On the chromatic number of some P_5-free graphs*

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Abstract

Let G be a graph. We say that G is perfectly divisible if for each induced subgraph H of G, $V(H)$ can be partitioned into A and B such that $H[A]$ is perfect and $\omega(H[B]) < \omega(H)$. We use P_t and C_t to denote a path and a cycle on t vertices, respectively. For two disjoint graphs F_1 and F_2, we use $F_1 \cup F_2$ to denote the graph with vertex set $V(F_1) \cup V(F_2)$ and edge set $E(F_1) \cup E(F_2)$, and use $F_1 + F_2$ to denote the graph with vertex set $V(F_1) \cup V(F_2)$ and edge set $E(F_1) \cup E(F_2) \cup \{xy \mid x \in V(F_1)$ and $y \in V(F_2)\}$. In this paper, we prove that (i) $(P_5, C_5, K_{2,3})$-free graphs are perfectly divisible, (ii) $\chi(G) \leq 2\omega^2(G) - \omega(G) - 3$ if G is $(P_5, K_{2,3})$-free with $\omega(G) \geq 2$, (iii) $\chi(G) \leq \frac{3}{2}(\omega^2(G) - \omega(G))$ if G is $(P_5, K_1 + 2K_2)$-free, and (iv) $\chi(G) \leq 3\omega(G) + 11$ if G is $(P_5, K_1 + (K_1 \cup K_3))$-free.

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1 Introduction

All graphs considered in this paper are finite, simple, and connected. Let G be a graph. The *clique number* $\omega(G)$ of G is the maximum size of the cliques of G, and the *independent number* $\alpha(G)$ of G is the maximum size of the independent sets of G. We use P_k and C_k to denote a *path* and a *cycle* on k vertices respectively. The complete bipartite graph with partite sets of size p and q is denoted by $K_{p,q}$, and the complete graph with l vertices is denoted by K_l.

Let G and H be two vertex disjoint graphs. The *union* $G \cup H$ is the graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. Similarly, the *join* $G + H$ is the

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graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{xy|$for each pair $x \in V(G)$ and $y \in V(H)\}$. For positive integer k, kG denotes the union of k copies of G.

We say that G induces H if G has an induced subgraph isomorphic to H, and say that G is *H-free* if G does not induce H. Let $\mathcal{H}$ be a family of graphs. We say that G is $\mathcal{H}$-free if G induces no member of $\mathcal{H}$. For a subset $X \subseteq V(G)$, let $G[X]$ denote the subgraph of G induced by X. A *hole* of G is an induced cycle of length at least 4, and a *k-hole* is a hole of length k. A *k-hole* is said to be an *odd (even) hole* if k is odd (even). An *antihole* is the complement of some hole. An *odd* (resp. *even*) antihole is defined analogously.

A coloring of G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. The minimum number of colors required to color G is said to be the *chromatic number* of G, denoted by $\chi(G)$. Obviously we have that $\chi(G) \geq \omega(G)$. However, determining the upper bound of the chromatic number of some family of graphs G, especially, giving a function of $\omega(G)$ to bound $\chi(G)$ is generally very difficult. Throughout the literature, plenty of work has been taken to investigate this problem. A family $\mathcal{G}$ of graphs is said to be *χ-bounded* if there is a function f such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$, and if such a function f does exist to $\mathcal{G}$, then f is said to be a *binding function* of $\mathcal{G}$ [14]. A graph G is said to be *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H. Thus the binding function for perfect graphs is $f(x) = x$. The famous *Strong Perfect Graph Theorem* [6] states that a graph is perfect if and only if it induces neither an odd hole nor an odd antihole. Erdős [11] showed that for any positive integers k and l, there exists a graph G with $\chi(G) \geq k$ and no cycles of length less than l. This result motivates the study of the chromatic number of $\mathcal{H}$-free graphs for some $\mathcal{H}$. Gyárfás [14,15], and Sumner [25] independently, proposed the following conjecture.

Conjecture 1.1 [15, 25] *For every tree T, T-free graphs are χ-bounded.*

Gyárfás [15] proved that $\chi(G) \leq (k-1)^{\omega(G)-1}$ for $k \geq 4$ if G is P_k-free and $\omega(G) \geq 2$. Gyárfás also suggested that there might exist χ-binding function for these classes of graphs with a better magnitude.

Since P_4-free graphs are perfect, determining an optimal binding function of P_5-free graphs attracts much attention. Sumner [25] showed that all (P_5, K_3)-free graphs are 3-colorable, and there exist many (P_5, K_3)-free graphs with chromatic number 3. Up to now, the best known upper bound for P_5-free graphs is due to Esperet *et al* [12], who showed that if G is P_5-free and $\omega(G) \geq 3$ then $\chi(G) \leq 5 \cdot 3^{\omega(G)-3}$, and the bound is sharp for $\omega(G) = 3$. A natural question is whether the exponential bound can be improved.

Problem 1.1 [20] *Are there polynomial functions f_{P_k} for $k \geq 5$ such that $\chi(G) \leq f_{P_k}(\omega(G))$ for every P_k-free graph G?*

Conjecture 1.2 [9] *There exists a constant c such that for every P_5-free graph G, $\chi(G) \leq c\omega^2(G)$.*

We say that a graph G admits a *perfect division* (A, B) if $V(G)$ can be partitioned into A and B such that $G[A]$ is perfect and $\omega(G[B]) < \omega(G)$. A graph G is said to be *perfectly divisible* if each of its induced subgraphs admits a perfect division [16]. Obviously, if G is perfectly divisible, then $\chi(G) \leq \omega(G) + (\omega(G) - 1) + \cdots + 2 + 1 = \binom{\omega(G)+1}{2}$.

Plenty of articles around the above topics have been published in the decades. Here we list some results related to (P_5, H)-free graphs for some small graph H, and refer the readers to [19, 22, 24] for more information on Conjecture 1.1 and related problems.

A *bull* is a graph consisting of a triangle with two disjoint pendant edges, a *cricket* is a graph consisting of a triangle with two adjacent pendant edges, a *diamond* is the graph $K_1 + P_3$, a *cochair* is the graph obtained from a diamond by adding a pendent edge to a vertex of degree 2, a *dart* is the graph $K_1 + (K_1 \cup P_3)$, a *hammer* is the graph obtained by identifying one vertex of a K_3 and one end vertex of a P_3, a *house* is just the complement of P_5, a *gem* is the graph $K_1 + P_4$, a *gem*$^+$ is the graph $K_1 + (K_1 \cup P_4)$, and a *paraglider* is the graph obtained from a diamond by adding a vertex joining to its two vertices of degree 2 (see Figure 1).

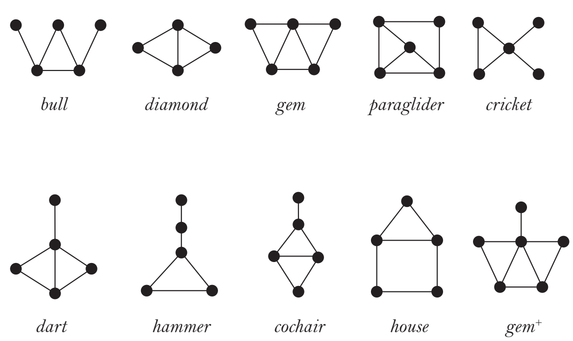

Figure 1: Illustration of some forbidden configurations

Fouquet *et al* [13] proved that (P_5, house)-free graphs are perfectly divisible. Schiermeyer [20] proved that $\chi(G) \leq \omega^2(G)$ for (P_5, H)-free graphs G, where H is a graph in {cricket, dart, diamond, gem, gem$^+$, $K_{1,3}$}. Brause *et al* [3] proved that $\chi(G) \leq \binom{\omega(G)+1}{2}$ if G is (P_5, hammer)-free, Chudnovsky and Sivaraman [7] showed that (P_5, bull)-free graphs and (odd hole, bull)-free graphs are both perfectly divisible, and Hoáng [16] showed that every (odd holes, banner)-free graph is perfectly divisible. Dong and Xu [10] proved that (P_5, F)-free graphs are perfectly divisible, where F is either a cochair or a cricket. Chudnovsky *et al* [8] proved that $\chi(G) \leq \lceil\frac{5\omega(G)}{4}\rceil$ if G is (P_5, gem)-free, which improves the results of [4] and [9]. Char and Karthick [5] showed that if G is (P_5, $K_1 + C_4$)-free, then $\chi(G) \leq \frac{3\omega(G)}{2}$. Huang and Karthick [18] showed that if G is (P_5, paraglider)-free, then $\chi(G) \leq \lceil\frac{3\omega(G)}{2}\rceil$.

