The maximum size of a nonhamiltonian graph with given order and connectivity*

Xingzhi Zhan and Leilei Zhang[†]

Department of Mathematics, East China Normal University, Shanghai 200241, China

Abstract

Motivated by work of Erdős, Ota determined the maximum size g(n, k) of a k-connected nonhamiltonian graph of order n in 1995. But for some pairs n, k, the maximum size is not attained by a graph of connectivity k. For example, g(15,3) = 77 is attained by a unique graph of connectivity 7, not 3. In this paper we obtain more precise information by determining the maximum size of a nonhamiltonian graph of order n and connectivity k, and determining the extremal graphs. Consequently we solve the corresponding problem for nontraceable graphs.

Key words. Connectivity; hamiltonian graph; traceable graph; size; extremal graph Mathematics Subject Classification. 05C35, 05C40, 05C30

1 Introduction

We consider finite simple graphs, and use standard terminology and notations. The order of a graph is its number of vertices, and the size its number of edges. For graphs we will use equality up to isomorphism, so $G_1 = G_2$ means that G_1 and G_2 are isomorphic. \overline{G} denotes the complement of a graph G. For two graphs G and G0 and G1 denotes the join of G2 and G3 and G4 denotes the isomorphical edges joining every vertex of G5 to every vertex of G6. We denote the complete graph of order G6.

^{*}E-mail addresses: zhan@math.ecnu.edu.cn(X.Zhan), mathdzhang@163.com(L.Zhang).

[†]Corresponding author.

One way to understand hamiltonian graphs is to investigate nonhamiltonian graphs. In 1961 Ore [8] determined the maximum size of a nonhamiltonian graph with a given order and also determined the extremal graphs.

Lemma 1. (Ore [8]) The maximum size of a nonhamiltonian graph of order n is $\binom{n-1}{2} + 1$ and this size is attained by a graph G if and only if $G = K_1 \vee (K_{n-2} + K_1)$ or $G = K_2 \vee \overline{K_3}$.

Bondy [1] gave a new proof of Lemma 1. It is natural to ask the same question by putting constraints on the graphs. In 1962 Erdős [6] determined the maximum size of a nonhamiltonian graph of order n and minimum degree at least k, while in 1995 Ota [9] determined the maximum size g(n, k) of a k-connected nonhamiltonian graph of order n. But for some pairs n, k, the maximum size is not attained by a graph of connectivity k. For example, g(15, 3) = 77 is attained by a unique graph of connectivity 7, not 3.

In this paper we obtain more precise information by determining the maximum size of a nonhamiltonian graph of order n and connectivity k, and determining the extremal graphs, from which Ota's result can be deduced. Consequently we solve the corresponding problem for nontraceable graphs.

2 Main results

Denote by V(G) and E(G) the vertex set and edge set of a graph G respectively. For $S \subseteq V(G)$, we denote by G[S] the subgraph of G induced by G(V) denotes the degree of a vertex V, and S(V) denotes the minimum degree of a graph G. Let S(V) denote the complete bipartite graph on S(V) and S(V) denote the complete bipartite graph on S(V) and S(V) denote the complete bipartite graph on S(V) denote the complete graph of S(V) denote graph of S(V) denote graph S(V) denote

We denote by $\kappa(G)$ and $\alpha(G)$ the connectivity and independence number of a graph G respectively.

We will need the following lemmas.

Lemma 2. (Chvátal [4]) Let G be a graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ where $n \geq 3$. If there is no integer k with $1 \leq k < n/2$ such that $d_k \leq k$ and $d_{n-k} < n-k$, then G is hamiltonian.

Lemma 2 can also be found in [3, p.488].

Lemma 3. (Chvátal-Erdős [5]) Let G be a graph of order at least three. If $\kappa(G) \geq$

 $\alpha(G)$, then G is hamiltonian.

Lemma 3 can also be found in [3, p.488] and [10, p.292].

