Anti-Ramsey numbers for trees in complete multi-partite graphs

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Abstract

Let G be a complete multi-partite graph of order n. In this paper, we consider the anti-Ramsey number $ar(G, \mathcal{T}_q)$ with respect to G and the set \mathcal{T}_q of trees with q edges, where $2 \leq q \leq n-1$. For the case q = n-1, the result has been obtained by Lu, Meier and Wang. We will extend it to q < n-1. We first show that $ar(G, \mathcal{T}_q) = \ell_q(G) + 1$, where $\ell_q(G)$ is the maximum size of a disconnected spanning subgraph H of G with the property that any two components of H together have at most q vertices. Using this equality, we obtain the exact values of $ar(G, \mathcal{T}_q)$ for $n-3 \leq q \leq n-1$. We also compute $ar(G, \mathcal{T}_q)$ by a simple algorithm when $(4n-2)/5 \leq q \leq n-1$.

1 Introduction

In this article, we consider simple graphs only. Given any graph G, let V(G) and E(G) denote the vertex set and edge set of G, and let com(G) denote the number of its components. If G_1, G_2, \dots, G_s are the components of G, where s = com(G), with $|V(G_1)| \ge \dots \ge |V(G_s)|$, let $or_i(G) = |V(G_i)|$ for all $i = 1, 2, \dots, s$. Thus, $or_1(G) \ge or_2(G) \ge \dots \ge or_s(G)$ and $or_1(G) + or_2(G) + \dots + or_s(G) = |V(G)|$. Let K_{p_1, p_2, \dots, p_k} denote the complete k-partite graph whose partite sets' sizes are p_1, p_2, \dots, p_k respectively. For any $1 \le r \le |V(G)|$, let $\mathscr{P}_r(G)$ be the family of r-element subsets of V(G). For any vertex $v \in V(G)$, let $E_G(v)$ be the set of edges in G which are incident with $v, N_G(v)$ be the set of vertices in G which are adjacent to v and $d_G(v)$ be the degree of v in G, i.e., the cardinality of $N_G(v)$. For any $S \subseteq V(G)$, let $E_G(S) = \bigcup_{v \in S} E_G(v)$ and let G[S] be the subgraph of G induced by S.

For a positive integer t, a *t*-edge-coloring of G is a surjective map from E(G) to $\{1, 2, \dots, t\}$. Note that an edge coloring here is actually a partition of E(G), and it is probably not a proper edge coloring of G. In an edge coloring of G, a subgraph H of G is called a *rainbow* subgraph if the colors assigned to the edges in H are pairwise distinct.

For a graph G and a family \mathcal{C} of graphs, the *anti-Ramsey number* with respect to G and \mathcal{C} , denoted by $ar(G, \mathcal{C})$, is the maximum integer t such that there is a t-edge-coloring of G in which every rainbow subgraph is not isomorphic to any graph in \mathcal{C} . If no such edge coloring

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exists, define $ar(G, \mathcal{C})$ to be zero. The study of anti-Ramsey numbers was initiated by Erdős, Simonovits and Sós [2]. Since then, a lot of research papers on this topic have been published. See [4] for a survey and [1, 3, 10, 11] for some recent development on specific G or \mathcal{C} .

In particular, a common type of C is related to trees. For example, the anti-Ramsey number for edge disjoint spanning trees has been studied thoroughly, the exact value of which has been obtained when G is a complete graph [5], a complete bipartite graph [6], and a complete multipartite graph in [9] very recently. Moreover, the anti-Ramsey number for edge disjoint spanning trees in general graphs has also been determined in [9].

From a different perspective, we will focus on another class of trees in the following.

This paper is motivated by the known results of $ar(G, \mathcal{T}_q)$ (see [7, 8]), where G is a complete graph K_n or a complete bipartite graph K_{p_1,p_2} and \mathcal{T}_q is the set of subtrees in G with exactly q edges. In [7], Jiang and West determined and obtained the exact value of $ar(K_n, \mathcal{T}_q)$ by proving the equality that for $2 \le q \le n-1$,

$$ar(K_n, \mathcal{T}_q) = \ell_q(K_n) + 1, \tag{1.1}$$

where $\ell_q(G)$ is the maximum size of a disconnected spanning subgraph H of G in which every two components together have at most q vertices (i.e., $or_1(H) + or_2(H) \leq q$). A result similar to (1.1) for $ar(K_{p_1,p_2}, \mathcal{T}_q)$ was obtained by Jin and Li [8], accompanying with exact values for certain cases.

As a generalization, we consider the anti-Ramsey number $ar(G, \mathcal{T}_q)$, where G is a complete multi-partite graph K_{p_1,p_2,\cdots,p_k} , $n = \sum_{i=1}^k p_i$ and $2 \le q \le n-1$. Note that the exact value of $ar(G, \mathcal{T}_{n-1})$ can be obtained as a corollary of the result in [9]. We will study the problem in a different approach. We first extend the result (1.1) to $ar(G, \mathcal{T}_q)$ for any complete multi-partite graph G and $2 \le q \le n-1$:

$$ar(G, \mathcal{T}_q) = \ell_q(G) + 1, \tag{1.2}$$

transforming the study of $ar(G, \mathcal{T}_q)$ to that of $\ell_q(G)$, which is a seemingly more numerical invariant.

In this article, we calculate $ar(G, \mathcal{T}_q)$ via determining $\ell_q(G)$ for the two cases $n-3 \leq q \leq n-1$ and $(4n-2)/5 \leq q \leq n-1$. For both cases, we show that

$$\ell_q(G) = |E(G)| - \min_{S \in \mathscr{P}_{n-q+1}(G)} |E_G(S)|,$$
(1.3)

unless (G,q) is one of the ordered pairs below in the first case:

$$(K_{3,3},4), (K_{4,3},4), (K_{3,3,3},6).$$
 (1.4)

The equality of (1.2) is established in Section 2. Let $S_q(G)$ be the set of disconnected spanning subgraphs H of G such that every two components of H together have at most q vertices and $\overline{S_q}(G)$ be the subset of graphs in $S_q(G)$ with the maximum size (i.e., $\ell_q(G)$). In Section 3, we show that $\overline{S_q}(G)$ contains a graph H with $or_1(H) + or_2(H) = q$ and $or_2(H) = or_3(H)$. In Section 4, we consider the case $n - 3 \leq q \leq n - 1$ and prove that (1.3) holds, unless (G, q) is an ordered pair in (1.4). In Section 5, we get a conclusion that $\min_{S \in \mathscr{P}_r(G)} |E_G(S)|$ can be determined by a simple algorithm (i.e., Algorithm A in Section 5) of repeatedly choosing vertices with the minimum degree. Applying the results obtained in Sections 4 and 5, we explicitly express $ar(G, \mathcal{T}_q)$ for the case $n - 3 \leq q \leq n - 1$ in Section 6. The case $(4n - 2)/5 \leq q \leq n - 1$ is studied in the last section (i.e. Section 7). For this case, $\ell_q(G)$ can be determined by (1.3) and thus it can be calculated by Algorithm A.

Let $\chi(G)$ denote the *chromatic number* of graph G and $\delta(G)$ be the *minimum degree* among all vertices of G. If H is a proper subgraph of G, write $G \setminus H$ for $G[V(G) \setminus V(H)]$. Especially, write $G \setminus x$ for $G[V(G) \setminus \{x\}]$. Given a subset E' of E(G), let $G \setminus E'$ be the spanning subgraph of G with edge set $E(G) \setminus E'$. If $E' = \{e'\}$, write $G \setminus e'$ for $G \setminus E'$.

2 The anti-Ramsey number $ar(K_{p_1,\dots,p_k},\mathcal{T}_q)$

In this section, we show that $ar(K_{p_1,\dots,p_k},\mathcal{T}_q) = \ell_q(K_{p_1,\dots,p_k}) + 1$. We first introduce three lemmas.

Lemma 2.1. Let G be any connected graph of order $n \ge 3$. For any q with $2 \le q \le n-1$,

$$ar(G, \mathcal{T}_q) \ge \ell_q(G) + 1. \tag{2.1}$$

Proof. Let $H \in \overline{S_q}(G)$, i.e., $H \in S_q(G)$ and $|E(H)| = \ell_q(G)$. Consider an $(\ell_q(G) + 1)$ -edgecoloring of G which assigns distinct colors to all edges in H and a new color to all the edges in $E(G) \setminus E(H)$. We will show that it is an edge coloring without any rainbow subtree of q edges.

Let T be a subtree of G with q edges and q + 1 vertices. By the definition of $S_q(G)$, for any two components H_1 and H_2 of H, $|V(H_1)| + |V(H_2)| \le q$. Then T contains vertices from at least three components of H, thus $|E(T) \cap (E(G) \setminus E(H))| \ge 2$ and T is not rainbow.