Chudnovsky and Sivaraman [7] showed that $\chi(G) \leq 2^{\omega(G)-1}$ if G is (P_5, C_5)-free, Brause *et al* [1] proved that $\chi(G) \leq d \cdot \omega^3(G)$ for some constant d if G is $(P_5, K_{2,3})$-free, and Schiermeyer [21] proved that $\chi(G) \leq c \cdot \omega^3(G)$ for some constant c if G is $(P_5, K_1 + 2K_2)$-

free. In this paper, we study a subclasses of P_5-free graphs, and prove the following theorems, which improve some results of [1, 21, 26].

Theorem 1.1 *Every $(P_5, C_5, K_{2,3})$-free graph is perfectly divisible.*

Theorem 1.2 *If G is $(P_5, K_{2,3})$-free then $\chi(G) \leq 2\omega^2(G) - \omega(G) - 3$.*

Theorem 1.3 *If G is $(P_5, K_1 + 2K_2)$-free with $\omega(G) \geq 2$ then $\chi(G) \leq \frac{3}{2}(\omega^2(G) - \omega(G))$.*

Theorem 1.4 *If G is $(P_5, K_1 + (K_1 \cup K_3))$-free then $\chi(G) \leq 3\omega(G) + 11$.*

Theorem 1.2 improves a result of Brause *et al* [1] and the upper bound $2\omega^2(G)-\omega(G)-3$ is sharp in the sense that all (P_5, K_3)-free graphs are 3-colorable and there are (P_5, K_3)-free graphs with chromatic number 3, Theorem 1.3 improves a result of Schiermeyer [21], and Theorem 1.4 improves a result of [26] which states that $\chi(G) \leq \frac{1}{2}(\omega^2(G) + \omega(G))$ for $\{2K_2, K_1 + (K_1 \cup K_3)\}$-free graphs.

It is known (see **Theorem** 14 of [3]) that the class of $2K_2 \cup 3K_1$-free graphs does not admit a linear binding function, and so one can not expect a linear binding function for $(P_5, K_{2,3})$-free graphs or for $(P_5, K_1 + 2K_2)$-free graphs.

In Section 2, we introduce a few more notations, and list several useful lemmas. Section 3 is devoted to the proof of Theorem 1.1. Theorems 1.2, 1.3, and 1.4 are proved in Sections 4, 5, and 6 respectively.

2 Preliminary and Notations

Let G be a graph, and let A be an antihole of G with $V(A) = \{v_1, v_2, \cdots, v_h\}$. We always enumerate the vertices of A cyclically such that $v_iv_{i+1} \notin E(G)$, and simply write $A = v_1v_2 \cdots v_h$. In this paper, the summations of subindex are taken modulo h for some h, and we always set $h + 1 \equiv 1$.

Observation 2.1 *The vertices of an odd antihole cannot be the union of two cliques.*

For two vertices x and y of G, an xy-path is an induced path with ends x and y. Throughout this paper, *all paths considered are induced paths.* The *distance* $d(x, y)$ between x and y is the length of the shortest xy-path of G.

Let P be a path, and let u and v be two vertices of P. We use P^* to denote the set of *internal vertices* of P (i.e., those vertices of degree 2 in P), and use $P[u, v]$ to denote the segment of P between u and v.

Let $v \in V(G)$, and let X be a subset of $V(G)$. We use $N_X(v)$ to denote the set of neighbors of v in X. We say that v is *complete* to X if $N_X(v) = X$, and say that v is *anticomplete* to X if $N_X(v) = \emptyset$. For two subsets X and Y of $V(G)$, we say that X is *complete* to Y if each vertex of X is complete to Y, and say that X is *anticomplete* to Y if each vertex of X is anticomplete to Y. If $2 \leq |X| \leq |V(G)| - 1$ and every vertex in $V(G)\backslash X$ is either complete to X or anticomplete to X, then X is said to be a *homogeneous set.*

Lemma 2.1 [7] *A minimal nonperfectly divisible graph admits no homogeneous sets.*

Let $d(v, X) = \min_{x \in X} d(v, x)$, and call $d(v, X)$ *the distance* of a vertex v to a subset X. Let i be a positive integer, and $N_G^i(X) = \{y \in V(G) \backslash X | d(y, X) = i\}$. We call $N_G^i(X)$ the *i-neighborhood* of X, and simply write $N_G^1(X)$ as $N_G(X)$. If no confusion may occur, we write $N^i(X)$ instead of $N_G^i(X)$, and $N^i(\{v\})$ is denoted by $N^i(v)$ for short.

Suppose that $C = v_1v_2v_3v_4v_5v_1$ is a 5-hole of G. For a subset $T \subseteq \{1, 2, 3, 4, 5\}$, let

$$N_T(C) = \{x \mid x \in N(C), \text{ and } v_ix \in E(G) \text{ if and only if } i \in T\}.$$

It is easy to check that for $k \in \{1, 2, 3, 4, 5\}$ and $l = k + 2$, $N_{\{k,k+2\}}(C) = N_{\{l,l+3\}}(C)$ and $N_{\{k,k+2,k+3\}}(C) = N_{\{l,l+1,l+3\}}(C)$.

The next lemma is devoted to the structure of P_5-free graphs. It holds trivially by the P_5-freeness of G, and so we omit its proof.

Lemma 2.2 *Suppose that G is a P_5-free graph and $C = v_1v_2v_3v_4v_5v_1$ is a 5-hole of G. Then,*

(*a*) *for $i \in \{1, 2, 3, 4, 5\}$, $N_{\{i\}}(C) = N_{\{i,i+1\}}(C) = \emptyset$, and $N_{\{i,i+2\}}(C) \cup N_{\{i,i+1,i+2\}}(C)$ is anticomplete to $N^2(C)$,*

(*b*) *if $x \in N(C)$ and $N^2(x) \cap N^3(C) \neq \emptyset$ then $x \in N_{\{1,2,3,4,5\}}(C)$, and*

(*c*) *for each vertex $x \in N^2(C)$ and each component B of $G[N^3(C)]$, x is either complete or anticomplete to B.*

We end this section by the following two lemmas which are also very useful in the proofs of the main results. A *clique cut set* is a cut set and is a clique.

Lemma 2.3 *A minimal nonperfectly divisible graph has no clique cut sets.*

Proof. If it is not the case, let G be a minimal nonperfectly divisible graph, and let S be a clique cut set of G. Let C_1 be a component of $G - S$, let $G_1 = G[V(C_1) \cup S]$, and let $G_2 = G - V(C_1)$. Then, both G_1 and G_2 are perfectly divisible. For $i \in \{1, 2\}$, let (A_i, B_i) be a perfect division of G_i with $G[A_i]$ perfect and $\omega(G[B_i]) < \omega(G_i)$. Since S is a clique, we see that both $A_1 \cap A_2$ and $B_1 \cap B_2$ are cliques as they are subsets of S, and thus $G[A_1 \cup A_2]$ is perfect and $\omega(B_1 \cup B_2) < \omega(G)$, a contradiction. ∎

Let G be a graph with $\alpha(G) = 2$, and let v be a vertex of G. Notice that $V(G) \setminus (N(v) \cup \{v\})$ is a clique, which implies that $G - N(v)$ is perfect. Thus the next lemma follows directly.