Given a graph G and a positive integer s with $s \leq \alpha(G)$, denote

$$\sigma_s(G) = \min \left\{ \left. \sum_{v \in T} \deg(v) \right| \ T \subseteq V(G) \ \text{is an independent set and} \ |T| = s \right\}.$$

The following result is a special case of Ota's theorem.

Lemma 4. (Ota [9, Theorem 1]) Let G be a k-connected graph of order n where $2 \le k < \alpha(G)$. If for every integer p with $k \le p \le \alpha(G) - 1$ we have $\sigma_{p+1}(G) \ge n + p^2 - p$, then G is hamiltonian.

A bipartite graph with partite sets X and Y is called balanced if |X| = |Y|. For $n \ge 3$, we denote by $K_{n,n-2} + 4e$ the bipartite graph obtained from $K_{n,n-2}$ by adding two vertices which are adjacent to two common vertices of degree n-2.

Lemma 5. (Liu-Shiu-Xue [7]) Given an integer $n \geq 4$, let $\Omega(n)$ denote the set of all nonhamiltonian balanced bipartite graphs of order 2n with minimum degree at least 2, and let $\Omega(3)$ denote the set of all nonhamiltonian balanced bipartite graphs of order 6. Then for any $n \geq 3$, the maximum size of a graph in $\Omega(n)$ is $n^2 - 2n + 4$ and this maximum size is uniquely attained by the graph $K_{n,n-2} + 4e$.

The case $n \ge 4$ of Lemma 5 is proved in [7, p.257] and the case n = 3 can be verified easily. We will use this lemma with all $n \ge 3$ cases.

Lemma 6. (Bondy [2]) Let G be a graph of order n with degree sequence $d_1 \le d_2 \le \cdots \le d_n$ and let k be an integer with $0 \le k \le n-2$. If for each integer j with $1 \le j \le n-1-d_{n-k}$ we have $d_j \ge j+k$, then G is (k+1)-connected.

Lemma 7. Let $G = K_s \vee \overline{K_t}$ or $G = K_s \vee (K_2 + \overline{K_t})$ where $t \geq 2$ in both cases, and let $F \subseteq E(G)$ with $|F| = f \leq s$. Then $\kappa(G - F) \geq s - f$, with equality if and only if all the edges in F are incident to one common vertex in $\overline{K_t}$.

Proof. We prove the case when $G = K_s \vee \overline{K_t}$. The case when $G = K_s \vee (K_2 + \overline{K_t})$ can be proved similarly.

It is easy to see that $\kappa(G) = s$. Since deleting one edge reduces the connectivity by at most one [10, p.169], we have $\kappa(G - F) \ge s - f$.

Next we use induction on f to prove the equality condition. First consider the case f = 1. Let $e \in E(G)$. It is easy to check that $\kappa(G - e) = s - 1$ if and only if e has one

endpoint in K_s and the other endpoint in $\overline{K_t}$. Now let $F \subseteq E(G)$ with $|F| = f \ge 2$ and suppose that for any $A \subseteq E(G)$ with |A| = f - 1, $\kappa(G - A) = s - (f - 1)$ if and only if all the edges in A are incident to one common vertex in $\overline{K_t}$.

If all edges in F are incident to one common vertex in $\overline{K_t}$, it is easy to verify that $\kappa(G-F)=s-f$. Conversely, suppose $\kappa(G-F)=s-f$. Let $F=\{e_1,e_2,\ldots,e_f\}$ and denote $F'=F\setminus\{e_f\}$. Then $\kappa(G-F')=s-f+1$. By the induction hypothesis, the edges e_1,\ldots,e_{f-1} are incident to one common vertex w in $\overline{K_t}$. The degree sequence of G-F' is

$$s - f + 1$$
, $\underbrace{s, \dots, s}_{t-1}$, $\underbrace{n - 2, \dots, n - 2}_{f-1}$, $\underbrace{n - 1, \dots, n - 1}_{s-f+1}$

where n=s+t and $s-f+1=\deg(w)$. We assert that e_f is incident to w and consequently all the edges in F are incident to one common vertex in $\overline{K_t}$. Let the degree sequence of G-F be $d_1 \leq \cdots \leq d_n$. By the above degree sequence of G-F' we deduce that $d_{n-s+f} \geq n-2$. Thus $n-1-d_{n-s+f} \leq 1$. If e_f is not incident to w, then we would have $d_1=s-f+1=1+(s-f)$. By Lemma 6, G-F is (s-f+1)-connected, contradicting the assumption $\kappa(G-F)=s-f$. This proves that e_f is incident to w.

