New upper bounds for the crossing numbers of crossing-critical graphs^{*}

Zongpeng Ding[†], Zhangdong Ouyang[‡]

School of mathematics and Computational Sciences Hunan First Normal University, Changsha 410205, P.R China

Yuanqiu Huang[§]

Department of Mathematics Hunan Normal University, Changsha 410081, P.R China

Fengming Dong[¶]

National Institute of Education, Nanyang Technological University, Singapore

Abstract

A graph G is k-crossing-critical if $cr(G) \ge k$, but $cr(G \setminus e) < k$ for each edge $e \in E(G)$, where cr(G) is the crossing number of G. It is known that for any k-crossing-critical graph G, $cr(G) \le 2.5k + 16$ holds, and in particular, if $\delta(G) \ge 4$, then $cr(G) \le 2k + 35$ holds, where $\delta(G)$ is the minimum degree of G. In this paper, we improve these upper bounds to 2.5k + 2.5 and 2k + 8 respectively. In particular, for any k-crossing-critical graph G with n vertices, if $\delta(G) \ge 5$, then $cr(G) \le 2k - \sqrt{k}/2n + 35/6$ holds.

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[†]Email: dzppxl@163.com.

[‡]Corresponding author. Email: oymath@163.com.

[§]Email: hyqq@hunnu.edu.cn.

[¶]Email: fengming.dong@nie.edu.sg (expired on 24/03/2027) and donggraph@163.com.

1 Introduction

All graphs considered here are simple, connected, finite and undirected unless otherwise specified. For any graph G, let V(G), E(G) and $\delta(G)$ denote its vertex set, edge set and minimum degree. A *drawing* of a graph G is a mapping D that assigns to each vertex in V(G) a distinct point in the plane, and to each edge uv in G a continuous arc connecting D(u) and D(v), not passing through the image of any other vertex. For any drawing D of G, let cr(D) denote the number of crossings in D, and the *crossing number* of G, denoted by cr(G), is the minimum value of cr(D)'s among all possible drawings D of G. For more on crossing numbers of graphs, we refer to [6] and the references therein.

A graph G is k-crossing-critical if $cr(G) \ge k$, but $cr(G \setminus e) < k$ for every edge $e \in E(G)$ (e.g. see [2]). A graph is crossing-critical if it is k-crossing-critical for some k.

Crossing-critical graphs give insight into structural properties of the crossing number invariant and have thus generated considerable interest. Let \mathscr{M}_k denote the set of *k*-crossing-critical graphs. Richter and Thomassen [4] showed that $cr(G) \leq 2.5k+16$ holds for each $G \in \mathscr{M}_k$. Salazar [5] improved this result to $cr(G) \leq 2k + 35$ for the case that $\delta(G) \geq 4$. Lomelí and Salazar [3] showed that, for each integer k > 0, there is an integer n_k such that for any $G \in \mathscr{M}_k$ with at least n_k vertices of degree greater than two, $cr(G) \leq 2k + 23$ holds.

In this paper, we further improve Richter and Thomassen's result in [4] and Salazar's result in [5] to $cr(G) \leq 2.5k + 2.5$ and $cr(G) \leq 2k + 8$ respectively. Furthermore, we show that, for any $G \in \mathcal{M}_k$ with *n* vertices, if $\delta(G) \geq 5$, then $cr(G) \leq 2k - \sqrt{k/2n} + 35/6$ holds.

2 Choose a suitable cycle in a graph

For any positive integers l and Δ , a cycle C in a graph G is called (l, Δ) -nearly-light if the length of C is at most l and at most one vertex of C has degree greater than Δ (see [3]).

For any cycle C in a graph G, define

$$\mu(C) = \min_{v \in V(C)} \sum_{u \in V(C) \setminus \{v\}} (d_G(u) - 2).$$

Clearly, if C is an (l, Δ) -nearly-light cycle in G, then $\mu(C) \leq (l-1)(\Delta - 2)$. In

[3], Lomelí and Salazar showed that if a k-crossing-critical graph G contains a cycle C with $\mu(C) \leq s$, then $cr(G) \leq 2(k-1) + s/2$, and therefore, if G contains an (l, Δ) -nearly-light cycle, then $cr(G) \leq 2(k-1) + (l-1)(\Delta - 2)/2$ holds.

The *skewness* of a graph G, denoted by sk(G), is the minimum integer t such that $G \setminus E_0$ is planar for a subset E_0 of E(G) with $|E_0| = t$. In this section, we show that for any graph G with $\delta(G) \geq 3$, G contains a cycle C with $\mu(C) \leq sk(G) + 10$. In Section 4, we apply this fact to find an upper bound for cr(G) in terms of sk(G) and $\delta(G)$, where G is crossing-critical.

