^T & 0 \end{bmatrix}$, $H = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$ and $H^{\#} = \begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix}$. Note that the matrices $L \triangleq \begin{bmatrix} 0 & V \\ V^T & 0 \end{bmatrix}$ and $L \triangleq H^{\#} = \begin{bmatrix} 0 & V \\ B & 0 \end{bmatrix}$ are of orders mp + nq and mq + np, respectively.

For an $n \times n$ symmetric (0, 1) matrix A with zero diagonal entries, let G_A denote the simple graph whose adjacency matrix is A. The (0, 1) matrices V, B and B^T are the biadjacency matrices for the bipartite graphs G_L , G_H and $G_{H^{\#}}$, respectively. Since H and $H^{\#}$ are permutationally similar, the graphs G_H and $G_{H^{\#}}$ are isomorphic. The next theorem is about a construction of cospectral graphs for the adjacency matrices. The proof of this theorem could found in [4]. Our proof gives an alternate simple proof for this result, and the proof idea could be extended for normalized Laplacian matrices as well.

Theorem 3.1. The bipartite graphs $G_{L \otimes H}$ and $G_{L \otimes H^{\#}}$ are cospectral for the adjacency matrices if and only if at least one of the bipartite graphs G_L or G_H is balanced.

Proof. First let us show if either m = n or p = q, the matrices $L \otimes H$ and $L \otimes H^{\#}$ are orthogonally similar, and hence they are cospectral.

Case 1: Let m = n. Then V is a square matrix and V is orthogonally equivalent to V^T . Thus there exist two orthogonal matrices R_1 and R_2 such that $R_1^T V R_2 = V^T$. Define

 $R = \mathcal{P}(R_1, R_2)$. Now,

$$R^{T}LR = \begin{bmatrix} 0 & R_{1} \\ R_{2} & 0 \end{bmatrix}^{T} \begin{bmatrix} 0 & V \\ V^{T} & 0 \end{bmatrix} \begin{bmatrix} 0 & R_{1} \\ R_{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & R_{2}^{T}V^{T}R_{1} \\ R_{1}^{T}VR_{2} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & V \\ V^{T} & 0 \end{bmatrix} = L.$$

Let $Q = \mathcal{P}(I_p, I_q)$ and $P = R \otimes Q$. Then Q is a permutation matrix, P is an orthogonal matrix and Q satisfies $Q^T H Q = H^{\#}$. Now

$$P^{T}(L\underline{\otimes}H)P = (R\underline{\otimes}Q)^{T}(L\underline{\otimes}H)(R\underline{\otimes}Q)$$
$$= (R^{T}LR)\underline{\otimes}(Q^{T}HQ)$$
$$= L\underline{\otimes}H^{\#}.$$

Note that the second step uses Proposition 2.1. Thus $G_{L \otimes H}$ and $G_{L \otimes H}$ are cospectral. **Case 2:** Let p = q. Then the matrix B is orthogonally equivalent to B^T . Hence, there exist two orthogonal matrices Q_1 and Q_2 such that $Q_1^T B Q_2 = B^T$. Let $Q = \mathcal{I}(Q_1, Q_2)$. Then $Q^T H Q = H^{\#}$. Let $R = \mathcal{I}(I_m, I_n)$ and $P = R \otimes Q$. Then $R^T L R = L$. Rest of the proof of this case is similar to that of Case 1.

Conversely, let the matrices $L \underline{\otimes} H$ and $L \underline{\otimes} H^{\#}$ have the same spectrum. Then mp + nq = mq + np, and hence (m - n)(p - q) = 0. Thus the result follows.

Next, we establish an identity for the partitioned tensor product for the normalized Laplacian matrices, which is useful in the construction of cospectral graphs for the normalized Laplacian matrices.

Lemma 3.2. Let G_1 and G_2 be two bipartite graphs with no isolated vertices. If the matrices $A(G_1)$ (resp., $A(G_2)$) and $\mathcal{L}(G_1)$ (resp., $\mathcal{L}(G_2)$) are partitioned into 2×2 matrices conformally with the partite sets, then $\mathcal{L}(G_{A(G_1)\underline{\otimes}A(G_2)}) = 2I - (\mathcal{L}(G_1)\underline{\otimes}\mathcal{L}(G_2))$.

