Gallai-Ramsey numbers for fans *

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March 1, 2019

Abstract

Given a graph G and a positive integer k, define the *Gallai-Ramsey number* to be the minimum number of vertices n such that any k-edge coloring of K_n contains either a rainbow (all different colored) triangle or a monochromatic copy of G. In this paper, we obtain general upper and lower bounds on the Gallai-Ramsey numbers for fans $F_m = K_1 + mK_2$ and prove the sharp result for $m = 2$ and for $m = 3$ with k even.

1 Introduction

In this work, we consider only edge-colorings of graphs. A coloring of a graph is called *rainbow* if no two edges have the same color.

Edge colorings of complete graphs that contain no rainbow triangle have very interesting and somewhat surprising structure. In 1967, Gallai [4] first examined this structure under the guise of transitive orientations of graphs and it can also be traced back to [1]. For this reason, colored complete graphs containing no rainbow triangle are called *Gallai colorings*. Gallai's result was restated in [6] in the terminology of graphs. For the following statement, a trivial partition is a partition into only one part.

Theorem 1 ([1, 4, 6]). *In any coloring of a complete graph containing no rainbow triangle, there exists a nontrivial partition of the vertices (called a Gallai partition) such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.*

*Supported by the National Science Foundation of China (Nos. 11601254, 11551001, 11161037, 61763041, 11661068, and 11461054) and the Science Found of Qinghai Province (Nos. 2016-ZJ-948Q, and 2014-ZJ-907) and the Qinghai Key Laboratory of Internet of Things Project (2017-ZJ-Y21).

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The induced subgraph of a Gallai colored complete graph constructed by selecting a single vertex from each part of a Gallai partition is called the *reduced graph*. By Theorem 1, the reduced graph is a 2-colored complete graph.

Given two graphs G and H, let $R(G, H)$ denote the 2-color Ramsey number for finding a monochromatic G or H, that is, the minimum number of vertices n needed so that every red-blue coloring of K_n contains either a red copy of G or a blue copy of H. Although the reduced graph of a Gallai partition uses only two colors, the original Gallai-colored complete graph could certainly use more colors. With this in mind, we consider the following generalization of the Ramsey numbers. Given two graphs G and H, the *general k-colored Gallai-Ramsey number $gr_k(G : H)$* is defined to be the minimum integer m such that every k-coloring of the complete graph on m vertices contains either a rainbow copy of G or a monochromatic copy of H. With the additional restriction of forbidding the rainbow copy of G, it is clear that $gr_k(G : H) \leq R_k(H)$ for any G.

The Gallai-Ramsey numbers have been studied for a few choices of the rainbow graph G and a variety of choices of the monochromatic graph H. In light of Theorem 1, many (perhaps most) of the results have involved a rainbow triangle. Recent breakthroughs include the Gallai-Ramsey numbers for the K_4, the K_5, and all odd cycles, as seen in the following results.

Theorem 2 ([9])**.** *For $k \geq 1$,*

$$gr_k(K_3 : K_4) = \begin{cases} 17^{k/2} + 1 & \text{if } k \text{ is even,} \\ 3 \cdot 17^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Theorem 3 ([10])**.** *Let $R = R(K_5, K_5) - 1$. For any integer $k \geq 2$,*

$$gr_k(K_3 : K_5) = \begin{cases} R^{k/2} + 1 & \text{if } k \text{ is even,} \\ 4 \cdot R^{(k-1)/2} + 1 & \text{if } k \text{ is odd} \end{cases}$$

unless $R = 42$, in which case we have

$$\begin{cases} gr_k(K_3 : K_5) = 43 & \text{if } k = 2, \\ 42^{k/2} + 1 \leq gr_k(K_3 : K_5) \leq 43^{k/2} + 1 & \text{if } k \geq 4 \text{ is even,} \\ 169 \cdot 42^{(k-3)/2} + 1 \leq gr_k(K_3 : K_5) \leq 4 \cdot 43^{(k-1)/2} + 1 & \text{if } k \geq 3 \text{ is odd.} \end{cases}$$

Theorem 4 ([12, 13])**.** *For integers $\ell \geq 3$ and $k \geq 1$, we have*

$$gr_k(K_3 : C_{2\ell+1}) = \ell \cdot 2^k + 1.$$

We refer the interested reader to the survey [2] for a catalog of results on this subject with a dynamically updated version available at [3].

In keeping with the trend of studying monochromatic subgraphs in Gallai colorings, we consider the fan graphs in this work. The *fan* graph with n triangles is denoted by F_n, where $F_n = K_1 + n\overline{K_2}$. Note that $F_1 = K_3$ and

F_2 is a graph obtained from two triangles by sharing one vertex, often called a "bowtie". The main results of this work, the precise result for F_2, nearly sharp bounds for F_3, and general bounds for F_n, are contained in the following three theorems. First our sharp result for F_2.

Theorem 5.

$$gr_k(K_3; F_2) = \begin{cases} 9, & \text{if } k = 2; \\ \frac{83}{2} \cdot 5^{\frac{k-4}{2}} + \frac{1}{2}, & \text{if } k \textit{ is even}, \ k \geq 4; \\ 4 \cdot 5^{\frac{k-1}{2}} + 1, & \text{if } k \textit{ is odd}. \end{cases}$$

Next our general bounds (and sharp result for any even number of colors) for F_3.

Theorem 6. *For $k \geq 2$,*

$$\begin{cases} gr_k(K_3; F_3) = 14 \cdot 5^{\frac{k-2}{2}} - 1, & \text{if } k \textit{ is even}; \\ gr_k(K_3; F_3) = 33 \cdot 5^{\frac{k-3}{2}}, & \text{if } k = 3, 5; \\ 33 \cdot 5^{\frac{k-3}{2}} \leq gr_k(K_3; F_3) \leq 33 \cdot 5^{\frac{k-3}{2}} + \frac{3}{4} \cdot 5^{\frac{k-5}{2}} - \frac{3}{4}, & \text{if } k \textit{ is odd}, \ k \geq 7. \end{cases}$$

In particular, we conjecture the following, which claims that the lower bound in Theorem 6 is the sharp result.

Conjecture 1. *For $k \geq 2$,*

$$gr_k(K_3; F_3) = \begin{cases} 14 \cdot 5^{\frac{k-2}{2}} - 1, & \textit{if } k \textit{ is even;} \\ 33 \cdot 5^{\frac{k-3}{2}}, & \textit{if } k \textit{ is odd.} \end{cases}$$

Finally our general bound for all fans.

Theorem 7. *For $k \geq 2$,*

$$\begin{cases} 4n \cdot 5^{\frac{k-2}{2}} + 1 \leq gr_k(K_3; F_n) \leq 10n \cdot 5^{\frac{k-2}{2}} - \frac{5}{2}n + 1, & \text{if } k \textit{ is even}; \\ 2n \cdot 5^{\frac{k-1}{2}} + 1 \leq gr_k(K_3; F_n) \leq \frac{9}{2}n \cdot 5^{\frac{k-1}{2}} - \frac{5}{2}n + 1, & \text{if } k \textit{ is odd}. \end{cases}$$

In our proofs, we make heavy use of the following known results for the 2-color Ramsey numbers of fans.

Proposition 1 ([5, 7, 8, 11])**.**

(1) $R(F_2, F_2) = 9$*;*

(2) $R(F_3, F_3) = 13$*;*

(3) $4n + 1 \leq R(F_n, F_n) \leq 6n$.

