PED AND POD PARTITIONS: COMBINATORIAL PROOFS OF RECURRENCE RELATIONS

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ABSTRACT. PED partitions are partitions with even parts distinct while odd parts are unrestricted. Similarly, POD partitions have distinct odd parts while even parts are unrestricted. In [16] several recurrence relations for the number of PED partitions of n are proved analytically. They are similar to the recurrence relation for the number of partitions of n given by Euler's pentagonal number theorem. We provide combinatorial proofs for all of these theorems and also for the pentagonal number theorem for PED partitions proved analytically in [11], which motivated the theorems in [16]. Moreover, we prove combinatorially a recurrence for POD partitions given in [7], Beck-type identities involving PED and POD partitions, and several other results about PED and POD partitions.

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1. INTRODUCTION

A partition λ of n is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of positive integers that add up to n. We refer to the integers λ_i as the parts of λ . As usual, we denote by p(n) the number of partitions of n. Note that p(x) = 0 if x is not a non-negative integer, and since the empty partition \emptyset is the only partition of 0, we have that p(0) = 1.

In this article we consider partitions in which parts of fixed parity are distinct: PED partitions have no repeated even parts and POD partitions have no repeated odd parts. We denote by ped(n), respectively pod(n), the number of PED, respectively POD, partitions of n.

One of the most celebrated theorems in the theory of partitions is Euler's pentagonal number theorem, which states that, for $n \ge 1$,

$$\sum_{j=-\infty}^{\infty} (-1)^k p(n-k(3k+1)/2) = 0.$$

Fink, Guy and Krusemeyer [11] proved an analogous result for PED partitions.

Theorem 1.1. For $n \ge 0$,

$$\sum_{j=-\infty}^{\infty} (-1)^j ped(n-j(3j+1)/2) = \begin{cases} (-1)^k & \text{ if } n = 2k(3k+1) \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{ otherwise.} \end{cases}$$

Inspired by this theorem, Merca [16] proved analogous recurrences for ped(n) involving triangular and square numbers. Let $T_k = k(k+1)/2$ denote the k^{th} triangular number.

Theorem 1.2. For $n \ge 0$,

$$\sum_{j>0} (-1)^{\lceil j/2 \rceil} ped(n-T_j) = \begin{cases} 1 & \text{if } n = 2T_k \text{ for some } k \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.3. For $n \ge 0$,

$$\sum_{j=-\infty}^{\infty} (-1)^j ped(n-2j^2) = \begin{cases} 1 & \text{if } n = T_k \text{ for some } k \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

In the articles cited, Theorems 1.1, 1.2, and 1.3 are proved using generating functions. We give combinatorial proofs of these theorems.

In [7], analogous results were derived analytically for pod(n). We denote by $Q_k(n)$ the number of distinct partitions of n into parts $\neq k \mod 4$.

Theorem 1.4. For $n \ge 0$ the following hold.

(i)
$$Q_0(n) = pod(n) + 2\sum_{k=1}^{\infty} (-1)^k pod(n-4k^2)$$

(ii) $Q_2(n) = \sum_{k=0}^{\infty} (-1)^{k(k+1)/2} pod(n-k(k+1))$

In [7], the authors also proved Theorem 1.4 (ii) combinatorially. Here we give a combinatorial proof of Theorem 1.4 (i).

We also give combinatorial proofs of the remaining theorems in [16] listed below. We prove the following theorem relating ped(n) and pod(n).

Theorem 1.5. For any non-negative integer n, we have

$$ped(n) = \sum_{k=0}^{\infty} pod(n-2T_k).$$

We then prove another recurrence for pod(n) involving triangular numbers.

Theorem 1.6. For $n \ge 0$,

$$\sum_{j=0}^{\infty} (-1)^{T_j} pod(n-T_j) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

We give a combinatorial proof of Merca's theorem relating ped(n) and $\overline{p}(n)$, the number of overpartitions of n. Overpartitions were introduced in [10]. They are partitions in which the first occurrence of a part may be overlined.

Theorem 1.7. For $n \ge 0$,

$$ped(n) = \sum_{k \ge 0} \overline{p}\left(\frac{n}{2} - \frac{T_k}{2}\right),$$

where $\overline{p}(x) = 0$ if x is not an integer.

Let o(n) denote the number of partitions of n into odd parts and denote by $o_{e-o}(n)$ the number of partitions of n with an even number of odd parts minus the number of partitions of n with an odd number of odd parts (in each case there are no even parts). We give a combinatorial theorem of the following result stated as Corollary 5.4 in [16].

Theorem 1.8. For $n \ge 0$,

$$\sum_{j=0}^{\infty} o_{e-o}(n-T_j) = \begin{cases} (-1)^k & \text{ if } n = 2k(3k+1) \text{ for some } k \in \mathbb{Z} \\ 0 & \text{ otherwise.} \end{cases}$$

Combining Theorem 1.8 with Theorem 1.1 we obtain a combinatorial proof of Merca's Theorem 5.3.

Corollary 1.9. For $n \ge 0$,

$$\sum_{j=-\infty}^{\infty} (-1)^{j} ped(n-j(3j+1)/2) = \sum_{j=0}^{\infty} o_{e-o}(n-T_{j}).$$

In the last section of the article we prove Beck-type theorems for ped(n) and pod(n). As we will explain in the next section, ped(n) is equal to the number of partitions of n into parts $\neq 0 \mod 4$, and pod(n) is equal to the number of partitions of n into parts $\neq 2 \mod 4$. The Beck-type identities give combinatorial interpretations for the excess in the number of parts in all partitions of n; and also for the excess in the number of parts in all PED partitions of n; and also for the excess in the number of parts in all partitions of n; and also for the excess in the number of parts in all partitions of n into parts $\neq 2 \mod 4$ over the number of parts in all POD partitions of n. These are analogous to the interpretations given in [2] for the excess in the number of parts in all partitions of n.

2. Preliminaries and Notation

If λ is a partition of n, we write $\lambda \vdash n$. We refer to n as the size of λ and also write $|\lambda| = n$. When we work with vectors of partitions, we write $(\lambda, \mu, \dots) \vdash n$ to mean $|\lambda| + |\mu| + \dots = n$. The length of λ is the number of parts of λ , denoted by $\ell(\lambda)$. For convenience, we abuse notation and use λ to denote either the multiset of its parts or the non-increasing sequence of parts. We write $a \in \lambda$ to mean the positive integer a is a part of λ . The number of times a > 0 appears in λ is denoted by m(a) and is called the multiplicity of a. We use the convention that $\lambda_k = 0$ for all $k > \ell(\lambda)$.

The Ferrers diagram of a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is an array of left justified boxes such that the *i*th row from the top contains λ_i boxes. We abuse notation and use λ to mean a partition or its Ferrers diagram.

Example 1. The Ferrers diagram of $\lambda = (4, 3, 3, 2)$ is shown below.



We define the following operations on partitions. If λ and μ are partitions, by $\lambda \cup \mu$ and $\lambda \setminus \mu$ we mean the obvious operations on the multisets of parts of λ and μ . Moreover $\lambda \setminus \mu$ is only defined if $\mu \subseteq \lambda$ as multisets. For a positive integer k, we denote by $k\lambda$ the partition whose parts are the parts of λ multiplied by k. If all parts of λ are divisible by k, we denote by $\lambda_{/k}$ the partition whose parts are the parts of λ divided by k.

We often write a partition λ as (λ^e, λ^o) , where λ^e , respectively λ^o , consists of the even, respectively odd, parts of λ .

We denote by calligraphy style capital letters the set of partitions enumerated by the function denoted by the same letters. For example, $\mathcal{P}(n)$ is the set of unrestricted partitions of n, which are enumerated by p(n), and $\mathcal{PED}(n)$ is the set of partitions of n in which even parts are distinct and thus $|\mathcal{PED}(n)| = ped(n)$. If the variable n is omitted, we mean the set of all partitions with the obvious restrictions. For example, $\mathcal{PED} = \bigcup_{n\geq 0} \mathcal{PED}(n)$. When necessary, we will clarify the notation.

For a thorough and detailed introduction to the theory of integer partitions, we refer the reader to [1].

We first derive a few basic facts about PED and POD partitions. Most are well-known and can be found for example in [3, 16, 7] and the references therein.

We use the Pochhammer symbol notation

$$(a;q)_n = \begin{cases} 1 & \text{for } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{for } n > 0; \\ (a;q)_{\infty} = \lim_{n \to \infty} (a;q)_n. \end{cases}$$

Throughout, we assume |q| < 1 so that all series converge absolutely.

The generating function for ped(n) is given by

$$\sum_{n \ge 0} ped(n)q^n = \frac{(-q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}.$$

This can be rewritten as

(1)
$$\sum_{n\geq 0} ped(n)q^n = \frac{(-q^2;q^2)_{\infty}}{(q;q^2)_{\infty}} \cdot \frac{(q^2;q^2)_{\infty}}{(q^2;q^2)_{\infty}} = \frac{(q^4;q^4)_{\infty}}{(q;q)_{\infty}} = \sum_{n\geq 0} b_4(n)q^n,$$

where $b_4(n)$ is the number of 4-regular partitions of n, i.e., partitions with no parts congruent to 0 mod 4.

One can also see combinatorially that $ped(n) = b_4(n)$. Start with $\lambda \in \mathcal{PED}(n)$ and, similar to Glaisher's transformation, split each part of the form $2^k c$ with $k \ge 2$ and c odd into 2^{k-1} parts equal to 2c to obtain a partition $\mu \in \mathcal{B}_4(n)$. Thus, each part of λ congruent to 0 mod 4 is split into equal parts congruent to 2 mod 4. The inverse transformation is defined by starting with $\mu \in \mathcal{B}_4(n)$ and repeatedly merging equal even parts until all even parts are distinct to obtain a partition $\lambda \in \mathcal{PED}(n)$.

Similarly, the generating function for pod(n) is given by

$$\sum_{n \ge 0} pod(n)q^n = \frac{(-q;q^2)_{\infty}}{(q^2;q^2)_{\infty}},$$

which can be rewritten as

(2)
$$\sum_{n\geq 0} pod(n)q^n = \frac{(-q;q^2)_{\infty}}{(q^2;q^2)_{\infty}} \cdot \frac{(q;q^2)_{\infty}}{(q;q^2)_{\infty}} = \frac{(q^2;q^4)_{\infty}}{(q;q)_{\infty}} = \sum_{n\geq 0} p_2(n)q^n,$$

where $p_2(n)$ is the number of partitions of n with no parts congruent to 2 mod 4.

To show combinatorially that $pod(n) = p_2(n)$, start with $\lambda \in \mathcal{POD}(n)$ and split each part congruent to 2 mod 4 into two equal odd parts to obtain a partition in $\mathcal{P}_2(n)$. For the inverse, start with a partition $\mu \in \mathcal{P}_2(n)$. For each repeated odd part c of μ with multiplicity $m(c) \geq 2$, merge $\lfloor m(c)/2 \rfloor$ pairs of parts equal to c to obtain $\lfloor m(c)/2 \rfloor$ parts equal to 2c. The obtained partition is in $\mathcal{POD}(n)$.

Next, we observe that $ped(n) - pod(n) \ge 0$ for all $n \ge 0$ and we give a combinatorial interpretation of the difference.

Proposition 1. Let $n \ge 0$. Then $b_4(n) - p_2(n)$ equals the number of 4-regular partitions of n such that the number of parts equal to 1 is less than twice the number of even parts.

Proof. We define a bijection from $\mathcal{P}_2(n)$ to the set of 4-regular partitions of n such that m(1) is at least twice the number of even parts.

If $\lambda = (\lambda^e, \lambda^o) \in \mathcal{P}_2(n)$, then all parts of λ^e are divisible by 4. Subtract 2 from each part in λ^e and insert $2\ell(\lambda^e)$ parts equal to 1 to obtain a 4-regular partition $\mu = (\mu^e, \mu^0)$ with $m(1) \ge 2\ell(\mu^e)$.