The next elementary result will be applied later.

Lemma 1 Let d_1, d_2, \dots, d_5 be integers with $3 \leq d_1 \leq d_2 \leq \dots \leq d_5$. Then, the following statements hold:

- (1). if $\sum_{i=1}^{3} \frac{1}{d_i} > \frac{1}{2}$, then $\sum_{i=1}^{2} (d_i 2) \le 10$;
- (2). if $\sum_{i=1}^{4} \frac{1}{d_i} > 1$, then $\sum_{i=1}^{3} (d_i 2) \le 5$; and
- (3). if $\sum_{i=1}^{5} \frac{1}{d_i} > \frac{3}{2}$, then $\sum_{i=1}^{4} (d_i 2) \le 4$.

Proof. (1) Note that the conclusion holds for $(d_1, d_2, d_3) = (3, 11, 11)$.

Suppose that $\sum_{i=1}^{2} (d_i - 2) \ge 11$. Then, $d_2 \ge 8$. Let $d_1 = 3 + \alpha_1$ and $d_2 = 8 + \alpha_2$, where $\alpha_1, \alpha_2 \ge 0$. As $\sum_{i=1}^{2} (d_i - 2) \ge 11$, $\alpha_1 + \alpha_2 \ge 4$. As $d_3 \ge d_2$, we have

$$\frac{1}{2} < \sum_{i=1}^{3} \frac{1}{d_i} \le \frac{2}{8 + \alpha_2} + \frac{1}{3 + \alpha_1}$$

which implies that $4\alpha_1 + \alpha_2 + \alpha_1\alpha_2 < 4$, a contradiction with $\alpha_1 + \alpha_2 \ge 4$.

(2) and (3) can be proved similarly.

Proposition 1 For any planar graph G with $\delta(G) \geq 3$, there exists a cycle C in G with $\mu(C) \leq 10$.

Proof. For any face f of a drawing of G, the weight of f, denoted by $\omega(f)$, is defined as follows:

$$\omega(f) = \sum_{v \sim f} \frac{1}{d_G(v)},$$

where $v \sim f$ means that v is a vertex on the boundary of f. We have $\omega(f) \leq \ell(f)/3$, where $\ell(f)$ denotes the length of the boundary of f. Obviously, $|V(G)| = \sum_{f \in F(G)} \omega(f)$ and $2|E(G)| = \sum_{f \in F(G)} \ell(f)$.

By Euler's polyhedron formula,

$$\sum_{f \in F(G)} (\omega(f) - \ell(f)/2 + 1) = 2,$$

implying that $\omega(f_0) - \ell(f_0)/2 + 1 > 0$ holds for some face f_0 . As $\omega(f_0) \le \ell(f_0)/3$,

$$0 < \omega(f_0) - \ell(f_0)/2 + 1 \le \ell(f_0)/3 - \ell(f_0)/2 + 1 = 1 - \ell(f_0)/6,$$

implying that $\ell(f_0) \leq 5$. Let C denote the boundary of face f_0 . Since $\delta(G) \geq 3$, C must be a cycle of G.

First consider the case that $\ell(f_0) = 3$. The inequality $\omega(f_0) - \ell(f_0)/2 + 1 > 0$ implies that $\omega(f_0) > 1/2$, i.e., $1/d_1 + 1/d_2 + 1/d_3 > 1/2$, where d_1, d_2, d_3 denote the degrees of the three vertices on cycle C with $d_1 \leq d_2 \leq d_3$. By Lemma 1 (1), $\mu(C) = d_1 - 2 + d_2 - 2 \leq 10$.

In the case $\ell(f_0) \in \{4, 5\}$, $\omega(f_0) - \ell(f_0)/2 + 1 > 0$ implies that $\omega(f) > \ell(f_0)/2 - 1$. By Lemma 1, there exists $v \in V(C)$ such that $\sum_{u \in V(C) \setminus \{v\}} (d_G(u) - 2) \leq 5$. Thus $\mu(C) \leq 5$.

Proposition 2 For any graph G with $\delta(G) \ge 3$, G contains a cycle C with $\mu(C) \le sk(G) + 10$.

Proof. Let t = sk(G) and E_0 be a subset of E(G) with $|E_0| = t$ such that $G \setminus E_0$ is planar. The proof is by induction on t. If t = 0, then G is planar, and the result follows from Proposition 1.