Lemma 2.4 *Graphs of independent number at most 2 are perfectly divisible.*

3 Perfect divisibility of $(P_5, C_5, K_{2,3})$-free graphs

This section is aim to prove Theorem 1.1. A cut set S is said to be a *minimal cut set* if any proper subset of S is not a cut set of G. We first prove a lemma on the structure of $(P_5, C_5, K_{2,3})$-free graphs.

Lemma 3.1 *Suppose that G is a $(P_5, C_5, K_{2,3})$-free graph without clique cut sets, and S is a minimal cut set of G. Then*

- *(a) $G-S$ has exactly two components, and for each pair of non-adjacent vertices $s_1, s_2 \in S$, each s_1s_2-path with interior in exactly one component has length 2,*
- *(b) each vertex of S is complete to at least one component of $G-S$, and*
- *(c) $\alpha(G[S]) = 2$.*

Proof. Let $C_1, C_2, \ldots, C_t$ be the components of $G-S$. It is certain that $t \geq 2$. Since S is a minimal cut set, we see that for each $i \in \{1, 2, \ldots, t\}$,

$$N_{V(C_i)}(x) \neq \emptyset \text{ for each vertex } x \in S. \tag{1}$$

Let $V_1 = V(C_1)$ and $G_1 = G[S \cup V_1]$, let $G_2 = G - V_1$, and let $V_2 = V(G_2) \setminus S$.

Since G has no clique cut set, we arbitrarily choose s_1 and s_2 to be two non-adjacent vertices in S. Suppose that $G-S$ has at least 3 components, then $G_2 - S$ is not connected as $G_1 - S = C_1$. Let C_2 and C_3 be two components of $G_2 - S$. For $i \in \{1, 2, 3\}$, let P_i be an s_1s_2-path with interior in C_i (recall that all paths considered are induced paths).

If one of P_1, P_2 and P_3 has length at least 3, then a C_5 or a P_5 appears. Otherwise, a $K_{2,3}$ appears. Hence, $G-S$ has two components $G[V_1]$ and $G[V_2]$. This also implies that each s_1s_2-path with interior in V_1 or V_2 has length 2.

Let $s \in S$. It follows from (1) that s has neighbors in both V_1 and V_2. Since both $G[V_1]$ and $G[V_2]$ are connected and G is P_5-free, we have that each vertex of S is complete to either V_1 or V_2.

Now it is left to show that $\alpha(G[S]) = 2$. Suppose to its contrary that s_3 is a vertex in $S \setminus \{s_1, s_2\}$ anticomplete to $\{s_1, s_2\}$. Thus we have that, for each pair of $i, j \in \{1, 2\}$, each s_is_3-path with interior in V_j has length 2. Since G induces no $K_{2,3}$, we have that $N_{V_i}(s_1) \cap N_{V_i}(s_2) \cap N_{V_i}(s_3) = \emptyset$ for some $i \in \{1, 2\}$, and so we may assume that $N_{V_1}(s_1) \cap N_{V_1}(s_2) \cap N_{V_1}(s_3) = \emptyset$. Let $w_1 \in V_1$ be a common neighbor of s_1 and s_2, let $w_2 \in V_1$ be a common neighbor of s_2 and s_3, and let $x \in V_2$ be a common neighbor of s_1 and s_3. If $w_1w_2 \notin E(G)$, then $G[\{s_1, w_1, s_2, w_2, s_3\}] = P_5$; otherwise, $G[\{s_1, s_3, w_1, w_2, x\}] = C_5$. This contradiction implies that $\alpha(G[S]) = 2$, which completes the proof of Lemma 3.1. ∎

Proof of **Theorem** 1.1. Let G be a $(P_5, C_5, K_{2,3})$-free graph. Suppose that G is not perfectly divisible but every proper induced subgraph of G is perfectly divisible. It is certain that G is connected and not perfect. Let S be a minimal cut set of G. By Lemma 2.3, S is not a clique. It follows from Lemma 3.1 that $\alpha(G[S]) = 2$, $G-S$ has exactly two components, say C_1 and C_2, and each vertex of S is either complete to $V(C_1)$ or $V(C_2)$. For $i \in \{1,2\}$, let $V_i = V(C_i)$, and let $G_i = G[V_i \cup S]$.

Let $S_0 \subseteq S$ be the set of vertices complete to $V_1 \cup V_2$. For $i \in \{1,2\}$, let $S_i \subseteq S \setminus S_0$ be the set of vertices only complete to V_i. Clearly $S = S_0 \cup S_1 \cup S_2$.

We claim that

$$\text{at least one of } V_1 \text{ and } V_2 \text{ is a clique.} \tag{2}$$

Suppose to its contrary that both V_1 and V_2 are not cliques. Since S is not a clique, we may choose s_1 and s_2 to be two non-adjacent vertices of S. Suppose that $\{s_1, s_2\} \cap S_0 \neq \emptyset$. If $\{s_1, s_2\} \cap S_i \neq \emptyset$ for some $i \in \{1,2\}$, then V_i is a clique, otherwise an induced $K_{2,3}$ is obtained. Similarly, if $\{s_1, s_2\} \subseteq S_0$, then both V_1 and V_2 must be cliques. Thus we may assume that $\{s_1, s_2\} \cap S_0 = \emptyset$. Note that $N_{V_i}(x) \neq \emptyset$ for each vertex $x \in S$ as S is a minimal cut set. If $\{s_1, s_2\} \subset S_1$, then G induces a $K_{2,3}$ whenever $N_{V_2}(s_1) \cap N_{V_2}(s_2) \neq \emptyset$, and G induces a P_5 or a C_5 whenever $N_{V_2}(s_1) \cap N_{V_2}(s_2) = \emptyset$, both are contradictions. Similar contradiction happens if $\{s_1, s_2\} \subset S_2$. Therefore, we may suppose that $s_1 \in S_1$ and $s_2 \in S_2$, that is, both $S_0 \cup S_1$ and $S_0 \cup S_2$ are cliques. Now by Observation 2.1, we have that $G[S]$ is perfect. Since $\omega(G - S) < \omega(G)$, it contradicts the minimal nonperfect divisibility of G, and which proves (2).

Next we claim that

$$\text{exact one of } V_1 \text{ and } V_2 \text{ is a clique.} \tag{3}$$

To prove (3), we will show that if V_1 and V_2 are both cliques then $\alpha(G) = 2$, and hence deduce a contradiction to Lemma 2.4 claiming that all graphs G with $\alpha(G) \leq 2$ are perfectly divisible.

Suppose to its contrary that V_1 and V_2 are both cliques but $\alpha(G) > 2$. Let $T = \{t_1, t_2, t_3\}$ be an independent set of G. It follows from Lemma 3.1 that $|T \cap S_i| = 2$ and $|T \cap V_{3-i}| = 1$ for some $i \in \{1,2\}$. Without loss of generality, we assume that $t_1, t_2 \in S_1$ and $t_3 \in V_2$.

Note that V_1 and V_2 are both cliques, and V_1 is complete to $S_0 \cup S_1$. If $S_2 = \emptyset$, then $(V_1 \cup V_2, S_0 \cup S_1)$ is a perfect division of G, contradicting the minimal nonperfect divisibility of G. Hence $S_2 \neq \emptyset$.

Let x be a vertex in S_2. Since no vertex of S_2 is complete to V_1, we may choose a vertex, say v_1, in V_1 with $xv_1 \notin E(G)$. Since $\alpha(G[S]) = 2$, we have that x cannot be anticomplete to $\{t_1, t_2\}$. Suppose $xt_1 \in E(G)$. To avoid a $P_5 = t_2 v_1 t_1 x t_3$, we have that x must be adjacent to t_2 as well. Hence we have that $\{t_1, t_2\}$ is complete to S_2.

If S_2 is not a clique, let x and x' be two non-adjacent vertices of S_2, then $G[\{T \cup \{x, x'\}]$ is a $K_{2,3}$. This implies that S_2 must be a clique.