Conversely, if $\mu = (\mu^e, \mu^o)$ is a 4-regular partition of n with $m(1) \ge 2\ell(\mu^e)$, remove $2\ell(\mu^e)$ parts equal to 1 and add 2 to each even part to obtain a partition $\lambda \in \mathcal{P}_2(n)$.

We use the bijection between $\mathcal{B}_4(n)$ and $\mathcal{PED}(n)$ described above to express ped(n) - pod(n) as the cardinality of a subset of $\mathcal{PED}(n)$. If $\lambda = (\lambda^e, \lambda^o) \in \mathcal{PED}(n)$ is obtained from $\mu = (\mu^e, \mu^o) \in \mathcal{B}_4(n)$, then

$$\ell(\mu^e) = \sum_{a \in \lambda^e} 2^{val_2(a) - 1}$$

where $val_2(a)$ is the 2-adic valuation of a, i.e., $val_2(a) = k$ such that $a = 2^k c$ with c odd. Moreover, $\lambda^o = \mu^o$. Thus,

$$ped(n) - pod(n) = \left| \left\{ \lambda \in \mathcal{PED}(n) \mid m(1) < \sum_{a \in \lambda^e} 2^{val_2(a)} \right\} \right|.$$

For any set of partitions $\mathcal{A}(n)$, let

$$a_{e-o}(n) := |\{\lambda \in \mathcal{A}(n) \mid \ell(\lambda) \text{ even}\}| - |\{\lambda \in \mathcal{A}(n) \mid \ell(\lambda) \text{ odd}\}|.$$

By a distinct partition we mean a partition in which no part is repeated. By an odd partition we mean a partition in which all parts are odd. Recall that, for $k \in \{0, 2\}$, we denote by $Q_k(n)$ the number of distinct partitions with no parts congruent to $k \mod 4$.

In [7, Theorem 1.2] it is shown that

$$p_{2,e-o}(n) = (-1)^n Q_0(n)$$

and

$$b_{4,e-o}(n) = (-1)^n Q_2(n).$$

Here, we give a somewhat analogous result.

Theorem 2.1. For all $n \ge 0$,

$$ped_{e-o}(n) = Q_{2,e-o}(n)$$

and

$$pod_{e-o}(n) = Q_{0,e-o}(n).$$

Proof. It is fairly straightforward to see that

$$\sum_{n\geq 0} ped_{e-o}(n)q^n = \frac{(q^2;q^2)_{\infty}}{(-q;q^2)_{\infty}} = \frac{(q;q^2)_{\infty}(-q;q^2)_{\infty}(q^4;q^4)_{\infty}}{(-q;q^2)_{\infty}}$$
$$= (q;q^2)_{\infty}(q^4;q^4)_{\infty} = \sum_{n\geq 0} Q_{2,e-o}(n)q^n.$$

Similarly,

$$\sum_{n\geq 0} pod_{e-o}(n)q^n = \frac{(q;q^2)_{\infty}}{(-q^2;q^2)_{\infty}} = (q;q^2)_{\infty}(q^2;q^4)_{\infty} = \sum_{n\geq 0} Q_{0,e-o}(n)q^n.$$

For the middle equality, we used Euler's identity $\frac{1}{(q;q^2)_{\infty}} = (-q;q)_{\infty}$ with q replaced by q^2 .

We will give combinatorial proofs for the identities in Theorem 2.1 at the end of the next section.

3. Involutions and bijections for partition identities

To prove combinatorially the theorems stated in the introduction, we make use of several involutions and bijections from the literature. In this section, we survey these transformations. The descriptions of most of these mappings are fairly involved and we do not reproduce them here. To help the reader keep track of the different involutions and bijections, we adopt the notation φ_* for the transformation due to authors with initials *. When defining an involution, there will be an exceptional set of partitions on which the involution is not defined. If the domain of the involution is (most of) the set of partitions $\mathcal{A}(n)$, we denote the exceptional set by $\mathcal{E}_{\mathcal{A}}(n)$. To prove identities in which a partition λ is counted with weight $(-1)^{\ell(\lambda)}$, the involutions described below reverse the parity of the length of partitions. We refer to them as sign reversing involutions.

1. The Kolitsch-Kolitsch transformation φ_K for the combinatorial proof of

(3)
$$(q^4; q^4)_{\infty} (q^3; q^4)_{\infty} (q; q^4)_{\infty} = \sum_{n=0}^{\infty} (-1)^{T_n} q^{T_n}$$

This is a special case of Jacobi's triple product. In the literature, there are many combinatorial proofs of the general form of the Jacobi triple product. We found the proof by Kolitsch-Kolitsch [15] to be most intuitive.

Recall that $\mathcal{Q}_2(n)$ is the set of distinct partitions of n with parts not congruent to 2 mod 4. Let $\mathcal{E}_{\mathcal{Q}_2}(n)$ be the subset of $\mathcal{Q}_2(n)$ defined by

$$\mathcal{E}_{Q_2}(n) = \begin{cases} \{(4i-1,4(i-1)-1,\ldots,7,3)\} & \text{if } n = 2i^2 + i \text{ for some } i > 0, \\ \{(4(i-1)+1,4(i-2)+1\ldots,5,1)\} & \text{if } n = 2i^2 - i \text{ for some } i > 0, \\ \{\emptyset\} & \text{if } n = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that above $\{\emptyset\}$ means that the given set contains the empty partition.

If $n = 2i^2 \pm i$ for some nonnegative integer *i*, the partition in $\mathcal{E}_{Q_2}(n)$ has length *i*. Moreover, $2i^2 + i = T_{2i} \equiv i \mod 2$ and $2i^2 - i = T_{2i-1} \equiv i \mod 2$.

Then, if n > 0, the transformation φ_K defined by L. W. Kolitsch and S. Kolitsch in [15] (with r = 4 and s = 1), is a sign reversing involution on $\mathcal{Q}_2(n) \setminus \mathcal{E}_{\mathcal{Q}_2}(n)$. It proves combinatorially that

$$Q_{2,e-o}(n) = \begin{cases} (-1)^{T_i} & \text{if } n = T_i \text{ for some } i \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

2. Franklin's transformation φ_F for the combinatorial proof of Euler's pentagonal number theorem.

Let $a(i) := (3i^2 + i)/2$, $i \in \mathbb{Z}$. We denote by $\mathcal{Q}(n)$ the set of distinct partitions of n and let $\mathcal{E}_{\mathcal{Q}}(n)$ be the subset of $\mathcal{Q}(n)$ defined by

$$\mathcal{E}_{\mathcal{Q}}(n) = \begin{cases} \{\pi_i^+ := (2i, 2i - 1, \dots, i + 1)\} & \text{if } n = a(i) \text{ for some } i > 0, \\ \{\pi_i^- := (2i - 1, 2i - 2, \dots, i)\} & \text{if } n = a(-i) \text{ for some } i > 0, \\ \{\pi_0^- := \{\emptyset\}\} & \text{if } n = a(0) = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

We refer to π_i^+ and π_i^- as pentagonal partitions and notice that each has *i* parts. Then, the transformation φ_F defined by Franklin (see for example [1]) is a sign reversing involution on $\mathcal{Q}(n) \setminus \mathcal{E}_{\mathcal{Q}}(n)$. It proves combinatorially that

$$Q_{e-o}(n) = \begin{cases} (-1)^i & \text{if } n = a(i) \text{ for some } i \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

3. If n > 0, the Bressoud-Zeilberger transformation φ_{BZ} defined in [9] is an involution on $\bigcup_{i=-\infty}^{\infty} \mathcal{P}(n-a(i))$ that reverses the parity of *i*. It proves combinatorially that

$$\sum_{i=-\infty}^{\infty} (-1)^{i} p(n-a(i)) = \begin{cases} 0 & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

To ease notation, when $\lambda \vdash n - a(i)$, we write $\varphi_{BZ}(\pi, \lambda)$, with π the pentagonal partition of size a(i), instead of $\varphi_{BZ}(\lambda)$.

4. The Ballantine-Merca transformation defined in [6] is a bijection

$$\varphi_{BM}: \bigcup_{k\geq 0} \mathcal{P}(n-T_k) \to \mathcal{QQ}(n),$$

where $\mathcal{QQ}(n) = \{(\lambda, \mu) \vdash n \mid \lambda, \mu \in \mathcal{Q}\}$ is the set of distinct partitions in two colors.

Next we make note of two sign reversing involutions defined by Andrews in [4] to prove Gauss' theta identities.

5. The Andrews transformation φ_{A_1} for the combinatorial proof of Gauss' theta identity

$$\frac{(q^2;q^2)_{\infty}}{(-q;q^2)_{\infty}} = \sum_{n=0}^{\infty} (-1)^{T_n} q^{T_n}.$$

The left hand side of the above identity is the generating function for $ped_{e-o}(n)$. Let $\mathcal{E}_{\mathcal{PED}}(n)$ be the subset of $\mathcal{PED}(n)$ defined by

$$\mathcal{E}_{\mathcal{PED}}(n) = \begin{cases} \{((2i+1)^i)\} & \text{if } n = 2i^2 + i \text{ for some } i > 0, \\ \{((2i-1)^i)\} & \text{if } n = 2i^2 - i \text{ for some } i > 0, \\ \{\emptyset\} & \text{if } n = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then the transformation φ_{A_1} defined by Andrews in [4] is a sign reversing involution on $\mathcal{PED}(n) \setminus \mathcal{E}_{\mathcal{PED}}(n)$. It proves combinatorially that

$$ped_{e-o}(n) = \begin{cases} (-1)^{T_i} & \text{if } n = T_i \text{ for some } i \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

6. The Andrews transformation φ_{A_2} for the combinatorial proof of Gauss' second theta identity

$$\frac{(q;q)_{\infty}}{(-q;q)_{\infty}} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}.$$

Let $\overline{\mathcal{P}}(n)$ be the set of overpartitions of n. Then the left hand side of the above identity is the generating function for $\overline{p}_{e-o}(n)$. Let $\mathcal{E}_{\overline{\mathcal{P}}}(n)$ be the subset of $\overline{\mathcal{P}}(n)$ defined by

$$\mathcal{E}_{\overline{\mathcal{P}}}(n) = \begin{cases} \{(\overline{m}, m, m, \dots, m), (m, m, m, \dots, m)\} & \text{ if } n = m^2 \text{ for some } m > 0, \\ \{\emptyset\} & \text{ if } n = 0, \\ \emptyset & \text{ otherwise.} \end{cases}$$

Then the transformation φ_{A_2} defined by Andrews in [4] is a sign reversing involution on $\overline{\mathcal{P}}(n) \setminus \mathcal{E}_{\overline{\mathcal{P}}}(n)$. It proves combinatorially that

$$\overline{p}_{e-o}(n) = \begin{cases} 2(-1)^m & \text{ if } n = m^2 \text{ for some } m \geq 0, \\ 1 & \text{ if } n = 0, \\ 0 & \text{ otherwise.} \end{cases}$$

Remark 1. In fact, the transformation φ_{A_2} also reverses the parity of the number of overlined parts in overpartitions.

7. Gupta's transformation φ_G for the combinatorial proof of

$$\frac{1}{(-q^2;q^2)_{\infty}(q;q^2)_{\infty}} = (-q;q^2)_{\infty}.$$

Let $\mathcal{Q}_{odd}(n)$ be the set of partitions of n with all parts odd and distinct. Gupta [14] defined an involution φ_G on $\mathcal{P}(n) \setminus \mathcal{Q}_{odd}(n)$ that reverses the parity of the number of even parts. It shows combinatorially that

$$p_e(n,2) - p_o(n,2) = q_{odd}(n),$$

where $p_{e/o}(n,2)$ is the number of partitions of n with an even (respectively odd) number of even parts.

Gupta's transformation also shows combinatorially that

$$p_e(n) - p_o(n) = (-1)^n q_{odd}(n)$$

where $p_{e/o}(n)$ is the number of partitions of n with an even (respectively odd) number of parts.