Notation 1. e(G) denotes the size of a graph G.

Notation 2. For positive integers n and k with n odd and $n \geq 2k + 1$, $G_1(n, k)$ denotes the graph obtained from $K_{(n-1)/2} \vee \overline{K_{(n+1)/2}}$ by deleting (n-1)/2 - k edges that are incident to one common vertex in $\overline{K_{(n+1)/2}}$; for positive integers n and k with n even and $n \geq 2k + 2$, $G_2(n, k)$ denotes the graph obtained from $K_{(n-2)/2} \vee (K_2 + \overline{K_{(n-2)/2}})$ by deleting (n-2)/2 - k edges that are incident to one common vertex in $\overline{K_{(n-2)/2}}$.

Note that by Dirac's theorem [3, p.485], for the existence of a nonhamiltonian graph of order n and connectivity k we necessarily have $n \ge 2k + 1$. Now we are ready to state and prove the main result.

Theorem 8. Let f(n,k) denote the maximum size of a nonhamiltonian graph of order n and connectivity k. Then

$$f(n,k) = \begin{cases} \binom{n-k}{2} + k^2 & \text{if } n \text{ is odd and } n \ge 6k - 5 \text{ or } n \text{ is even and } n \ge 6k - 8, \\ \frac{3n^2 - 8n + 5}{8} + k & \text{if } n \text{ is odd and } 2k + 1 \le n \le 6k - 7, \\ \frac{3n^2 - 10n + 16}{8} + k & \text{if } n \text{ is even and } 2k + 2 \le n \le 6k - 10. \end{cases}$$

If n = 6k - 5, then f(n, k) is attained by a graph G if and only if $G = K_k \vee (K_{n-2k} + \overline{K_k})$ or $G = G_1(n, k)$. If n = 6k - 8, then f(n, k) is attained by a graph G if and only if

 $G = K_k \vee (K_{n-2k} + \overline{K_k})$ or $G = G_2(n, k)$. If n is odd and $n \geq 6k - 3$ or n is even and $n \geq 6k - 6$, then f(n, k) is attained by a graph G if and only if $G = K_k \vee (K_{n-2k} + \overline{K_k})$. If n is odd and $2k + 1 \leq n \leq 6k - 7$, then f(n, k) is attained by a graph H if and only if $H = G_1(n, k)$. If n is even and $2k + 2 \leq n \leq 6k - 10$, then f(n, k) is attained by a graph Z if and only if $Z = G_2(n, k)$.

Proof. The case k = 1 of Theorem 8 follows from Lemma 1. Note that the extremal graph $K_2 \vee \overline{K_3}$ of order 5 in Lemma 1 has connectivity 2 and hence it should be excluded.

Next suppose $k \geq 2$. It is easy to verify that the extremal graphs stated in Theorem 8 are nonhamiltonian graphs of order n and connectivity k with size f(n,k). They are nonhamiltonian since any hamiltonian graph must be tough [3, pp.472-473]. Thus it remains to show that f(n,k) is an upper bound on the size and it can only be attained by these extremal graphs.

Let Q be a nonhamiltonian graph of order n and connectivity k with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. By Lemma 3, $k < \alpha(Q)$ and by Lemma 4, there exists an integer p with $k \leq p \leq \alpha(Q) - 1$ such that $\sigma_{p+1}(Q) \leq n + p^2 - p - 1$. Let S be an independent set of Q with cardinality p+1 whose degree sum is $\sigma_{p+1}(Q)$. Then $e(Q[V(Q) \setminus S]) \leq \binom{n-p-1}{2}$. We distinguish four cases.