Given a coloring G of K_n, let $k' = k'(G)$ be the number of colors inducing a subgraph with maximum degree at least 2. Call each of these k' colors *useful* and call any remaining colors *wasted*. Define $gr'_{k'}(K_3 : H)$ to be the minimum integer n so that every coloring of K_n in which at most k' colors are useful must contain either a rainbow triangle or a monochromatic copy of H. We will use the easy fact that for any positive integer k and graph H,

$$gr_k(K_3 : H) \leq gr'_k(K_3 : H).$$

2 The case $n = 2$

In order to help prove Theorem 5, we first show the following.

Theorem 8. *For all $k' \geq 0$, we have*

$$gr'_{k'}(K_3 : F_2) = \begin{cases} 2 \cdot 5^{\frac{k'}{2}} + 1 & \text{if } k' \text{ is even, or} \\ 4 \cdot 5^{\frac{k'-1}{2}} + 1 & \text{if } k' \text{ is odd.} \end{cases}$$

Theorem 8 follows immediately from Lemmas 1 and 2 and implies the upper bound for the odd case in Theorem 5. The upper bound for the even case in Theorem 5 follows from Lemma 3. The lower bounds in Theorems 8 and 5 are proven in the following lemma.

Lemma 1. *For any $i \geq 1$, there exists a Gallai coloring of the complete graph on:*

- $4 \cdot 5^i$ *vertices using $2i + 1$ colors which contains no monochromatic copy of F_2.*
- $2 \cdot 5^i$ *vertices with $2i$ useful colors which contains no monochromatic copy of F_2.*
- $\frac{83}{2} \cdot 5^{\frac{2i-4}{2}} - \frac{1}{2}$ *vertices using $2i$ colors which contains no monochromatic copy of F_2.*

Proof. For the first item in the statement, define G_0 to be a monochromatic copy of K_4, say with color 1. Suppose that we have constructed G_i, a coloring of a complete graph on $4 \cdot 5^i$ vertices using colors from $[2i + 1]$ with no rainbow traingle and no monochromatic copy of F_2. We construct G_{i+1} by making 5 copies of G_i and inserting all edges between these copies to form a blow-up of the unique 2-coloring of K_5 with no monochromatic triangle using colors $2i + 2$ and $2i + 3$. This is a Gallai coloring of the complete graph on $4 \cdot 5^{i+1}$ vertices using colors from $[2i + 3]$ with no monochromatic copy of F_2.

For the second item in the statement, define G_0 to be a monochromatic copy of K_2 with a wasted color, say color 0. Using this G_0 in place of G_0 in the previous construction, after i iterations, we obtain a Gallai coloring of a complete graph G_i of order $2 \cdot 5^i$ with $2i$ useful colors which contains no monochromatic copy of F_2. This construction makes heavy use of the wasted color, color 0.

For the third item in the statement, we use the following inductive construction. Let G_1 be a copy of K_4 colored entirely by color 1 and let G_2 be a copy of K_8 consisting of two copies of G_1 joined by all edges of color 2. Now suppose we have constructed a Gallai colored complete graph G_{2i-2} using $2i-2$ colors which contains no monochromatic copy of F_2. For $k = 2i$, we construct the graph G_{2i} by making five copies of G_{2i-2} and inserting edges of colors $2i$ and $2i-1$ between the copies to form a blow-up of the unique 2-coloring of K_5 with no monochromatic triangle. This coloring contains no rainbow triangle and no monochromatic copy of F_2. In this way, we construct G_4', a colored complete graph on 40 vertices.

For $k = 2i \geq 6$, we then extend this construction as follows. Let A_4' be a complete graph of order 9 consisting of four copies of K_2 with colors $1, 2, 3, 4$, and K_1, and inserting edges of colors 1 and 2 between these copies to form a blow-up of the unique 2-colored K_5 containing no monochromatic triangle.

For $j \geq 5$, let A_j be a complete graph of order 10 consisting of five copies of K_2 with colors $1, 2, 3, 4$, and j, and inserting edges of colors 1 and 2 between these copies to form a blow-up of the unique 2-colored K_5 containing no monochromatic triangle. See Figure 1 for a diagram of this construction.

Within G_4' as defined above, there are 5 copies of G_2. We replace one of these copies by A_4' to create a new graph G_4 with $|G_4| = 41$. Within G_6 as defined above (using G_4 in the construction), there are 5 copies of G_4. We replace two of the copies of A_4' by A_5 and A_6 respectively. Note that these copies of A_4' to be replaced must be chosen from different copies of G_4 from the construction of G_6 to avoid creating a monochromatic copy of F_2. This replacement adds 2 vertices to G_6 resulting in $|G_6| = 5 \times 41 + 2$. In the induction step, it is easy to see that the same replacement can always be made to replace two further copies of A_4' by A_{2i+1} and A_{2i+2} so

$$|G_{2i+2}| = 5\left(41 \cdot 5^{\frac{2i-4}{2}} + \frac{1}{2} \cdot 5^{\frac{2i-4}{2}} - \frac{1}{2}\right) + 2 = 41 \cdot 5^{\frac{2i-2}{2}} + \frac{1}{2} \cdot 5^{\frac{2i-2}{2}} - \frac{1}{2},$$

as claimed. □

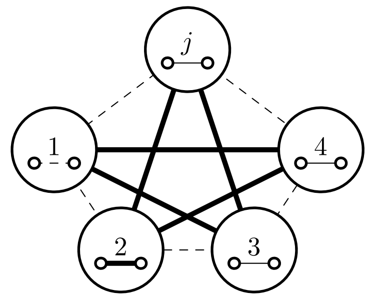

Figure 1: The coloring A_j of K_{10}

Lemma 2. *For all $k \geq 3$ and $0 \leq k' \leq k$, we have*

$$gr'_{k'}(K_3 : F_2) \leq \begin{cases} 2 \cdot 5^{\frac{k}{2}} + 1 & \text{if } k \text{ is even, or} \\ 4 \cdot 5^{\frac{k-1}{2}} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. For $k' \leq k$, let

$$n = n(k') = \begin{cases} 2 \cdot 5^{\frac{k'}{2}} + 1 & \text{if } k' \text{ is even, or} \\ 4 \cdot 5^{\frac{k'-1}{2}} + 1 & \text{if } k' \text{ is odd.} \end{cases}$$

Consider a k-coloring G of K_n in which at most k' colors are useful and suppose for a contradiction that G contains no rainbow triangle and no monochromatic copy of F_2. By Theorem 1, there is a Gallai partition of G, say with t parts and choose such a partition with t minimum. Let $H_1, H_2, \dots, H_t$ be the parts of this partition. Note that to avoid a rainbow triangle, at most one wasted color can appear incident to each vertex, meaning that the wasted colors all together induce a matching.

Claim 1. *If one vertex has all red edges to a part A, then there is no red copy of $2K_2$ as a subgraph of A. If at least two vertices have red edges to a part A, then red appears on at most one edge in A.*

Proof. First suppose u has all red edges to a part A and there is a red copy of $2K_2$ within A. Then u is the center of a red copy of F_2 using the two disjoint red edges within A. Otherwise let u and v be two vertices with red edges to a part A and suppose a vertex $w \in A$ has two incident red edges within A, say to x and y. Then w is the center of a red copy of F_2 with triangles wux and wvy, a contradiction. □

If $k' = 0$, then any coloring of K_n with $n \geq 3$ contains a rainbow triangle. If $k' = 1$, then the monochromatic K_5 minus a (wasted) matching must contain a monochromatic copy of F_2.