8. Glaisher's bijection (see for example [5, Section 2.3]) $\varphi_{Gl} : \mathcal{O}(n) \to \mathcal{Q}(n)$ for the combinatorial proof of Euler's identity

$$\frac{1}{(q;q^2)_{\infty}} = (-q;q)_{\infty}.$$

We already alluded to this transformation in the previous section. 9. Finally we prove combinatorially that

$$(-q;q)_{\infty}(q;q)_{\infty} = (q^2;q^2)_{\infty}.$$

Let $\mathcal{Q}_{even}(n)$ be the set of partitions of n with all parts even and distinct. Recall that $\mathcal{QQ}(n) = \{(\lambda, \mu) \vdash n \mid \lambda, \mu \in \mathcal{Q}\}$. We have the following combinatorial interpretation of the above identity.

Proposition 2. For $n \ge 0$ we have

$$|\{(\lambda,\mu) \in \mathcal{QQ}(n) \mid \ell(\lambda) \ even\}| - |\{(\lambda,\mu) \in \mathcal{QQ}(n) \mid \ell(\lambda) \ odd\}| = Q_{even,e-o}(n).$$

The statement of this proposition can also be interpreted as the excess in the number of distinct partitions in two colors, red and blue, with an even number of red parts over the number of distinct partitions in two colors with an odd number of red parts equals the excess in the number of partitions into an even number of distinct even parts over the number of partitions into an odd number of distinct even parts.

Proof. Let $\mathcal{E}_{QQ}(n)$ be the subset of QQ(n) defined by

$$\mathcal{E}_{\mathcal{Q}\mathcal{Q}}(n) = \{ (\lambda, \mu) \in \mathcal{Q}\mathcal{Q}(n) \mid \lambda = \mu \}.$$

We define a transformation ζ on $\mathcal{QQ}(n) \setminus \mathcal{E}_{\mathcal{QQ}}(n)$ as follows. If $(\lambda, \mu) \in \mathcal{QQ}(n)$ with $\ell(\lambda) \not\equiv \ell(\mu) \mod 2$, let $\zeta(\lambda, \mu) = \zeta(\mu, \lambda)$. If $\ell(\lambda) \equiv \ell(\mu) \mod 2$ and $\lambda \neq \mu$, let *i* be the smallest positive integer such that $\lambda_i \neq \mu_i$.

(i) If $\lambda_i > \mu_i$, let $\zeta(\lambda, \mu) = (\lambda \setminus (\lambda_i), \mu \cup (\lambda_i))$.

(ii) If $\lambda_i < \mu_i$, let $\zeta(\lambda, \mu) = (\lambda \cup (\mu_i), \mu \setminus (\mu_i))$.

Then, ζ is an involution on $\mathcal{QQ}(n) \setminus \mathcal{E}_{\mathcal{QQ}}(n)$ mapping pairs from case (i) to pairs from case (ii) and vice-versa. Moreover, $\zeta(\lambda, \mu)$ reverses the parity of the length of λ . Thus,

$$\begin{split} |\{(\lambda,\mu) \in \mathcal{QQ}(n) \mid \ell(\lambda) \text{ even}\}| &- |\{(\lambda,\mu) \in \mathcal{QQ}(n) \mid \ell(\lambda) \text{ odd}\}| \\ &= |\{(\lambda,\lambda) \in \mathcal{QQ}(n) \mid \ell(\lambda) \text{ even}\}| - |\{(\lambda,\lambda) \in \mathcal{QQ}(n) \mid \ell(\lambda) \text{ odd}\}|. \end{split}$$

Mapping (λ, λ) to 2λ completes the proof.

For the rest of the article, whenever we refer to the zeta transformation, we mean the transformation of Proposition 2.

Remark 2. If in the proof above, the zeta transformation is applied to pairs (λ, μ) with $\lambda, \mu \in \mathcal{Q}_{odd}$, we obtain a combinatorial proof for

$$(q;q^2)_{\infty}(-q;q^2)_{\infty} = (q^2;q^4)_{\infty}.$$

To simplify explanations, for the remainder of the article, when we write that an involution is applied to a given set of partitions, it is implied that the exceptional set has been removed.

We end this section by noting that using the involutions φ_K and φ_{A_1} , we obtain a combinatorial proof of the first identity in Theorem 2.1.

To prove the second identity in Theorem 2.1, i.e., $pod_{e-o}(n) = Q_{0,e-o}(n)$ for $n \ge 0$, we use Gupta's involution φ_G . First note that, if $\lambda = (\lambda^e, \lambda^o)$, then $\ell(\lambda) \equiv \ell(\lambda^e) + n \mod 2$. Fix a partition λ^o with distinct odd parts and size at most n. Consider the subset of $\mathcal{POD}(n)$ consisting of partitions whose odd parts are precisely the parts of λ^o , i.e.,

$$\mathcal{POD}_{\lambda^o}(n) = \{ (\lambda^e, \lambda^o) \vdash n \mid \lambda^e \text{ has even parts} \}.$$

Then, $\lambda_{/2}^e$ is any partition in $\mathcal{P}((n - |\lambda^o|)/2)$. The transformation $(\lambda^e, \lambda^o) \rightarrow (2\varphi_G(\lambda_{/2}^e), \lambda^o)$ is a sign reversing involution on

$$\mathcal{POD}_{\lambda^o}(n) \setminus \{ (\lambda^e, \lambda^o) \in \mathcal{POD}_{\lambda^o}(n) \mid \lambda^e \text{ distinct parts} \equiv 2 \mod 4 \}.$$

Observing that $\mathcal{POD}(n) = \bigcup_{\lambda^o \in \mathcal{O}} \mathcal{POD}_{\lambda^o}(n)$, it follows that $pod_{e-o}(n) = Q_{0,e-o}(n)$.

4. Combinatorial proof of Theorem 1.1

In this section we give two combinatorial proofs of Theorem 1.1 which states that for $n \ge 0$,

$$\sum_{j=-\infty}^{\infty} (-1)^j ped(n-j(3j+1)/2) = \begin{cases} (-1)^k & \text{if } n = 2k(3k+1) \text{ for some } k \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

First proof. Let $P_{<4}(n)$ be the number of partitions of n with parts occurring at most thrice. In Section 2 we showed combinatorially that $ped(n) = b_4(n)$. A well known generalization of Glaisher's bijection [13] shows that $b_4(n) = P_{<4}(n)$. In fact, in [11] the theorem is stated in terms of $P_{<4}(n)$.

Let $\mathcal{QP}_{<4}(n) = \{(\lambda, \mu) \vdash n \mid \lambda \in \mathcal{P}_{<4}, \mu \in \mathcal{Q}\}$. The transformation $(\lambda, \mu) \rightarrow (\lambda, \varphi_F(\mu))$ on $\mathcal{QP}_{<4}(n)$ shows that

(4)
$$\sum_{j=-\infty}^{\infty} (-1)^{j} ped(n-j(3j+1)/2)$$

is the generating function for

(5)
$$|\{(\lambda,\mu) \in \mathcal{QP}_{<4}(n) \mid \ell(\mu) \text{ even}\}| - |\{(\lambda,\mu) \in \mathcal{QP}_{<4}(n) \mid \ell(\mu) \text{ odd}\}|.$$

Let $(\lambda, \mu) \in \mathcal{QP}_{<4}(n)$ and define $\alpha = \lambda \cup \mu$. Then α has parts occurring at most four times. We examine the pairs $(\lambda, \mu) \in \mathcal{QP}_{<4}(n)$ that result in the same partition α . Let γ be the partition into distinct parts that consists of one copy of each part of α that occurs exactly four times. Then all parts of γ appear in $\alpha \setminus \gamma$ exactly three times. Let ν be the partition into distinct parts that consists of one copy of each part of $\alpha \setminus \gamma$ that is not a part of γ . For each subset β of ν , the pair $(\alpha \setminus (\gamma \cup \beta), \gamma \cup \beta)$ is in $\mathcal{QP}_{<4}(n)$ and results in α . If $\nu \neq \emptyset$, it is easily seen that the number of subsets of ν with even cardinality is equal to the number of subsets of ν with odd cardinality (fix part a of ν and map a subset β of ν with $a \notin \beta$ to $\beta \cup (a)$). Thus, if $\nu \neq \emptyset$, the set of pairs $(\alpha \setminus (\gamma \cup \beta), \gamma \cup \beta)$ with β a subset of ν contributes zero to (5). If $\nu = \emptyset$, then each part of α has multiplicity 4. Then there is only one pair in $(\lambda, \mu) \in \mathcal{QP}_{<4}(n)$ such that $\alpha = \lambda \cup \mu$, namely $(^{3}\mu, \mu)$, where $^{3}\mu$ is the partition whose parts are the parts of μ each with multiplicity 3. Thus (5) equals

$$|\{({}^{3}\mu,\mu) \mid \mu \in \mathcal{Q}, \ell(\mu) \text{ even}\}| - |\{({}^{3}\mu,\mu) \mid \mu \in \mathcal{Q}, \ell(\mu) \text{ odd}\}|.$$

Finally, using the transformation $({}^{3}\mu, \mu) \rightarrow ({}^{3}\varphi_{F}(\mu), \varphi_{F}(\mu))$ shows that the only partition that contributes to (5) is $({}^{3}\mu, \mu)$ with μ a pentagonal partition. This completes the proof.

Second proof. Let $\mathcal{QPED}(n) = \{(\lambda, \mu) \vdash n \mid \lambda \in \mathcal{PED}, \mu \in Q\}$. The transformation $(\lambda, \mu) \to (\lambda, \varphi_F(\mu))$ shows that (4) is the generating function for

(6)
$$|\{(\lambda,\mu) \in \mathcal{QPED}(n) \mid \ell(\mu) \text{ even}\}| - |\{(\lambda,\mu) \in \mathcal{QPED}(n) \mid \ell(\mu) \text{ odd}\}|$$

Let $(\lambda, \mu) = (\lambda^e, \lambda^o, \mu^e, \mu^o) \in QPED(n)$. The zeta transformation applied to (λ^e, μ^e) shows that (6) equals

$$\begin{aligned} |\{(\lambda^{e}, \lambda^{o}, \lambda^{e}, \mu^{o}) \in \mathcal{QPED}(n) \mid \ell(\lambda^{e}) + \ell(\mu^{o}) \text{ even}\}| \\ &- |\{(\lambda^{e}, \lambda^{o}, \lambda^{e}, \mu^{o}) \in \mathcal{QPED}(n) \mid \ell(\lambda^{e}) + \ell(\mu^{o}) \text{ odd}\}|. \end{aligned}$$

Let $\eta = \lambda^e \cup \mu^o$, identify $(\lambda^e, \lambda^o, \lambda^e, \mu^o)$ with $(\lambda^e, \lambda^o, \eta)$, and map λ^o to $\nu = \varphi_{Gl}(\lambda^o) \in \mathcal{Q}$. Then (6) equals

$$\begin{aligned} |\{(\lambda^{e},\nu,\eta)\vdash n\mid \lambda^{e}\in\mathcal{Q}_{even},\,\nu,\eta\in\mathcal{Q},\ell(\eta)\text{ even}\}|\\ -\left|\{(\lambda^{e},\nu,\eta)\vdash n\mid \lambda^{e}\in\mathcal{Q}_{even},\,\nu,\eta\in\mathcal{Q},\ell(\eta)\text{ odd}\}\right|\end{aligned}$$

We apply the zeta transformation to (ν, η) and proceed as in Proposition 2 to see that (6) equals

$$|\{(\lambda^{e}, 2\eta) \vdash n \mid \lambda^{e}, 2\eta \in \mathcal{Q}_{even}, \ell(\eta) \text{ even}\}| - |\{(\lambda^{e}, 2\eta) \vdash n \mid \lambda^{e}, 2\eta \in \mathcal{Q}_{even}, \ell(\eta) \text{ odd}\}|.$$

Next, we apply ζ to $(\lambda^e, 2\eta)$ and proceed as in Proposition 2 to see that (6) equals

$$|\{4\eta \vdash n \mid \eta \in \mathcal{Q}, \ell(\eta) \text{ even}\}| - |\{4\eta \vdash n \mid \eta \in \mathcal{Q}, \ell(\eta) \text{ odd}\}|.$$

Finally applying the transformation $4\eta \to 4\varphi_F(\eta)$ concludes the proof.