As there is no rainbow subtree of q edges in this coloring, the conclusion holds. \Box

Given a t-edge-coloring c of a graph G, for any edge $e \in E(G)$, denote the color of e by c(e). A representing graph of the t-edge-coloring c is a rainbow spanning subgraph of G with precisely t edges.

Lemma 2.2. Let G be any connected graph and c be a t-edge-coloring of G. Assume that H is a representing graph of coloring c such that H has a component H_1 with $|V(H_1)| \ge or_1(H')$ for every representing graph H' of this coloring. If $com(H) \ge 2$, then H_1 has a bridge b.

Proof. Let H_1, \dots, H_s be all the components of H, where $s \ge 2$.

Since G is connected, there exists an edge $e \in E(G) \setminus E(H)$ such that e joins a vertex u in H_1 to a vertex v in H_i , where $2 \leq i \leq s$. As e is not in H and H is a representing graph of

coloring c, there must be a unique edge b in E(H) such that c(b) = c(e). We will show b is a bridge of H_1 .

Let H' be the spanning subgraph of G obtained from H by removing edge b and adding edge e. H' is also a representing graph for coloring c. If either b is not in H_1 or b is in H_1 but b is not a bridge of H_1 , then H' has a component including vertex v and all the vertices of H_1 , whose order is strictly larger than $|V(H_1)|$, a contradiction to the selection of H.

Hence H_1 has a bridge b.

The following result due to Jiang and West [7] will also be applied in the proof of Theorem 2.4.

Lemma 2.3 ([7]). Every connected graph G contains a vertex w such that for each $e \in E(G)$, the component of G - e containing w has at least |V(G)|/2 vertices.

Now we can prove the main result in this section.

Theorem 2.4. For any positive integers $k, p_1, p_2, \cdots, p_k, q$ with $k \ge 2$ and $2 \le q \le \sum_{i=1}^k p_i - 1$,

$$ar(K_{p_1,\dots,p_k},\mathcal{T}_q) = \ell_q(K_{p_1,\dots,p_k}) + 1.$$
(2.2)

Proof. Let $n = \sum_{i=1}^{k} p_i$ and X_1, X_2, \dots, X_k be the partite sets of K_{p_1,\dots,p_k} with $|X_i| = p_i$ for all $i = 1, 2, \dots, k$. According to Lemma 2.1, $ar(K_{p_1,\dots,p_k}, \mathcal{T}_q) \ge \ell_q(K_{p_1,\dots,p_k}) + 1$. Thus, we only need to prove $ar(K_{p_1,\dots,p_k}, \mathcal{T}_q) \le \ell_q(K_{p_1,\dots,p_k}) + 1$.

Let $t = ar(K_{p_1,\dots,p_k}, \mathcal{T}_q)$ and G denote K_{p_1,\dots,p_k} . We will show that for any t-edge-coloring of G, if there is no rainbow subtree of q edges, then $t \leq \ell_q(G) + 1$.

Now let c be any t-edge-coloring of G without rainbow trees of q edges and choose a representing graph H of coloring c such that $or_1(H) \ge or_1(H')$ for every representing graph H' of this coloring.

Let H_1 be a component of H with $|V(H_1)| = or_1(H)$. By Lemma 2.1, $|E(H)| = t \ge 1$, thus $|V(H_1)| \ge 2$. Moreover, since there is no rainbow tree of q edges in coloring c, $|V(H_1)| \le q \le n-1$, implying that H has more than one component. Let H_2, \dots, H_s be the components of H other than H_1 , with $|V(H_i)| = or_i(H)$ for $i = 2, \dots, s$, where $s \ge 2$.

We will show $t \leq \ell_q(G) + 1$ by the following two cases.

Claim 1: If $|V(H_2)| = 1$, then $t \le \ell_q(G) + 1$.

Proof. As G is connected, there must be an edge $e = xz \in E(G) \setminus E(H)$, where $z \in V(H_1)$ and $x \in V(G) \setminus V(H_1)$.

By Lemma 2.2, there is a bridge $b \in E(H_1)$ with c(e) = c(b). Denote the two connected components of $H_1 \setminus b$ by $H_1^{(1)}$ and $H_1^{(2)}$, as shown in Figure 1.

Let H^* be the spanning subgraph of G obtained from H by deleting edge b. Note that the components of H^* are $H_1^{(1)}, H_1^{(2)}, H_2, \cdots, H_s$.

Since $or_2(H) = |V(H_2)| = 1$, we have $|V(H_p)| = 1$ for all $p = 2, 3, \dots, s$. Thus,

$$or_1(H^*) + or_2(H^*) = |V(H_1^{(1)})| + |V(H_1^{(2)})| = |V(H_1)| \le q.$$
 (2.3)



Figure 1: $H_1^{(1)}$ and $H_1^{(2)}$ are the components of $H_1 \setminus b$, and $x \in V(G) \setminus V(H_1)$

Hence $H^* \in S_q(G)$ and $t = |E(H)| = |E(H^*)| + 1 \le \ell_q(G) + 1$. Claim 1 holds. Claim 2: If $|V(H_2)| \ge 2$, then $t \le \ell_q(G) + 1$.

Proof. By Lemma 2.3, H_1 contains a vertex u, such that for each $e \in E(H_1)$, the component of $H \setminus e$ containing u has at least $|V(H_1)|/2$ vertices. Assume $u \in V(H_1) \cap X_i$ for some i, where $1 \leq i \leq k$.

Since H_2 is connected and $|V(H_2)| \ge 2$, we have $V(H_2) \not\subseteq X_i$. Let $w \in V(H_2) \setminus X_i$. Then $uw \in E(G)$.

By Lemma 2.2, there is a bridge $b \in E(H_1)$ with c(uw) = c(b). Denote the two connected components of $H_1 \setminus b$ by $H_1^{(1)}$ and $H_1^{(2)}$ with $u \in H_1^{(1)}$, as shown in Figure 2.



Figure 2: $H_1^{(1)}$ and $H_1^{(2)}$ are the components of $H_1 \setminus b$, and $w \in V(H_2)$

Let H' be the graph obtained from H by deleting edge b and adding edge uw. H' is also a representing graph of coloring c with a component which consists of vertices in both $H_1^{(1)}$ and H_2 . By the assumptions of H and H_1 , we have

$$|V(H_1^{(1)})| + |V(H_2)| \le |V(H_1)| = |V(H_1^{(1)})| + |V(H_1^{(2)})|,$$

implying that $|V(H_1^{(2)})| \ge or_2(H)$.

Let H^* be the spanning subgraph of G obtained from H by deleting edge b. Note that H^* has components $H_1^{(1)}, H_1^{(2)}, H_2, \cdots, H_s$.

Due to the choice of u, $|V(H_1^{(1)})| \ge |V(H_1^{(2)})|$. Hence $or_1(H^*) = |V(H_1^{(1)})|$ and $or_2(H^*) = |V(H_1^{(2)})|$. Together with the fact that $|V(H_1^{(1)})| + |V(H_1^{(2)})| = |V(H_1)| \le q$, we conclude that $H^* \in S_q(G)$. Hence $t = |E(H)| = |E(H^*)| + 1 \le \ell_q(G) + 1$ and Claim 2 holds.

By Claims 1 and 2, $ar(K_{p_1,\dots,p_k},\mathcal{T}_q) \leq \ell_q(K_{p_1,\dots,p_k}) + 1$, and thus the result holds. \Box

3 Preparation work

To get the exact value of $\ell_q(K_{p_1,\dots,p_k})$, we shall give some lemmas based on the properties of multi-partite graphs and $\ell_q(G)$ in this section.

Lemma 3.1. Let G be a complete multi-partite graph, and let V_1 and V_2 be disjoint subsets of V(G). If $|V_1| \ge |V_2|$ and $V_1 \cup V_2$ is not an independent set of G, then there exists a vertex $u \in V_2$ such that $|N_G(u) \cap V_1| \ge |N_G(u) \cap V_2|$, where the inequality is strict when either $|V_1| > |V_2|$ or $\chi(G[V_2]) < \chi(G[V_1 \cup V_2])$.

Proof. Since G is a complete multi-partite graph, $G' := G[V_1 \cup V_2]$ is also a complete multipartite graph. Assume that X_1, X_2, \dots, X_k are the partite sets of G', where $k = \chi(G')$, and $Y_i = V_1 \cap X_i$ and $Z_i = V_2 \cap X_i$ for all $i = 1, 2, \dots, k$. Since $V_1 \cup V_2$ is not an independent set of $G', k \ge 2$.

Let $r := \chi(G'[V_2])$. We may assume that $Z_i \neq \emptyset$ for all $i = 1, 2, \dots, r$. Then $Z_i = \emptyset$ and $Y_i = X_i$ for all $r + 1 \le i \le k$.