Next suppose $k' = 2$ so $n = 11$, say with red and blue being the useful colors. Let p be the number of edges in wasted colors. If we contract the edges in wasted colors, we obtain a 2-colored graph of order $11 - p$ so if $p \leq 2$, this graph immediately contains a monochromatic copy of F_2 by Proposition 1. Thus, we must have $3 \leq p \leq 5$. With $k' = 2$, there is a Gallai partition of $V(G)$ (not necessarily the same one as considered before) in which each wasted edge is a part of order 2 and all other vertices are parts of order 1. It is this partition that we will consider for the remainder of the analysis of the case $k' = 2$.

If $p = 5$, then the reduced graph is a copy of K_6 where 5 of the vertices represent parts of order 2. In this reduced graph, there is a monochromatic triangle, say in red. Since at least two of the parts used in this triangle must have order 2, this produces a red copy of F_2.

Next suppose $p = 3$ and let H_1, H_2, and H_3 be the parts of order 2 formed by these wasted edges. To avoid creating a monochromatic copy of F_2, not all edges between these parts can be the same color so suppose H_2 is joined by red edges to $H_1 \cup H_3$ and the edges between H_1 and H_3 are blue. Let $R = V(G) \setminus (\cup_{i=1}^{3} V(H)_i)$ and observe that no vertex in R has blue edges to both H_1 and H_3 and no vertex in R has blue edges to both of either $\{H_1, H_2\}$ or $\{H_2, H_3\}$. Hence, every vertex in R has blue edges to H_2, so by Claim 1, there is at most one blue edge within R. This means that R induces a red K_5 minus an edge, which contains a red copy of F_2.

Finally suppose $p = 4$. Following the proof of the case $p = 3$ above by defining the parts H_1, H_2, and H_3 and the set R, we again see that R contains at most one blue edge along with one wasted edge. This means that R induces a red copy of K_5 minus the two disjoint edges, which again contains a red copy of F_2, completing the proof in the case $k' = 2$.

Thus, suppose for induction that $k' \geq 3$ and the result holds for smaller values of k'. By the minimality of t, if $t \leq 3$, then $t = 2$ so first suppose $t = 2$ say with red edges between the two parts. If both parts have order at least 2, then by Claim 1, red is wasted within each part. Then by induction on k', we have

$$|G| = |H_1| + |H_2| \leq 2(n(k' - 1) - 1) < n,$$

a contradiction. If one part has order 1, then by Claim 1, the other part A contains no red copy of $2K_2$. By removing a single vertex from A, we can remove all but at most one red edge from A, leaving red wasted within A. Then apply induction within A (minus that one vertex) to arrive at a contradiction. We may therefore assume $t \geq 4$ and there are two colors, say red and blue, that are both connected in the reduced graph.

Certainly $t \leq 8$ by Proposition 1 so $4 \leq t \leq 8$. If $t \geq 6$, then there exists a monochromatic triangle in the reduced graph of G. To avoid a copy of F_2, at least two of the parts represented in this monochromatic triangle must have order 1. Thus, if $t \geq 6$, then at most 4 parts have order at least 2, and red and blue are both wasted (possibly after the removal of one vertex) so we may apply induction on k' within these parts to arrive at a contradiction.

Finally suppose $t = 5$. If all parts have order at least 2, then red and blue are both wasted within all the parts. If a part has order 1, then by removing at most one vertex from all other parts, red and blue are both wasted within the remaining parts. In either case, we have

$$|G| = \sum_{i=1}^{t} |H_i| \leq 5(n(k'-2)-1)+5 < n(k'),$$

a contradiction, completing the proof of Lemma 2. $\square$

Lemma 3. *For all $k \geq 2$, we have*

$$gr_k(K_3 : F_2) \leq n_k = \begin{cases} 9 & \textit{if } k=2, \textit{ or} \\ \frac{83}{2}\cdot 5^{\frac{k-4}{2}} + \frac{1}{2} & \textit{if } k \geq 4 \textit{ is even, or} \\ 4 \cdot 5^{\frac{k-1}{2}} + 1 & \textit{if } k \textit{ is odd, } \ k \geq 5. \end{cases}$$

Proof. The case where $k = 2$ follows from the 2-color Ramsey number and the case where k is odd follows from Lemma 2 so suppose $k \geq 4$ is even. Let G be a Gallai coloring of the complete graph of order

$$n = n_k = \begin{cases} 9 & \text{if } k=2, \text{ or} \\ \frac{83}{2}\cdot 5^{\frac{k-4}{2}} + \frac{1}{2} & \text{if } k \geq 4 \text{ is even, or} \\ 4 \cdot 5^{\frac{k-1}{2}} + 1 & \text{if } k \text{ is odd, } \ k \geq 5. \end{cases}$$

which contains no monochromatic copy of F_2. Consider a Gallai partition, say with t parts where the partition is chosen so that t is minimum. Let red and blue be the colors used between parts of the partition. Then certainly $2 \leq t \leq 8$ since $R(F_2, F_2) = 9$. Suppose the parts of this partition are H_i for $1 \leq i \leq t$ and that $|H_i| \geq |H_{i+1}|$ for all i.

First suppose $k = 4$, so $n = 42$. If $t \leq 3$, then by the minimality of t, we may assume $t = 2$, say with all red edges in between the two parts. If $|H_2| \geq 2$, then by Claim 1, there is at most one red edge within either H_1 or H_2, meaning that the removal of at most one vertex leaves both parts with no red edges inside. This implies that

$$|G| = |H_1| + |H_2| \leq 2(n_{k-1} - 1) + 1 = 2n_{k-1} - 1 < n,$$

a contradiction. On the other hand, if $|H_2| = 1$, then there is no red copy of $2K_2$ within H_1 so the removal of at most 2 vertices within H_1 destroys all red edges within H_1. This implies that

$$|G| = |H_1| + 1 \leq [(n_{k-1} - 1) + 2] + 1 < n,$$

again a contradiction. We may therefore assume that $t \geq 4$. Since $R(F_2, F_2) = 9$, we have $4 \leq t \leq 8$. Let r be the number of "large" parts with order at least 2. To avoid creating a monochromatic copy of F_2, there can be no monochromatic triangle within the reduced graph corresponding to these large parts so this immediately means that $r \leq 5$.

Next we show a claim that will be helpful in the remainder of the proof.

Claim 2. *If G is a Gallai coloring of K_{10} containing at most one edge in each of two colors and all remaining edges in two other colors, then G contains a monochromatic copy of F_2. If G is a Gallai coloring of K_9 containing at most one edge in one color and all remaining edges in two other colors, then G contains a monochromatic copy of F_2.*

Proof. For the first statement, contract one of the two edges with a color that appears on only that edge to arrive in the situation of the second statement. For the second statement, let G be a Gallai coloring of K_9 in which there is at most one edge in one color (say green) and all remaining edges are either red or blue. Since $R(F_2, F_2) = 9$, there must exist such a green edge, say uv. Let A be the set of vertices with red edges to u and v and let B be the set of vertices with blue edges to u and v, and suppose, without loss of generality, that $|A| \geq |B|$. By Claim 1, there is at most one red edge in A and at most one blue edge in B.