5. Combinatorial proof of Theorem 1.2

The goal of this section is to prove combinatorially that for $n \ge 0$,

$$\sum_{j\geq 0} (-1)^{T_j} ped(n-T_j) = \begin{cases} 1 & \text{if } n = 2T_k \text{ for some } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We begin by establishing two helpful results. First, we provide a combinatorial proof for

(7)
$$(-q;q)_{\infty}(-q;q)_{\infty}(q;q)_{\infty} = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

To state the combinatorial interpretation of (7) let $\mathcal{D}_3(n)$ be the set of distinct partitions in three colors, i.e.,

$$\mathcal{D}_3(n) = \{ (\lambda, \mu, \eta) \vdash n \mid \lambda, \mu, \eta \in \mathcal{Q} \}.$$

and define

$$D'_{3,e-o}(n) := |\{(\lambda,\mu,\eta) \in \mathcal{D}_3(n) \mid \ell(\lambda) \text{ even}\}| - |\{(\lambda,\mu,\eta) \in \mathcal{D}_3(n) \mid \ell(\lambda) \text{ odd}\}|.$$

Proposition 3. For $n \ge 0$ we have

$$D'_{3,e-o}(n) = \begin{cases} 1 & \text{if } n = T_k \text{ for some } k \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\delta_k = (k, k - 1, k - 2, ..., 2, 1)$ be the staircase partition of length k and size T_k . The transformation that maps $(\lambda, \mu, \eta) \in \mathcal{D}_3(n)$ to $(\lambda, \varphi_{BM}^{-1}(\mu, \eta), \delta_k)$ is a bijection between $\mathcal{D}_3(n)$ and $\mathcal{QP}(n)$, where

$$\mathcal{QP}(n) = \{ (\lambda, \gamma, \delta_k) \vdash n \mid \lambda \in \mathcal{Q}, \gamma \in \mathcal{P}, k \in \mathbb{Z}_{\geq 0} \}.$$

Denote by $\mathcal{E}_1(n)$ the following disjoint union of subsets of $\mathcal{QP}(n)$

$$\mathcal{E}_1(n) = \bigcup_{m,k \ge 0} \{ (\pi, \gamma, \delta_k) \vdash m \mid \pi \in \mathcal{E}_Q, \gamma \in \mathcal{P} \}.$$

Note that π is a pentagonal partition, $\mathcal{E}_{\mathcal{Q}}$ is the exceptional set for φ_F , and γ is an unrestricted partition.

The transformation that maps $(\lambda, \gamma, \delta_k) \in \mathcal{QP}(n)$ to $(\varphi_F(\lambda), \gamma, \delta_k)$ is an involution on $\mathcal{QP}(n) \setminus \mathcal{E}_1(n)$ that reverses the parity of $\ell(\lambda)$. Thus

$$D'_{3,e-o}(n) = E_{1,e-o}(n),$$

where $E_{1,e-o}(n) = |\{(\pi,\gamma,\delta_k) \in \mathcal{E}_1(n) \mid \ell(\pi) \text{ even}\}| - |\{(\pi,\gamma,\delta_k) \in \mathcal{E}_1(n) \mid \ell(\pi) \text{ odd}\}|.$

Let $\mathcal{E}_2(n)$ be the subset of $\mathcal{E}_1(n)$ defined by

$$\mathcal{E}_2(n) = \begin{cases} \{(\emptyset, \emptyset, \delta_k)\}, & \text{if } n = T_k \text{ for some } k \ge 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

The transformation that maps $(\pi, \gamma, \delta_k) \in \mathcal{E}_1(n)$ to $(\varphi_{BZ}(\pi, \gamma), \delta_k)$ is an involution of $\mathcal{E}_1(n) \setminus \mathcal{E}_2(n)$ that reverses the parity of $\ell(\pi)$. Hence

$$D'_{3,e-o}(n) = |\mathcal{E}_2(n)| = \begin{cases} 1 & \text{if } n = T_k \text{ for some } k \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 5.1. For $n \ge 0$ we have

$$DE'_{3,e-o}(n) = \begin{cases} 1 & \text{if } n = 2T_k \text{ for some } k \ge 0, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$DE'_{3,e-o}(n) = |\{(\lambda,\mu,\nu) \vdash n \mid \lambda,\mu,\nu \in \mathcal{Q}_{even}, \ell(\lambda) even\}| - |\{(\lambda,\mu,\nu) \vdash n \mid \lambda,\mu,\nu \in \mathcal{Q}_{even}, \ell(\lambda) odd\}|.$$

Proof. The proof is obtained by doubling each part in all partitions involved in the proof of Proposition 3. \Box

The next result is analogous to Proposition 2. Its combinatorial proof uses similar ideas to the zeta transformation but it is much more involved. Recall that $\mathcal{O}(n)$ is the set of odd partitions of n. Let $\mathcal{OO}(n) = \{(\lambda, \mu) \vdash n \mid \lambda, \mu \in \mathcal{O}\}.$

Proposition 4. For $n \ge 0$ we have

$$(8) \quad |\{(\lambda,\mu) \in \mathcal{OO}(n) \mid \ell(\lambda) \ even\}| - |\{(\lambda,\mu) \in \mathcal{OO}(n) \mid \ell(\lambda) \ odd\}| = Q_{even}(n).$$

The statement of this proposition can also be interpreted as the excess in the number of odd partitions in two colors, red and blue, with an even number of red parts over the number of odd partitions in two colors with an odd number of red parts equals the number of partitions into distinct even parts. In terms of generating functions, we obtain a combinatorial proof for

$$\frac{1}{(-q;q^2)_{\infty}(q;q^2)_{\infty}} = (-q^2;q^2)_{\infty}$$

Proof. Let $\mathcal{E}_{\mathcal{OO}}(n)$ be the subset of $\mathcal{OO}(n)$ defined by

 $\mathcal{E}_{\mathcal{OO}}(n) = \{ (\lambda, \mu) \in \mathcal{OO}(n) \mid \lambda = \emptyset, \text{ parts of } \mu \text{ have even multiplicity} \}.$

We will construct an involution ψ on: $\mathcal{OO}(n) \setminus \mathcal{E}_{\mathcal{OO}}(n)$ that reverses the parity of $\ell(\lambda)$.

If $(\lambda, \mu) \in \mathcal{OO}(n)$ with $\ell(\lambda) \not\equiv \ell(\mu) \mod 2$, let $\psi(\lambda, \mu) = (\mu, \lambda)$.

Next, suppose that $(\lambda, \mu) \in \mathcal{OO}(n) \setminus \mathcal{E}_{\mathcal{OO}}(n)$ with $\ell(\lambda) \equiv \ell(\mu) \mod 2$. We define the following subsets of $\mathcal{OO}(n)$:

$$\mathcal{O}_{e,e}(n) = \{ (\lambda, \mu) \in \mathcal{OO}(n) \mid \ell(\lambda), \ell(\mu) \text{ even} \}$$

and

$$\mathcal{O}_{o,o}(n) = \{ (\lambda, \mu) \in \mathcal{OO}(n) \mid \ell(\lambda), \ell(\mu) \text{ odd} \}$$

Thus, $\mathcal{E}_{\mathcal{OO}}(n) \subseteq \mathcal{O}_{e,e}(n)$ and $(\lambda, \mu) \in \mathcal{O}_{o,o}(n) \cup (\mathcal{O}_{e,e}(n) \setminus \mathcal{E}_{\mathcal{OO}}(n))$.

Before we define $\psi(\lambda, \mu)$, we introduce some necessary notation. We denote by λ_s , respectively μ_s , the smallest part of λ , respectively μ . If $\lambda = \emptyset$ (or $\mu = \emptyset$), we define $\lambda_s = \infty$ (or $\mu_s = \infty$). Let $\mu^<$ be the partition consisting of the parts of μ that are less than λ_s . Then $\mu \setminus \mu^<$ is the partition consisting of the parts of μ larger or equal to λ_s . Similarly, let $\lambda^<$ be the partition consisting of the parts of λ that are less than μ_s .

We write $\mathcal{O}_{o,o}(n) = \mathcal{A}_o \sqcup \mathcal{B}_o \sqcup \mathcal{C}_o \sqcup \mathcal{D}_o$, where

$$\mathcal{A}_{o} = \{ (\lambda, \mu) \in \mathcal{O}_{o,o}(n) \mid \lambda_{s} \geq \mu_{s} \}, \\ \mathcal{B}_{o} = \{ (\lambda, \mu) \in \mathcal{O}_{o,o}(n) \mid \lambda_{s} < \mu_{s}, \ell(\lambda^{<}) \text{ even} \}, \\ \mathcal{C}_{o} = \{ (\lambda, \mu) \in \mathcal{O}_{o,o}(n) \mid \lambda_{s} < \mu_{s}, \ell(\lambda^{<}) \text{ odd}, \end{cases}$$

only the largest part of λ has odd multiplicity},

$$\mathcal{D}_o = \{ (\lambda, \mu) \in \mathcal{O}_{o,o}(n) \mid \lambda_s < \mu_s, \ell(\lambda^{<}) \text{ odd}$$

some part other than the largest part of λ has odd multiplicity}.

Similarly, we write $\mathcal{O}_{e,e}(n) \setminus \mathcal{E}_{\mathcal{OO}}(n) = \mathcal{A}_e \sqcup \mathcal{B}_e \sqcup \mathcal{C}_e \sqcup \mathcal{D}_e$, where

$$\mathcal{A}_{e} = \{ (\lambda, \mu) \in \mathcal{O}_{e,e}(n) \mid \lambda \neq \emptyset, \lambda_{s} \leq \mu_{s} \}, \\ \mathcal{B}_{e} = \{ (\lambda, \mu) \in \mathcal{O}_{e,e}(n) \mid \lambda \neq \emptyset, \lambda_{s} > \mu_{s}, \ell(\mu^{<}) \text{ even} \}, \\ \mathcal{C}_{e} = \{ (\emptyset, \mu) \in \mathcal{O}_{e,e}(n) \mid \text{not all multiplicities in } \mu \text{ are even} \}, \\ \mathcal{D}_{e} = \{ (\lambda, \mu) \in \mathcal{O}_{e,e}(n) \mid \lambda \neq \emptyset, \lambda_{s} > \mu_{s}, \ell(\mu^{<}) \text{ odd} \}.$$

We now define bijections $\psi_{\mathcal{I}} : \mathcal{I}_o \to \mathcal{I}_e$, where $\mathcal{I} \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$. Examples for each mapping are provided following the proof of Theorem 1.2.

Define $\psi_{\mathcal{A}} : \mathcal{A}_o \to \mathcal{A}_e$ by $\psi_{\mathcal{A}}(\lambda, \mu) = (\lambda \cup (\mu_s), \mu \setminus (\mu_s))$. Then the inverse, $\psi_{\mathcal{A}}^{-1} : \mathcal{A}_e \to \mathcal{A}_o$, is given by $\psi_{\mathcal{A}}^{-1}(\lambda, \mu) = (\lambda \setminus (\lambda_s), \mu \cup (\lambda_s))$. Define $\psi_{\mathcal{B}} : \mathcal{B}_o \to \mathcal{B}_e$ by $\psi_{\mathcal{B}}(\lambda, \mu) = (\lambda \setminus \lambda^{<} \cup (\mu_s), \mu \setminus (\mu_s) \cup \lambda^{<})$. Note $\lambda \setminus \lambda^{<} \neq \emptyset$ because $\ell(\lambda)$ is odd but $\ell(\lambda^{<})$ is even. The inverse $\psi_{\mathcal{B}}^{-1} : \mathcal{B}_e \to \mathcal{B}_o$ is given by $\psi_{\mathcal{B}}^{-1}(\lambda, \mu) = (\lambda \setminus (\lambda_s) \cup \mu^{<}, \mu \setminus \mu^{<} \cup (\lambda_s))$.