We now assume that $t \ge 1$. Let $e = v_1 v_2 \in E_0$ and $G' = G \setminus e$. Then sk(G') = t - 1. For i = 1, 2, if $d_G(v_i) = 3$, then $d_{G'}(v_i) = 2$ and there exists an edge e_i in G' incident with v_i and some vertex u_i . Note that u_1 and u_2 may be not distinct.

Let E_1 be the subset of $\{e_1, e_2\}$ such that $e_i \in E_1$ if and only if $d_G(v_i) = 3$. Clearly, $|E_1| = 2$ if and only if $d_G(v_1) = d_G(v_2) = 3$, and $E_1 = \emptyset$ if and only if $d_G(v_1) > 3$ and $d_G(v_2) > 3$. Let $H = G'/E_1$, i.e., H is the graph obtained from G' by contracting all edges in E_1 . Thus, if $E_1 = \emptyset$, then H is G' itself; if $E_1 = \{e_1, e_2\}$, then H is as shown in Figure 1 (c).

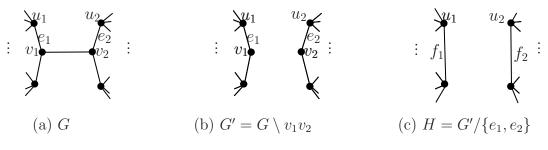


Figure 1: Graphs G, G' and H when $d_G(v_1) = d_G(v_2) = 3$

As $H = G'/E_1$ and each edge e_i in E_1 is incident with a vertex of degree 2 in G', we have sk(H) = sk(G') = t - 1. By the constriction of H,

$$\delta(H) = \min\{d_{G'}(u) : u \in V(G'), d_{G'}(u) > 2\} \ge 3.$$

By the inductive assumption, H contains a cycle C_1 with $\mu_H(C_1) \leq t - 1 + 10 = t + 9$. Assume that v is a vertex in C_1 such that

$$\mu_H(C_1) = \sum_{u \in V(C_1) \setminus \{v\}} (d_H(u) - 2).$$

As v must be a vertex in C_1 with the maximum degree in H, we have $d_H(v) \ge 3$.

Let C be the cycle of G' obtained from C_1 by replacing edge f_i by path P_i whenever f_i is an edge in C_1 for i = 1, 2, where P_i is the path in G' of length 2 which has v_i as its center vertex in the case $d_{G'}(v_i) = 2$.

Note that C is also a cycle in G. Recall that $G' = G \setminus e$ and e joins v_1 and v_2 . There are two cases to consider.

Case 1: $\{v_1, v_2\} \not\subseteq V(C)$.

In this case,

$$\mu_G(C) \le 1 + \mu_H(C_1) \le t + 10.$$

Case 2: $\{v_1, v_2\} \subseteq V(C)$.

Note that $C \cup e$ is a subgraph in G, as shown in Figure 2. Let C_1 and C_2 be the two cycles in $C \cup e$ with $e \in E(C_i)$ for i = 1, 2 and $v \in V(C_1)$.

Observe that

$$\mu_{G}(C_{1}) \leq \sum_{u \in V(C_{1}) \setminus \{v\}} (d_{G}(u) - 2) \\
\leq \sum_{u \in V(C) \setminus \{v\}} (d_{G}(u) - 2) - 1 - (|V(C_{2})| - 2)(\delta(G) - 2) \\
= \mu_{G'}(C) + 2 - 1 - (|V(C_{2})| - 2)(\delta(G) - 2) \\
\leq \mu_{H}(C_{1}) + 2 - 1 - 1 \\
= t + 9.$$
(2.1)

Thus, the result holds.

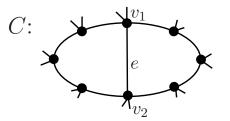


Figure 2: Edge e is a chord of cycle C

3 Upper bound for cr(G) in terms of sk(G)

It is readily checked that $sk(G) \leq cr(G)$. In this section, we shall establish a new relationship between the crossing number and skewness of a graph.

Lemma 2 Let G be a graph with $n \ge 4$ vertices, and e be an edge in G such that $G \setminus e$ is planar. For any planar drawing D_1 of $G \setminus e$, D_1 can be extended to a drawing D of G with $cr(D) \le \frac{2n-7}{3}$.

Proof. It suffices to prove this result for the case that $G \setminus e$ is a maximal planar graph.

Let G_1 denote $G \setminus e$ and D_1 be a planar drawing of G_1 . As G_1 is a maximal planar graph, G_1 is 3-connected. Let D_1^* denote the dual of D_1 . Thus, D_1^* is 3-connected [1, Exercise 10.2.7].