If $|A| \geq 5$, then A contains a blue copy of K_5 minus at most one edge, which contains a blue copy of F_2, for a contradiction. This means we may assume that $|A| = 4$ and $|B| = 3$, say with $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3\}$. Since B contains at most one blue edge, suppose b_3 has red edges to $\{b_1, b_2\}$.

First suppose A contains no red edge. Then no vertex in B has two blue edges to A to avoid creating a blue copy of F_2. In particular, b_3 has at least 3 red edges to A. Then b_1 and b_2 each have at least two red neighbors in common with b_3, creating a red copy of F_2 centered at b_3. We may therefore assume that A contains a red edge, say a_1a_2, so all other edges within A are blue.

Next suppose B contains no blue edge. Then no vertex in B can have red edges to both a_1 and a_2, meaning that one of a_1 or a_2 (suppose a_1) has two blue edges to B, say to b_1 and b_2. To avoid creating a blue copy of F_2, this means that b_1 and b_2 must both have all red edges to $\{a_3, a_4\}$. Then to avoid a red copy of F_2, the vertex b_3 must have blue edges to $\{a_3, a_4\}$, meaning that b_3 must have red edges to both a_1 and a_2, making a red copy of F_2 for a contradiction. This means that B must contain a blue edge, say b_1b_2.

If b_3 has a red edge to either a_1 or a_2 (say a_1), then to avoid a red copy of F_2, a_1 has blue edges to both b_1 and b_2, creating a blue copy of F_2 centered at a_1. Thus, b_3 must have blue edges to both a_1 and a_2. To avoid a blue copy of F_2, this also implies that b_3 has red edges to $\{a_3, a_4\}$.

If b_1 has red edges to both a_3 and a_4, then to avoid a red copy of F_2 centered at b_3, we see that b_2 must have blue edges to $\{a_3, a_4\}$. This forms a blue copy of F_2 centered at b_2, for a contradiction. This means that b_1, and similarly b_2, must have exactly one red edge to $\{a_3, a_4\}$ and to avoid a red copy of F_2, these edges must go to the same vertex, say a_3. Both b_1 and b_2 must then have blue edges to a_4, forming a blue copy of F_2 centered at a_4, a contradiction completing the proof of Claim 2 □

We consider cases based on the value of r. Since $n > 9$, we certainly have $r \geq 1$.

Case 1. *$r = 5$.*

In this case, we must also have $t = 5$ since otherwise there must be a monochromatic triangle in the reduced graph which contains at least two vertices corresponding to parts of order at least 2, making a monochromatic copy of F_2 in G. Then the reduced graph must be the unique 2-coloring of K_5 with no monochromatic triangle. By Claim 1, there is at most one red edge and at most one blue edge within each part H_i. If there is a red (or blue) edge in a part H_i, then to avoid a red (respectively blue) copy of F_2, there can be no red (respectively blue) edge in any other part. Thus, at most one part can have one edge in each of red and blue and all other parts have at most one such edge. By Claim 2, we have

$$|G| = \sum_{i=1}^{t} |H_i| \leq 9 + 4 \cdot 8 = 41 < n,$$

a contradiction.

Case 2. $r = 4$.

To avoid a monochromatic copy of F_2 among the large parts, each of these parts must have red edges to some other large part and blue edges to some other large part. By Claim 1, this means that each part has at most one red and at most one blue edge. By Claim 2, this means that each part has order at most 9 so with at most 4 other parts (of order 1 each), we have

$$|G| = \sum_{i=1}^{t} |H_i| \leq 4 \cdot 9 + 4 = 40 < n,$$

a contradiction.

Case 3. $r = 3$.

To avoid a monochromatic copy of F_2, the subgraph of the reduced graph induced on the vertices corresponding to the three large parts must not be a monochromatic triangle. Without loss of generality, suppose all edges from H_2 to H_3 are blue while all edges from H_1 to $H_2 \cup H_3$ are red. By Claim 1, each of H_2 and H_3 contains at most one red and at most one blue edge so by Claim 2, we have $|H_2|, |H_3| \leq 9$.

By the minimality of t, there is at least one part of order 1 with blue edges to H_1, so by Claim 1, H_1 contains at most one red edge and no blue copy of $2K_2$. By removing at most one vertex from H_1, we can obtain a subgraph with at most one blue edge, meaning that $|H_1| \leq 10$. With at most 5 other parts (of order 1 each), this means that

$$|G| = \sum_{i=1}^{t} |H_i| \leq 2 \cdot 9 + 10 + 5 = 33 < n,$$

a contradiction.

Case 4. $r = 2$.

Suppose the edges between H_1 and H_2 are red, so by Claim 1, each of H_1 and H_2 contains at most one red edge. By the minimality of t, there exists at least one part of order 1 with blue edges to H_1 (and similarly to H_2). By removing at most one vertex from H_1, we can obtain a subgraph with at most one blue edge, meaning that $|H_1| \leq 10$ and similarly $|H_2| \leq 10$. With at most 6 other parts (of order 1 each), this means that

$$|G| = \sum_{i=1}^{t} |H_i| \leq 2 \cdot 10 + 6 = 26 < n,$$

a contradiction.

Case 5. $r = 1$.

By the minimality of t, there is at least one part of order 1 with red edges to H_1 and at least one part of order 1 with blue edges to H_1. By Claim 1, there is no red or blue copy of $2K_2$ within H_1 so by removing at most 2 vertices from H_1, we can obtain a subgraph with at most one red and at most one blue edge. By Claim 2, this means that $|H_1| \leq 9 + 2 = 11$. With at most 7 other parts (of order 1 each), this means that

$$|G| = \sum_{i=1}^{t} |H_i| \leq 11 + 7 = 18 < n,$$

a contradiction.

This completes the proof of the situation where $k = 4$. We may therefore assume that $k \geq 6$ for the remainder of the proof.

First suppose $t \leq 3$, so we may assume $t = 2$ by the minimality of t, say with parts H_1 and H_2 with all red edges in between them. Assume, for a moment, that $|H_i| \geq 2$ for each $i \in \{1, 2\}$. Then within H_1 and H_2, there can be a total of at most one red edge to avoid creating a red copy of F_2. Then by removing a single vertex from G, this red edge can be destroyed, leaving behind two parts each with no red edges. This means that

$$|G| = |H_1| + |H_2| \leq 2(n_{k-1} - 1) + 1 = 2\left(4 \cdot 5^{\frac{k-2}{2}}\right) < n_k,$$

a contradiction. On the other hand, if $|H_1| = 1$, then H_2 contains no red copy of $2K_2$ so by deleting at most two vertices from H_2, we can destroy all red edges from within H_2. This means that

$$|G| = |H_1| + |H_2| \leq 1 + (n_{k-1} - 1) + 2 = 4 \cdot 5^{\frac{k-2}{2}} + 3 < n_k,$$

again a contradiction. This means we may assume $4 \leq t \leq 8$.

Let r be the number of parts of the Gallai partition with order at least 2 and so $|H_r| \geq 2$ while $|H_{r+1}| = 1$. Certainly any monochromatic triangle among the parts of order at least 2 would create a monochromatic copy of F_2 so this means that $r \leq 5$. At the other extreme, if $r = 0$, then G is simply a 2-coloring so this is the case $k = 2$. We consider cases based on the value of r.