If $(\lambda, \mu) \in C_e \sqcup D_e$, then $\ell(\mu)$ is even and not all the multiplicities of parts in μ are even, so there must be at least two odd multiplicities. Let μ_i be the smallest part of μ such that the number of parts of μ smaller than μ_i is odd. Denote by $\mu^{<<}$ the partition whose parts are the parts of μ smaller than μ_i . Similarly, if $(\lambda, \mu) \in D_o$, then there exists some part of λ with odd multiplicity that is not the largest part, so we can define λ_i and $\lambda^{<<}$. Note that by definition $\lambda^{<<} \neq \lambda$.

Define $\psi_{\mathcal{C}} : \mathcal{C}_o \to \mathcal{C}_e$ by $\psi_{\mathcal{C}}(\lambda, \mu) = (\emptyset, \mu \cup \lambda)$. Note that only the largest part of λ has odd multiplicity, so $\ell(\lambda^{<})$ being odd implies that $\lambda^{<} = \lambda$, which means μ_s is larger than all the parts of λ . This also shows that $\mu \cup \lambda$ cannot have all parts with even multiplicity. The inverse $\psi_{\mathcal{C}}^{-1} : \mathcal{C}_e \to \mathcal{C}_o$ is given by $\psi_{\mathcal{C}}^{-1}(\emptyset, \mu) = (\mu^{<<}, \mu \setminus \mu^{<<})$.

even multiplicity. The inverse $\psi_{\mathcal{C}}^{-1} : \mathcal{C}_e \to \mathcal{C}_o$ is given by $\psi_{\mathcal{C}}^{-1}(\emptyset, \mu) = (\mu^{<<}, \mu \setminus \mu^{<<})$. Define $\psi_{\mathcal{D}} : \mathcal{D}_o \to \mathcal{D}_e$ by $\psi_{\mathcal{D}}(\lambda, \mu) = (\lambda \setminus \lambda^{<<}, \mu \cup \lambda^{<<})$. Note that $\lambda_i \leq \mu_s$ as $\ell(\lambda^{<})$ is odd and λ_i is minimal. Recall, $\lambda \setminus \lambda^{<<} \neq \emptyset$ as $\lambda_i \in \lambda \setminus \lambda^{<<}$. The inverse $\psi_{\mathcal{D}}^{-1} : \mathcal{D}_e \to \mathcal{D}_o$ is given by $\psi_{\mathcal{D}}^{-1}(\lambda, \mu) = (\lambda \cup \mu^{<<}, \mu \setminus \mu^{<<})$.

Each of these mappings reverses the parity of $\ell(\lambda)$. We define ψ on $\mathcal{O}_{o,o}(n) \cup (\mathcal{O}_{e,e}(n) \setminus \mathcal{E}_{\mathcal{OO}}(n))$ by the corresponding $\psi_{\mathcal{I}}$ or $\psi_{\mathcal{I}}^{-1}$, where $\mathcal{I} \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$.

This shows that that the left hand side of (8) equals $|\mathcal{E}_{\mathcal{OO}}(n)|$. To finish the proof, we construct a bijection $\psi_{\mathcal{E}} : \mathcal{E}_{\mathcal{OO}}(n) \to \mathcal{Q}_{even}(n)$.

Given $(\emptyset, \mu) \in \mathcal{E}_{\mathcal{OO}}(n)$, let $\widetilde{\mu}$ be the partition consisting of the different parts of μ each with half its multiplicity in μ . We use Glaisher's transformation on $\widetilde{\mu}$ to obtain a partition with distinct parts. Then $2\varphi_{Gl}(\widetilde{\mu}) \in \mathcal{Q}_{even}(n)$ and the transformation $(\emptyset, \mu) \to 2\varphi_{Gl}(\widetilde{\mu})$ gives the desired bijection.

To complete the combinatorial proof of Theorem 1.2, we need to show combinatorially that

(9)
$$DE'_{3,e-o}(n) = \sum_{j\geq 0} (-1)^{T_j} ped(n-T_j).$$

Denote by $\mathcal{PPED}(n)$ the set

$$\mathcal{PPED}(n) = \{ (\lambda, \mu) \vdash n \mid \lambda, \mu \in \mathcal{PED} \}.$$

Using the transformation φ_{A_1} on λ , the right hand side of (9) is equal to

$$|\{(\lambda,\mu)\in \mathcal{PPED}(n)\mid \ell(\lambda) \text{ even}\}|-|\{(\lambda,\mu)\in \mathcal{PPED}(n)\mid \ell(\lambda) \text{ odd}\}|.$$

Given $(\lambda, \mu) \in \mathcal{PPED}(n)$, we write $(\lambda, \mu) = (\lambda^e, \lambda^o, \mu^e, \mu^o)$ as usual. Fix a partition λ^e with distinct even parts and size at most n. Consider the subset of $\mathcal{PPED}(n)$ such that the even parts of the first partition are precisely the parts of λ^e , i.e.,

$$\mathcal{PPED}_{\lambda^{e}}(n) = \{ (\lambda^{e}, \lambda^{o}, \mu^{e}, \mu^{o}) \vdash n \mid \mu^{e} \in \mathcal{Q}_{even}, \lambda^{o}, \mu^{o} \in \mathcal{O} \}.$$

Then, applying the involution ψ to (λ^o, μ^o) and proceeding as in the proof of Proposition 4, we see that

$$\begin{aligned} |\{(\lambda^{e}, \lambda^{o}, \mu^{e}, \mu^{o}) \in \mathcal{PPED}_{\lambda^{e}}(n) \mid \ell(\lambda^{o}) \text{ even}\}| \\ &- |\{(\lambda^{e}, \lambda^{o}, \mu^{e}, \mu^{o}) \in \mathcal{PPED}_{\lambda^{e}}(n) \mid \ell(\lambda^{o}) \text{ odd}\}| \\ &= |\{(\lambda^{e}, \mu^{e}, \gamma) \vdash n \mid \mu^{e}, \gamma \in \mathcal{Q}_{even}\}| \end{aligned}$$

Summing after all partitions $\lambda^e \in \mathcal{Q}_{even}$, shows that the right hand side of (9) equals $DE'_{3,e-o}(n)$.

We conclude this section with examples illustrating the bijections $\psi_{\mathcal{A}}, \psi_{\mathcal{B}}, \psi_{\mathcal{C}}$, and $\psi_{\mathcal{D}}$. Below, marked boxes in one partition of the pair are moved to the other partition.

Example 2. Let $\lambda = (3, 1, 1), \mu = (5, 3, 1)$. Then $\mu_s \leq \lambda_s$ and $\psi_A(\lambda, \mu) = (\delta, \gamma)$, where $\delta = (3, 1, 1, 1), \gamma = (5, 3)$.

$$(\lambda,\mu)=\left(\begin{array}{c} \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \bullet \end{array}\right) \mapsto \left(\begin{array}{c} \hline \\ \hline \\ \hline \\ \bullet \end{array}\right), \begin{array}{c} \hline \\ \hline \\ \hline \\ \bullet \end{array}\right)=(\delta,\gamma)$$

Example 3. Let $\lambda = (3, 1, 1), \mu = (5, 3, 3)$. Then $\mu_s > \lambda_s$ and $\ell(\lambda^{<}) = 2$ is even. Then $\psi_{\mathcal{B}}(\lambda, \mu) = (\delta, \gamma)$, where $\delta = (3, 3), \gamma = (5, 3, 1, 1)$.

$$(\lambda,\mu) = \left(\fbox{\bullet}, \intercal{\bullet} \right) \mapsto \left(\fbox{\bullet}, \intercal{\bullet} \right) \mapsto \left(\fbox{\bullet}, \intercal{\bullet} \right) = (\delta,\gamma)$$

Example 4. Let $\lambda = (3, 1, 1), \mu = (7, 5, 5)$. Then $\mu_s > \lambda_s, \ell(\lambda^{<}) = 3$ is odd (in fact, $\lambda^{<} = \lambda$), and the only part of λ with odd multiplicity is the first part. Then $\psi_{\mathcal{C}}(\lambda, \mu) = (\delta, \gamma)$, where $\delta = \emptyset, \gamma = (7, 5, 5, 3, 1, 1)$.



Example 5. Let $\lambda = (5, 5, 3, 1, 1), \mu = (7)$. Then $\mu_s > \lambda_s, \ell(\lambda^{<}) = 5$ is odd, and there is a part of λ with odd multiplicity that is not the first part. Then $\lambda_i = 5$ and $\psi_{\mathcal{D}}(\lambda, \mu) = (\delta, \gamma)$, where $\delta = (5, 5), \gamma = (7, 3, 1, 1)$.



For the remainder of the article, the transformation ψ refers to the transformation of Proposition 4.

6. Combinatorial Proof of Theorem 1.3

In this section we prove combinatorially that for $n \ge 0$,

(10)
$$\sum_{j=-\infty}^{\infty} (-1)^{j} ped(n-2j^{2}) = \begin{cases} 1 & \text{if } n = T_{k} \text{ for some } k \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

Doubling parts in each overpartition involved in Andrews proof of Gauss's second theta identity, one obtains a combinatorial proof for

$$\overline{pe}_{e-o}(n) = \begin{cases} 2(-1)^m & \text{if } n = 2m^2 \text{ for some } m \ge 0, \\ 1 & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{pe}(n)$ is the number of overpartitions of n with even parts. Thus, if we define

$$\overline{\mathcal{P}}\mathcal{PED}(n) = \{(\overline{\lambda}, \mu) \vdash n \mid \overline{\lambda} \in \overline{\mathcal{PE}}, \ \mu \in \mathcal{PED}\},\$$

to prove (10), we need to show that

(11)
$$|\{(\overline{\lambda},\mu)\in\overline{\mathcal{PPED}}(n)\mid\ell(\overline{\lambda}) \text{ even}\}|-|\{(\overline{\lambda},\mu)\in\overline{\mathcal{PPED}}(n)\mid\ell(\overline{\lambda}) \text{ odd}\}|$$

equals 1 if n is a triangular number and 0 otherwise.

As before, we write $\mu = (\mu^e, \mu^o)$, and we also write $\overline{\lambda} = (\alpha, \beta)$, where α consists of the overlined parts of $\overline{\lambda}$ and β consists of the nonoverlined parts of $\overline{\lambda}$. Then, we write $(\overline{\lambda}, \mu) \in \overline{\mathcal{PPED}}(n)$ as $(\alpha, \beta, \mu^e, \mu^o)$. We note that the order on the partitions in the quadruple is important. In the quadruple, the parts of the first partition α are not overlined. However, when we create $(\overline{\lambda}, \mu) \in \overline{\mathcal{PPED}}(n)$ from the quadruple, we overline the parts of α . Thus, in $(\alpha, \beta, \mu^e, \mu^o) \in \overline{\mathcal{PPED}}(n)$, partitions α and μ^e have parts even and distinct, β is a partition with even parts (possibly repeated), and μ^o is a partition with odd parts (possibly repeated).