Since both V_1 and $S_2 \cup V_2$ are cliques and V_1 is complete to $S_0 \cup S_1$, we have that $G[V_1 \cup V_2 \cup S_2]$ is perfect by Observation 2.1, and $\omega(G[S_0 \cup S_1]) < \omega(G)$. Thus $(V_1 \cup V_2 \cup S_2, S_0 \cup S_1)$ is a perfect division of G, which leads to a contradiction and proves (3).

Now we may assume that V_1 is a clique and V_2 is not.

Since V_2 is not a clique, we must have that $S_0 \cup S_2$ is a clique, otherwise an induced $K_{2,3}$ appears. Thus $V_1 \cup S_0$ is also a clique. Since G is $(P_5, C_5, K_{2,3})$-free, by Observation 2.1 we have that

$$G[V_1 \cup S_0 \cup S_2] \text{ is perfect.} \tag{4}$$

Suppose that $S_0 \cup S_2 \neq \emptyset$, and let $v \in S_0 \cup S_2$. If $\omega(G[V_2 \cup S_1]) < \omega(G)$, then $(V_1 \cup S_0 \cup S_2, V_2 \cup S_1)$ is a perfect division of G by (4). Thus we may further assume that $\omega(G[V_2 \cup S_1]) = \omega(G)$. Let $K_1, \ldots, K_r$ be all the cliques of $G[V_2 \cup S_1]$ of size $\omega(G)$, and let $L_i = K_i \cap S_1$ for each $i \in \{1, 2, \ldots, r\}$. Since $|K_i| = \omega(G)$ and $\omega(G[V_2]) < \omega(G)$, we have that $L_i \neq \emptyset$, and v is not complete to L_i. Let $M_i \subseteq L_i$ be the set of vertices which are non-adjacent to v for $i \in \{1, 2, \ldots, r\}$, and let $M = \bigcup_{i=1}^{r} M_i$. Since $\alpha(G[S]) = 2$, we have that M is a clique, and so $V_1 \cup M$ is a clique. Notice that $S_0 \cup S_2$ is a clique. Thus $G[V_1 \cup S_0 \cup S_2 \cup M]$ induces no odd antihole by Observation 2.1, and so it is perfect. Now we have that G is perfectly divisible as $\omega(G[(V_2 \cup S_1) \setminus M]) < \omega(G[V_2 \cup S_1]) = \omega(G)$, which contradicts the minimal nonperfect divisibility of G.

Hence we may suppose that, for each minimal cut set S of G, $S_0 \cup S_2 = \emptyset$. Consequently, we must have that $|V_1| = 1$ (as otherwise V_1 is a homogeneous set, contradicting Lemma 2.1), that is,

$$\text{every minimal cut set of } G \text{ equal } N(x) \text{ for some vertex } x \text{ of } G. \tag{5}$$

Let v be a vertex of G. It is certain that $N(v)$ is a cut set. Next we show that

$$N(v) \text{ is a minimal cut set,} \tag{6}$$

which implies that the converse of (5) holds as well.

Suppose to its contrary that w is a vertex such that $N(w)$ is not a minimal cut set. Let $T_1, T_2, \ldots, T_r$ be all the subsets of $N(w)$ where each one is a minimal cut set. It follows from (5) that there are some vertices, say $w_1, w_2, \ldots, w_r$, such that $T_i = N(w_i)$ for each $i \in \{1, 2, \ldots, r\}$. We claim that

$$\{w, w_1, w_2, \ldots, w_r\} = V(G) \setminus N(w). \tag{7}$$

If it is not the case, then let C be a component of $G - N(w) - \{w, w_1, w_2, \ldots, w_r\}$, and let $X \subseteq N(w)$ be the set of vertices where each one has a neighbor in C. It is certain that X is a cut set. Thus there must be some i such that $T_i \subseteq X$. Without loss of generality, we suppose that $T_1 \subseteq X$. Since G has no clique cut set by Lemma 2.3, we have that T_1 is not a clique, and so has two non-adjacent vertices, say t_1 and t_1'. Let P be a shortest t_1t_1'-path with interior in C. If P has length 2, then $t_1wt_1', t_1w_1t_1'$ and P form an induced $K_{2,3}$ in G. If P has length 3, then t_1wt_1' and P form a C_5 in G. Otherwise, we have that P has length greater than 3, and then we can find a P_5 in G. These contradictions proves (7).

Since $\{w, w_1, w_2, \dots, w_r\}$ is independent, it follows from (7) that $(\{w, w_1, w_2, \dots, w_r\}, N(w))$ forms a perfect division of G, and so (6) holds. Thus by Lemma 3.1, we have that

$$\alpha(G[N(x)]) = 2 \text{ for each vertex } x \text{ of } G. \tag{8}$$

We choose $v_1 \in V(G)$ and let $S_1 = N(v_1)$. Then, $\alpha(G[S_1]) = 2$, and S_1 is a minimal cut set by (6). Let s_1 and s_2 be two non-adjacent vertices in S_1. Let $V_1 = \{v_1\}$, and let $V_2 = V(G) \setminus \{S_1 \cup \{v_1\})$. We have that $G[V_2]$ is connected by Lemma 3.1(a).

Let $M = N_{V_2}(s_1) \cap N_{V_2}(s_2)$, and let $M_i = N_{V_2}(s_i) \setminus M$ for $i \in \{1, 2\}$. Since G induces no $K_{2,3}$, we have that M must be a clique. If both M_1 and M_2 are not empty, let $m_i \in M_i$ for $i \in \{1, 2\}$, then $G[\{m_1, m_2, s_1, s_2, v_1\}] = C_5$ whenever $m_1m_2 \in E(G)$, and $G[\{m_1, m_2, s_1, s_2, v_1\}] = P_5$ whenever $m_1m_2 \notin E(G)$. Without loss of generality, we assume that $M_2 = \emptyset$. Now we claim that

$$V_2 = M \cup M_1. \tag{9}$$

Suppose that (9) does not hold. Since $G[V_2]$ is connected, we may choose a vertex, say z, in $V_2 \setminus (M \cup M_1)$ that is adjacent to some vertex of $M \cup M_1$. If $zz_1 \in E(G)$ for some $z_1 \in M$, then $\{s_1, s_2, z\}$ is an independent set contained in $N(z_1)$, contradicting (8). If $zz_2 \in E(G)$ for some vertex $z_2 \in M_1$, then $G[\{s_1, s_2, v_1, z, z_2\}] = P_5$. This proves (9).

Note that $S_1 = N(v_1)$. By (8), we have that no vertex of $N(v_1)\setminus\{s_1, s_2\}$ is anticomplete to $\{s_1, s_2\}$. Let $T = N_{S_1}(s_1) \cap N_{S_1}(s_2)$, let $T_1 = N_{S_1}(s_1) \setminus T$, and let $T_2 = N_{S_1}(s_2) \setminus T$. If T_i is not a clique for some $i \in \{1, 2\}$, let $t_{i,1}$ and $t_{i,2}$ be two non-adjacent vertices of T_i, then $\{s_{3-i}, t_{i,1}, t_{i,2}\}$ is an independent set in S_1, which leads to a contradiction to (8). Thus both T_1 and T_2 are cliques.

Let $A = T_2 \cup \{s_1, s_2\}$, and let $B = V(G) \setminus A$. It is certain that $G[A]$ is perfect. Since s_1 is complete to B by (9), we see that $\omega(G[B]) < \omega(G)$, which implies that (A, B) is a perfect division of G. This contradicts the minimal nonperfect divisibility of G and proves Theorem 1.1. ∎

4 $(P_5, K_{2,3})$-free graphs

In this section, we prove Theorem 1.2.