Case 1. $r = 5$.

In this case, we must also have $t = 5$ since otherwise there must be a monochromatic triangle in the reduced graph which contains at least two vertices corresponding to parts of order at least 2, making a monochromatic copy of F_2 in G. Then the subgraph of the reduced graph induced on the parts of order at least 2 must be the unique 2-coloring of K_5 containing no monochromatic triangle. Inside the parts $H_1, H_2, \ldots, H_5$, by Claim 1, there is a total of at most one red edge and at most one blue edge. Then by removing a total of at most two vertices, all red and blue edges can be removed from within the parts. By induction on k, this means that

$$\begin{aligned} |G| &= \sum_{i=1}^{5} |H_i| \\ &\leq 5(n_{k-2} - 1) + 2 \\ &= 5\left(\frac{83}{2} \cdot 5^{\frac{k-6}{2}} - \frac{1}{2}\right) + 2 \\ &= \frac{83}{2} \cdot 5^{\frac{k-4}{2}} - \frac{1}{2} \\ &< n, \end{aligned}$$

a contradiction.

Case 2. $r = 4$.

Within the reduced graph restricted to the parts of order at least 2, each vertex must have at least one incident red edge and at least one incident blue edge. By claim 1, inside the parts H_1, H_2, H_3, H_4, there can be a total of at most two red edges and at most two blue edges and the two cannot be adjacent. This means that by removing at most 4 vertices, all red and blue edges can be removed from within H_i for $i \leq 4$. With $t \leq 8$, this means

$$\begin{aligned} |G| &= \sum_{i=1}^{t} |H_i| \\ &\leq 4(n_{k-2} - 1) + 4 + 4 \\ &= 4\left(\frac{83}{2} \cdot 5^{\frac{k-6}{2}} - \frac{1}{2}\right) + 8 \\ &< n, \end{aligned}$$

a contradiction.

Case 3. $r = 3$.

Certainly all edges between the three parts of order at least 2 cannot have the same color to avoid a monochromatic copy of F_2. Without loss of generality (since we may safely disregard the relative orders among these large sets), suppose all edges from H_1 to $H_2 \cup H_3$ are red while all edges between H_2 and

H_3 are blue. Then by Claim 1, there is at most one red edge in either H_1 or in $H_2 \cup H_3$ so the removal of at most one vertex can destroy all red edges from within the parts. Similarly, there is also at most one blue edge within either H_2 or H_3 so the removal of at most one vertex can destroy all blue edges from within the parts H_2 and H_3.

By the minimality of t, there is a part (clearly of order 1 by the assumed structure) with blue edges to H_1. By Claim 1, this means that H_1 contains no blue copy of $2K_2$, so the removal of at most two vertices can destroy all blue edges from within H_1.

Together, the removal of at most 4 vertices can destroy all red and blue edges from within the parts. Since $t \leq 8$, there are at most 5 vertices in $G \setminus (H_1 \cup H_2 \cup H_3)$, meaning that

$$\begin{aligned} |G| &= \sum_{i=1}^{t} |H_i| \\ &\leq [3(n_{k-2} - 1) + 4] + 5 \\ &= 3\left(\frac{83}{2} \cdot 5^{\frac{k-6}{2}} - \frac{1}{2}\right) + 9 \\ &< n, \end{aligned}$$

a contradiction.

Case 4. $r = 2$.

Suppose red is the color of all edges between H_1 and H_2. By Claim 1, there can be at most one red edge within either H_1 or H_2 so the removal of a single vertex can destroy all red edges from within these parts. By the minimality of t, there is at least one part (single vertex) with all blue edges to H_1 and similarly at least one part with all blue edges to H_2. By Claim 1, there can be no blue copy of $2K_2$ within H_i for any $i \leq 2$ so that means the removal of at most 2 vertices from within each part H_i can destroy all blue edges within these parts. Finally, since $t \leq 8$, there can be at most 6 additional vertices in $G \setminus (H_1 \cup H_2)$ so this means that

$$|G| \leq |H_1| + |H_2| + 6 \leq [2(n_{k-2} - 1) + 5] + 6 < n,$$

a contradiction.

Case 5. $r = 1$.

Much like the arguments in the previous case, by the minimality of t, there is at least one part (single vertex) with all red edges to H_1 and at least one part with all blue edges to H_1. By Claim 1, this means that the removal of a total of at most 4 vertices can destroy all red and blue edges from within H_1. This yields

$$|G| \leq |H_1| + 7 \leq [(n_{k-2} - 1) + 4] + 7 < n,$$

a contradiction, completing the proof of Lemma 3. □

3 The case $n = 3$

We first prove a lower bound in the following lemma.

Lemma 4.

$$gr_k(K_3, F_3) \geq \begin{cases} 14 \times 5^{\frac{k-2}{2}} - 1, & \text{if } k \textit{ is even;} \\ 33 \times 5^{\frac{k-3}{2}}, & \text{if } k \textit{ is odd.} \end{cases}$$

Proof. We prove this result by inductively constructing a k-colored copy of K_n where

$$n = \begin{cases} 14 \times 5^{\frac{k-2}{2}} - 2, & \text{if } k \textit{ is even;} \\ 33 \times 5^{\frac{k-3}{2}} - 1, & \text{if } k \textit{ is odd,} \end{cases}$$

which contains no rainbow triangle and no monochromatic copy of F_3. For the base of this induction, let G_1 be a 1-colored copy of K_6, which clearly contains no monochromatic copy of F_3, say using color 1. Let G_2 be a 2-colored copy of K_{12}, constructed by making 2 copies of G_1 and inserting all edges of colors 2 in between the two copies. Let G_3^3 be a 3-colored copy of K_8 consisting of a monochromatic copy of C_5 in color 2, a monochromatic triangle in color 3, and all remaining edges in color 1. Then let G_3 be a 3-colored copy of K_{32} constructed by taking the union of G_3^3 with 4 disjoint copies of G_1 and inserting all edges of colors 2 and 3 in between these copies to form a blow-up of the unique 2-colored K_5 containing no monochromatic triangle. Despite the extra edges of colors 2 and 3 within the copy of G_3^3, it can be easily verified that G_3 contains no monochromatic copy of F_3 and certainly no rainbow triangle. See Figure 2 for a diagram of G_3 where solid edges are color 2, dashed edges are color 3, and all edges not pictured are color 1.

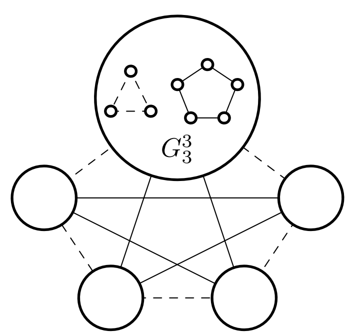

Figure 2: The coloring G_3

Let G_4^1 be a 3-colored copy of K_{14} constructed by taking a copy of G_1 and a copy of G_3^3 and inserting all edges of color 2 in between the two graphs. Let G_3^4 be a 3-colored copy of K_8 constructed by replacing the edges of color 3 in G_3^3 with edges of color 4. Let G_4^2 be a 3-colored copy of K_{14} constructed by taking a copy of G_1 and copy of G_3^4 and inserting all edges of colors 2 in between the two graphs. Finally let G_4 be a 4-colored copy of K_{68} constructed by making 2 copies of G_4^1, 2 copies of G_4^2, and one copy of G_2 and inserting edges of colors 3 and 4 between the five graphs to form the unique 2-colored K_5 with no monochromatic triangle. It can be easily verified that G_4 is a 4-coloring of K_{68} containing no monochromatic copy of F_3 and no rainbow triangle.