We apply the zeta transformation to (α, μ^e) and proceed as in Proposition 2 to see that

$$\begin{split} |\{(\alpha, \beta, \mu^{e}, \mu^{o}) \in \overline{\mathcal{P}}\mathcal{PED}(n) \mid \ell(\alpha) + \ell(\beta) \text{ even}\}| \\ &- |\{(\alpha, \beta, \mu^{e}, \mu^{o}) \in \overline{\mathcal{P}}\mathcal{PED}(n) \mid \ell(\alpha) + \ell(\beta) \text{ odd}\}| \\ &= |\{(2\alpha, \beta, \mu^{o}) \vdash n \mid \alpha \in \mathcal{Q}_{even}, \beta \text{ even parts}, \mu^{o} \in \mathcal{O}, \ell(\alpha) + \ell(\beta) \text{ even}\}| \\ &- |\{(2\alpha, \beta, \mu^{o}) \vdash n \mid \alpha \in \mathcal{Q}_{even}, \beta \text{ even parts}, \mu^{o} \in \mathcal{O}, \ell(\alpha) + \ell(\beta) \text{ odd}\}|. \end{split}$$

Next, we fix the partition $\alpha \in \mathcal{Q}_{even}$ and apply Gupta's transformation φ_G to $\beta \cup \mu^o$ (which is an arbitrary partition of size $n - 2|\alpha|$) to see that

$$\begin{split} |\{(2\alpha,\beta,\mu^o) \vdash n \mid \beta \text{ even parts}, \mu^o \in \mathcal{O}, \ell(\beta) \text{ even}\}| \\ &- |\{(2\alpha,\beta,\mu^o) \vdash n \mid \beta \text{ even parts}, \mu^o \in \mathcal{O}, \ell(\beta) \text{ odd}\}| \\ &= |\{(2\alpha,\nu) \vdash n \mid \nu \in \mathcal{Q}_{odd}\}|. \end{split}$$

Summing over all distinct partitions α , we see that (11) equals

$$\begin{aligned} |\{(2\alpha,\nu) \vdash n \mid \alpha \in \mathcal{Q}_{even}, \nu \in \mathcal{Q}_{odd}, \ell(\alpha) \text{ even}\}| \\ &- |\{(2\alpha,\nu) \vdash n \mid \alpha \in \mathcal{Q}_{even}, \nu \in \mathcal{Q}_{odd}, \ell(\alpha) \text{ odd}\}| \end{aligned}$$

Since the number of odd parts in a partition of n is congruent to $n \mod 2$, it follows that (11) equals $(-1)^n Q_{2,e-o}(n)$.

Finally, the transformation φ_K gives a combinatorial proof for

$$(-1)^n Q_{2,e-o}(n) = \begin{cases} 1 & \text{if } n = T_i \text{ for some } i \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

This concludes the proof.

7. Combinatorial proof of Theorem 1.4(i)

In this section we prove combinatorially that for $n \ge 0$,

(12)
$$Q_0(n) = pod(n) + 2\sum_{k=1}^{\infty} (-1)^k pod(n-4k^2).$$

Recall that $Q_0(n)$ is the number of distinct partitions of n with parts $\not\equiv 0 \pmod{4}$.

We begin as in the combinatorial proof of Theorem 1.3. Let $\overline{\mathcal{P}}_4$ be the set of overpartitions with all parts $\equiv 0 \mod 4$ and define

$$\overline{\mathcal{P}}_4\mathcal{POD}(n) := \{ (\overline{\lambda}, \mu) \vdash n \mid \overline{\lambda} \in \overline{\mathcal{P}}_4, \ \mu \in \mathcal{POD} \}.$$

We use Andrews' transformation φ_{A_2} for the proof of the second Gauss identity and quadruple the parts in all partitions in the proof to see that the right hand side of (12) equals

$$|\{(\overline{\lambda},\mu)\in\overline{\mathcal{P}}_{4}\mathcal{POD}(n)\mid\ell(\overline{\lambda})\text{ even}\}|-|\{(\overline{\lambda},\mu)\in\overline{\mathcal{P}}_{4}\mathcal{POD}(n)\mid\ell(\overline{\lambda})\text{ odd}\}|.$$

As before, we write $\overline{\lambda} = (\alpha, \beta)$, where α consists of the overlined parts of $\overline{\lambda}$ and β consists of the nonoverlined parts of $\overline{\lambda}$. We write $\mu = (\mu^{e,2}, \mu^{e,4}, \mu^o)$, where $\mu^{e,2}$, respectively $\mu^{e,4}$, is the partition consisting of the parts of μ congruent to 2, respectively 0, mod 4, and μ^o consists of the odd parts of μ .

We denote by 4Q, respectively $2Q_{odd}$, the set of all partitions into distinct parts congruent to 0, respectively 2, modulo 4. We fix $\alpha \in 4\mathcal{Q}$ and consider the subset of $\overline{\mathcal{P}}_4 \mathcal{POD}(n)$ with the overlined parts of $\overline{\lambda}$ exactly the parts of α , i.e.

$$\overline{\mathcal{P}}_{4}\mathcal{POD}_{\alpha}(n) = \left\{ (\alpha, \beta, \mu^{e,2}, \mu^{e,4}, \mu^{o}) \vdash n \middle| \begin{array}{l} \beta \text{ parts} \equiv 0 \mod 4, \\ \mu^{e,2/4} \text{ parts} \equiv 2/0 \mod 4, \mu^{o} \in \mathcal{Q}_{odd} \end{array} \right\}.$$

The transformation $(\beta, \mu^{e,2}) \to 2\varphi_G((\beta \cup \mu^{e,2})_{/2})$ shows that

$$\begin{aligned} |\{(\alpha,\beta,\mu^{e,2},\mu^{e,4},\mu^{o})\in\overline{\mathcal{P}}_{4}\mathcal{POD}_{\alpha}(n)\mid\ell(\beta)\text{ even}\}|\\ &-|\{(\alpha,\beta,\mu^{e,2},\mu^{e,4},\mu^{o})\in\overline{\mathcal{P}}_{4}\mathcal{POD}_{\alpha}(n)\mid\ell(\beta)\text{ odd}\}|\end{aligned}$$

$$= |\{(\alpha, \gamma, \mu^{e,4}, \mu^o) \vdash n \mid \gamma \in 2\mathcal{Q}_{odd}, \mu^{e,4} \text{ parts} \equiv 0 \mod 4, \mu^o \in \mathcal{Q}_{odd}\}|$$
$$=: \overline{\mathcal{P}}_4 \mathcal{POD}'_\alpha(n).$$

Summing over all α , shows that the right hand side of (12) equals

$$\begin{split} |\{(\alpha, \gamma, \mu^{e, 4}, \mu^{o}) \in \overline{\mathcal{P}}_{4} \mathcal{POD}'(n) \mid \ell(\alpha) \text{ even}\}| \\ &- |\{(\alpha, \gamma, \mu^{e, 4}, \mu^{o}) \in \overline{\mathcal{P}}_{4} \mathcal{POD}'(n) \mid \ell(\alpha) \text{ odd}\}|, \end{split}$$

where $\overline{\mathcal{P}}_4 \mathcal{POD}'(n) = \bigcup_{\alpha \in 4\mathcal{Q}} \overline{\mathcal{P}}_4 \mathcal{POD}'_{\alpha}(n)$. Next, we map $(\alpha, \gamma, \mu^{e,4}, \mu^o)$ to $(4\varphi_F(\alpha_{/4}), \gamma, \mu^{e,4}, \mu^o)$ to see that the right hand side of (12) equals

$$\begin{aligned} |\{(4\pi,\gamma,\mu^{e,4},\mu^o)\in\mathcal{P}_4\mathcal{POD}'(n)\mid\ell(\pi)\text{ even}\}|\\ &-|\{(4\pi,\gamma,\mu^{e,4},\mu^o)\in\overline{\mathcal{P}}_4\mathcal{POD}'(n)\mid\ell(\pi)\text{ odd}\}|,\end{aligned}$$

where π is a pentagonal partition. Finally, applying φ_{BZ} to $(\pi, \mu_{/4}^{e,4})$ we see that the right hand side of (12) equals

$$|\{(\gamma, \mu^o) \vdash n \mid \gamma \in 2\mathcal{Q}_{odd}, \mu^o \in \mathcal{Q}_{odd}\}| = Q_0(n).$$

8. Combinatorial proofs of Theorems 1.5, 1.6, 1.7, and 1.8

In this section, we provide combinatorial proofs for the remaining theorems in [16].

8.1. Proof of Theorem 1.5. We prove combinatorially that for $n \ge 0$ we have

$$ped(n) = \sum_{k=0}^{\infty} pod(n-2T_k).$$

Remark 3. In [16], the theorem is stated with $pod(n-2T_k)$ replaced by $p_2(n-2T_k)$. However, in Section 2, we gave the bijection between $\mathcal{POD}(n)$ and $\mathcal{P}_2(n)$.

We create a bijection $h: \mathcal{PED}(n) \to \bigcup_{k=0}^{\infty} \mathcal{POD}(n-2T_k).$

Start with $\lambda = (\lambda^e, \lambda^o) \in \mathcal{PED}(n)$. Let η^o be the partition into odd distinct parts whose parts are precisely the parts of λ^o that have odd multiplicity. Then each part in $\lambda^o \setminus \eta^o$ is odd and has even multiplicity. We transform $\lambda^o \setminus \eta^o$ into a partition μ^e with distinct even parts as follows. Let ν^o be the partition whose parts are the different parts of $\lambda^o \setminus \eta^o$ each occurring with half its multiplicity in $\lambda^o \setminus \eta^o$. Then μ^e is defined as $2\varphi_{Gl}(\nu^o)$ and clearly $|\lambda^e| + |\eta^o| + |\mu^e| = n$. Next we define η^e to be the partition $2\varphi_{BM}^{-1}(\lambda_{2,2}^e, \mu_{2,2}^e)$. Thus, η^e is a partition into even parts of size $|\lambda^e| + |\mu^e| - 2T_k$ for some $k \ge 0$. We define $h(\lambda) = (\eta^e, \eta^o) \in \mathcal{POD}(n - 2T_k)$.

To define the inverse of h, start with $(\eta^e, \eta^o) \in \mathcal{POD}(n - 2T_k)$ for some $k \geq 0$. Then $2\varphi_{BM}(\eta_{/2}^e)$ is a partition of $n - |\eta^o|$ with distinct even parts in two colors. We write this partition as (λ^e, α) . Let β be the partition obtained by doubling the multiplicity of each part of $\varphi_{Gl}^{-1}(\alpha_{/2})$. Hence, β has odd parts each with even multiplicity and we let $\lambda^o = \eta^o \cup \beta$. Then $h^{-1}(\eta^e, \eta^o) = (\lambda^e, \lambda^o) \in \mathcal{PED}(n)$.

Remark 4. It follows from Theorem 1.5 that for $n \ge 0$ we have

$$ped(n) - pod(n) = \sum_{k=1}^{\infty} pod(n - 2T_k).$$

This gives another proof that $ped(n) - pod(n) \ge 0$. Moreover, from the discussion in Section 2, it follows that

$$\sum_{k=1}^{\infty} pod(n-2T_k) = \left| \left\{ \lambda \in \mathcal{PED}(n) \mid m(1) < \sum_{a \in \lambda^e} 2^{val_2(a)} \right\} \right|.$$

8.2. **Proof of Theorem 1.6.** We prove combinatorially that for $n \ge 0$ we have

(13)
$$\sum_{j=0}^{\infty} (-1)^{T_j} pod(n-T_j) = \begin{cases} 1 & \text{if } n=0, \\ 0 & \text{if } n>0. \end{cases}$$

The statement is clearly true if n = 0 since $\mathcal{POD}(0) = \{\emptyset\}$. Let n > 0, and define

$$\mathcal{Q}_2 \mathcal{POD}(n) = \{(\eta, \lambda) \vdash n \mid \eta \in \mathcal{Q}_2, \lambda \in \mathcal{POD}\}.$$

Using φ_K , shows that proving (13) is equivalent to showing that

(14)
$$|\{(\eta, \lambda) \in \mathcal{Q}_2 \mathcal{POD}(n) \mid \ell(\eta) \text{ even}\}| - |\{(\eta, \lambda) \in \mathcal{Q}_2 \mathcal{POD}(n) \mid \ell(\eta) \text{ odd}\}|$$

equals 0. We write $(\eta, \lambda) \in \mathcal{Q}_2 \mathcal{POD}(n)$ as $(\eta^e, \eta^o, \lambda^e \lambda^o)$. Thus η^o, λ^o have distinct odd parts, η^e has distinct parts $\equiv 0 \mod 4$, and λ^e has even parts. We apply the

zeta transformation to (η^o, λ^o) and proceed as in Proposition 2 to see that (14) is equal to

$$|\{(\eta^e, 2\eta^o, \lambda^e) \vdash n \mid \eta^o \in \mathcal{Q}_{odd}, \eta^e \in 4\mathcal{Q}, \lambda^e \text{ even parts}, \ell(\eta^o) + \ell(\eta^e) \text{ even}\}|$$

 $-|\{(\eta^e, 2\eta^o, \lambda^e) \vdash n \mid \eta^o \in \mathcal{Q}_{odd}, \eta^e \in 4\mathcal{Q}, \lambda^e \text{ even parts}, \ell(\eta^o) + \ell(\eta^e) \text{ odd}\}|.$

We map the triple $(\eta^e, 2\eta^o, \lambda^e)$ to the overpartition (α, β) of n/2 whose overlined parts are the parts of $\alpha = \eta^o \cup \eta^e_{/2}$ and the non-ovelined parts are the parts of $\beta = \lambda_{/2}^e$. It is easy to see that this transformation is invertible. Thus, (14) equals

$$|\{(\alpha,\beta)\in\overline{\mathcal{P}}(n/2)\mid\ell(\alpha)\text{ even}\}|-|\{(\alpha,\beta)\in\overline{\mathcal{P}}(n/2)\mid\ell(\alpha)\text{ odd}\}|.$$

It remains to show that if n is even, the number of overpartitions of n/2 with an even number of overlined parts is equal to the number of overpartitions of n/2 with an odd number of overlined parts. As remarked in Section 3, Andrews' involution φ_{A_2} on $\overline{\mathcal{P}}(n) \setminus \mathcal{E}_{\overline{\mathcal{P}}(n)}$ reverses the parity of the number of overlined parts. Moreover, the set $\mathcal{E}_{\overline{\mathcal{P}}(n)}$ is either empty or consists of two overpartitions: one with no overlined parts and one with one overlined part. This concludes the proof.