Proof. Let $D(G_1)$ and $D(G_2)$ denote the degree matrices corresponding to the graphs G_1 and G_2 respectively. Then,

$$\mathcal{L}(G_{A(G_1)\underline{\otimes}A(G_2)}) = I - \left(D(G_{A(G_1)\underline{\otimes}A(G_2)})^{-1/2} (A(G_1)\underline{\otimes}A(G_2)) D(G_{A(G_1)\underline{\otimes}A(G_2)})^{-1/2}\right)$$
$$= I - \left(D(G_1)^{-1/2} A(G_1) D(G_1)^{-1/2}\right) \underline{\otimes} \left(D(G_2)^{-1/2} A(G_2) D(G_2)^{-1/2}\right)$$
$$= I - \left((I - \mathcal{L}(G_1))\underline{\otimes}(I - \mathcal{L}(G_2))\right)$$
$$= 2I - (\mathcal{L}(G_1)\underline{\otimes}\mathcal{L}(G_2)).$$

In the next theorem, we establish the cospectrality for the normalized Laplacian matrices of partitioned tensor products graphs $G_{L\otimes H}$ and $G_{L\otimes H^{\#}}$.

Theorem 3.3. Let G_L and G_H be two bipartite graphs with no isolated vertices. The bipartite graphs $G_{L \otimes H}$ and $G_{L \otimes H^{\#}}$ are cospectral for the normalized Laplacian matrices if and only if at least one of G_L or G_H is balanced.

Proof. Let $D(G_L)$, $D(G_H)$ and $D(G_{H^{\#}})$ denote the degree matrices for the graphs G_L , G_H and $G_{H^{\#}}$, respectively. Let either m = n or p = q.

Case 1: Suppose m = n. Then V is an $n \times n$ matrix. Let $D(G_L) = \mathcal{I}(C_1, C_2)$ where C_1 and C_2 are $n \times n$ diagonal degree matrices of the respective partite sets. Since G_L does not have any isolated vertices, $C_1^{-1/2}$ and $C_2^{-1/2}$ exist. Let $E = C_1^{-1/2}VC_2^{-1/2}$. Then there exist two orthogonal matrices R_1 and R_2 such that $E = R_2^T E^T R_1$. Set $R = \mathcal{P}(R_1, R_2)$. Now,

$$\mathcal{L}(G_L) = I - D(G_L)^{-1/2} A(G_L) D(G_L)^{-1/2}$$

$$= I - \begin{bmatrix} 0 & E \\ E^T & 0 \end{bmatrix}$$

$$= I - \begin{bmatrix} 0 & R_2^T E^T R_1 \\ R_2 E R_1^T & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & R_1 \\ R_2 & 0 \end{bmatrix}^T \left(I - \begin{bmatrix} 0 & E \\ E^T & 0 \end{bmatrix} \right) \begin{bmatrix} 0 & R_1 \\ R_2 & 0 \end{bmatrix}$$

$$= R^T \mathcal{L}(G_L) R.$$

The permutation matrix $Q = \mathcal{P}(I_p, I_q)$ satisfies $Q^T \mathcal{L}(G_H)Q = \mathcal{L}(G_{H^{\#}})$, and $P = R \underline{\otimes} Q$ is an orthogonal matrix. By Proposition 2.1, it follows that $P^T(\mathcal{L}(G_L)\underline{\otimes}\mathcal{L}(G_H))P = \mathcal{L}(G_L)\underline{\otimes}\mathcal{L}(G_{H^{\#}})$.

Case 2: Suppose p = q. Then B is a $p \times p$ matrix. Let $D(G_H) = \mathcal{I}(D_1, D_2)$ where D_1 and D_2 are $p \times p$ diagonal matrices. Then, $D(G_{H^{\#}}) = \mathcal{I}(D_2, D_1)$. Since G_H does not have any isolated vertices, $D_1^{-1/2}$ and $D_2^{-1/2}$ exist. Let $F = D_1^{-1/2} B D_2^{-1/2}$. Then there exist two orthogonal matrices Q_1 and Q_2 such that $Q_1^T F Q_2 = F^T$. Let $Q = \mathcal{I}(Q_1, Q_2)$, R be the identity matrix such that $R^T \mathcal{L}(G_L)R = \mathcal{L}(G_L)$ and $P = R \otimes Q$. Then $P^T(\mathcal{L}(G_L) \otimes \mathcal{L}(G_H))P =$ $\mathcal{L}(G_L) \otimes \mathcal{L}(G_{H^{\#}})$.