For $i \in \{4, 5\}$, let G_4^i be a colored copy of K_{34} constructed by taking a copy of G_3 and replacing one of its copies of G_1, one with edges of color 2 to the copy of G_3^3, with a copy of K_8 containing a monochromatic copy of C_5 in color i, a monochromatic triangle in color 3, and all remaining edges in color 1. Note that if a copy of G_1 with edges of color 3 to the copy of G_3^3 was replaced, then a monochromatic copy of F_3 would be created in the process (see Figure 3 where all edges pictured have color 3, the thicker ones producing the copy of F_3). The graph G_5, a copy of K_{164} using colors $\{1, 2, 3, 4, 5\}$, is then constructed by taking one copy of G_4^4, one copy of G_4^5, and 3 copies of G_3 and inserting edges of colors 4 and 5 between these 5 graphs to form a blow-up of the unique 2-colored K_5 with no monochromatic triangle. It is not difficult to verify that G_5 contains no monochromatic copy of F_3 and no rainbow triangle.

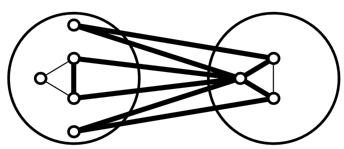

Figure 3: A copy of F_3 appearing in color 3

In fact, the argument above using Figure 3 yields the following easy fact.

Fact 1. *If X and Y are two parts of a Gallai partition with all red edges in between, each with at least 3 vertices, then if there is a vertex in X with at least two incident red edges inside X, then Y contains no red edges. Similarly, X cannot contain a vertex with 3 incident red edges.*

Another similar argument yields yet another easy fact to be used later.

Fact 2. *If X and Y are two parts of a Gallai partition with all red edges in between and there are two disjoint red edges inside X, then Y contains no red edges.*

First suppose k is even and suppose we have constructed a coloring G_{2i-2} of a complete graph where i is a positive integer and $2 \leq 2i - 2 < k$, using the $2i - 2$ colors $\{1, 2, \ldots, 2i - 2\}$ and having order $n_{2i-2} = 14 \times 5^{i-2} - 2$ such that G_{2i-2} contains no rainbow triangle and no monochromatic copy of F_3. For $j \in \{2i - 1, 2i\}$, let G_{2i-1}^j be a graph constructed from G_{2i-2} by changing one copy of G_1 used in the construction of G_2 (as part of the construction of G_{2i-2}) into a colored copy of K_8 containing a monochromatic C_5 in color 2 and a triangle in color j and all other edges in color 1. We then construct G_{2i} by taking 2 copies of G_{2i-1}^{2i-1}, 2 copies of G_{2i-1}^{2i}, and a copy of G_{2i-2}, and inserting all edges of colors $2i-1$ and $2i$ between the five graphs to form a blow-up of the unique 2-colored K_5 with no monochromatic triangle in such a way that the copies of G_{2i-1}^{2i-1} are connected by edges of color $2i$ and the copies of G_{2i-1}^{2i} are connected by edges of color $2i-1$. Then G_{2i} is a colored complete graph of order

$$n_{2i} = 2\left(14 \times 5^{i-2}\right) + 2\left(14 \times 5^{i-2}\right) + \left(14 \times 5^{i-2} - 2\right) = 14 \times 5^{i-1} - 2$$

containing no rainbow triangle and no monochromatic copy of F_3, as desired.

Finally suppose k is odd and again suppose we have constructed a coloring G_{2i-1} of a complete graph where i is a positive integer and $3 \leq 2i-1 < k$, using the $2i-1$ colors $\{1, 2, \ldots, 2i-1\}$ and having order $n_{2i-1} = 33 \times 5^{i-2} - 1$ such that G_{2i-1} contains no rainbow triangle and no monochromatic copy of F_3. For $j \in \{2i, 2i+1\}$, let G_{2i}^{j} be a graph constructed from G_{2i-1} by changing one copy of G_1 used in the construction of G_{2i-1} into a colored copy of K_8 containing a monochromatic C_5 in color 2 and a triangle in color j and all other edges in color 1. We then construct G_{2i+1} by taking a copy of G_{2i}^{2i+1}, a copy of G_{2i}^{2i}, and 3 copies of G_{2i-1}, and inserting all edges of colors $2i$ and $2i+1$ between the five graphs to form a blow-up of the unique 2-colored K_5 with no monochromatic triangle. Then G_{2i+1} is a colored complete graph of order

$$n_{2i+1} = \left(33 \times 5^{i-2} + 1\right) + \left(33 \times 5^{i-2} + 1\right) + 3\left(33 \times 5^{i-2} - 1\right) = 33 \times 5^{i-1} - 1$$

containing no rainbow triangle and no monochromatic copy of F_3, as desired. $\square$

Lemma 5. *For $k \geq 3$,*

$$gr_k(K_3; F_3) \leq \begin{cases} 14 \times 5^{\frac{k-2}{2}} - 1, & \text{if } k \text{ is even;} \\ 33 \times 5^{\frac{k-3}{2}}, & \text{if } k = 3, 5; \\ 33 \times 5^{\frac{k-3}{2}} + \frac{3}{4} \times 5^{\frac{k-5}{2}} - \frac{3}{4}, & \text{if } k \text{ is odd, } k \geq 7. \end{cases}$$

The proof of this lemma is similar (albeit more tiresome) to the proof of the upper bound presented in Lemma 3. We therefore omit the proof and provide it in an appendix for the interested reader.

4 For General F_n

First an easy lemma.

Lemma 6. *If G is a Gallai colored complete graph of order at least $4n-3$ in which all parts of a Gallai partition have order at most $n-1$ and all edges in between the parts of G have one color, say red, then G contains a red copy of F_n.*

Proof. Let $H_1, H_2, \ldots, H_t$ be the parts of the assumed partition, so since $|G| \geq 4n-3$, we see that $t \geq 5$. Since $|H_i| \leq n-1$, there exists an integer r and corresponding set of parts $H_2, H_3, \ldots, H_r$ such that $n \leq |H_2 \cup H_3 \cup \cdots \cup H_r| \leq 2n-2$. This, in turn, implies that $|H_{r+1} \cup H_{r+2} \cup \cdots \cup H_t| \geq n$. Then a single vertex from H_1 along with n red edges from $H_2 \cup H_3 \cup \cdots \cup H_r$ to $H_{r+1} \cup H_{r+2} \cup \cdots \cup H_t$ produces a red copy of F_n. $\square$

Theorem 7 is proven by the following two lemmas, one for the upper bound and one for the lower bound.