If r = 1, then for each $u \in V_2$, $|N_{G'}(u) \cap V_2| = 0$ and $|N_{G'}(u) \cap V_1| = |V_1 \setminus X_1| > 0$, and the result is trivial. Now assume that $r \ge 2$. Note that

$$\frac{1}{r-1} \sum_{i=1}^{r} (|V_2 \setminus Z_i|) = \sum_{i=1}^{r} |Z_i| = |V_2| \\
\leq |V_1| \\
= \sum_{i=1}^{r} |Y_i| + \sum_{i=r+1}^{k} |Y_i| \\
= \frac{1}{r-1} \sum_{i=1}^{r} (|V_1 \setminus Y_i|) - \frac{1}{r-1} \sum_{i=r+1}^{k} |Y_i| \\
\leq \frac{1}{r-1} \sum_{i=1}^{r} (|V_1 \setminus Y_i|),$$
(3.1)

where the inequality above is strict when either $|V_1| > |V_2|$ or r < k. Thus, there exists $i: 1 \le i \le r$ such that

$$|V_1 \setminus Y_i| \ge |V_2 \setminus Z_i|,$$

where the inequality is strict when either $|V_1| > |V_2|$ or r < k. Let $u \in Z_i$. Observe that

$$|N_{G'}(u) \cap V_1| - |N_{G'}(u) \cap V_2| = |V_1 \setminus Y_i| - |V_2 \setminus Z_i| \ge 0,$$
(3.2)

where the inequality is strict when either $|V_1| > |V_2|$ or r < k.

The result holds.

Lemma 3.2. Let G be a complete multi-partite graph. For disjoint subsets V_1 and V_2 of V(G), if $\chi(G[V_2]) = r$ and $\chi(G[V_1 \cup V_2]) = k$, then there exists $u \in V_2$ such that $|N_G(u) \cap V_1| \ge \frac{(r-1)|V_1|+k-r}{r} \ge \frac{(r-1)|V_1|}{r}$. *Proof.* Note that $G' := G[V_1 \cup V_2]$ is a complete k-partite graph. Assume that X_1, X_2, \dots, X_k are the partite sets of G'. Let $Y_i := V_1 \cap X_i$ and $Z_i := V_2 \cap X_i$ for all $i = 1, 2, \dots, k$. Since $\chi(G[V_2]) = r$, we may assume that $Z_i \neq \emptyset$ for all $i = 1, 2, \dots, r$. Then $Z_i = \emptyset$ and $Y_i = X_i$ for $r+1 \leq i \leq k$. Note that

$$|V_1| = \sum_{i=1}^{k} |Y_i| \ge (k-r) + \sum_{i=1}^{r} |Y_i|,$$

by which $|Y_i| \leq \frac{|V_1|-k+r}{r}$ holds for some *i* with $1 \leq i \leq r$. Let $u \in Z_i$. As G' is a complete *k*-partite graph with partite sets X_1, X_2, \dots, X_k and $u \in X_i$, *u* is adjacent to all vertices in $V_1 \setminus Y_i$, implying that

$$|N_G(u) \cap V_1| = |V_1| - |Y_i| \ge |V_1| - \frac{|V_1| - k + r}{r}.$$

The result holds.

Lemma 3.3. For any graph Q with components Q_1, Q_2, \dots, Q_s , if $\chi(Q) = t$, then there exists one vertex u_i in Q_i for each $i \in [s]$ such that

$$\sum_{i=1}^{s} d_Q(u_i) \le \frac{t-1}{t} |V(Q)|.$$
(3.3)

Proof. We first show it for s = 1. Let U_1, U_2, \dots, U_t be the color classes of a proper t-coloring with $|U_1| \ge |U_2| \ge \dots \ge |U_t|$. Then, for each vertex $u \in U_1$, we have

$$d_Q(u) \le \sum_{i=2}^t |U_i| \le \frac{t-1}{t} |V(Q)|.$$
(3.4)

By (3.4), there exists vertex u_i in Q_i for all $i \in [s]$ such that

$$\sum_{i=1}^{s} d_Q(u_i) = \sum_{i=1}^{s} d_{Q_i}(u_i) \le \sum_{i=1}^{s} \frac{\chi(Q_i) - 1}{\chi(Q_i)} |V(Q_i)| \le \sum_{i=1}^{s} \frac{t - 1}{t} |V(Q_i)| \le \frac{t - 1}{t} |V(Q)|.$$
(3.5)

Lemma 3.4. Let G be a connected graph of order $n \ge 3$ and q be an integer with $2 \le q \le n-1$. For any $H \in S_q(G)$, $com(H) \ge 3$ holds, and if $H \in \overline{S_q}(G)$, then

- (i) each component of H is a vertex-induced subgraph of H; and
- (ii) when $or_1(H) \ge or_1(H')$ for every graph $H' \in \overline{\mathcal{S}_q}(G)$, then either $or_1(H) + or_3(H) = q$ or $|N_G(u) \cap V(H_2)| > |N_G(u) \cap V(H_1)|$ for each $u \in V(H_2)$, where H_1 and H_2 are components of H with $|V(H_i)| = or_i(H)$ for i = 1, 2.

Proof. Let s = com(H) and H_1, H_2, \cdots, H_s be the components of H with $|V(H_i)| = or_i(H)$ for $i = 1, 2, \cdots, s$. As $H \in \mathcal{S}_q(G)$, $|V(H_1)| + |V(H_2)| \le q < n$, implying that $s \ge 3$.

Now assume that $\ell_q(G) = |E(H)|$. (i) is trivial.

(ii). Assume $|V(H_1)| \ge or_1(H')$ for every $H' \in \overline{\mathcal{S}_q}(G)$. Suppose (ii) fails. Then $|V(H_1)| + |V(H_3)| < q$ and $|N_G(u) \cap V(H_1)| \ge |N_G(u) \cap V(H_2)|$ for some $u \in V(H_2)$.

Note that $|N_G(u) \cap V(H_2)| \geq 1$, i.e., $|V(H_2)| \geq 2$. Otherwise, $|V(H_2)| = |V(H_3)| = 1$, $|V(H_1)| \leq q-2$ and $|E(H)| = |E(H_1)|$. Since G is connected, there exists an edge $uv \in E(G)$, where $u \in V(H_1)$ and $v \in V(G) \setminus V(H_1)$. Let H' be a spanning subgraph of G with edge set $E(G[V(H_1) \cup \{v\}])$. Then $or_1(H') \leq q-1$ and $or_2(H') = 1$, thus $H' \in \mathcal{S}_q(G)$, a contradiction to the assumption of $H \in \overline{\mathcal{S}_q}(G)$ as |E(H')| > |E(H)|.

Let H'_1 denote the subgraph $G[V(H_1) \cup \{u\}]$ and H' be the graph obtained from H by adding all edges in $\{uv : v \in N_G(u) \cap V(H_1)\}$ and deleting all edges in $\{uv : v \in N_G(u) \cap V(H_2)\}$. Since $|N_G(u) \cap V(H_1)| \ge |N_G(u) \cap V(H_2)| \ge 1$, H'_1 is connected and $or_1(H') = |V(H'_1)| = |V(H_1)| + 1$. Moreover,

$$|E(H')| - |E(H)| = |N_G(u) \cap V(H_1)| - |N_G(u) \cap V(H_2)| \ge 0,$$

and $or_2(H') \le \max\{|V(H_2)| - 1, |V(H_3)|\}$. Thus,

$$or_1(H') + or_2(H') \le \max\{|V(H_1)| + |V(H_2)|, |V(H_1)| + 1 + |V(H_3)|\} \le q,$$

implying that $H' \in S_q(G)$. As $|E(H')| \ge |E(H)| = \ell_q(G)$, we have $H' \in \overline{S_q}(G)$, a contradiction to the assumption of H as $or_1(H') > or_1(H)$.

(ii) holds.

For any two finite sequences (a_1, a_2, \dots, a_s) and (b_1, b_2, \dots, b_t) , write $(a_1, a_2, \dots, a_s) \succeq (b_1, b_2, \dots, b_t)$ if either s = t and $a_i = b_i$ for all $i = 1, 2, \dots, s$, or there exists $i : 1 \le i \le \min\{s, t\}$ such that $a_i > b_i$ and $a_j = b_j$ for all $1 \le j < i$. If $a_1 + a_2 + \dots + a_s = b_1 + b_2 + \dots + b_t$ and all a_i 's and b_j 's are positive, then either $(a_1, a_2, \dots, a_s) \succeq (b_1, b_2, \dots, b_t)$ or $(b_1, b_2, \dots, b_t) \succeq (a_1, a_2, \dots, a_s)$.

For any graph H, let $Seq_{-}or(H)$ denote the sequence $(or_1(H), or_2(H), \cdots, or_s(H))$, where s = com(H). If H and H' are spanning subgraphs of G, then either $Seq_{-}or(H) \succeq Seq_{-}or(H')$ or $Seq_{-}or(H') \succeq Seq_{-}or(H)$. Obviously, if $Seq_{-}or(H) \succeq Seq_{-}or(H')$, then $or_1(H) \ge or_1(H')$.