Since perfectly divisible graphs G has chromatic number at most $\binom{\omega(G)+1}{2}$, it follows from Theorem 1.1 that we only need to consider the $(P_5, K_{2,3})$-free graphs with a 5-hole. Let G be a $(P_5, K_{2,3})$-free graph, and let $C = v_1v_2v_3v_4v_5v_1$ be a 5-hole of G. Recall that for $T \subseteq \{1, 2, 3, 4, 5\}$, $N_T(C)$ consists of the vertices not on C but each has exactly $\{v_i \mid i \in T\}$ as its neighbors on C, and for integer $i \geq 1$, $N^i(C)$ consists of the vertices of distance i apart from C. Let u and v be two non-adjacent vertices in $N(C)$. We say that $\{u, v\}$ is a *bad pair* if there is an $i \in \{1, 2, 3, 4, 5\}$ such that $u \in N_{\{i,i+1,i+3\}}(C)$ and $v \in N_{\{i,i+1,i+2,i+4\}}(C)$.

Lemma 4.1 *Suppose that G is a $(P_5, K_{2,3})$-free graph with a 5-hole $C = v_1v_2v_3v_4v_5v_1$, and u, v are two vertices in $N(C)$. Then all the followings hold.*

(*a*) *If there exist three consecutive vertices of C, named v_i, v_{i+1}, and v_{i+2}, such that $\{v_i, v_{i+2}\} \subseteq N(u) \cap N(v)$ and $v_{i+1} \notin N(u) \cup N(v)$, then $uv \in E(G)$.*

(*b*) *$N_{\{i,i+2\}}(C)$, $N_{\{i,i+1,i+3\}}(C)$, and $N_{\{i,i+1,i+2,i+3\}}(C)$ are all cliques for $1 \leq i \leq 5$.*

(*c*) *$\alpha(G[N_{\{1,2,3,4,5\}}(C)]) \leq 2$, and for each $i \in \{1, 2, 3, 4, 5\}$, $\alpha(G[N_{\{i,i+1,i+2\}}(C)]) \leq 2$, and $N_{\{i,i+1,i+2\}}(C)$ is complete to $N_{\{i+1,i+2,i+3\}}(C)$.*

(*d*) *If $uv \notin E(G)$ and $\{u, v\}$ is not a bad pair, then $N(u) \cap N(v) \cap N^2(C) = \emptyset$.*

(*e*) *$N(C)\backslash(N_{\{1,2,3,4,5\}}(C) \bigcup_{1 \leq i \leq 5} N_{\{i,i+1,i+2\}}(C))$ can be partitioned into five cliques S_1, S_2, S_3, S_4 and S_5 such that $|S_i| \leq \omega(G) - 1$ and v_i is anticomplete to S_i for each i.*

Proof. If $\{v_i, v_{i+2}\} \subseteq N(u) \cap N(v)$ and $v_{i+1} \notin N(u) \cup N(v)$ for some i, then $uv \in E(G)$ to avoid a $K_{2,3}$ on $\{u, v, v_i, v_{i+1}, v_{i+2}\}$. Hence ($a$) holds, and ($b$) follows directly from (a).

Now we come to prove (c). If either $G[N_{\{i,i+1,i+2\}}(C)]$ or $G[N_{\{1,2,3,4,5\}}(C)]$ has an independent set of size 3, say $\{u, v, w\}$, then $\{u, v, w, v_i, v_{i+2}\}$ induces a $K_{2,3}$. If $N_{\{i,i+1,i+2\}}(C)$ is not complete to $N_{\{i+1,i+2,i+3\}}(C)$ for some i, we may choose $x \in N_{\{i,i+1,i+2\}}(C)$ and $y \in N_{\{i+1,i+2,i+3\}}(C)$ with $xy \notin E(G)$, then a $P_5 = xv_{i+1}yv_{i+3}v_{i+4}$ appears. Hence (c) holds.

Next we prove (d). Suppose that $uv \notin E(G)$ and $\{u, v\}$ is not a bad pair, and suppose that $N(u) \cap N(v) \cap N^2(C)$ has a vertex w. If one of u and v is in $N_{\{1,2,3,4,5\}}(C)$, then there exists some $i \in \{1, 2, 3, 4, 5\}$ such that $G[\{u, v, v_i, v_{i+2}, w\}] = K_{2,3}$, which leads to a contradiction. Thus $\{u, v\} \subseteq W = \bigcup_{1 \leq i \leq 5} (N_{\{i,i+1,i+3\}}(C) \cup N_{\{i,i+1,i+2,i+3\}}(C))$ by Lemma 2.2(a).

Now we first suppose that $u \in N_{\{k,k+1,k+3\}}(C)$ for some k, and by symmetry we may assume that $k = 1$. Since $\{u, v\}$ is not a bad pair, we see that $v \in W \setminus N_{\{1,2,3,5\}}(C)$. It follows from ($a$) that, $v \notin N_{\{1,2,4\}}(C) \cup N_{\{1,3,4\}}(C) \cup N_{\{2,4,5\}}(C) \cup N_{\{1,2,3,4\}}(C) \cup N_{\{4,5,1,2\}}(C)$. But $G[\{u, v, w, v_2, v_5\}] = P_5$ if $v \in N_{\{1,3,5\}}(C)$, $G[\{u, v, w, v_1, v_3\}] = P_5$ if $v \in N_{\{2,3,5\}}(C) \cup N_{\{2,3,4,5\}}(C)$, and $G[\{u, v, w, v_1, v_4\}] = P_5$ if $v \in N_{\{3,4,5,1\}}(C)$. Hence we have, by symmetry, that

$$\{u, v\} \cap (\bigcup_{1 \leq i \leq 5} (N_{\{i,i+1,i+3\}}(C)) = \emptyset,$$

that is, $\{u, v\} \subseteq \bigcup_{1 \leq i \leq 5} N_{\{i,i+1,i+2,i+3\}}(C)$. Without loss of generality, we may assume that $u \in N_{\{1,2,3,4\}}(C)$. Thus $v \notin \bigcup_{1 \leq i \leq 5} N_{\{i,i+1,i+2,i+3\}}(C)$ by (a) of this lemma. This contradiction proves (d).

Finally, we prove (e). Let $S_1 = N_{2,5}(C) \cup N_{\{2,3,5\}}(C) \cup N_{\{2,4,5\}}(C) \cup N_{\{2,3,4,5\}}(C)$, $S_2 = N_{\{1,3\}}(C) \cup N_{\{1,3,4\}} \cup N_{\{1,3,5\}} \cup N_{\{1,3,4,5\}}$, $S_3 = N_{\{2,4\}} \cup N_{\{1,2,4,5\}}(C)$, $S_4 = N_{\{3,5\}}(C) \cup N_{\{1,2,3,5\}}(C)$, and $S_5 = N_{\{1,4\}}(C) \cup N_{\{1,2,4\}}(C) \cup N_{\{1,2,3,4\}}(C)$. It is certain that v_i is anticomplete to S_i for each i, and one can check easily from (a) that each S_i is a clique of size at most $\omega(G)-1$. ∎

Lemma 4.2 *Let G be a $(P_5, K_{2,3})$-free graph and $C = v_1v_2v_3v_4v_5v_1$ be a 5-hole of G. If G contains no clique cut sets, then $N^3(C) = \emptyset$, and for each component B of $N^2(C)$, $\alpha(B) \leq 2$ and $N(B) \cap N(C) \subseteq N_{\{1,2,3,4,5\}}(C)$ whenever $\omega(B) = \omega(G)$.*

Proof. Suppose that G has no clique cut sets. Let $S' \subseteq N(C)$ be the set of vertices having no neighbors in $N^2(C)$, and let $S = N(C)\backslash S'$. By Lemma 2.2(a), if a vertex x of $N(C)$ has neighbors in $N^2(C)$, then $|N_C(x)| \geq 3$. Hence for any two vertices u and v of S we have that

$$N_C(v) \cap N_C(u) \neq \emptyset. \tag{10}$$

Next we prove this lemma by considering the connectedness of $G[N^2(C)]$.

Case 1. Suppose that $G[N^2(C)]$ is connected. Since G contains no clique cut sets, there must exist two non-adjacent vertices, say u and v, in S. By (10), we may choose z to be a common neighbor of u and v on C.