Lemma 7. *For $k \geq 2$,*

$$gr_k(K_3 : F_n) \leq \begin{cases} 10n \times 5^{\frac{k-2}{2}} - \frac{5}{2}n + 1, & \text{if } k \textit{ is even;} \\ \frac{9}{2}n \times 5^{\frac{k-1}{2}} - \frac{5}{2}n + 1, & \text{if } k \textit{ is odd.} \end{cases}$$

Proof. From Proposition 1, we have $4n+1 \leq R(F_n, F_n) \leq 6n$, and hence the result is true for $k=2$. We therefore suppose $k \geq 3$ and let G be a coloring of K_m where

$$m = m(k,n) = \begin{cases} 10n \times 5^{\frac{k-2}{2}} - \frac{5}{2}n + 1, & \text{if } k \textit{ is even;} \\ \frac{9}{2}n \times 5^{\frac{k-1}{2}} - \frac{5}{2}n + 1, & \text{if } k \textit{ is odd.} \end{cases}$$

Since G is a G-coloring, it follows from Theorem 1 that there is a Gallai partition of $V(G)$. Suppose that the two colors appearing in the Gallai partition are red and blue. Let t be the number of parts in this partition and choose such a partition where t is minimized. Let $H_1, H_2, \ldots, H_t$ be the parts of this partition, say with $|H_1| \geq |H_2| \geq \cdots \geq |H_t|$. When the context is clear, we also abuse notation and let H_i denote the vertex of the reduced graph corresponding to the part H_i.

If $2 \leq t \leq 3$, then by the minimality of t, we may assume $t=2$. Let H_1 and H_2 be the corresponding parts. Suppose all edges from H_1 to H_2 are red. To avoid creating a red copy of F_n, there are at most $n-1$ disjoint red edges in each H_i with $i=1,2$. Delete all the vertices of these maximum red matchings within H_1 and H_2 to create graphs H_1' and H_2', leaving no red edge within either H_i'. This means that

$$|G| = |H_1| + |H_2| \leq 2(m(k-1,n)-1) + (2n-2) < m,$$

a contradiction.

Let r be the number of parts of the Gallai partition with order at least n and call these parts "large" while other parts are called "small". Then $|H_r| \geq n$ and $|H_{r+1}| \leq n-1$. To avoid a monochromatic copy of F_n, there can be no monochromatic triangle within the reduced graph restricted to these r large parts, leading to the following immediate fact.

Fact 3. $r \leq 5$.

The remainder of the proof is broken into cases based on the value of r.

Case 1. $r = 0$.

Let A be the set of parts with blue edges to H_1, and B be the set of parts with red edges to H_1. Note that by minimality of t, we have $A \neq \emptyset$ and $B \neq \emptyset$. To avoid a blue copy of F_n, there are at most $n-1$ disjoint blue edges within A and similarly at most $n-1$ disjoint red edges within B. By removing at most $2n-2$ vertices from A (and at most $2n-2$ vertices from B), we remove all blue edges from A (respectively all red edges from B). Denote the resulting subgraphs by A' and B'. Then all the edges in between the parts of the Gallai partition of G that are contained in A' are red and all the edges in between the respective parts of B' are blue. From Lemma 6, we have $|A'| \leq 4n-4$ and $|B'| \leq 4n-4$. Then

$$|G| \leq |A'| + |B'| + |H_1| + 2(2n-2) \leq 8n-8+n-1+4n-4 = 13n-13 < m,$$

a contradiction.

Case 2. $r = 1$.

Let A be the set of parts with blue edges to H_1, and B be the set of parts with red edges to H_1. By the same argument as in Case 1, we may remove at most $2n-2$ vertices from each of A and B to produce sets A' and B' with containing no blue or red edges respectively, where $|A'| \leq 4n-4$ and $|B'| \leq 4n-4$. Since $A \neq \emptyset$ and $B \neq \emptyset$ and to avoid a monochromatic copy of F_n, there are at most $n-1$ disjoint red edges and at most $n-1$ disjoint blue edges within H_1. By removing at most $4n-4$ vertices from H_1, we eliminate all red and blue edges from H_1, leaving a new subgraph H_1'. This means that $|H_1'| \leq m(k-2,n)-1$ so

$$|G| \leq [|A'|+|B'|+(4n-4)]+[|H_1'|+(4n-4)] \leq 16n-17+[m(k-2,n)-1] < m,$$

a contradiction.

Case 3. $r = 2$.

Suppose all edges from H_1 to H_2 are red. To avoid creating a monochromatic copy of F_n, there is no part outside H_1 and H_2 with red edges to all of $H_1 \cup H_2$. Also since neither H_1 nor H_2 can contain more than $n-1$ disjoint red edges or more than $n-1$ disjoint blue edges, we have $|H_i| \leq m(k-2,n)-1+(4n-4)$, for $i = 1, 2$. Now a claim about parts other than H_1 and H_2.

Claim 3. *There exists a part, say H_3, such that the edges between H_1 and H_3 have a different color from the edges between H_2 and H_3.*

Proof. Assume, to the contrary, that for each part H_i $(3 \leq i \leq t)$, such that the edges between H_1 and H_3 and the edges between H_2 and H_3 receive same color (and therefore blue). Then we can regard $H_1 \cup H_2$ as one part, and the union of other parts as another part, of a new Gallai partition with only 2 parts, which contradicts the assumption that t is minimum and $t \geq 4$. □

By Claim 3, there exists a small part, say H_3, such that the edges between H_1 and H_3 receive different colors from the edges between H_2 and H_3. Let A be the set of parts with blue edges to H_3, and B be the set of parts with red edges to H_3. Without loss of generality, we assume that A contains H_1 and B contains H_2. There are at most $n-1$ disjoint blue edges within A and at most $n-1$ disjoint red edges within B. Let A' be the vertex set from A obtained by deleting at most $2n-2$ vertices from $A \setminus H_1$ on this blue matching, and let B' be the vertex set from B by deleting at most $2n-2$ vertices from $B \setminus H_2$ on this red matching. All edges in between the parts within A' are red and all edges in between the parts within B' are blue. Then we have the following claim.

Claim 4. *$|A'| - |H_1| \leq 2n-2$.*

Proof. Assume, to the contrary, that $|A'|-|H_1| \geq 2n-1$. Then there are at least 3 small parts in $A'-H_1$. Choose one of them, say X, and let $Y = A' - H_1 - X$. Clearly $|X| \leq n-1$, $|H_1| \geq n$, and $|Y| \geq n$. The edges between H_1 and X, the edges between H_1 and Y, the edges between X and Y are all red, and hence there is a blue F_n centered at a vertex of X, a contradiction. □

From Claim 4, we have $|A'| - |H_1| \leq 2n-2$ and symmetrically $|B'| - |H_2| \leq 2n-2$, so

$$\begin{aligned}
|A| + |B| &\leq |A'| + |B'| + (4n-4) \\
&\leq |H_1| + |H_2| + 8n - 8 \\
&\leq 2m(k-2,n) + 2(4n-4) + 8n - 8 \\
&\leq 2m(k-2,n) + 16n - 16.
\end{aligned}$$

Since $|H_3| \leq n-1$, it follows that

$$|G| \leq 17n - 17 + 2m(k-2,n) - 1] < m,$$

a contradiction.

Case 4. $r = 5$.