Assume that e is an edge of G joining vertices v_1 and v_2 in G. As G_1 is 3-connected, $\delta(G_1) \geq 3$. Thus, each vertex v_i is incident with at least three faces of D_1 . Assume that f_1, f_2, f_3 are faces of D_1 which are incident with v_1 and f'_1, f'_2, f'_3 are faces of D_1 which are incident with v_2 .

If $\{f_1, f_2, f_3\} \cap \{f'_1, f'_2, f'_3\} \neq \emptyset$, say $f_1 = f'_1$, then D_1 can be extended to a planar drawing D of G by adding an edge within face f_1 joining v_1 and v_2 in D_1 . Thus $cr(D) = 0 \leq (2n-7)/3$ holds.

Now assume that $\{f_1, f_2, f_3\} \cap \{f'_1, f'_2, f'_3\} = \emptyset$. As D_1^* is 3-connected, there exist 3 vertex-disjoint paths P_1, P_2 and P_3 in D_1^* connecting vertices in $\{f_1, f_2, f_3\}$ to vertices in $\{f'_1, f'_2, f'_3\}$. Observe that

$$\sum_{i=1}^{3} |E(P_i)| = \sum_{i=1}^{3} (|V(P_i)| - 1) \le |V(D_1^*)| - 3 = 2n - 4 - 3 = 2n - 7, \quad (3.1)$$

where $|V(D_1^*)| = 2n - 4$ follows from the fact that D_1 is a maximal plane graph and $|V(D_1^*)|$ is equal to the number of faces of D_1 .

Assume that $|E(P_1)| \leq |E(P_i)|$ for i = 2, 3. By (3.1), $|E(P_1)| \leq (2n-7)/3$. Assume that P_1 is a path in D_1^* joining vertices f_1 and f'_1 . As f_1 is a face of D_1 incident with v_1 and f'_1 is a face of D_1 incident with v_2 , P_1 actually generates a curve on the plane connecting v_1 and v_2 which crosses with exactly $|E(P_1)|$ edges in D_1 . This curve represents a way of drawing edge e in D_1 . Thus, we get a drawing D of Gwith $cr(D) = |E(P_1)| \leq (2n-7)/3$.

According to the Lemma 2, we now reveal the relationship between cr(G) and sk(G).

Theorem 1 Let G be a graph with n vertices. Then,

$$cr(G) \le \frac{3sk(G)^2 + (4n - 17)sk(G)}{6}$$

Moreover, the upper bound is tight.

Proof. If sk(G) = 0, then G is planar and cr(G) = 0.

Now assume that sk(G) = t > 0. By definition, there exists a set E_0 of edges in G with $|E_0| = t$ such that $G \setminus E_0$ is planar.

Let G_1 denote the subgraph $G \setminus E_0$ and D_1 be a planar drawing of G_1 . Applying Lemma 2 to each edge in E_0 , we get a drawing D of G such that

$$cr(D) \le t\frac{2n-7}{3} + \binom{t}{2}.$$

As t = sk(G) and $cr(G) \leq cr(D)$, the claim follows. It can be verified easily that the upper bound is tight for the complete graph K_5 .

4 Main results on crossing-critial graphs

Recall that \mathcal{M}_k denotes the set of k-crossing-critical graphs.

Theorem 2 Let G be a graph in \mathcal{M}_k with minimum degree δ . If G contains a cycle C with $\mu(C) = s$, then,

$$cr(G) \leq \begin{cases} 2k + \frac{s-5}{2}, & \text{if } \delta = 3;\\ 2k - sk(G) + \frac{\delta(s-\delta+2)}{2(\delta-2)}, & \text{if } \delta \ge 4. \end{cases}$$

Proof. As $\mu(C) = s$, there exists some vertex v in C such that

$$|E(P)| = |V(C)| - 1 \le \frac{1}{\delta - 2} \sum_{u \in V(C) \setminus \{v\}} (d_G(u) - 2) = \frac{s}{\delta - 2}.$$

Let e be an edge of C with ends v and w and P denote the path of $C \setminus e$. As G is k-crossing-critical, $cr(G \setminus e) \leq k - 1$. Let D be a drawing of $G \setminus e$ with at most k - 1 crossings. Note that edges in P may cross each other in the drawing D. We regard the drawing of P as a planar graph H with vertices of degrees 2 and 4. Let P' be a shortest path in H joining v and w. There are two ways of reconnecting v and w close to P', one for each side of P'.