Case 1. n is odd and $n \ge 6k - 5$.

Subcase 1.1. $p \le (n-3)/2$.

The conditions $p \le (n-3)/2$ and $n \ge 6k-5$ imply 3p+3k+1<2n. This, together with the condition $p \ge k$, yields $(p-k)(3p+3k+1-2n) \le 0$. It follows that

$$e(Q) \le n + p^2 - p - 1 + \binom{n - p - 1}{2} \le \binom{n - k}{2} + k^2$$
 (1)

and equality holds in the second inequality in (1) if and only if p = k.

Now suppose that Q has size $\binom{n-k}{2} + k^2$. Then p = k, S has cardinality k+1 and degree sum $n + k^2 - k - 1$, and $V(Q) \setminus S$ is a clique. Since k+1 < (n+1)/2, we have $d_{(n+1)/2} \ge n - k - 2$. By Lemma 2, there exists i with i < n/2 such that $d_i \le i$ and $d_{n-i} \le n - i - 1$. Since n is odd, the condition i < n/2 means $i \le (n-1)/2$. We have

$$e(Q) = {\binom{n-k}{2}} + k^2 \le [i^2 + (n-2i)(n-i-1) + i(n-1)]/2, \tag{2}$$

where the inequality is equivalent to $(i-k)(2n-3i-3k-1) \le 0$. Since $i \ge d_i \ge \delta(Q) \ge k$, we obtain i = k or $n \le (3i+3k+1)/2$.

If i = k, equality holds in (2) and hence the degree sequence of Q is

$$\underbrace{k,\ldots,k}_{k},\underbrace{n-k-1,\ldots,n-k-1}_{n-2k},\underbrace{n-1,\ldots,n-1}_{k},$$

implying that $Q = K_k \vee (K_{n-2k} + \overline{K_k})$.

Now suppose $i \neq k$. Then we have $n \leq (3i+3k+1)/2$. If $i \leq (n-3)/2$, then $n \leq 6k-7$, contradicting our assumption $n \geq 6k-5$. Thus i=(n-1)/2. We have $n-k-2 \leq d_{(n+1)/2} \leq (n-1)/2$. Hence $6k-5 \leq n \leq 2k+3$, which, together with the condition $k \geq 2$, yields k=2 and n=7. It is easy to check that there are exactly four graphs of order 7 and size 14 with $d_4=3$, among which $G_1(7,2)$ is the only graph that is nonhamiltonian with connectivity 2. Hence $Q=G_1(7,2)$.

Subcase 1.2. $p \ge (n-1)/2$.

Clearly Q is a spanning subgraph of $R = K_{n-p-1} \vee \overline{K_{p+1}}$. If $p \geq (n+1)/2$, then

$$e(Q) \le \binom{n-p-1}{2} + (n-p-1)(p+1) < \binom{n-k}{2} + k^2$$

where the second inequality follows from the condition $n \ge 6k - 5$. If p = (n - 1)/2, we have $\kappa(R) = n - p - 1 > k$. Let $F \subseteq E(R)$ such that Q = R - F. Since $\kappa(Q) = k$, by Lemma 7 we have $|F| \ge n - p - 1 - k$. Thus

$$e(Q) \le \binom{n-p-1}{2} + (n-p-1)(p+1) - (n-p-1-k) \le \binom{n-k}{2} + k^2$$
 (3)

and equality holds in the second inequality of (3) if and only if n = 6k - 5.

Suppose $e(Q) = \binom{n-k}{2} + k^2$. Then n = 6k - 5 and |F| = n - p - 1 - k. By Lemma 7, all the edges in F are incident to one common vertex in $\overline{K_{p+1}}$. Since p = (n-1)/2, we have n - p - 1 = (n-1)/2, p + 1 = (n+1)/2 and |F| = (n-1)/2 - k. It follows that $Q = G_1(n, k)$.