In both the cases, the matrices $\mathcal{L}(G_L) \underline{\otimes} \mathcal{L}(G_H)$ and $\mathcal{L}(G_L) \underline{\otimes} \mathcal{L}(G_{H^{\#}})$ are orthogonally similar. By Lemma 3.2, we have $\mathcal{L}(G_{L\underline{\otimes}H}) = 2I - \mathcal{L}(G_L) \underline{\otimes} \mathcal{L}(G_H)$ and $\mathcal{L}(G_{L\underline{\otimes}H^{\#}}) = 2I - \mathcal{L}(G_L) \underline{\otimes} \mathcal{L}(G_{H^{\#}})$. Then the matrices $\mathcal{L}(G_{L\underline{\otimes}H})$ and $\mathcal{L}(G_{L\underline{\otimes}H^{\#}})$ are orthogonally similar, and hence they are cospectral.

Converse is easy to verify.

Remark 3.4. By taking $V = J_{m,n}, J_{1,n}$, and $J_{1,2}$, in Theorem 3.1 and Theorem 3.3, we get

the constructions of Ji et al. given in [9, Theorem 2.1], Kannan and Pragada given in [10, Theorem 3.1], and Butler given in [2, Theorem 2.1], respectively.

4 Property η and Isomorphism

We assume from here on that the bipartite graphs G_L and G_H have no isolated vertices, and they are not necessarily connected. In this section, we investigate the existence of isomorphism between the cospectral pair obtained in Theorem 3.1 and Theorem 3.3. We show that, under appropriate restrictions, the isomorphism is closely related to the PET matrices and the automorphisms of bipartite graphs.

If a bipartite graph G is disconnected, then G has more than one bipartition. Next we extend the property π for disconnected bipartite graphs. Let G be a bipartite graph with biadjacency matrix B. We say that G has property π with respect to B, if B is a PET matrix. We define the canonical partite sets of the partitioned tensor product graph $G_{L \otimes H}$ (resp., $G_{L \otimes H^{\#}}$) to be the bipartition induced by the biadjacency matrix $V \otimes B$ (resp., $V \otimes B^T$), where V and B are biadjacency matrices of G_L and G_H , respectively.

The next theorem connects the property π with the isomorphism between the graphs $G_{L\otimes H}$ and $G_{L\otimes H^{\#}}$.

Theorem 4.1. Let G_L and G_H be bipartite graphs (not necessarily connected) with biadjacency matrices V and B, respectively. Then the following statements are equivalent:

- (a) There exists an isomorphism between the graphs $G_{L \otimes H}$ and $G_{L \otimes H^{\#}}$ which respects the partite sets of the canonical bipartitions.
- (b) At least one of G_L or G_H has property π .

Proof. (a) \implies (b) By assumption, there exists a permutation matrix P of the form either $\mathcal{I}(P_1, P_4)$ or $\mathcal{P}(P_2, P_3)$, where P_i is a permutation matrix for $i \in \{1, 2, 3, 4\}$, such that

$$P^{T}\begin{bmatrix}0 & V \otimes B\\V^{T} \otimes B^{T} & 0\end{bmatrix}P = \begin{bmatrix}0 & V \otimes B^{T}\\V^{T} \otimes B & 0\end{bmatrix}.$$

If $P = \mathcal{I}(P_1, P_4)$, then $P_1^T(V \otimes B)P_4 = V \otimes B^T$. The matrices $V \otimes B$ and $V \otimes B^T$ are permutationally equivalent. Then, by Theorem 2.2, B is PET. If $P = \mathcal{P}(P_2, P_3)$, then $P_3^T(V^T \otimes B^T)P_2 = V \otimes B^T$. Hence V is PET.