Theorem 3.5. Let G be a complete multi-partite graph of order n with at least two partite sets. Assume that $2 \le q \le n - 1$ and H is a graph in $\overline{S_q}(G)$ such that $or_1(H) \ge or_1(H')$ for every $H' \in \overline{S_q}(G)$. The following hold:

- (i) $or_1(H) + or_3(H) = q$; and
- (ii) if $Seq_{-}or(H) \succeq Seq_{-}or(H')$ holds for every $H' \in \overline{S_q}(G)$, then $Seq_{-}or(H)$ is a sequence of the following form:

$$(h_1, \underbrace{h_2, \cdots, h_2}_t, h_3)$$

where $t \ge 1, h_1 \ge h_2 \ge h_3$, and $h_3 = h_2$ when t = 1; or a sequence of the following form:

$$(h_1, \underbrace{h_2, \cdots, h_2}_{t_1}, \underbrace{1, \cdots, 1}_{t_2})$$

where $t_1, t_2 \ge 2$ and $h_1 \ge h_2 \ge 2$.

Proof. Let s = com(H) and H_1, H_2, \dots, H_s be the components of H with $|V(H_i)| = or_i(H)$ for $i = 1, 2, \dots, s$, where $s \ge 3$ by Lemma 3.4.

(i). The result is trivial when q = 2. Assume $q \ge 3$ in the following.

Suppose that $or_1(H) + or_3(H) < q$. Then, Lemma 3.4 (ii) implies that $|N_G(u) \cap V(H_2)| > |N_G(u) \cap V(H_1)|$ for each $u \in V(H_2)$. But, as $|V(H_1)| \ge |V(H_2)|$, due to Lemma 3.1, $V(H_1) \cup V(H_2)$ is an independent set in G. Thus $|V(H_1)| = |V(H_2)| = 1$ and for $u \in V(H_2)$, $|N_G(u) \cap V(H_2)| = |N_G(u) \cap V(H_1)| = 0$, a contradiction. (i) holds.

(ii). We first prove the following claim.

Claim 1: for any $5 \le b \le s$, if $or_{b-1}(H) < or_2(H)$, then $or_{b-1}(H) = 1$.

Proof. Suppose the claim fails. Then, there exists b with $5 \le b \le s$ such that $2 \le or_{b-1}(H) < or_2(H)$, i.e., $2 \le |V(H_{b-1})| < |V(H_2)|$.

By Lemma 3.1, there exists $u \in V(H_b)$ such that $|N_G(u) \cap V(H_b)| \leq |N_G(u) \cap V(H_{b-1})|$. Let H' be the graph obtained from H by adding all edges in $\{uv : v \in N_G(u) \cap V(H_{b-1})\}$ and deleting all edges in $\{uv : v \in N_G(u) \cap V(H_b)\}$. Obviously, $|E(H')| \geq |E(H)|$.

Let H'_{b-1} be the subgraph $G[V(H_{b-1}) \cup \{u\}]$. Since $|V(H_{b-1})| \geq 2$ and G is a complete k-partite graph, H'_{b-1} is connected.

Thus, the components of H' are $H_1, H_2, \dots, H_{b-2}, H'_{b-1}, H_{b+1}, \dots, H_s$ together with components of $H_b \setminus \{u\}$, implying that $Seq_{-}or(H) \not\succeq Seq_{-}or(H')$.

By the given condition, $|V(H'_{b-1})| = |V(H_b)| + 1 \leq |V(H_2)|$. Thus, $H' \in S_q(G)$. As $|E(H')| \geq |E(H)|$, $H \in \overline{S_q}(G)$. However, $Seq_or(H) \not\geq Seq_or(H')$, a contradiction to the assumption of H.

Hence Claim 1 holds.

Let $or_i(H) = h_i$ for i = 1, 2. By the result in (i), we have $or_3(H) = h_2$. If $or_{s-1}(H) = h_2$, then $Seq_-or(H) = (h_1, \underbrace{h_2, \cdots, h_2}_{t}, h_3)$, where $h_3 = or_s(H) \leq h_2$. If $or_{s-1}(H) < h_2$, then $s \geq 5$ and there exists $5 \leq b \leq s$ such that $or_b(H) \leq or_{b-1}(H) < or_2(H) = h_2$ and $or_{b-2}(H) = or_2(H)$. In this case, due to Claim 1, $or_{b-1}(H) = 1$, implying that $Seq_-or(H) = (h_1, \underbrace{h_2, \cdots, h_2}_{t_1}, \underbrace{1, 1, \cdots, 1}_{t_2})$, where $t_1, t_2 \geq 2$.

Hence (ii) holds.

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For a complete multi-partite graph G of order n and $2 \le q \le n-1$, let H be a member in $\overline{\mathcal{S}_q}(G)$ such that $Seq_-or(H) \succeq Seq_-or(H')$ holds for every $H' \in \overline{\mathcal{S}_q}(G)$. By Theorem 3.5, $h_1 + h_2 = q$ and $h_1 + 2h_2 \le n$, thus $1 \le h_2 \le \min\{q/2, n-q\}$ and we can express all the possible sequences according to the value of h_2 . For example, if n = 13 and q = 8, then, $1 \le h_2 \le \min\{8/2, 13 - 8\} = 4$, and $Seq_-or(H)$ is one of the sequences below:

$$(4, 4, 4, 1), (5, 3, 3, 2), (5, 3, 3, 1, 1), (6, 2, 2, 2, 1), (6, 2, 2, 1, 1, 1), (7, 1, 1, 1, 1, 1, 1).$$

4 To find $\ell_q(K_{p_1,\dots,p_k})$ for $n-3 \le q \le n-1$

In this section, we consider the case $n-3 \le q \le n-1$, and show that (1.3) holds unless (G,q) is an ordered pair in (1.4).

Let G be a connected graph of order n and q be an integer with $2 \le q \le n-1$. Recall that for any $S \subseteq V(G)$, $E_G(S) = \bigcup_{v \in S} E_G(v)$. Clearly,

$$|E(G)| - \min_{S_0 \in \mathscr{P}_r(G)} |E_G(S_0)| = \max_{S \in \mathscr{P}_{n-r}(G)} |E(G[S])|.$$
(4.1)

For any $S \in \mathscr{P}_{q-1}(G)$, the spanning subgraph H of G with edge set E(G[S]) has at least n-q+2 components, i.e., G[S] and n-q+1 trivial components, and thus it belongs to $\mathcal{S}_q(G)$. Hence

$$\ell_q(G) \ge \max_{S \in \mathscr{P}_{q-1}(G)} |E(G[S])| = |E(G)| - \min_{S_0 \in \mathscr{P}_{n-q+1}(G)} |E_G(S_0)|$$
(4.2)

It can be verified that the inequality of (4.2) is strict when (G,q) is one of the following ordered pairs:

$$(K_{3,3},4), (K_{4,3},4), (K_{3,3,3},6).$$
 (4.3)

For example, if $G = K_{3,3,3}$ and q = 6, then $\ell_q(G) \ge 9$, as $\mathcal{S}_q(G)$ has a graph consisting of three components each of which is isomorphic to K_3 , while each 5-vertex subgraph of G has at most 8 edges. Similarly, if $G = K_{3,3}$ or $K_{4,3}$ and q = 4, each 3-vertex subgraph of G has at most 2 edges, while $\mathcal{S}_q(G)$ has a graph containing 3 edges.

Now we shall show that if G is a complete multi-partite graph of order n and $\max\{2, n-3\} \le q \le n-1$, then the equality of (4.2) holds if and only if (G, q) is not an ordered pair in (4.3). Thus, by applying Theorem 2.4, $ar(G, \mathcal{T}_q)$ can be determined.

Theorem 4.1. Let G denote K_{p_1,p_2,\dots,p_k} and $n = p_1 + p_2 + \dots + p_k$, where $k \ge 2$ and $p_1 \ge p_2 \ge \dots \ge p_k \ge 1$. For $n-3 \le q \le n-1$, if $q \ge 2$ and (G,q) is not an ordered pair in (4.3), then

$$\ell_q(G) = |E(G)| - \min_{S \in \mathscr{P}_{n-q+1}(G)} |E_G(S)|.$$
(4.4)

Proof. Suppose that (4.4) is not true. Then, the inequality of (4.2) is strict.

Assume that (G, q) is not one of the ordered pairs in (4.3) and H is a graph in $\overline{\mathcal{S}_q}(G)$ such that $or_1(H) \ge or_1(H')$ for every $H' \in \overline{\mathcal{S}_q}(G)$. Let H_1, H_2, \cdots, H_s be the components of H with $|V(H_i)| = or_i(H)$ for $i = 1, 2, \cdots, s$, where $s \ge 3$ by Lemma 3.4.

Let r = n - q. Then $1 \le r \le 3$.

Claim 1: $|E(H)| > |E(H_0)|$ holds for every subgraph H_0 of G with $|V(H_0)| \le q-1$ (= n-r-1). *Proof.* As $|E(H)| = \ell_q(G)$ and (4.4) fails, the inequality of (4.2) is strict, and thus the claim follows.