Case 2. n is odd and $2k + 1 \le n \le 6k - 7$.

Subcase 2.1. $k \le p < (n-1)/2$.

We have

$$e(Q) \le n + p^2 - p - 1 + \binom{n - p - 1}{2} < \frac{3n^2 - 8n + 5}{8} + k.$$

Subcase 2.2. p = (n-1)/2.

In this case Q is a spanning subgraph of $K_p \vee \overline{K_{p+1}}$. Since $\kappa(Q) = k$, by Lemma 7 we obtain

$$e(Q) \le \binom{p}{2} + p(p+1) - (p-k) = \frac{3n^2 - 8n + 5}{8} + k$$

and equality holds if and only if $Q = G_1(n, k)$.

Subcase 2.3. p > (n-1)/2.

In this case Q is a spanning subgraph of $K_{n-p-1} \vee \overline{K_{p+1}}$. Then

$$e(Q) \le \binom{n-p-1}{2} + (n-p-1)(p+1) < \frac{3n^2 - 8n + 5}{8} + k,$$

where we have used the condition $n \leq 2p - 1$.

Case 3. n is even and $n \ge 6k - 8$.

Subcase 3.1. p < (n-2)/2.

The assumptions imply 3p + 3k + 1 - 2n < 0. We have

$$e(Q) \le n + p^2 - p - 1 + \binom{n - p - 1}{2} \le \binom{n - k}{2} + k^2$$
 (4)

where the second inequality is equivalent to

$$(p-k)(3p+3k+1-2n) \le 0.$$

Thus equality holds in the second inequality in (4) if and only if p = k.

Suppose $e(Q) = \binom{n-k}{2} + k^2$. Then p = k and Q has a clique of cardinality n - p - 1 and an independent set of cardinality p + 1 whose degree sum equals $n + p^2 - p - 1$. Also $d_{(n+2)/2} \ge n - k - 2$. By Lemma 2, there exists i < n/2 such that $d_i \le i$ and $d_{n-i} \le n - i - 1$. We have

$$e(Q) = {\binom{n-k}{2}} + k^2 \le [i^2 + (n-2i)(n-i-1) + i(n-1)]/2,$$

where the inequality is equivalent to

$$(i-k)(2n-3i-3k-1) \le 0. (5)$$

Note that $i \geq k$ since $i \geq d_i \geq \delta(Q) \geq k$. If i = k, then the degree sequence of Q is

$$\underbrace{k,\ldots,k}_{k}$$
, $\underbrace{n-k-1,\ldots,n-k-1}_{n-2k}$, $\underbrace{n-1,\ldots,n-1}_{k}$,

implying $Q = K_k \vee (K_{n-2k} + \overline{K_k}).$

Next suppose $i \neq k$. Then the inequality (5) implies $2n-3i-3k-1 \leq 0$. If $i \leq (n-4)/2$, we deduce that $n \leq 6k-10$, a contradiction. Hence i=(n-2)/2. Now the conditions $d_{n-i} \leq n-i-1$ and $d_{(n+2)/2} \geq n-k-2$ yield $n-k-2 \leq d_{(n+2)/2} \leq n/2$. Thus $6k-8 \leq n \leq 2k+4$. It follows that k=2 and $4 \leq n \leq 8$ or k=3 and $10 \leq n \leq 10$. The possibility n=4 contradicts k < n/2 and n=6 contradicts $i \neq k$. Only the two pairs (k,n)=(2,8), (3,10) can occur.

If k=2 and n=8, by the conditions n=8, e=19, $d_5=4$ and being nonhamiltonian we deduce that $Q=K_3\vee (K_2+\overline{K_3})$; if k=3 and n=10, the conditions n=10, e(Q)=30, k=3, $d_6=5$ and being nonhamiltonian force $Q=G_2(10,3)$.

Subcase 3.2. p = (n-2)/2.

Clearly $\alpha(Q) \geq p + 1$. We further distinguish two cases.