(b) \implies (a) Let either G_L or G_H has property π . Suppose G_L has the property π , that is, G_L admits an automorphism that interchanges its partite sets. Then, there exist

two permutation matrices R_2 and R_3 such that $R^T L R = L$ where $R = \mathcal{P}(R_2, R_3)$. Set $Q = \mathcal{P}(I_p, I_q)$ and $P = R \otimes Q$. Then $Q^T H Q = H^{\#}$, and

$$P^{T}(L\underline{\otimes}H)P = (R\underline{\otimes}Q)^{T}(L\underline{\otimes}H)(R\underline{\otimes}Q)$$
$$= (R^{T}LR)\underline{\otimes}(Q^{T}HQ)$$
$$= L\underline{\otimes}H^{\#}.$$

Thus the graphs $G_{L \otimes H}$ and $G_{L \otimes H^{\#}}$ are isomorphic, and the isomorphism induced by the permutation matrix P respects the partite sets.

Proof idea of the other case is similar.

Weichsel proved that if G_L and G_H are two connected bipartite graphs, then $G_L \times G_H$ has exactly two connected bipartite components [15]. Godsil and McKay noted that the disjoint union of the bipartite graphs $G_{L \otimes H}$ and $G_{L \otimes H^{\#}}$ is the direct product $G_L \times G_H$ [4]. Hence, if G_L and G_H are connected, then the two connected components of $G_{L \otimes H}$ are $G_{L \otimes H}$ and $G_{L \otimes H^{\#}}$. Using these observations, we obtain Hammack's result as a corollary of Theorem 4.1.

Corollary 4.2. [7, Theorem 1] Suppose G_1 and G_2 are connected bipartite graphs. The two components of $G_1 \times G_2$ are isomorphic if and only if at least one of G_1 or G_2 has the property π .

The bipartite graphs constructed by Ji, Gong and Wang have the property that any isomorphism between them respects the partite sets [9, Lemma 3.2]. From Theorem 4.1, we observe that respecting partite sets is the key property for characterizing isomorphism in terms of PET matrices (property π). To this end, we define property η , which relaxes the connectedness assumption in the Hammack's result and includes a broader class of bipartite graphs.

Definition 4.3. Two bipartite graphs G_L and G_H are said to have property η , if whenever the bipartite graphs $G_{L \otimes H}$ and $G_{L \otimes H^{\#}}$ are isomorphic, there exists an isomorphism between $G_{L \otimes H}$ and $G_{L \otimes H^{\#}}$ that respects the partite sets of the canonical bipartitions.

Next we show that any pair of connected bipartite graphs have property η and thus property η is relaxation of connectedness requirement from Hammack's result [7, Theorem 1].

Theorem 4.4. Let G_L and G_H be connected bipartite graphs. Then, they have property η .

Proof. Suppose G_L and G_H are connected, and $G_{L \otimes H}$ and $G_{L \otimes H^{\#}}$ are isomorphic. Since $G_{L \otimes H}$ and $G_{L \otimes H^{\#}}$ are connected bipartite graphs, thus any isomorphism between them respects the partite sets. Hence G_L and G_H have property η .

A biregular bipartite graph is a bipartite graph G for which any two vertices in the same partite sets have the same degree as each other. If degree of the vertices in one of the partite sets is k and degree of the vertices in the other partite set is l, then the graph is said to be (k, l)-biregular. We say that a biregular bipartite graph has distinct degrees if $k \neq l$. Next we prove that property η is satisfied if one of the bipartite graphs is biregular with distinct degrees.

Theorem 4.5. Let G_L and G_H be bipartite graphs. If G_L is a non-empty (k, l)-regular bipartite graph with $k \neq l$, then G_L and G_H have property η .

Proof. Let the graphs $\Gamma_1 = G_{L \otimes H}$ and $\Gamma_2 = G_{L \otimes H^{\#}}$ be isomorphic. Then, Γ_1 and Γ_2 are cospectral. By Theorem 3.1, the graph G_H is balanced, since G_L cannot be balanced as $k \neq l$.