Let $r_D(P)$ denotes the number of crossings of edges of P in D. It is not hard to verify that the total number of crossings in these two drawings of e is at most

$$\lambda = \sum_{u \in V(C) \setminus \{v,w\}} (d_G(u) - 2) + 2r_D(P)$$

$$\leq s - (\delta - 2) + 2r_D(P)$$

Therefore, one of the two drawings for e crosses at most $\lambda/2$ edges of D. Note that $r_D(P) \leq k-1$, implying that

$$cr(G) \le \frac{\lambda}{2} + k - 1 \le \frac{s - \delta + 2}{2} + 2k - 2 = \frac{s - \delta}{2} + 2k - 1.$$
 (4.1)

If $\delta = 3$, then it follows that $cr(G) \le 2k + (s-5)/2$.

Now we consider the case that $\delta \geq 4$. Removing from D the edges (at most $s/(\delta-2)$) of P leaves a drawing with at most $k - 1 - r_D(P)$ crossings. Therefore, there is a set of at most $1 + s/(\delta - 2) + k - 1 - r_D(P)$ edges whose removal from G leaves a planar graph. Thus, $sk(G) \leq 1 + s/(\delta - 2) + k - 1 - r_D(P)$, implying that

$$r_D(P) \le \frac{s}{\delta - 2} + k - sk(G).$$

As
$$\lambda \leq s - (\delta - 2) + 2r_D(P)$$
, by (4.1),

$$cr(G) \leq \frac{\lambda}{2} + k - 1$$

$$\leq \frac{s - \delta + 2}{2} + r_D(P) + k - 1$$

$$\leq \frac{s - \delta + 2}{2} + k - 1 + \frac{s}{\delta - 2} + k - sk(G).$$
(4.2)

The result holds.

Theorem 3 Let $G \in \mathcal{M}_k$ with n vertices and minimum degree δ . Then,

$$cr(G) \leq \begin{cases} 2.5(k+1), & \text{if } \delta = 3;\\ 2(k+4), & \text{if } \delta = 4;\\ 2k - \sqrt{k}/2n + 35/6, & \text{if } \delta \ge 5. \end{cases}$$

Proof. Let t = sk(G). By Proposition 2, G contains a cycle C with $\mu(C) \le t + 10$. If $\delta = 3$, by Theorem 2, $cr(G) \le 2k + (t + 10 - 5)/2 = 2k + 2.5 + 0.5t \le 2.5k + 2.5$, as $t \le k$.

If $\delta = 4$, by Theorem 2, $cr(G) \le 2k - t + (t + 10 - 4 + 2) = 2k + 8$.

Now consider the case that $\delta \geq 5$. by Theorem 2,

$$cr(G) \le 2k - t + \frac{\delta(t + 10 - \delta + 2)}{2(\delta - 2)} \le 2k - t + \frac{5(t + 7)}{6} = 2k + \frac{35 - t}{6}.$$
 (4.3)

By (4.3), if $t \ge \frac{3\sqrt{k}}{n}$, the result holds. In the following, assume that $t < \frac{3\sqrt{k}}{n}$. By Theorem 1,

$$cr(G) \le \frac{3t^2 + (4n - 17)t}{6} < \frac{3(9k/n^2) + (4n - 17)\frac{3\sqrt{k}}{n}}{6} = \frac{9k + (4n - 17)n\sqrt{k}}{2n^2}.$$
(4.4)

If k = 1, then, by (4.4),

$$cr(G) \le \frac{9 + (4n - 17)n}{2n^2} < 2,$$
(4.5)

and the result holds.

If $k \ge 2$, then, by (4.4),

$$cr(G) - \left(2k + 35/6 - \frac{1}{2n}\sqrt{k}\right)$$

$$\leq \frac{9k + (4n - 17)n\sqrt{k}}{2n^2} - \left(2k + 35/6 - \frac{1}{2n}\sqrt{k}\right)$$

$$< \frac{9k + (4n - 0)n\sqrt{k}}{2n^2} - \left(2k + 0 - \frac{1}{2n}\sqrt{k}\right)$$

$$= \frac{(4.5 - 2n^2)k + (2n + 0.5)n\sqrt{k}}{n^2}$$

$$< 0, \qquad (4.6)$$

where the last inequality follows from the facts that the solution of the inequality $(4.5 - 2n^2)k + (2n + 0.5)n\sqrt{k} < 0$ is $k > n^2(4n + 1)^2/(4n^2 - 9)^2$ and that $2 > n^2(4n + 1)^2/(4n^2 - 9)^2$ holds for all $n \ge 5$. Thus, we complete the proof. \Box

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