8.3. Proof of Theorem 1.7. We prove combinatorially that for $n \ge 0$ we have

$$ped(n) = \sum_{k \ge 0} \overline{p}\left(rac{n}{2} - rac{T_k}{2}
ight)$$

We will use the bijective proof of Fu and Tang [12] for Watson's identity

$$|\mathcal{Q}(n)| = \sum_{k \ge 0} p\left(\frac{n - T_k}{2}\right)$$

We denote their bijection by φ_{FT} .

Let $n \geq 0$. We create a bijection between $\mathcal{PED}(n)$ and $\bigcup_{k=0}^{\infty} \overline{\mathcal{P}}\left(\frac{n}{2} - \frac{T_k}{2}\right)$. Start with $\lambda = (\lambda^e, \lambda^o) \in \mathcal{PED}(n)$. Let $\alpha = \lambda_{/2}^e$ and $\beta = \varphi_{FT}(\varphi_{Gl}(\lambda^o))$. Thus, $\beta \in \mathcal{P}((|\lambda^o| - T_k)/2)$ for some $k \geq 0$. We map λ to the overpartition whose overlined parts are the parts of α and the non-overlined parts are the parts of β . Then $(\alpha, \beta) \in \overline{\mathcal{P}}\left(\frac{n}{2} - \frac{T_k}{2}\right)$. The transformation is clearly invertible as we used bijections to define it.

8.4. Proof of Theorem 1.8. We prove combinatorially that for $n \ge 0$ we have

(15)
$$\sum_{j=0}^{\infty} o_{e-o}(n-T_j) = \begin{cases} (-1)^k, & \text{if } n = 2k(3k+1), \ k \in \mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}$$

First notice that if $\lambda \in \mathcal{O}(n)$, then $\ell(\lambda) \equiv n \mod 2$, so $o_{e-o}(n) = (-1)^n o(n)$. Thus, if we define $S(n) = \sum_{j=0}^{\infty} (-1)^{T_j} o(n-T_j)$, the left hand side of (15) equals $(-1)^n S(n)$. Let

$$\mathcal{OPED}(n) = \{(\lambda, \mu) \vdash n \mid \lambda \in \mathcal{PED}, \mu \in \mathcal{O}\}.$$

As in the proof of Theorem 1.2,

 $S(n) = |\{(\lambda, \mu) \in \mathcal{OPED}(n) \mid \ell(\lambda) \text{ even}\}| - |\{(\lambda, \mu) \in \mathcal{OPED}(n) \mid \ell(\lambda) \text{ odd}\}|.$

We write $(\lambda, \mu) = (\lambda^e, \lambda^o, \mu)$, and apply the transformation ψ to (λ^o, μ) . Then, Proposition 4 implies

$$S(n) = |\{(\lambda^e, \eta) \vdash n \mid \lambda^e, \eta \in \mathcal{Q}_{even}, \ell(\lambda^e) \text{ even}\}|$$

 $-|\{(\lambda^e,\eta) \vdash n \mid \lambda^e, \eta \in \mathcal{Q}_{even}, \ell(\lambda^e) \text{ odd}\}|.$

By Proposition 2,

$$S(n) = |\{\nu \in 4\mathcal{Q} \mid \ell(\nu) \text{ even}\}| - |\{\nu \in 4\mathcal{Q} \mid \ell(\nu) \text{ odd}\}|.$$

Finally, mapping ν to $4\varphi_F(\nu_{/4})$ gives

$$S(n) = \begin{cases} (-1)^k & \text{if } n = 2k(3k+1), \ k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Observing that S(n) = 0 if n is odd completes the proof.

9. Beck-Type Identities

Beck-type identities are companions to partition identities. If the original identity states that the number of partitions of n satisfying condition X equals the number of partitions of n satisfying condition Y, then the Beck-type companion identity gives a combinatorial interpretation for the excess in the number of parts in all partitions of n subject to X over the number of parts in all partitions of n subject to Y. The first such identity, a companion for Euler's identity, was conjectured by Beck [18] and proved by Andrews [2].

In this section we prove Beck-type companion identities to the identities given by (1) and (2), i.e., for all $n \ge 0$ we have

$$ped(n) = b_4(n)$$

and

$$pod(n) = p_2(n).$$

Before we state and prove the Beck-type identities, we recall a useful generalization of the original work in [2].

Let S_1 and S_2 be subsets of the positive integers. Denote by $\mathcal{O}_r(n)$ the set of partitions of n with parts from S_2 and by $\mathcal{D}_r(n)$ the set of partitions of n with parts from S_1 and no part repeated r or more times. If $|\mathcal{O}_r(n)| = |\mathcal{D}_r(n)|$ for all $n \ge 0$, the pair (S_1, S_2) is called an Euler pair of order r. Subbarao [17] proved that (S_1, S_2) is an Euler pair of order r if and only if $rS_1 \subseteq S_1$ and $S_2 = S_1 \setminus rS_1$. Ballantine and Welch [8] proved the following theorem analytically and combinatorially, which gives a Beck-type identity for Euler pairs.

Theorem 9.1. Suppose (S_1, S_2) is an Euler pair of order r and let $n \ge 0$. The excess in the number of parts in all partitions of $\mathcal{O}_r(n)$ over the number of parts in all partitions of $\mathcal{D}_r(n)$ is equal to the number of partitions with parts from S_1 and exactly one part repeated at least r times. The excess also equals the number of partitions with exactly one part (possibly repeated) from rS_1 and all other parts from S_2 .

In particular, if S_1 is the set of even positive integers and S_2 is the set of positive integers congruent to 2 mod 4, then (S_1, S_2) is an Euler pair of order 2. Thus, if $\mathcal{B}_{4,even}(n)$ is the set of 4-regular partitions of n with even parts, then for $n \ge 0$ we have

(16)
$$|\mathcal{B}_{4,even}(n)| = |\mathcal{Q}_{even}(n)|,$$

and we obtain the following corollary to Theorem 9.1.

Corollary 9.2. Let $n \ge 0$. The excess in the number of parts in all partitions in $\mathcal{B}_{4,even}(n)$ over the number of parts in all partitions in $\mathcal{Q}_{even}(n)$ is equal to the number of partitions with even parts and exactly one part repeated. The excess also equals the number of partitions with exactly one part (possibly repeated) congruent to 0 mod 4 and all other parts congruent to 2 mod 4.

9.1. A Beck-type companion identity to $ped(n) = b_4(n)$. Before we state the theorem, we introduce some notation. Let b(n) be the excess in the number of parts in all partitions in $\mathcal{B}_4(n)$ over the number of parts in all partitions in $\mathcal{PED}(n)$. Let $\mathcal{PED}_1(n)$ be the set of partitions of n with one even part repeated, all other even parts distinct, and odd parts unrestricted. Let $\mathcal{B}_{4,1}(n)$ be the set of partitions of n with exactly one part (possibly repeated) $\equiv 0 \mod 4$ and all other parts $\not\equiv 0 \mod 4$.

Theorem 9.3. For all $n \ge 0$, $b(n) = |\mathcal{PED}_1(n)| = |\mathcal{B}_{4,1}(n)|$.

Using Corollary 9.2 and its proofs in [8], we obtain both analytic and combinatorial proofs of the theorem.

Analytic Proof. Let $z, q \in \mathbb{C}$, |z|, |q| < 1 (so that all series converge absolutely). Let $b_4(m, n)$, respectively ped(m, n), be the number of partitions in $\mathcal{B}_4(n)$, respectively $\mathcal{PED}(n)$, with m parts. We define

$$F(z,q) := \sum_{m \ge 0} \sum_{n \ge 0} b_4(m,n) z^m q^n,$$

$$G(z,q) := \sum_{m \ge 0} \sum_{n \ge 0} ped(m,n) z^m q^n.$$

We have

$$F(z,q) = \frac{1}{(zq^2;q^4)_{\infty}(zq;q^2)_{\infty}},$$

$$G(z,q) = \frac{(-zq^2;q^2)_{\infty}}{(zq;q^2)_{\infty}}.$$

Then,

$$\begin{split} \sum_{n=0}^{\infty} b(n)q^n &= \frac{\partial}{\partial z} \bigg|_{z=1} (F(z,q) - G(z,q)) \\ &= \frac{\partial}{\partial z} \bigg|_{z=1} \frac{1}{(zq;q^2)_{\infty}} \left(\frac{1}{(zq^2;q^4)_{\infty}} - (-zq^2;q^2)_{\infty} \right) \\ &= \frac{\partial}{\partial z} \left[\frac{1}{(zq;q^2)_{\infty}} \right]_{z=1} \cdot \left(\frac{1}{(q^2;q^4)_{\infty}} - (-q^2;q^2)_{\infty} \right) \\ &+ \frac{1}{(q;q^2)_{\infty}} \cdot \frac{\partial}{\partial z} \left[\frac{1}{(zq^2;q^4)_{\infty}} - (-zq^2;q^2)_{\infty} \right]_{z=1} \\ &= \frac{1}{(q;q^2)_{\infty}} \cdot \frac{\partial}{\partial z} \left[\frac{1}{(zq^2;q^4)_{\infty}} - (-zq^2;q^2)_{\infty} \right]_{z=1}, \end{split}$$

where we invoked (16) in generating functions form:

$$\frac{1}{(q^2;q^4)_{\infty}} = (-q^2,q^2)_{\infty}$$

To complete the proof, we use the analytic proof of Corollary 9.2 which can be expressed as

$$\begin{aligned} \frac{\partial}{\partial z} \bigg|_{z=1} \left(\frac{1}{(zq^2; q^4)_{\infty}} - (-zq^2; q^2)_{\infty} \right) &= \frac{1}{(q^2; q^4)_{\infty}} \sum_{i=1}^{\infty} \frac{q^{4i}}{1 - q^{4i}} \\ &= (-q^2; q^2)_{\infty} \sum_{i=1}^{\infty} \frac{q^{2(2i)}}{1 - q^{2(2i)}} \end{aligned}$$

Since

$$\sum_{n=0}^{\infty} |\mathcal{PED}_1(n)| q^n = \frac{(-q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \cdot \sum_{i=1}^{\infty} \frac{q^{2(2i)}}{1 - q^{2(2i)}}$$

and

$$\sum_{n=0}^{\infty} |\mathcal{B}_{4,1}(n)| q^n = \frac{1}{(q;q^2)_{\infty}(q^2;q^4)_{\infty}} \cdot \sum_{i=1}^{\infty} \frac{q^{4i}}{1-q^{4i}},$$

it follows that

$$\sum_{n=0}^{\infty} b(n)q^n = \sum_{n=0}^{\infty} |\mathcal{B}_{4,1}(n)|q^n = \sum_{n=0}^{\infty} |\mathcal{PED}_1(n)|q^n.$$

Combinatorial Proof. As explained in Section 2, the transformation that maps $\lambda = (\lambda^e, \lambda^o) \in \mathcal{B}_4(n)$ to $(2\varphi_{Gl}(\lambda_{/2}^e), \lambda^o) \in \mathcal{PED}(n)$ is a bijection. Moreover, this bijection preserves the number of odd parts of λ . Thus, by Corollary 9.2, for a fixed partition $\lambda^o \in \mathcal{O}$, the number of parts in all partitions in $\{(\lambda^e, \lambda^o) \vdash n \mid \lambda^e \in \mathcal{B}_{4,even}\}$ minus the number of parts in all partitions in $\{(\lambda^e, \lambda^o) \vdash n \mid \lambda^e \in \mathcal{Q}_{even}\}$ equals the the number of partitions of $n - |\lambda^o|$ with even parts and exactly one part repeated. This excess also equals the number of partitions of $n - |\lambda^o|$ with exactly one part (possibly repeated) congruent to 0 mod 4 and all other parts congruent to 2 mod 4. We map each partition μ in the sets described above to $\mu \cup \lambda^o$ to obtain a partition of n in $\mathcal{PED}_1(n)$, respectively $\mathcal{B}_{4,1}(n)$. Considering all $\lambda^o \in \mathcal{O}$ completes the proof.