Let $V(\Gamma_i) = X_i \cup Y_i$ be the canonical vertex partitions of the graphs Γ_i for i = 1, 2. Without loss of generality, assume that k < l. Let f be an isomorphism from Γ_1 to Γ_2 . Let b_i and b'_i denote the i^{th} row sum of the matrices B and B^T , respectively. Let x_1 be the vertex of maximum degree in X_1 . Suppose that $f(x_1) \in Y_2$. Then $d_{\Gamma_1}(x_1) = lb_i$ for some $1 \le i \le p$, and $d_{\Gamma_2}(f(x_1)) = kb_j$ for some $1 \le j \le p$. Since the isomorphism preserves the degrees, we have $lb_i = kb_j$. Since x_1 has maximum degree in X_1 , $b_i \ge b_j$ for any $1 \le j \le p$, and hence $kb_j \ge lb_j$. If $b_j \ne 0$, then $k \ge l$, a contradiction to the initial assumption that k < l. Hence, if $x_1 \in X_1$, then $f(x_1) \in X_2$. If $b_j = 0$ then $lb_i = kb_j$, $b_i = 0$. But x_1 is a vertex of maximum degree lb_i in the set X_1 and thus B = 0. So we could choose $f(x_1) \in X_2$. In any case, $f(x_1) \in X_2$.

Let x_1, \ldots, x_{rm} be the vertices of X_1 with the same maximum degree such that $d_{\Gamma_1}(x_{1+(s-1)m}) = \ldots = d_{\Gamma_1}(x_{m+(s-1)m}) = lb_{i_s}$ for $s \in \{1, 2, \ldots, r\}$ where $b_{i_1} = \ldots = b_{i_r}$ for $1 \leq i_1, \ldots, i_r \leq p$. Then, using the previous argument, $f(x_1), \ldots, f(x_{rm}) \in X_2$ such that $d_{\Gamma_2}(f(x_{1+(s-1)m})) = \ldots = d_{\Gamma_2}(f(x_{m+(s-1)m})) = lb'_{j_s}$ for $s \in \{1, 2, \ldots, r\}$ where $b'_{j_1} = \ldots = b'_{j_r}$ for $1 \leq j_1, \ldots, j_r \leq p$. Define B' to be the matrix obtained by removing the i_s^{th} row and j_s^{th} column of B for all $s \in \{1, 2, \ldots, r\}$. Define Γ'_1 and Γ'_2 to be the induced bipartite graphs corresponding to the biadjacency matrices $V \otimes B'$ and $V \otimes B'^T$, respectively. Since Γ'_1 and Γ'_2 are isomorphic as well, apply the same argument for Γ'_1 and Γ'_2 until all the rows and columns of B are exhausted. Thus $f(X_1) = X_2$ and hence $f(Y_1) = Y_2$. Similarly, if k > l, then consider the set of vertices of maximum degree in Y_1 and show that $f(Y_1) = Y_2$ and hence $f(X_1) = X_2$. Hence, G_L and G_H satisfy property η .

Note that at each step, the vertices from both the partite sets of the induced bipartite graphs of Γ_1 and Γ_2 are being removed. This is justified since our motive is to first show just $f(X_1) = X_2$. Ji, Gong and Wang in Lemma 3.2. [9] remove vertices from only X_1 and X_2 at each step.

Since we are interested in the construction of cospectral nonisomorphic graphs, we use this result to construct cospectral graphs that are not isomorphic.

Theorem 4.6. Let G_L and G_H be bipartite graphs. Let G_L be a biregular bipartite graph with distinct degrees and let G_H be balanced. Then the graphs $G_{L \otimes H}$ and $G_{L \otimes H^{\#}}$ are nonisomorphic if and only if G_H does not admit an automorphism that interchanges its partite sets.

Proof. Since G_L is a biregular bipartite graph with distinct degrees, the corresponding $m \times n$ biadjacency matrix V has constant row sum k and constant column sum l. Since the sum of row sums must be the same as the sum of column sums, we have km = ln. But $k \neq l$, hence $m \neq n$. Hence, G_L has unequal partition sizes. Since G_H has equal partition sizes, by Theorem 3.1 and Theorem 3.3, the graphs $G_{L \otimes H}$ and $G_{L \otimes H^{\#}}$ are cospectral. Now, as G_L has unequal partitions sizes, it doesn't admit an automorphism that interchanges its partite sets. Hence, the condition for non-isomorphism follows from Theorems 4.1 and 4.5.