Claim 2: $|V(H_2)| \ge 2$.

Proof. Suppose that $|V(H_2)| = 1$. Observe that $|E(H_1)| = |E(H)|$. As $H \in \mathcal{S}_q(G)$, we have

 $|V(H_1)| \le q - |V(H_2)| = q - 1$. It contradicts Claim 1. Claim 2 holds.

Claim 3: $|V(H_1)| + |V(H_2)| = q$ and $|V(H_2)| = |V(H_3)| \le r$.

Proof. Claim 3 follows from Theorem 3.5 (i).

Claim 4: r > 1.

Proof. If r = 1, then by Claim 3, $|V(H_2)| = 1$, a contradiction to Claim 2, thus Claim 4 holds. \natural Claim 5: $r \neq 2$.

Proof. Suppose that r = 2. By Claims 2 and 3, $|V(H_2)| = |V(H_3)| = 2$ and $|V(H_1)| + |V(H_2)| + |V(H_3)| = n$, thus s = 3, both H_2 , H_3 are isomorphic to K_2 and $|E(H)| = |E(H_1)| + 2$.

We are now going to show that $|V(H_1)| = 2$. Suppose that $|V(H_1)| > |V(H_2)|$. As $|V(H_2)| = 2$, by Lemma 3.1, there exists a vertex $u \in V(H_2)$ such that $|N_G(u) \cap V(H_1)| \ge 1 + |N_G(u) \cap V(H_2)| = 2$.

Let H_0 be the subgraph $G[V(H_1) \cup \{u\}]$. Observe that $|V(H_0)| = |V(H_1)| + 1 = n - 3 = n - r - 1$ and

$$|E(H_0)| \ge |E(H_1)| + 2 = |E(H)|,$$

a contradiction to Claim 1.

Hence $|V(H_1)| = 2$, implying that |E(H)| = 3 and n = |V(H)| = 6. Thus, n - r - 1 = 3. If $k \ge 3$, then G contains a subgraph H_0 isomorphic to K_3 , a contradiction to Claim 1. Thus k = 2. The three edges in H form a perfect matching of G, implying that $G \cong K_{3,3}$, a contradiction to the assumption of G.

Claim 5 holds.

Claim 6: If r = 3, then $s \leq 3$.

Proof. Suppose that r = 3 and $s \ge 4$. By Claims 2 and 3, $2 \le |V(H_2)| = |V(H_3)| \le 3$ and $n - |V(H_1)| - |V(H_2)| = r = 3$, implying $s \le 4$. Thus s = 4, $|V(H_2)| = |V(H_3)| = 2$ and $|V(H_4)| = 1$.

We are now going to show that $|V(H_1)| = 2$. Suppose that $|V(H_1)| \ge 3 > |V(H_2)|$.

As $|V(H_2)| = 2$, by Lemma 3.1, there exists a vertex $u \in V(H_2)$ such that $|N_G(u) \cap V(H_1)| \ge 1 + |N_G(u) \cap V(H_2)| = 2$. Let H_0 be the subgraph $G[V(H_1) \cup \{u\}]$. Observe that $|V(H_0)| = |V(H_1)| + 1 = n - 4 = n - r - 1$ and

$$|E(H_0)| \ge |E(H_1)| + 2 = |E(H)|,$$

a contradiction to Claim 1.

Hence $|V(H_1)| = 2$, implying that |E(H)| = 3 and n = 7.

Suppose that $k \ge 3$. Let H_0 be a subgraph isomorphic to K_3 . Observe that $|E(H_0)| = 3 = |E(H)|$ and $|V(H_0)| = 3 = n - 4 = n - r - 1$, a contradiction to Claim 1.

Thus, k = 2. As *H* has a matching of size 3 and n = 7, *G* is isomorphic to $K_{4,3}$, a contradiction to the assumption of *G*.

Claim 6 holds.

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Claim 7: $r \neq 3$.

Proof. Suppose that r = 3. By Claim 6, s = 3 and $|V(H_2)| = |V(H_3)| = 3$ by Claims 2 and 3, implying that H_i is isomorphic to K_3 or P_3 (the path of order 3) for i = 2, 3.

We first show that $|V(H_1)| = 3$. Suppose that $|V(H_1)| \ge 4$.

For $2 \leq i \leq 3$, as $|V(H_1)| > |V(H_i)| = 3$, by Lemma 3.1, there exists a vertex $u_i \in V(H_i)$ such that $|N_G(u_i) \cap V(H_1)| \ge 1 + |N_G(u_i) \cap V(H_i)| \ge |E(H_i)|$. Let $H_0 = G[V(H_1) \cup \{u_2, u_3\}]$. Observe that

$$|E(H_0)| = |E(H_1)| + \sum_{i=2}^{3} |N_G(u_i) \cap V(H_1)| \ge |E(H_1)| + |E(H_2)| + |E(H_3)| = |E(H)|$$

and $|V(H_0)| = |V(H_1)| + 2 = n - 4 = n - r - 1$, a contradiction to Claim 1.

Hence $|V(H_1)| = 3$, implying that n = 9 and $6 \le |E(H)| \le 9$.

If $k \ge 4$, then G must have a subgraph H_0 isomorphic to $K_5 - e$, the graph obtained from K_5 by removing one edge. Note that $|V(H_0)| = 5 = n - 4 = n - r - 1$ and $|E(H_0)| = 9 \ge |E(H)|$, a contradiction to Claim 1.

Thus, $k \leq 3$. If $p_2 = 1$, then $p_3 = 1$, a contradiction to the fact that H has three vertexdisjoint paths of length 2. Hence $p_2 \ge 2$.

If |E(H)| = 9, then $H_i \cong K_3$ for all i = 1, 2, 3, implying that k = 3 and $G \cong K_{3,3,3}$, a contradiction to the assumption of G.

Now assume that $|E(H)| \leq 8$. If k = 3, as $p_2 \geq 2$, G has a subgraph H_0 isomorphic to $K_{2,2,1}$, implying that $|E(H_0)| \ge 8 \ge |E(H)|$. But, $|V(H_0)| = 5 = n - 4 = n - r - 1$, a contradiction to Claim 1.

Hence k = 2, and each H_i is isomorphic to P_3 (i.e., the path graph of length 2), implying that |E(H)| = 6. As $p_2 \ge 2$, G has a subgraph H_0 isomorphic to $K_{3,2}$, implying that $|E(H_0)| =$ 6 = |E(H)| and $|V(H_0)| = 5 = n - 4 = n - r - 1$, a contradiction to Claim 1.

Claim 7 holds. þ

As r = n - q, the result follows from Claims 4, 5, 6 and 7.

To find $\min_{S \in \mathscr{P}_r(G)} |E_G(S)|$ for a complete multi-partite graph G 5

In this section, we shall show that for a complete multi-partite graph G, $\min_{S \in \mathscr{P}_r(G)} |E_G(S)|$ is equal to $|E_G(S_0)|$ for every set $S_0 \in \mathscr{P}_r(G)$ obtained by a simple algorithm (i.e. Algorithm A).

For any graph G of order n and any integer r with $1 \le r \le n$, let $\mathscr{P}_r^*(G)$ denote the family of subsets $S \in \mathscr{P}_r(G)$ obtained by the following "greedy" algorithm. Note that this algorithm chooses one vertex by one vertex and always chooses a vertex of the minimum degree in the remaining graph.

Algorithm A:

- Step 1 Set i := 1, $S := \emptyset$ and $G_1 := G$;
- Step 2 Choose a vertex u from G_i of degree $\delta(G_i)$;
- Step 3 Set $S := S \cup \{u\};$
- Step 4 If i = r, then output S and STOP; otherwise, set i := i + 1 and $G_i := G_{i-1} \setminus \{u\}$, and return to Step 2.

In the following, we shall show that if G is a complete multi-partite graph, then $\mathscr{P}_r^*(G)$ is exactly the set of elements $S_0 \in \mathscr{P}_r(G)$ such that

$$|E_G(S_0)| = \min_{S \in \mathscr{P}_r(G)} |E_G(S)|.$$
(5.1)

We first establish the following conclusion.

Lemma 5.1. Let G be a complete k-partite graph of order n with partite sets X_1, X_2, \dots, X_k , where $k \ge 2$. Assume that $T_0 \in \mathscr{P}_b(G)$, where $1 \le b \le n$. Then, $|E(G[T_0])| \ge |E(G[T])|$ holds for every $T \in \mathscr{P}_b(G)$ if and only if for each pair of distinct numbers $i, j \in \{1, 2, \dots, k\}$, if $|X_i \cap T_0| \ge |X_j \cap T_0| + 2$, then $X_j \subseteq T_0$.