If $\alpha(Q) \geq p+2$, then Q is a spanning subgraph of $K_p \vee \overline{K_{p+2}}$. By Lemma 7 we have

$$e(Q) \le \binom{p}{2} + p(p+2) - (p-k) < \binom{n-k}{2} + k^2.$$
 (6)

The second inequality in (6) is equivalent to $p^2 + (5 - 4k)p + 3k^2 - 5k + 2 > 0$ which is guaranteed by $p = (n-2)/2 \ge 3k - 5$.

If $\alpha(Q) = p + 1$, then Q is a spanning subgraph of $K_{p+1} \vee \overline{K_{p+1}}$. Let Q' denote the graph obtained from Q by deleting all the edges in K_{p+1} . Then Q' is a nonhamiltonian balanced bipartite graph. There are two cases.

(a) Suppose $n \ge 8$ and $\delta(Q') \ge 2$ or n = 6. By Lemma 5, $e(Q') \le (p+1)^2 - 2(p+1) + 4 = p^2 + 3$. Hence

$$e(Q) \le \binom{p+1}{2} + p^2 + 3 \le \binom{n-k}{2} + k^2.$$
 (7)

The second inequality in (7) is equivalent to

$$p^{2} + (5 - 4k)p + 3k^{2} - 3k - 4 \ge 0$$
(8)

which is implied by the condition $p \geq 3k - 5$. Equality holds in (8) if and only if k = 2 and p = 2, i.e., Q' is the extremal graph of order 6 defined in Lemma 5. Hence Q has size $\binom{n-k}{2} + k^2$ if and only if $Q = K_2 \vee (K_2 + \overline{K_2})$.

(b) Now suppose $n \geq 8$ and $\delta(Q') \leq 1$. Let $x \in V(Q')$ with $\deg_{Q'}(x) = \delta(Q')$. Starting with the structure $K_{p+1} \vee \overline{K_{p+1}}$, we deduce that x lies in K_{p+1} , since $\delta(Q) \geq 2$. In this

case Q is a spanning subgraph of $K_p \vee (K_2 + \overline{K_p})$. By Lemma 7 and using the fact that $p \geq 3k - 5$ we have

$$e(Q) \le \binom{p+2}{2} + p^2 - (p-k) \le \binom{n-k}{2} + k^2.$$
 (9)

Equality in the second inequality in (9) holds if and only if p = 3k - 5; i.e, n = 6k - 8. Thus by (9) and Lemma 7, Q has size $\binom{n-k}{2} + k^2$ if and only if $Q = G_2(n, k)$ with n = 6k - 8.

Subcase 3.3. p > (n-2)/2.

Q is a spanning subgraph of $K_{n-p-1} \vee \overline{K_{p+1}}$. If p = n/2, we have $n-p-1 \geq k$. By Lemma 7

$$e(Q) \le \binom{n-p-1}{2} + (n-p-1)(p+1) - (n-p-1-k) < \binom{n-k}{2} + k^2.$$
 (10)

The second inequality in (10) is equivalent to $p^2 + (3-4k)p + 3k^2 - k - 2 > 0$, which is implied by $p = n/2 \ge 3k - 4$.

If $p \ge (n+2)/2$, we have

$$e(Q) \le \binom{n-p-1}{2} + (n-p-1)(p+1) < \binom{n-k}{2} + k^2.$$
 (11)

The second inequality in (11) is equivalent to $p^2 + p + 3k^2 + k - 2nk > 0$. To prove this inequality it suffices to show $p^2 + (1 - 4k)p + 3k^2 + 5k > 0$, which is implied by $p \ge (n+2)/2 \ge 3k - 3$.

Case 4. n is even and $2k + 2 \le n \le 6k - 10$.

Denote m = (n-2)/2. Then $3 \le k \le m$. We distinguish three subcases.

Subcase 4.1. p < m.