Now as a corollary, we obtain the result of Ji, Gong and Wang.

Corollary 4.7. [9, Theorem 3.1] Let $V = J_{m,n}$ such that $m \neq n$ and let B is a square matrix. Then, the bipartite graphs $G_{L \otimes H}$ and $G_{L \otimes H^{\#}}$ are cospectral for the adjacency as well as the normalized Laplacian matrices, and they are isomorphic if and only if B is PET.

Proof. Since $V = J_{m,n}$ and $m \neq n$, the corresponding bipartite graph G_L is a biregular bipartite graph with distinct degrees. Hence, the result follows from Theorem 4.6.

Now, let us illustrate the construction given in Theorem 4.6 with an example.

Example. The following pair of matrices V and B satisfy all the conditions stated in Theorem 4.5 and corresponding graphs are illustrated below.

$$V = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

In the following figure, G_L and G_H denote the bipartite graphs with biadjacency matrices V and B, respectively. The graphs $G_{L\underline{\otimes}H}$ and $G_{L\underline{\otimes}H^{\#}}$ are the cospectral nonisomorphic pairs. Note that this example of cospectral bipartite graphs is not obtainable from the results of Ji et al. and Hammack.



Figure 1: G_L , G_H , $G_{L\otimes H}$ and $G_{L\otimes H^{\#}}$

Acknowledgment:

We are indebted to the referees for the comments and suggestions. M. Rajesh Kannan would like to thank the Department of Science and Technology, India, for financial support through the projects MATRICS (MTR/2018/000986).

References

- Andries E. Brouwer and Willem H. Haemers. Spectra of graphs. Universitext. Springer, New York, 2012.
- [2] Steve Butler. A note about cospectral graphs for the adjacency and normalized Laplacian matrices. *Linear Multilinear Algebra*, 58(3-4):387–390, 2010.
- [3] Fan R. K. Chung. Spectral graph theory, volume 92 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997.
- [4] Chris D. Godsil and Brendan D. McKay. Products of graphs and their spectra. In Combinatorial mathematics, IV (Proc. Fourth Australian Conf., Univ. Adelaide, Adelaide, 1975), pages 61–72. Lecture Notes in Math., Vol. 560, 1976.
- [5] Chris D. Godsil and Brendan D. McKay. Constructing cospectral graphs. Aequationes Math., 25(2-3):257-268, 1982.

- [6] Willem H. Haemers. Are almost all graphs determined by their spectrum? Not. S. Afr. Math. Soc., 47(1):42–45, 2016.
- [7] Richard H. Hammack. Proof of a conjecture concerning the direct product of bipartite graphs. *European J. Combin.*, 30(5):1114–1118, 2009.
- [8] Pranava K. Jha, Sandi Klavžar, and Blaž Zmazek. Isomorphic components of Kronecker product of bipartite graphs. *Discuss. Math. Graph Theory*, 17(2):301–309, 1997.
- [9] Yizhe Ji, Shicai Gong, and Wei Wang. Constructing cospectral bipartite graphs. *Discrete Math.*, 343(10):112020, 7, 2020.
- [10] M. Rajesh Kannan and Shivaramakrishna Pragada. On the construction of cospectral graphs for the adjacency and the normalized Laplacian matrices. *Linear and Multilinear Algebra*, 0(0):1–22, September 2020.
- [11] Allen J. Schwenk. Almost all trees are cospectral. In New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich., 1971), pages 275–307, 1973.
- [12] Edwin R. van Dam and Willem H. Haemers. Which graphs are determined by their spectrum? volume 373, pages 241–272. 2003. Special issue on the Combinatorial Matrix Theory Conference (Pohang, 2002).
- [13] Edwin R. van Dam and Willem H. Haemers. Developments on spectral characterizations of graphs. *Discrete Math.*, 309(3):576–586, 2009.
- [14] Hitesh Wankhede. Constructing cospectral graphs using partitioned tensor product. MS Thesis. IISER Pune, 2021.
- [15] Paul M. Weichsel. The Kronecker product of graphs. Proc. Amer. Math. Soc., 13:47–52, 1962.