Proof. Let $p_i = |X_i|$ for all $i = 1, 2, \dots, k$. Assume that $p_1 \ge p_2 \ge \dots \ge p_k \ge 1$. Let $a_0 = 0$, and for $j = 1, 2, \dots, p_1$, let $a_j = \max\{1 \le i \le k : p_i \ge j\}$. For example, if k = 3, $p_1 = 4$, $p_2 = 3$ and $p_3 = 1$, then, $a_1 = 3$, $a_2 = 2$, $a_3 = 2$ and $a_4 = 1$. In order to complete the proof, we shall show the equivalence of the following three statements for any $T_0 \in \mathscr{P}_b(G)$:

- (i) $|E(G[T_0])| \ge |E(G[T])|$ holds for every $T \in \mathscr{P}_b(G)$;
- (ii) for each pair of distinct numbers $i, j \in \{1, 2, \dots, k\}$, if $|X_i \cap T_0| \ge |X_j \cap T_0| + 2$, then $X_j \subseteq T_0$.
- (iii) let h be the unique number in $\{1, 2, \dots, p_1\}$ determined by the inequality: $a_1 + a_2 + \dots + a_{h-1} < b \le a_1 + a_2 + \dots + a_{h-1} + a_h$. Then, $G[T_0]$ is isomorphic to K_{q_1,q_2,\dots,q_k} , where $q_i = p_i$ when $p_i \le h-2$, and $h-1 \le q_i \le h$ otherwise. Furthermore, there are exactly $b (a_1 + \dots + a_{h-1})$ indices $i: 1 \le i \le k$ such that $q_i = h$.

(i) \Rightarrow (ii). Assume that $T_0 \in \mathscr{P}_b(G)$ such that $|E(G[T_0])| \geq |E(G[T])|$ holds for every $T \in \mathscr{P}_b(G)$. As G is a complete k-partite graph, for any $U \subseteq V(G)$, G[U] is also a complete multi-partite graph, and its size is

$$|E(G[U])| = \binom{|U|}{2} - \sum_{i=1}^{k} \binom{|U \cap X_i|}{2}.$$
(5.2)

Suppose that there is a pair of distinct numbers $i, j \in \{1, 2, \dots, k\}$, such that $X_j \not\subseteq T_0$ and $|X_j \cap T_0| \leq |X_i \cap T_0| - 2$. Let $v \in X_j \setminus T_0$ and $v' \in X_i \cap T_0$. For $T' = (T_0 \setminus \{v'\}) \cup \{v\} \in \mathscr{P}_b(G)$,

by (5.2),

$$|E(G[T'])| - |E(G[T_0])| = {\binom{|T_0 \cap X_i|}{2}} + {\binom{|T_0 \cap X_j|}{2}} - {\binom{|T_0 \cap X_i| - 1}{2}} - {\binom{|T_0 \cap X_j| + 1}{2}} = |T_0 \cap X_i| - |T_0 \cap X_j| - 1 > 0,$$
(5.3)

a contradiction to the given condition. Thus, (i) \Rightarrow (ii) holds.

(ii) \Rightarrow (iii). Without loss of generality, assume that $h := |X_1 \cap T_0| \ge |X_j \cap T_0|$ for all $j = 2, 3, \dots, k$. By the given condition, for any $2 \le j \le k$, $|X_j \cap T_0| \ge |X_1 \cap T_0| - 1 = h - 1$, unless $X_j \subseteq T_0$. Thus, if $p_j \le h - 2$, then $X_j \subseteq T_0$. Obviously, h is the unique number satisfying the inequality below:

$$a_1 + a_2 + \dots + a_{h-1} < b \le a_1 + a_2 + \dots + a_{h-1} + a_h,$$

and

$$|\{1 \le j \le k : |X_j \cap T_0| = h\}| = b - (a_1 + a_2 + \dots + a_{h-1}).$$
(5.4)

Note that $G[T_0]$ is a complete multi-partite graph with partite sets $T_0 \cap X_i$ for $i = 1, 2, \dots, k$, where $|T_0 \cap X_i| = |X_i| = p_i$ whenever $p_i \leq h - 2$, and $h - 1 \leq |T_0 \cap X_i| \leq h$ otherwise. Furthermore, by (5.4), $G[T_0]$ has exactly $b - (a_1 + a_2 + \dots + a_{h-1})$ partite sets $T_0 \cap X_j$ of size h. Thus (iii) holds.

(iii) \Rightarrow (i). Assume that condition (iii) is satisfied for T_0 . Assume that $T' \in \mathscr{P}_b(G)$ such that $|E(G[T'])| \ge |E(G[T])|$ holds every $T \in \mathscr{P}_b(G)$. As (iii) follows from (i), condition (iii) is satisfied for T'. As both T_0 and T' satisfy condition (iii), we have $G[T'] \cong G[T_0]$, implying that $|E(G[T'])| = |E(G[T_0])|$, and thus $|E(G[T_0])| \ge |E(G[T])|$ holds for every $T \in \mathscr{P}_b(G)$. Hence (iii) \Rightarrow (i) holds.

Therefore (i) \Leftrightarrow (ii) holds, and the result is proven.

Theorem 5.2. Let G be a complete multi-partite graph of order n with at least two partite sets. For any $1 \le r \le n$ and $S_0 \in \mathscr{P}_r(G)$, $S_0 \in \mathscr{P}_r^*(G)$ if and only if

$$|E_G(S_0)| = \min_{S \in \mathscr{P}_r(G)} |E_G(S)|.$$

$$(5.5)$$

Proof. Let X_1, X_2, \dots, X_k be the partite sets of G, where $k \geq 2$.

We need only to show that the following statements are equivalent for any $S \in \mathscr{P}_r(G)$:

- (i) $S \in \mathscr{P}_r^*(G);$
- (ii) for each pair of distinct numbers i, j in $\{1, 2, \dots, k\}$, if $|X_i \setminus S| \ge 2 + |X_j \setminus S|$, then $X_j \cap S = \emptyset$;
- (iii) $|E(G \setminus S)| \ge |E(G[T])|$ for each $T \in \mathscr{P}_{n-r}(G)$; and
- (iv) $|E_G(S)| \leq |E_G(S')|$ for each $S' \in \mathscr{P}_r(G)$.

The equivalence of (ii) and (iii) follows from Lemma 5.1 by taking $T_0 = V(G) \setminus S$, while (iii) and (iv) are equivalent by (4.1). It remains to prove that (i) and (ii) are equivalent.

Assume that $S = \{u_1, u_2, \dots, u_r\} \in \mathscr{P}_r^*(G)$, where u_s is the s-th vertex in S selected by Algorithm A for $s = 1, 2, \dots, r$, and let $S_s := \{u_1, u_2, \dots, u_s\}$ and $G_s := G \setminus \{u_1, \dots, u_{s-1}\}$.

Suppose that (ii) fails. Then, there exist distinct numbers i, j in $\{1, 2, \dots, k\}$ such that $X_j \cap S \neq \emptyset$ and $|X_i \setminus S| \ge 2 + |X_j \setminus S|$. As $X_j \cap S \neq \emptyset$, there exists $u_q \in S \cap X_j$. If $|X_j \cap S| \ge 2$, we assume u_q is chosen with the largest possible value of q.

As q is the largest number in $\{1, 2, \dots, r\}$ such that $u_q \in X_j$,

$$|X_i \setminus S_{q-1}| \ge |X_i \setminus S| \ge 2 + |X_j \setminus S| = 1 + |X_j \setminus S_{q-1}|$$

Thus, for any $u \in X_i \setminus S_{q-1}$, we have

$$d_{G_q}(u_q) = |V(G_q)| - |X_j \setminus S_{q-1}| \ge |V(G_q)| - |X_i \setminus S_{q-1}| + 1 = d_{G_q}(u) + 1,$$

a contradiction to the condition that u_q has the minimum degree in G_q . Thus (i) \Rightarrow (ii) holds.

Assume that condition (ii) is satisfied. Then for each pair of distinct numbers j, j' in $\{1, 2, \dots, k\}$ such that $X_j \cap S \neq \emptyset$ and $X_{j'} \cap S \neq \emptyset$, the difference between $|X_j \setminus S|$ and $|X_{j'} \setminus S|$ is at most one. Moreover, if $X_i \cap S = \emptyset$ for some $i \in \{1, 2, \dots, k\}$, then

$$|X_i| = |X_i \setminus S| \le 1 + \min\{|X_j \setminus S| : X_j \cap S \neq \emptyset, 1 \le j \le k\}.$$

Note that the vertices in S can be determined by the following Algorithm.

Algorithm B:

- Step 1. Set t := r and $S_t := S$;
- Step 2. choose $j : 1 \leq j \leq k$ such that $X_j \cap S_t \neq \emptyset$ and $|X_j \setminus S_t| \leq |X_{j'} \setminus S_t|$ for each $j' \neq j$ with $X_{j'} \cap S_t \neq \emptyset$;
- Step 3. let u_t be a vertex in $S_t \cap X_j$;

Step 4. if t = 1, then STOP; otherwise, set t := t - 1, $S_t := S_{t+1} \setminus \{u_{t+1}\}$, and go to Step 2.