We have

$$e(Q) \le \binom{n-p-1}{2} + n + p^2 - p - 1 < \frac{3n^2 - 10n + 16}{8} + k.$$
 (12)

The second inequality in (12) is equivalent to $3p^2 - (4m+3)p + m^2 + 5m - 2k < 0$, which is implied by the conditions $k \le p < m \le 3k - 6$.

Subcase 4.2. p = m.

If $\alpha(Q) \geq p+2$, then Q is a spanning subgraph of $K_p \vee \overline{K_{p+2}}$. Recall that $p \geq k$. By

Lemma 7 we have

$$e(Q) \le \binom{p}{2} + p(p+2) - (p-k) = (3p^2 + p + 2k)/2 < (3p^2 + p + 2k)/2 + 1$$
$$= \frac{3n^2 - 10n + 16}{8} + k.$$

Now suppose $\alpha(Q) = p + 1$. Then Q is a spanning subgraph of $K_{p+1} \vee \overline{K_{p+1}}$. Define the graph Q' as in Subcase 3.2 above.

If $\delta(Q') \geq 2$, by Lemma 5 we have $e(Q') \leq p^2 + 3$. Hence

$$e(Q) \le {p+1 \choose 2} + p^2 + 3 < \frac{3n^2 - 10n + 16}{8} + k.$$

If $\delta(Q') \leq 1$, then Q is a spanning subgraph of $K_p \vee (K_2 + \overline{K_p})$. By Lemma 7 we obtain

$$e(Q) \le {p+2 \choose 2} + p^2 - (p-k) = \frac{3n^2 - 10n + 16}{8} + k$$

and equality holds if and only if $Q = G_2(n, k)$.

Subcase 4.3. p > m.

Q is a spanning subgraph of $K_{n-p-1} \vee \overline{K_{p+1}}$. We further distinguish two cases.

(a) p = m + 1. Now the conditions $2k + 2 \le n$ and p = m + 1 = n/2 imply $n - p - 1 = (n-2)/2 \ge k$. By Lemma 7, we have

$$e(Q) \le {n-p-1 \choose 2} + (n-p-1)(p+1) - (n-p-1-k) < \frac{3n^2 - 10n + 16}{8} + k$$

where the second inequality is equivalent to $4p^2 - n^2 + 2n - 4p + 8 > 0$ which holds, since n = 2p.

(b) $p \ge m + 2$. In this case the following rough estimate suffices:

$$e(Q) \le {n-p-1 \choose 2} + (n-p-1)(p+1) < \frac{3n^2 - 10n + 16}{8} + k.$$
 (13)

The second inequality in (13) is equivalent to $4p^2 + 4p - n^2 - 6n + 8k + 16 > 0$, which holds, since $n \le 2p - 2$. This completes the proof.

The following corollary follows from Theorem 8 immediately.

Corollary 9. Let f(n,k) be defined as in Theorem 8. If G is a graph of order n and connectivity k with size larger than f(n,k), then G is hamiltonian.

Next we use Theorem 8 to deduce Ota's result.

Corollary 10. (Ota [9, p.209]) The maximum size of a k-connected nonhamiltonian graph of order n is

$$\max\left\{ \binom{n-k}{2} + k^2, \ \binom{\lfloor (n+2)/2 \rfloor}{2} + \lfloor \frac{n-1}{2} \rfloor^2 \right\}. \tag{14}$$

Proof. Denote the number in (14) by M. Let f(n,c) be defined as in Theorem 8. It is easy to verify that

$$\max\{f(n,c) | k \le c < n/2\} = M$$

and the result follows.

A graph is *traceable* if it contains a Hamilton path; otherwise it is *nontraceable*. Next we turn to nontraceable graphs.

The following trick is well-known (e.g. [4, p.166] or [5, p.112]).

Lemma 11. Let G be a graph and denote $H = G \vee K_1$. Then G is traceable if and only if H is hamiltonian, and $\kappa(G) = k$ if and only if $\kappa(H) = k + 1$.