9.2. A Beck-type companion identity to $pod(n) = p_2(n)$. Let b'(n) be the excess in the number of parts in all partitions in $\mathcal{P}_2(n)$ over the number of parts in all partitions in $\mathcal{POD}(n)$. Let $\mathcal{POD}_1(n)$ be the set of partitions of n with one odd part repeated, all other odd parts distinct, and even parts unrestricted. Let $\mathcal{P}_{2,1}(n)$ be the set of partitions of n with exactly one part (possibly repeated) $\equiv 2 \mod 4$ and all other parts $\not\equiv 2 \mod 4$.

Theorem 9.4. For all $n \ge 1$, $b'(n) = |\mathcal{POD}_1(n)| = |\mathcal{P}_{2,1}(n)|$.

Unlike the proof of Theorem 9.3, here we cannot use the work of [8]. There is no Euler pair of order 2 with S_1 being the set of positive odd integers since $2S_1 \not\subseteq S_1$.

Analytic Proof. Let $z, q \in \mathbb{C}$, |z|, |q| < 1 (so that all series converge absolutely). Let $p_2(m, n)$, respectively pod(m, n), be the number of partitions in $\mathcal{P}_2(n)$, respectively $\mathcal{POD}(n)$, with m parts. We define

$$F(z,q) := \sum_{m \ge 0} \sum_{n \ge 0} p_2(m,n) z^m q^n,$$

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$$G(z,q) := \sum_{m \ge 0} \sum_{n \ge 0} pod(m,n) z^m q^n.$$

We have

$$F(z,q) = \frac{1}{(zq^4;q^4)_{\infty}(zq;q^2)_{\infty}},$$

$$G(z,q) = \frac{(-zq;q^2)_{\infty}}{(zq^2;q^2)_{\infty}}.$$

Then,

$$\sum_{n=0}^{\infty} b'(n)q^n = \frac{\partial}{\partial z} \bigg|_{z=1} (F(z,q) - G(z,q)).$$

Using logarithmic differentiation,

$$\left. \frac{\partial}{\partial z} \right|_{z=1} F(z,q) = \frac{1}{(q^4;q^4)_{\infty}(q;q^2)_{\infty}} \cdot \sum_{i=1}^{\infty} \left(\frac{q^{4i}}{1-q^{4i}} + \frac{q^{2i-1}}{1-q^{2i-1}} \right),$$

 $\quad \text{and} \quad$

$$\frac{\partial}{\partial z} \bigg|_{z=1} G(z,q) = \frac{(-q;q^2)_{\infty}}{(q^2;q^2)_{\infty}} \cdot \sum_{i=1}^{\infty} \left(\frac{q^{2i-1}}{1+q^{2i-1}} + \frac{q^{2i}}{1-q^{2i}} \right).$$

Using (2), it follows that

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q^4; q^4)_{\infty}(q; q^2)_{\infty}} \cdot \left[\sum_{i=1}^{\infty} \left(\frac{q^{4i}}{1 - q^{4i}} + \frac{q^{2i-1}}{1 - q^{2i-1}} \right) - \sum_{i=1}^{\infty} \left(\frac{q^{2i-1}}{1 + q^{2i-1}} + \frac{q^{2i}}{1 - q^{2i}} \right) \right].$$

We will show that

$$\begin{split} \sum_{i=1}^{\infty} \left(\frac{q^{4i}}{1-q^{4i}} + \frac{q^{2i-1}}{1-q^{2i-1}} \right) &- \sum_{i=1}^{\infty} \left(\frac{q^{2i-1}}{1+q^{2i-1}} + \frac{q^{2i}}{1-q^{2i}} \right) \\ &= \sum_{i=1}^{\infty} \frac{q^{4i-2}}{1-q^{4i-2}}. \end{split}$$

First, note that

$$\sum_{i=1}^{\infty} \sum_{m=1}^{\infty} q^{m(2i)} = \sum_{i=1}^{\infty} \sum_{\substack{m \ge 1 \\ odd}} q^{m(2i)} + \sum_{i=1}^{\infty} \sum_{\substack{m \ge 1 \\ even}} q^{m(2i)}$$
$$= \sum_{i=1}^{\infty} \sum_{\substack{m \ge 1 \\ even}} q^{m(2i-1)} + \sum_{i=1}^{\infty} \sum_{\substack{m=1 \\ m=1}}^{\infty} q^{m(4i)}.$$

Then

$$\sum_{i=1}^{\infty} \left(\frac{q^{4i}}{1-q^{4i}} + \frac{q^{2i-1}}{1-q^{2i-1}} \right) - \sum_{i=1}^{\infty} \left(\frac{q^{2i-1}}{1+q^{2i-1}} + \frac{q^{2i}}{1-q^{2i}} \right)$$

$$\begin{split} &= \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} \left(q^{m(4i)} + q^{m(2i-1)} \right) - \left(\sum_{i=1}^{\infty} \frac{q^{2i-1}}{1+q^{2i-1}} + \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} q^{m(2i)} \right) \\ &= \sum_{i=1}^{\infty} \sum_{\substack{m \ge 1 \\ odd}} q^{m(2i-1)} - \sum_{i=1}^{\infty} \frac{q^{2i-1}}{1+q^{2i-1}} \\ &= \sum_{i=1}^{\infty} \sum_{\substack{m \ge 1 \\ odd}} q^{m(2i-1)} - \left(\sum_{i=1}^{\infty} \sum_{\substack{m \ge 1 \\ odd}} q^{m(2i-1)} - \sum_{i=1}^{\infty} \frac{q^{4i-2}}{1-q^{4i-2}} \right) \\ &= \sum_{i=1}^{\infty} \sum_{\substack{m \ge 1 \\ even}} q^{m(2i-1)} = \sum_{i=1}^{\infty} \frac{q^{4i-2}}{1-q^{4i-2}}. \end{split}$$

Since

$$\sum_{n=0}^{\infty} |\mathcal{P}_{2,1}(n)| q^n = \frac{1}{(q^4; q^4)_{\infty}(q; q^2)_{\infty}} \cdot \sum_{i=1}^{\infty} \frac{q^{4i-2}}{1-q^{4i-2}}$$

and

$$\sum_{n=0}^{\infty} |\mathcal{POD}_1(n)| q^n = \frac{(-q;q^2)_{\infty}}{(q^2;q^2)_{\infty}} \cdot \sum_{i=1}^{\infty} \frac{q^{2(2i-1)}}{1-q^{2(2i-1)}},$$

it follows that

$$\sum_{n=0}^{\infty} b'(n)q^n = \sum_{n=0}^{\infty} |\mathcal{P}_{2,1}(n)|q^n = \sum_{n=0}^{\infty} |\mathcal{POD}_1(n)|q^n.$$

Combinatorial Proof. Recall that when showing $pod(n) = p_2(n)$, we map $\lambda \in \mathcal{POD}(n)$ to $\mu \in \mathcal{P}_2(n)$ by splitting each part congruent to 2 mod 4 into two equal odd parts and leaving all other parts the same. Thus b'(n) is equal to the number of parts congruent to 2 mod 4 in all partitions of $\mathcal{POD}(n)$.

Denote by $\mathcal{POD}^*(n)$ the set of partitions $\mathcal{POD}(n)$ with a single part congruent to 2 mod 4 marked. Note that marked partitions differ from overpartitions since any part congruent to 2 mod 4 may be marked. For example, $(9, 7, 6^*, 6, 6, 4)$ and $(9, 7, 6, 6^*, 6, 4)$ are different marked partitions in $\mathcal{POD}^*(n)$. Thus, b'(n) = $|\mathcal{POD}^*(n)|$. We establish a bijection $f : \mathcal{POD}^*(n) \to \mathcal{POD}_1(n)$.

Start with $\lambda \in \mathcal{POD}^*(n)$ and suppose $\lambda_i = a^*$ is the marked part. Moreover, suppose $\lambda_{i-j+1} = a$ and $\lambda_{i-j} > a$ (we set $\lambda_0 = \infty$). Thus the marked part λ_i is the j^{th} part among the parts equal to a. We define $f(\lambda)$ to be the partition obtained from λ by removing the marking and splitting j of the parts equal to aeach into two equal parts. Since $a \equiv 2 \mod 4$, the new parts are equal and odd. Thus $f(\lambda) \in \mathcal{POD}_1(n)$.

To show that f is invertible, start with $\mu \in \mathcal{POD}_1(n)$ with repeated odd part b. Let $m(b) \geq 2$ be the multiplicity of b in μ . Then $f^{-1}(\mu)$ is the marked partition obtained from μ by merging $\lfloor m(b)/2 \rfloor$ pairs of parts equal to b to obtain $\lfloor m(b)/2 \rfloor$ parts equal to 2b. We mark the $\lfloor m(b)/2 \rfloor^{th}$ part equal to 2b in the obtained partition. Thus, $f^{-1}(\mu) \in \mathcal{POD}^*(n)$.

From the bijective proof of $pod(n) = p_2(n)$ we see that b'(n) is also equal to the number of odd parts in all partitions in $\mathcal{P}_2(n)$ each counted with (the integer part

of) half its multiplicity, i.e.

$$b'(n) = \sum_{\lambda \in \mathcal{P}_2(n)} \sum_{c \in \lambda, c \text{ odd}} \left\lfloor \frac{m_{\lambda}(c)}{2} \right\rfloor,$$

wher $m_{\lambda}(c)$ is the multiplicity of c in λ .

Denote by $\mathcal{P}_2^*(n)$ the set of partitions in $\mathcal{P}_2(n)$ with a single odd part marked. Moreover, part a in λ may be marked only if it is the j^{th} occurrence of a in λ with j even. Thus $b'(n) = |\mathcal{P}_2^*(n)|$. We establish a bijection $g : \mathcal{P}_2^*(n) \to \mathcal{P}_{2,1}(n)$.

Start with $\lambda \in \mathcal{P}_2^*(n)$ and suppose $\lambda_i = a^*$ is the marked part. Moreover, suppose λ_i is the j^{th} occurrence of a in λ . We define $g(\lambda)$ to be the partition obtained from λ by removing the marking and merging j/2 pairs of parts equal to a to obtain j/2 parts equal to 2a. Thus $g(\lambda) \in \mathcal{P}_{2,1}(n)$.

To show that g is invertible, start with $\mu \in \mathcal{P}_{2,1}(n)$ with repeated part $b \equiv 2 \mod 4$. Let $m(b) \geq 2$ be the multiplicity of b in μ . Then $g^{-1}(\mu)$ is the partition obtained from μ by splitting each part equal to b into two parts equal to b/2 and marking the $(2m(b))^{th}$ occurrence of b/2 in the obtained partition. Thus $g^{-1}(\mu) \in \mathcal{P}_2^*(n)$.

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