Assume that u_1, u_2, \dots, u_r are vertices in S determined by Algorithm B. Note that for each $t = 1, 2, \dots, r, S_t = \{u_1, \dots, u_t\}$ and u_t has the minimum degree in $G \setminus S_{t-1}$, where $S_0 = \emptyset$. Hence, $S \in \mathscr{P}_r^*(G)$ and (ii) \Rightarrow (i) holds.

The result is proven.

Corollary 5.3. Let G denote K_{p_1,p_2,\dots,p_k} and $n = p_1 + p_2 + \dots + p_k$, where $k \ge 2$. For $2 \le q \le n-1$, if $\ell_q(G) = |E(G)| - \min_{S \in \mathscr{P}_{n-q+1}(G)} |E_G(S)|$, then,

$$ar(G, \mathcal{T}_q) = 1 + \sum_{1 \le i < j \le k} p_i p_j - |E_G(S_0)|,$$
(5.6)

where $S_0 \in \mathscr{P}^*_{n-q+1}(G)$.

Proof. The result follows from Theorems 2.4 and 5.2.

6 To determine $ar(K_{p_1,\dots,p_k},\mathcal{T}_q)$ for $n-3 \leq q \leq n-1$

In this section, we shall give an explicit expression for $ar(K_{p_1,\dots,p_k},\mathcal{T}_q)$ when $\max\{2, n-3\} \leq q \leq n-1$, where $n = p_1 + \dots + p_k$.

If (G, q) is an ordered pair in (4.3), $ar(G, \mathcal{T}_q)$ can be determined by applying Theorems 2.4 and 3.5 (ii) that

$$ar(G, \mathcal{T}_q) = 1 + \ell_q(G) = \begin{cases} 4, & \text{if } (G, q) = (K_{3,3}, 4) \text{ or } (K_{4,3}, 4); \\ 10, & \text{if } (G, q) = (K_{3,3,3}, 6). \end{cases}$$
(6.1)

For example, if $(G,q) = (K_{4,3},4)$ and H is a member in $\overline{\mathcal{S}_4}(K_{4,3})$ such that $Seq_-or(H) \succeq Seq_-or(H')$ for every $H' \in \overline{\mathcal{S}_4}(K_{4,3})$, then, by Theorem 3.5 (ii), $Seq_-or(H)$ is either (2, 2, 2, 1) or (3, 1, 1, 1, 1), implying that $\ell_4(K_{4,3}) = 3$. Similarly, if $(G,q) = (K_{3,3,3}, 6)$ and H is a member in $\overline{\mathcal{S}_6}(K_{3,3,3})$ such that $Seq_-or(H) \succeq Seq_-or(H')$ for every $H' \in \overline{\mathcal{S}_6}(K_{3,3,3})$, then, by Theorem 3.5 (ii), $Seq_-or(H)$ is one of the sequences (3, 3, 3), (4, 2, 2, 1) or (5, 1, 1, 1, 1), implying that $\ell_6(K_{3,3,3}) = 9$.

In the following, we consider the case that G is a complete multi-partite graph of order n and $\max\{2, n-3\} \le q \le n-1$ such that (G, q) is not an ordered pair in (4.3).

Note that for any complete multi-partite graph G and $u \in V(G)$, $d_G(u) = \delta(G)$ if and only if u is contained in a partite set with the largest cardinality. Now let $G = K_{p_1,p_2,\cdots,p_k}$ with partite sets X_1, X_2, \cdots, X_k and n = |V(G)|, where $k \ge 2$, $p_1 \ge p_2 \ge \cdots \ge p_k \ge 1$ and $|X_i| = p_i$ for all $i = 1, 2, \cdots, k$. Clearly, for any $S = \{x_1, x_2, \cdots, x_s\} \subseteq V(G)$, if $x_i \in X_{j_i}$ for all $i \in [s]$, then

$$|E_G(S)| = \sum_{i=1}^{s} (n - p_{j_i}) - \sum_{1 \le i_1 < i_2 \le s} \sigma(j_{i_2} - j_{i_1}) = sn - \sum_{i=1}^{s} p_{j_i} - \sum_{1 \le i_1 < i_2 \le s} \sigma(j_{i_2} - j_{i_1}), \quad (6.2)$$

where $\sigma(x)$ is the function defined by $\sigma(0) = 0$ and $\sigma(x) = 1$ when $x \neq 0$. Then, for $S \in \mathscr{P}_t^*(G)$, where $2 \leq t \leq 4$, $|E_G(S)|$ can be determined by applying Algorithm A and (6.2) as follows. If $S \in \mathscr{P}_2^*(G)$,

$$|E_G(S)| = \begin{cases} 2n - 2p_1, & \text{if } p_1 > p_2;\\ 2n - p_1 - p_2 - 1, & \text{if } p_1 = p_2. \end{cases}$$
(6.3)

If $S \in \mathscr{P}_3^*(G)$,

$$|E_G(S)| = \begin{cases} 3n - 3p_1, & \text{if } p_1 \ge p_2 + 2; \\ 3n - 2p_1 - p_2 - 2, & \text{if } p_2 + 1 \ge p_1 \ge p_3 + 1; \\ 3n - p_1 - p_2 - p_3 - 3, & \text{if } p_1 = p_2 = p_3. \end{cases}$$
(6.4)

If $S \in \mathscr{P}_4^*(G)$,

$$|E_G(S)| = \begin{cases} 4n - 4p_1, & \text{if } p_1 \ge p_2 + 3; \\ 4n - 3p_1 - p_2 - 3, & \text{if } p_1 = p_2 + 2 \text{ or } p_1 = p_2 + 1 \ge p_3 + 2; \\ 4n - 2p_1 - 2p_2 - 4, & \text{if } p_1 = p_2 \ge p_3 + 1; \\ 4n - 2p_1 - p_2 - p_3 - 5, & \text{if } p_4 + 1 \le p_1 \le p_2 + 1 = p_3 + 1; \\ 4n - \sum_{i=1}^4 p_i - 6, & \text{if } p_1 = p_2 = p_3 = p_4. \end{cases}$$

$$(6.5)$$

Theorem 6.1. Let G denote K_{p_1,p_2,\dots,p_k} and $n = p_1 + p_2 + \dots + p_k$, where $k \ge 2$ and $p_1 \ge p_2 \ge \dots \ge p_k \ge 1$. For $\max\{2, n-3\} \le q \le n-1$, if (G,q) is an ordered pair in (4.3), then $ar(G, \mathcal{T}_q)$ is given in (6.1); otherwise,

$$ar(G, \mathcal{T}_q) = 1 + \sum_{1 \le i < j \le k} p_i p_j - |E_G(S_0)|,$$
(6.6)

where $S_0 \in \mathscr{P}_{n-q+1}^*(G)$ and $|E_G(S_0)|$ is given in (6.3), (6.4) and (6.5) respectively for q = n-1, n-2 and n-3.

Proof. As $G = K_{p_1, p_2, \dots, p_k}$ and (G, q) is not an ordered pair in (4.3), where $\max\{2, n-3\} \le q \le n-1$, by Theorem 4.1 and Corollary 5.3, we have

$$ar(G, \mathcal{T}_q) = 1 + E(G) - |E(G[S_0])|, \tag{6.7}$$

where $S_0 \in \mathscr{P}^*_{n-q+1}(G)$. Thus, the result follows from (6.3), (6.4) and (6.5).

7 To determine $ar(K_{p_1,\dots,p_k},\mathcal{T}_q)$ for $(4n-2)/5 \le q \le n-1$

Let G denote K_{p_1,\dots,p_k} . In this section, we consider the case when $(4n-2)/5 \le q \le n-1$, where $n = p_1 + p_2 + \dots + p_k$. We shall show that in this case, $ar(G, \mathcal{T}_q)$ can be calculated by $|E(G)| - |E_G(S_0)| + 1$ for any $S_0 \in \mathscr{P}^*_{n-q+1}(G)$ in Theorem 7.2. Moreover, we give the explicit expression for $ar(K_{p_1,\dots,p_k},\mathcal{T}_q)$ in Corollary 7.3 with a further condition that $p_1 - p_2 \ge (n+2)/5$.

Lemma 7.1. Let H be a spanning subgraph of a complete multi-partite graph G of order nand let r be a positive integer such that $n \ge 5r - 2$. If $com(H) \ge 2$, $or_2(H) \le r$ and $n - (or_1(H) + or_2(H)) \le r$, then there exists $S \subseteq V(H) \setminus V(H_1)$ with $|S| \le or_2(H) - 1$ such that $|E(G[V(H_1) \cup S])| \ge |E(H)|$, where H_1 is a largest component of H.

Proof. We prove this result by induction on $or_2(H)$. It is trivial if $or_2(H) = 1$. Assume that it holds when $or_2(H) < d$, where $d \ge 2$. Now consider the case that $or_2(H) = d$.