Notation 3. For positive integers n and k with n odd and $n \geq 2k + 3$, $H_1(n,k)$ denotes the graph obtained from $K_{(n-3)/2} \vee (K_2 + \overline{K_{(n-1)/2}})$ by deleting (n-3)/2 - k edges that are incident to one common vertex in $\overline{K_{(n-1)/2}}$; for positive integers n and k with n even and $n \geq 2k + 2$, $H_2(n,k)$ denotes the graph obtained from $K_{(n-2)/2} \vee \overline{K_{(n+2)/2}}$ by deleting (n-2)/2 - k edges that are incident to one common vertex in $\overline{K_{(n+2)/2}}$.

By Dirac's theorem [3, p.485] and Lemma 11, for the existence of a nontraceable graph of order n and connectivity k we must have $n \ge 2k + 2$. The following corollary follows from Theorem 8 and Lemma 11 immediately.

Corollary 12. Let $\varphi(n, k)$ denote the maximum size of a nontraceable graph of order n and connectivity k. Then

$$\varphi(n,k) = \begin{cases} \binom{n-k-1}{2} + k(k+1) & \text{if } n \text{ is odd and } n \ge 6k-3 \text{ or } n \text{ is even and } n \ge 6k, \\ \frac{3n^2 - 12n + 17}{8} + k & \text{if } n \text{ is odd and } 2k+3 \le n \le 6k-5, \\ \frac{3n^2 - 10n + 8}{8} + k & \text{if } n \text{ is even and } 2k+2 \le n \le 6k-2. \end{cases}$$

If n = 6k-3, then $\varphi(n,k)$ is attained by a graph G if and only if $G = K_k \vee (K_{n-2k-1} + \overline{K_{k+1}})$ or $G = H_1(n,k)$. If n = 6k, then $\varphi(n,k)$ is attained by a graph G if and only if $G = K_k \vee (K_{n-2k-1} + \overline{K_{k+1}})$

 $K_k \vee (K_{n-2k-1} + \overline{K_{k+1}})$ or $G = H_2(n,k)$. If n is odd and $n \geq 6k-1$ or n is even and $n \geq 6k+2$, then $\varphi(n,k)$ is attained by a graph G if and only if $G = K_k \vee (K_{n-2k-1} + \overline{K_{k+1}})$. If n is odd and $2k+3 \leq n \leq 6k-5$, then $\varphi(n,k)$ is attained by a graph G if and only if $G = H_1(n,k)$. If n is even and $2k+2 \leq n \leq 6k-2$, then $\varphi(n,k)$ is attained by a graph G if and only if $G = H_2(n,k)$.

Acknowledgement. This research was supported by the NSFC grants 11671148 and 11771148 and Science and Technology Commission of Shanghai Municipality (STCSM) grant 18dz2271000.

References

- [1] J.A. Bondy, Variations on the hamiltonian theme, Canad. Math. Bull., 15(1972), no. 1, 57-62.
- [2] J.A. Bondy, Properties of graphs with constraints on degrees, Studia Sci. Math. Hungar. 4(1969), 473-475.
- [3] J.A. Bondy and U.S.R. Murty, Graph Theory, GTM 244, Springer, 2008.
- [4] V. Chvátal, On Hamilton's ideals, J. Combin. Theory, Ser. B, 12(1972), 163-168.
- [5] V. Chvátal and P. Erdős, A note on Hamiltonian circuits, Discrete Math., 2(1972), 111-113.
- [6] P. Erdős, Remarks on a paper of Pósa, Magyar Tud. Akad. Math. Kutató. Int. Közl., 7(1962), 227-229.
- [7] R. Liu, W.C. Shiu and J. Xue, Sufficient spectral conditions on Hamiltonian and traceable graphs, Linear Algebra Appl., 467(2015), 254-266.
- [8] O. Ore, Arc coverings of graphs, Ann. Mat. Pura Appl., 55(1961), 315-321.
- [9] K. Ota, Cycles through prescribed vertices with large degree sum, Discrete Math., 145(1995), 201-210.
- [10] D.B. West, Introduction to Graph Theory, Prentice Hall, Inc., 1996.