Let $T_{uv} = N(u) \cap N(v) \cap N^2(C)$, and let $T_u = (N(u) \cap N^2(C)) \setminus T_{uv}$ and $T_v = (N(v) \cap N^2(C)) \setminus T_{uv}$.

First suppose that $T_{uv} = \emptyset$. Since G is $(P_5, K_{2,3})$-free, we have

$$T_u \text{ is complete to } T_v, \text{ and } \max\{\alpha(G[T_u]), \alpha(G[T_v]) \leq 2, \tag{11}$$

otherwise for any pair of non-adjacent vertices $t_u \in T_u$ and $t_v \in T_v$, t_uuzvt_v is a P_5, and $G[T_u \cup T_v \cup \{w\}]$ induces a $K_{2,3}$ whenever $\alpha(G[T_w]) \geq 3$ for any $w \in \{u, v\}$. This leads to a contradiction.

We further claim that

$$N^2(C) = T_u \cup T_v, \text{ and } N^3(C) = \emptyset. \tag{12}$$

If $N^2(C) \neq T_u \cup T_v$, since $G[N^2(C)]$ is connected, we may suppose by symmetry that $N^2(C)\backslash(T_u \cup T_v)$ has a vertex, say w, which has a neighbor t_u in T_u, and so $vzut_uw$ is a P_5. Thus $N^2(C) = T_u \cup T_v$. If $N^3(C) \neq \emptyset$, then let w be a vertex of $N^3(C)$, and suppose by symmetry that t'_u is a neighbor of w in T_u, again we have a $P_5 = vzut'_uw$. This proves (12).

From (11) and (12), we see that $\alpha(G[N^2(C)]) \leq 2$. If $\omega(G[N^2(C)]) = \omega(G)$, then $T_u \neq \emptyset$ and $T_v \neq \emptyset$. To avoid a P_5, we see that all neighbors of $N^2(C)$ in $N(C)$ must be contained in $N_{\{1,2,3,4,5\}}(C)$. So the lemma holds whenever $G[N^2(C)]$ is connected and $T_{uv} = \emptyset$.

Next we may assume that $T_{uv} \neq \emptyset$, and let t_{uv} be a vertex in T_{uv}. By Lemma 4.1(d), we have that $\{u, v\}$ is a bad pair. Suppose by symmetry that $u \in N_{\{1,2,4\}}(C)$ and $v \in N_{\{1,2,3,5\}}(C)$. Since u and v have a common neighbor z on C, and since G is $K_{2,3}$-free, we have that

$$T_{uv} \text{ is a clique.} \tag{13}$$

Suppose that $T_v \neq \emptyset$. Let t_v be a vertex of T_v. If t_v has a neighbor t'_{uv} in T_{uv}, then a $P_5 = v_5v_1ut'_{uv}t_v$ appears, and if t_v is anticomplete to T_{uv}, then a $P_5 = v_4ut_{uv}vt_v$ appears. This implies that $T_v = \emptyset$. Similarly we have that $T_u = \emptyset$.

Note that $G[N^2(C)]$ is connected. Thus $N^2(C) = T_{uv}$, otherwise we may choose a vertex w in $N^2(C)\setminus T_{uv}$ where w has a neighbor $t_{uv} \in T_{uv}$ which implies a $P_5 = v_5v_1ut_{uv}w$. If $N^3(C) \neq \emptyset$, let w be a vertex in $N^3(C)$ and w' be a neighbor of w in $N^2(C)$, then a $P_5 = ww'uv_1v_5$ appears. Thus $N^3(C) = \emptyset$, and by (13) we have that this lemma holds when $G[N^2(C)]$ is connected and $T_{uv} \neq \emptyset$. Hence Lemma 4.2 holds when $G[N^2(C)]$ is connected.

Case 2. Suppose that $G[N^2(C)]$ is not connected. Let $T_1, T_2, \ldots, T_k$ be the components of $G[N^2(C)]$. Recall that S consists of the vertices in $N(C)$ that have neighbors in $N^2(C)$. For each $i \in \{1, 2, \ldots, k\}$, let $S_i = N_S(T_i)$, and let Z_i be the subgraph of $G[N^3(C)]$ that is not anticomplete to T_i. If there exists some $i_0 \in \{1, 2, \ldots, k\}$ such that S_{i_0} is not a clique, by applying the same arguments to $G[V(C) \cup S_{i_0} \cup T_{i_0} \cup Z_{i_0}]$ as that used in Case 1, we can show that $Z_{i_0} = \emptyset$ and $\alpha(T_{i_0}) \leq 2$, and $N(T_{i_0}) \cap N(C) \subseteq N_{\{1,2,3,4,5\}}(C)$ whenever $\omega(T_{i_0}) = \omega(G)$. Hence the lemma holds if S_i is not a clique for all $1 \leq i \leq k$.

Thus by symmetry suppose that S_1 is a clique. Since S_1 is not a clique cut set, we have that $S_1 \neq S$. Without loss of generality, let $T_1, T_2, \cdots, T_l$ be the components such that $N_S(T_i) \subseteq S_1$ for each $i \in \{1, 2, \ldots, l\}$. It is obvious that $1 \leq l \leq k-1$, otherwise S_1 is a clique cut set. Now we have that T_i has neighbors in $S\setminus S_1$ for each $i \in \{l+1, \ldots, k\}$.

Since S_1 is not a clique cut set, we have that $N^3(C) \neq \emptyset$, and T_1 must have some neighbors in $N^3(C)$. Let R be a component of $G[N^3(C)]$ such that T_1 is not anticomplete to R. Since S_1 is not a clique cut set, we have that R cannot be anticomplete to $\cup_{i=l+1}^{k} T_i$. Without loss of generality, suppose that R is not anticomplete to T_k. Choose $t_1 \in T_1$, $t_k \in T_k$, and $r_1, r_k \in R$ such that $t_1r_1 \in E(G)$ and $t_kr_k \in E(G)$. By Lemma 2.2(c), $\{t_1, t_k\}$ is complete to R.

We can choose two adjacent vertices $s_k \in S\setminus S_1$ and $t'_k \in T_k$. Let P' be a shortest $t_kt'_k$-path in T_k. Let $r \in R$ and z be a neighbor of s_k on C. Thus a path $P = t_1rt_kP't'_ks_kz$ of length at least 5 appears. This contradiction proves Lemma 4.2. ∎

Now, we can prove Theorem 1.2.

Proof of Theorem 1.2. Let G be a $\{P_5, K_{2,3}\}$-free graph. We may assume that G is connected and contains no clique cut set. If G is $(P_5, C_5, K_{2,3})$-free, then G is perfectly divisible by Theorem 1.1, which implies that $\chi(G) \leq \frac{1}{2}(\omega^2(G) + \omega(G))$. Thus we suppose that G is $(P_5, K_{2,3})$-free and contains a 5-hole C. By Lemma 2.2(a), we have that

$$N(C) = N_{\{1,2,3,4,5\}}(C) \bigcup_{1\leq i\leq 5} (N_{\{i,i+2\}}(C) \cup N_{\{i,i+1,i+2\}}(C)$$
$$\cup N_{\{i,i+1,i+3\}}(C) \cup N_{\{i,i+1,i+2,i+3\}}(C)).$$

Note that $\omega(G[N_{\{1,2,3,4,5\}}(C)]) \leq \omega(G) - 2$, $\omega(G[N_{\{1,2,3\}}(C) \cup N_{\{2,3,4\}}(C)] \leq \omega(G) - 2$, $\omega(G[N_{\{3,4,5\}}(C) \cup N_{\{1,4,5\}}(C)] \leq \omega(G) - 2$, and $\omega(G[N_{\{1,2,5\}}(C)] \leq \omega(G) - 2$. By Lemma 2.4

and Lemma 4.1$(c)(e)$,

$$\begin{aligned}
\chi(G[N(C)]) &\leq \chi(G[N_{\{1,2,3\}}(C)\cup N_{\{2,3,4\}}(C)])+\chi(G[N_{\{3,4,5\}}(C)\cup N_{\{1,4,5\}}(C)])\\
&\quad +\chi(G[N_{\{1,2,5\}}(C)])+\chi(G[N_{\{1,2,3,4,5\}}(C)])+5(\omega(G)-1)\\
&\leq 4\cdot\frac{(\omega(G)-2)^2+\omega(G)-2}{2}+5(\omega(G)-1)\\
&= 2\omega^2(G)-\omega(G)-3. \qquad (14)
\end{aligned}$$

By Lemma 4.1(e), we can color the vertices of C with the colors used on the vertices of $N(C)\backslash(N_{\{1,2,3,4,5\}}(C)\bigcup\limits_{1\leq i\leq 5}N_{\{i,i+1,i+2\}}(C))$ (which is counted $5(\omega(G)-1)$ in (14)).