Let H_2, H_3, \dots, H_s be the components of H different from H_1 and let W denote the subgraph $H \setminus V(H_1)$. Clearly, W consists of components H_2, H_3, \dots, H_s and $|V(W)| = |V(H_2)| + (|V(H_3)| + \dots + |V(H_s)|) \le or_2(H) + r \le 2r$, implying that $|V(H_1)| \ge n - 2r \ge 3r - 2$. **Claim 1**: There exist u and U with $u \in U \subseteq V(W)$ such that $|U \cap V(H_i)| = 1$ for each $i \in [s] \setminus \{1\}$ and

$$|N_G(u) \cap V(H_1)| \ge \sum_{v \in U} d_W(v).$$

Proof. Let $t = \chi(W)$. If t = 1, then W is an empty graph and Claim 1 is trivial. Now assume that $t \ge 2$. By Lemma 3.2, there exists $u \in V(W)$ such that

$$|N_G(u) \cap V(H_1)| \ge \frac{t-1}{t} \times |V(H_1)|.$$
(7.1)

Assume that $u \in V(H_b)$, where $2 \leq b \leq s$. Clearly, $d_W(u) \leq |V(H_b)| - 1$. By Lemma 3.3, there exists $u_i \in V(H_i)$ for each $i \in [s] \setminus \{1, b\}$ such that

$$\sum_{\substack{2 \le i \le s \\ i \ne b}} d_W(u_i) \le \frac{t'-1}{t'} |V(W) \setminus V(H_b)| \le \frac{t-1}{t} (|V(W)| - |V(H_b)|),$$
(7.2)

where $t' = \chi(W \setminus V(H_b))$. Let $U = \{u\} \cup \{u_i : 2 \le i \le s, i \ne b\}$. Since $|V(H_b)| \le r$ and $|V(W)| \le 2r$, we have

$$\sum_{v \in U} d_W(v) \leq |V(H_b)| - 1 + \frac{t-1}{t} (|V(W)| - |V(H_b)|)$$

$$\leq \frac{t-1}{t} |V(W)| + \frac{1}{t} |V(H_b)| - 1$$

$$\leq \frac{t-1}{t} (2r) + \frac{1}{t} r - 1$$

$$= \frac{r(2t-1)}{t} - 1.$$
(7.3)

Since $|V(H_1)| \ge 3r - 2$,

$$\frac{r(2t-1)}{t} - 1 = \frac{t-1}{t}(3r-2) - \frac{(t-2)(r-1)}{t} \le \frac{t-1}{t}(3r-2) \le \frac{t-1}{t}|V(H_1)|.$$
(7.4)

Thus, Claim 1 follows from (7.1), (7.3) and (7.4).

By Claim 1, there exist u and U with $u \in U \subseteq V(W)$ such that the properties in Claim 1 hold. Let H' be the spanning subgraph of G obtained from H by adding all edges in $\{uv : v \in N_G(u) \cap V(H_1)\}$ and deleting all edges in $\bigcup_{v \in U} E_W(v)$.

As $d \ge 2$, it is clear that $or_2(H') = or_2(H) - 1 \le r - 1$ and $or_3(H') + \dots + or_{com(H')}(H') = or_3(H) + \dots + or_s(H) \le r$.

As $|V(H'_1)| \ge or_2(H) \ge 2$, $H'_1 := G[V(H_1) \cup \{u\}]$ is connected and the largest component of H'. Since $or_2(H') = or_2(H) - 1 = d - 1$, by inductive assumption, there exists $S' \subseteq V(H') \setminus V(H'_1)$ with $|S'| \le or_2(H') - 1 = or_2(H) - 2$ such that $|E(G[V(H'_1) \cup S'])| \ge |E(H')|$.

Let $S = S' \cup \{u\}$. Then $|S| \leq or_2(H) - 1$ and $V(H'_1) \cup S' = V(H_1) \cup S$. By Claim 1, we

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have $|E(H')| \ge |E(H)|$. Thus,

$$|E(G[V(H_1) \cup S])| = |E(G[V(H_1') \cup S'])| \ge |E(H')| \ge |E(H)|.$$

The result follows.

Theorem 7.2. Let G be a complete multi-partite graph of order n with at least two partite sets. If q is an integer with $(4n-2)/5 \le q \le n-1$, then

$$ar(G, \mathcal{T}_q) = |E(G)| + 1 - |E_G(S_0)|, \tag{7.5}$$

where S_0 is a member in $\mathscr{P}^*_{n-q+1}(G)$.

Proof. Let $H \in \overline{\mathcal{S}_q}(G)$ such that $or_1(H) \ge or_1(H')$ holds for every $H' \in \overline{\mathcal{S}_q}(G)$. Claim 1: $or_2(H) = 1$.

Proof. Suppose that $or_2(H) \ge 2$. Let r = n - q. As $(4n - 2)/5 \le q$, we have $n \ge 5r - 2$. Let s = com(H), then $s \ge 3$ by Lemma 3.4.

By Theorem 3.5 (i), $or_1(H) + or_2(H) = q$ and $or_2(H) = or_3(H)$. Clearly,

$$or_3(H) + \dots + or_s(H) = n - (or_1(H) + or_2(H)) = n - q = r$$

and $2 \leq or_2(H) = or_3(H) \leq r$.

Let H_1 be the largest component of H. Since $n \ge 5r - 2$, by Lemma 7.1, there exists $S \subseteq V(H) \setminus V(H_1)$ with $|S| \le or_2(H) - 1$ such that $|E(G[V(H_1) \cup S])| \ge |E(H)|$. Since $or_2(H) \ge 2$, $|E(G[V(H_1) \cup S])| \ge |E(H)|$ implies that $|S| \ge 1$. Note that $G[V(H_1) \cup S]$ is connected as $|V(H_1)| \ge or_2(H) \ge 2$, and

$$|V(H_1)| + |S| \le or_1(H) + or_2(H) - 1 = q - 1,$$

by which the spanning subgraph H' of G with edge set $E(G[V(H_1) \cup S])$ belongs to $S_q(G)$. As $|E(G[V(H_1) \cup S])| \ge |E(H)|, H' \in \overline{S_q}(G)$, while $or_1(H') = |V(H_1)| + |S| > |V(H_1)|$, a contradiction to the assumption that $or_1(H) \ge or_1(H')$. Hence $or_2(H) = 1$.

By Theorem 3.5 (i) and Claim 1, $|V(H_1)| = q - or_2(H) = q - 1$ and

$$\ell_q(G) = |E(H)| = |E(H_1)| = |E(G)| - |E_G(S_0)|$$

where $S_0 = V(H) \setminus V(H_1) \in \mathscr{P}_{n-q+1}(G)$. By (4.2), $|E_G(S_0)| \leq |E_G(S)|$ for each $S \in \mathscr{P}_{n-q+1}(G)$, thus by Theorem 5.2, $S_0 \in \mathscr{P}_{n-q+1}^*(G)$.

Due to Theorem 2.4, $ar(G, \mathcal{T}_q) = \ell_q(G) + 1$. The result then follows.

Corollary 7.3. Let G denote K_{p_1,p_2,\dots,p_k} and $n = p_1 + p_2 + \dots + p_k$, where $k \ge 2$ and $p_1 \ge p_2 \ge \dots \ge p_k \ge 1$. If $(4n-2)/5 \le q \le n-1$ and $p_1 - p_2 \ge (n+2)/5$, then

$$ar(G, \mathcal{T}_q) = |E(G)| + 1 - (n - q + 1)(n - p_1).$$

Proof. By Theorem 7.2, $ar(G, \mathcal{T}_q) = 1 + |E(G)| - |E_G(S)|$ for any $S \in \mathscr{P}_{n-q+1}^*(G)$. Since $p_1 - p_2 \ge (n+2)/5 = n - (4n-2)/5 \ge n-q$, by Algorithm A, there exists $S_0 \in \mathscr{P}_{n-q+1}^*(G)$ with $S_0 \subseteq X_1$, where X_1 is the partite set of G with $|X_1| = p_1$.

Since $S_0 \subseteq X_1$ and $|S_0| = n - q + 1$, we have $|E_G(S_0)| = (n - p_1)(n - q + 1)$. Thus, the result holds.

Remark. Note that the ordered pairs $(K_{3,3}, 4)$ and $(K_{3,3,3}, 6)$ in (4.3) imply that the condition $q \ge (4n-2)/5$ in Theorem 7.2 cannot be improved into $q \ge \lfloor (4n-2)/5 \rfloor$, but we guess it may be true when $q \ge (2n+1)/3$. In addition, when $q \le n-4$ and q is close to n, the value of $ar(K_{p_1,\dots,p_k},\mathcal{T}_q)$ might also be obtained by applying Corollary 5.3. However, the exact values of $ar(K_{p_1,\dots,p_k},\mathcal{T}_q)$ for all q are generally hard to compute, even for complete bipartite graphs, as also mentioned in [8].

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