Let B be a component of $G[N^2(C)]$. By Lemma 4.2, we see that $N^2(C)=G-N(C)-V(C)$, $\alpha(B)\leq 2$ and $N(B)\cap N(C)\subseteq N_{\{1,2,3,4,5\}}(C)$ if $\omega(B)=\omega(G)$. So, we have, by Lemmas 2.4, that $\chi(B)\leq\frac{(\omega(G)-1)^2+\omega(G)-1}{2}=\frac{\omega^2(G)-\omega(G)}{2}$ if $\omega(B)<\omega(G)$, and $\chi(B)\leq\frac{\omega^2(G)+\omega(G)}{2}$ otherwise.

Note that $N^2(C)$ is anticomplete to $\bigcup\limits_{1\leq i\leq 5}N_{\{i,i+1,i+2\}}(C)$ by Lemma 2.2(a), and B is anticomplete to $N(C)\backslash N_{\{1,2,3,4,5\}}(C)$ by Lemma 4.2 if $\omega(B)=\omega(G)$. If $\omega(B)<\omega(G)$, we can color the vertices in B with the colors used on the vertices of $\bigcup\limits_{1\leq i\leq 5}N_{\{i,i+1,i+2\}}(C)$ (which is counted no less than $\frac{\omega^2(G)-\omega(G)}{2}$ in (14)). If $\omega(B)=\omega(G)$, we can color the vertices in B with the colors used on the vertices of $N(C)\backslash N_{\{1,2,3,4,5\}}(C)$ (which is counted no less than $\frac{\omega^2(G)+\omega(G)}{2}$ in (14)).

Therefore, $\chi(G)\leq 2\omega^2(G)-\omega(G)-3$ as desired. ∎

5 (P_5, K_1+2K_2)-free graphs

For two subsets X and Y of $V(G)$, we say that X *dominates* Y if each vertex of Y has a neighbor in X. The next two lemmas are very useful in the proof of Theorem 1.3.

Lemma 5.1 [2] *Every connected P_5-free graph has a dominating clique or a dominating P_3.*

Lemma 5.2 [26] *Let G be a $2K_2$-free graph. Then $\chi(G)\leq\frac{1}{2}(\omega^2(G)+\omega(G))$.*

Proof of Theorem 1.3. Let G be a connected (P_5,K_1+2K_2)-free graph with at least two vertices. By Lemma 5.1, G has a dominating clique or a dominating P_3. If G has a dominating P_3, say $v_1v_2v_3$, then $N(v_i)$ induces a $2K_2$-free graph for each i, otherwise v_i and the $2K_2$ in $G[N(v_i)]$ induce a K_1+2K_2. Thus by Lemma 5.2 we have that $\chi(G)\leq\chi(G[N(v_1)])+\chi(G[N(v_2)])+\chi(G[N(v_3)])\leq\frac{3}{2}((\omega(G)-1)^2+\omega(G)-1)=\frac{3}{2}(\omega^2(G)-\omega(G))$. Thus we may assume that G has a dominating clique, say K_k on vertices $\{v_1,v_2,v_3,\cdots,v_k\}$. Let $S=N(v_1)\cup N(v_2)$ and $T=V(G)\backslash S$, that is, T consists of exactly those vertices anticomplete to $\{v_1,v_2\}$. Since G is (K_1+2K_2)-free, we have that $N(v_i)$ induces a $2K_2$-free subgraph for $i=1,2$. For each $i\in\{3,\ldots,k\}$, since the vertices of $N(v_i)\cap T$ are anticomplete to $\{v_1, v_2\}$, we have that $N(v_i) \cap T$ is an independent set. Hence by Lemma 5.2, $\chi(G) \leq \chi(G[N(v_1)]) + \chi(G[N(v_2)]) + \omega(G) - 2 \leq \omega^2(G) - 2$. Note that $\frac{3}{2}(\omega^2(G) - \omega(G)) \geq \omega^2(G) - 2$. Hence if G is $(P_5, K_1 + 2k_2)$-free, then $\chi(G) \leq \frac{3}{2}(\omega^2(G) - \omega(G))$ as desired. ∎

6 $(P_5, K_1 + (K_1 \cup K_3))$-free graphs

Sumner [25] (see also [12]) proved that a connected (P_5, K_3)-free graph is either bipartite or can be obtained from a 5-hole by replacing each vertex with an independent set and then replacing each edge by a complete bipartite graph.

Lemma 6.1 [25] *If G is (P_5, K_3)-free then $\chi(G) \leq 3$.*

If G is $(P_5, K_1 \cup K_3)$-free, then $G - N(v) - v$ is (P_5, K_3)-free for each vertex v of G, and so by a simple induction one can show the following lemma.

Lemma 6.2 *$\chi(G) \leq 3\omega(G) - 3$ for every $(P_5, K_1 \cup K_3)$-free graph G with at least one edge.*

Before proving Theorem 1.4, we first prove a few lemmas on the structure of $(P_5, K_1 + (K_1 \cup K_3))$-free graphs. From now on, we always suppose that G is a $(P_5, K_1 + (K_1 \cup K_3))$-free graph.

Lemma 6.3 *Suppose that G has a 5-hole $C = v_1v_2v_3v_4v_5v_1$ and has no clique cut set, and let T be a component of $G[N^2(C)]$. Then the followings hold.*

- (a) *For each $i \in \{1, 2, 3, 4, 5\}$, $G[N(v_i)]$ is $K_1 \cup K_3$-free, $G[N_{\{i,i+2\}}(C)]$ is K_3-free, and $N_{\{i,i+1,i+2\}}(C) \cup N_{\{i,i+1,i+3\}}(C) \cup N_{\{i,i+1,i+2,i+3\}}(C)$ is independent.*
- (b) *If no vertex in $N(C)$ dominates T, then there exist two non-adjacent vertices u and v in $N(C)$ such that both $N_T(u)$ and $N_T(v)$ are not empty.*

Proof. Statement (a) follows directly from the $K_1 + (K_1 \cup K_3)$-freeness of G.

To prove (b), let $S = N(T) \cap N(C)$, and suppose that no vertex in S dominates T. By Lemma 2.2(a), $S \subseteq N_{\{1,2,3,4,5\}}(C) \cup (\bigcup_{1 \leq i \leq 5} N_{\{i,i+2,i+3\}}(C) \cup N_{\{i,i+1,i+2,i+3\}}(C))$. If $G[S]$ is not connected, we are done. Thus suppose that $G[S]$ is connected. We choose an arbitrary vertex $u \in S$ and let $T_u = N_T(u)$. Since u does not dominate T, by the connectedness of T, we may choose a vertex $w \in T \backslash T_u$ such that w is not anticomplete to T_u. Let v be a neighbor of w in S. Since G is P_5-free, we have that $u \in N_{\{1,2,3,4,5\}}(C)$, and so $\{v_{i+2}, v_{i+3}\} \subseteq N(u) \cap N(v)$. Since $uv \in E(G)$ implies a $K_1 + (K_1 \cup K_3)$ on $\{w, u, v, v_{i+2}, v_{i+3}\}$, we have that $uv \notin E(G)$ as desired. ∎

Lemma 6.4 *Suppose that G has a 5-hole $C = v_1v_2v_3v_4v_5v_1$ and no clique cut set. Then $G[N^3(C)]$ is K_3-free, and $N^2(C)$ can be partition into two parts A and B such that both $G[A]$ and $G[B]$ are K_3-free.*

Proof. Let B be a component of $G[N^3(C)]$ and $u \in N^2(C)$ be a vertex that has a neighbor in B. By Lemma 2.2(c), we see that u must be complete to B, and so $G[N^3(C)]$ must be K_3-free to avoid a $K_1 + (K_1 \cup K_3)$.

Let $T = N^2(C)$. Without loss of generality, we suppose that $G[T]$ is connected, and let

$$S = \{v | v \in N(C) \text{ such that } N_T(v) \neq \emptyset\}.$$

If there exists some vertex in S that dominates T, then we are done as $G[T]$ is obviously K_3-free to avoid a $K_1 + (K_1 \cup K_3)$. Thus suppose that no vertex of S dominates T.

By Lemma 6.3(b), there exist two non-adjacent vertices, say u and v, in S such that both u and v have neighbors T. It follows from Lemma 2.2(a) that u and v have a common neighbor, say z, on C.

It is certain that both $G[N_T(u)]$ and $G[N_T(v)]$ are K_3-free. If $T = N(u) \cup N(v)$, then $(N_T(u), N_T(v) \setminus N_T(u))$ is a partition of T as desired. Thus suppose that $T \neq N(u) \cup N(v)$. Let $R = T \backslash (N(u) \cup N(v))$ and $R_1, R_2, \cdots, R_r$ be the components of $G[R]$.

Note that $G[T]$ is connected. For each $i \in \{1, 2, \ldots, r\}$, R_i has a neighbor, say t_i, in $N(u) \cup N(v)$. If t_i is not complete to R_i, we may choose two adjacent vertices x and y in R_i with $t_i x \in E(G)$ and $t_i y \notin E(G)$, then either zut_ixy or zvt_ixy is a P_5 of G. Therefore, t_i must be complete to R_i, and so $G[R]$ is K_3-free to avoid a $K_1 + (K_1 \cup K_3)$.

Let $T_v = N_T(v) \setminus N_T(u)$. If R is not anticomplete to T_v, let $r \in R$ and $t_v \in T_v$ be a pair of adjacent vertices, then rt_vvzu is a P_5 in G. Thus R is anticomplete to T_v, and consequently, $(N_T(u), R \cup T_v)$ is a partition of T as desired. This proves Lemma 6.4. ∎

Lemma 6.5 *Suppose that G is C_5-free and contains an odd antihole A with at least seven vertices. Let S be the set of vertices which are complete to A, and let $T = N(A) \backslash S$. Then $G[S]$ is $K_1 \cup K_3$-free, T can be partition into at most $2k+1$ independent sets, and $N^2(A) = \emptyset$.*

Proof. Suppose that $V(A) = \{v_1, v_2, \ldots, v_{2k+1}\}$, where $k \geq 3$ and $v_iv_{i+1} \notin E(G)$ for each $i \in \{1, 2 \ldots, 2k+1\}$. Since G is $K_1 + (K_1 \cup K_3)$-free, it is certain that $G[S]$ is $K_1 \cup K_3$-free.

Note that the vertex of T is neither complete nor anticomplete to A. For each vertex u of T, there must be an $i_u \in \{1, 2, \ldots, 2k+1\}$ such that $uv_{i_u} \notin E(G)$ and $uv_{i_u+1} \in E(G)$, and so $uv_{i_u+3} \in E(G)$ to avoid either a C_5 or a P_5 depending on whether $uv_{i_u+2} \in E(G)$ or not. For each $i \in \{1, 2, \ldots, 2k+1\}$, let

$$T_i = \{v | v \in T, vv_i \notin E(G) \text{ but } vv_{i+1} \in E(G) \text{ and } vv_{i+3} \in E(G)\} \ .$$

Thus $T = \cup_{1 \leq i \leq 2k+1} T_i$. Since G is $K_1 + (K_1 \cup K_3)$-free, each T_i is independent, otherwise $G[\{v_i, v_{i+1}, v_{i+3}, x, x'\}] = K_1 + (K_1 \cup K_3)$ for any two adjacent vertices x and x' of T_i. Hence T can be partition into at most $2k+1$ independent sets.

Suppose that $N^2(A) \neq \emptyset$. Let v be a vertex in $N(A)$ that has a neighbor, say x, in $N^2(A)$. It is obvious that $v \notin S$, otherwise a $K_1 + (K_1 \cup K_3)$ appears in G. Without loss of generality, we suppose that $v \in T_1$. Thus either a $K_1 + (K_1 \cup K_3)$ appears on

$\{v, v_2, v_4, v_{2k}, x\}$ whenever $vv_{2k} \in E(G)$, or a $P_5 = xvv_2v_{2k}v_1$ appears whenever $vv_{2k} \notin E(G)$. Therefore, $N^2(A) = \emptyset$. This proves Lemma 6.5. ∎

We are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let G be a $\{P_5, K_1 + (K_1 \cup K_3\}$-free graph. We may suppose that G is connected, contains no clique cut set, and is not perfect. Thus G contains a 5-hole or an odd antihole with at least 7 vertices.

First suppose that G contains a 5-hole $C = v_1v_2v_3v_4v_5v_1$. Since G is P_5-free, we have that $V(G) = V(C) \cup N(C) \cup N^2(C) \cup N^3(C)$. By Lemma 2.2($a$), we have that

$$N(C) = N_{\{1,2,3,4,5\}}(C) \bigcup_{1 \le i \le 5} (N_{\{i,i+2\}}(C) \cup N_{\{i,i+1,i+2\}}(C) \cup N_{\{i,i+1,i+3\}}(C) \cup N_{\{i,i+1,i+2,i+3\}}(C)).$$

By Lemmas 6.1, 6.2, and 6.3(a), we have that $\chi(G[N_{\{1,2,3,4,5\}}(C)]) \le 3(\omega(G) - 3)$, $\chi(G[\bigcup_{1 \le i \le 5} N_{\{i,i+2\}}(C)]) \le 15$, and $\chi(G[\bigcup_{1 \le i \le 5} (N_{\{i,i+1,i+2\}}(C) \cup N_{\{i,i+1,i+3\}}(C) \cup N_{\{i,i+1,i+2,i+3\}}(C))]) \le 5$. Therefore, $\chi(G[N(C)]) \le 3\omega(G) + 11$.

By Lemmas 6.1 and 6.4, $\chi(G[N^2(C)]) \le 6$ and $\chi(G[N^3(C)]) \le 3$. Since $\bigcup_{1 \le i \le 5} N_{\{i,i+2\}}(C)$ is anticomplete to $N^2(C) \cup N^3(C)$ by Lemma 6.3(a), we can color the vertices of $V(C) \cup N^2(C) \cup N^3(C)$ with the 15 colors used on $\bigcup_{1 \le i \le 5} N_{\{i,i+2\}}(C)$. Thus $\chi(G) \le 3\omega(G) + 11$.

Now we suppose that G is $\{P_5, C_5, K_1 + (K_1 \cup K_3\}$-free and contains an odd antihole A with $|A| = 2k + 1 \ge 7$. Let $S \subseteq N(A)$ be the set of all vertices that are complete to A, and let $T = N(A) \backslash S$. By Lemma 6.5, we have that $V(G) = A \cup N(A)$, $G[S]$ is $K_1 \cup K_3$-free, and T is the union of $2k+1$ independent sets. Hence $\chi(G[S]) \le 3(\omega(S)-1) \le 3(\omega(G)-k-1)$ by Lemma 6.2, and so $\chi(G) \le \chi(A)+\chi(G[S])+\chi(G[T]) \le (k+1)+3(\omega(G)-k-1)+(2k+1) < 3\omega(G)$. This proves Theorem 1.4. ∎

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