In this case, $t = 5$ since otherwise any monochromatic triangle in the reduced graph restricted to $H_1, H_2, \ldots, H_6$ would yield a monochromatic copy of F_n. To avoid the same construction, the reduced graph on the parts H_1, H_2, H_3, H_4, H_5 must be the unique 2-coloring of K_5 with no monochromatic triangle, say with $H_1H_2H_3H_4H_5H_1$ and $H_1H_3H_5H_2H_4H_1$ making two monochromatic cycles in red and blue respectively. In order to avoid a red copy of F_n with center vertex in H_1, it must be the case that $H_2 \cup H_5$ contains at most $n-1$ disjoint red edges. Similarly $H_1 \cup H_3$, $H_2 \cup H_4$, $H_3 \cup H_5$, and $H_4 \cup H_1$ each contain at most $n-1$ disjoint red edges. Putting these together, there are at most $\frac{5n-5}{2}$ disjoint red edges within the parts $H_1, H_2, \ldots, H_5$. Thus, by deleting at most $5n-5$ vertices, the resulting graph can be devoid of red edges and symmetrically, by deleting at most another $5n-5$ vertices, the resulting graph can also be devoid of blue edges. This means that

$$|G| \leq 10n - 10 + 5[m(k-2,n)-1] < m,$$

a contradiction.

Case 5. $r = 4$.

To avoid monochromatic triangle in K_4, the four large parts must form one of two structures:

- Type 1: There is a red cycle $H_1H_2H_3H_4H_1$ and a blue 2-matching $\{H_1H_3, H_2H_4\}$ in the reduced graph, or
- Type 2: There is a red path $H_2H_1H_4H_3$ and a blue path $H_1H_3H_2H_4$ in the reduced graph.

For Type 1, we first have the following claim.

Claim 5. *There is no small part outside* $\{H_1, H_2, H_3, H_4\}$.

Proof. Assume, to the contrary, that there exists a small part H_5 in G. This proof focuses on the reduced graph. Since H_1H_3 is blue, it follows that to avoid a blue triangle in the reduced graph and thereby a blue copy of F_n in G, at least one of H_1H_5 and H_3H_5 must be red, say H_1H_5 is red. Since H_1H_2 and H_1H_4 are red, it follows that H_2H_5 and H_4H_5 must be blue, and hence $H_2H_4H_5H_2$ is a blue triangle, a contradiction. □

By Claim 5, there are only four parts in G and they are large. Recall that $H_1H_2H_3H_4H_1$ is a red cycle and $\{H_1H_3, H_2H_4\}$ is a blue 2-matching. We can then regard $H_1 \cup H_3$ and $H_2 \cup H_4$ as two parts of a Gallai partition of G and the edges between these parts are all red, which contradicts the minimality of t.

For Type 2, we first consider the case where $t \geq 5$. Outside $\{H_1, H_2, H_3, H_4\}$, there are small parts $H_5, H_6, \ldots, H_t$. For each such part H_i with $5 \leq i \leq t$, since H_2H_3 is blue, to avoid a blue triangle of the form $H_2H_4H_iH_2$, at least one of the edges H_2H_i and H_3H_i must be red.

First suppose one is red, say H_2H_i is red and H_3H_i is blue. Since H_1H_2 and H_2H_i are red, it follows that H_1H_i must be blue, and hence $H_1H_3H_iH_1$ is a blue triangle, a contradiction.

We may therefore assume that for all H_i with $5 \leq i \leq t$, we have that the edges H_2H_i and H_3H_i are red. To avoid a red triangle, the edges H_1H_i and H_4H_i are blue. By minimality of t, we have $t = 5$ since all parts H_i for $i \geq 5$ have the same color on edges to H_j for $j \leq 4$. Clearly, $H_1H_2H_5H_3H_4H_1$ is a red cycle and $H_1H_5H_4H_2H_3H_1$ is a blue cycle. We may then apply the same arguments as in Case 4 to arrive at a contradiction.

We may therefore assume that $t = r = 4$. Since the edges H_1H_2 and H_1H_4 are red, there are at most $n-1$ independent red edges within $H_2 \cup H_4$, so by deleting $2n-2$ vertices in $H_2 \cup H_4$, there are no red edges remaining in H_2 and no red edges in H_4. Similarly, by deleting $2n-2$ vertices in $H_1 \cup H_3$, there are no red edges remaining in H_2 and no red edges remaining in H_4. Symmetrically, if we delete $4n-4$ vertices in $H_1 \cup H_2 \cup H_3 \cup H_4$, there are no blue edges in H_i for $1 \le i \le 4$. This means that

$$|G| \le 4[m(k-2, n) - 1] + 8n - 8 < m,$$

a contradiction.

Case 6. $r = 3$.

The triangle in the reduced graph cannot be monochromatic so without loss of generality, suppose all edges from H_1 to $H_2 \cup H_3$ are red, and H_2H_3 is blue. To avoid a red or blue triangle, any remaining parts are partitioned into the following sets.

- Let A be the set of parts outside H_1, H_2, H_3 each with all blue edges to H_1, H_3 and all red edges to H_2,
- Let B be the set of parts outside H_1, H_2, H_3 each with all red edges to H_2, H_3 and all blue edges to H_1,
- Let C be the set of parts outside H_1, H_2, H_3 each with all blue edges to H_1, H_2 and all red edges to H_3.

Note that $|G| = |A| + |B| + |C| + |H_1| + |H_2| + |H_3|$ (see Figure 4).

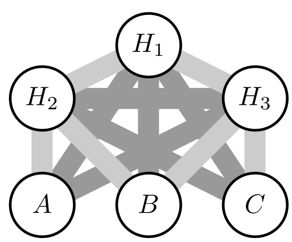

Figure 4: Structure of G

We first consider the subcase $B \neq \emptyset$. Then we have the following claims.

Claim 6. $|A| + |C| \le 2n - 2$.

Proof. Assume, to the contrary, that $|A| + |C| \ge 2n - 1$ and let $v \in B$. Note that each edge from B to $A \cup C$ is either red or blue. If there is a vertex $v \in B$ with at least n red edges to $A \cup C$, then these edges along with the red edges from H_3 to $B \cup C$ and the red edges from H_2 to $B \cup A$ form a red copy of F_n. This means that v must have at least n blue edges to $A \cup C$. Then these edges along with the blue edges from H_1 to $A \cup C$ form a blue copy of F_n for a contradiction. $\square$

Claim 7. $|B| \leq 2n-2$.

Proof. Assume, for a contradiction, that $|B| \geq 2n-1$. Then each edge from B to $A \cup C$ is red or blue. If $A \cup C \neq \emptyset$, let $v \in A \cup C$. By the same argument as in the proof of Claim 6, there is a monochromatic copy of F_n, so we may assume $A \cup C = \emptyset$. By minimality of t, we have that B must be a single (small) part of the partition, and so $|B| \leq n-1$, a contradiction. $\square$

By Claims 6 and 7, we have $|A| + |C| \leq 2n-2$ and $|B| \leq 2n-2$. Since all edges from H_1 to $H_2 \cup H_3$ are red, there can be at most $n-1$ disjoint red edges within $H_2 \cup H_3$. It follows that by deleting at most $2n-2$ vertices in $H_2 \cup H_3$, there will be no red edges remaining in $H_2 \cup H_3$. Similarly, by deleting at most $2n-2$ vertices in each of H_2 and H_3, there will be no blue edges remaining in H_2 or H_3. Also, by deleting at most $4n-4$ vertices from H_1, there will be no red edges or blue edges remaining in H_1. Putting these together, by deleting at most $10n-10$ vertices from $H_1 \cup H_2 \cup H_3$, there are no red and blue edges remaining in any of H_1, H_2 or H_3. This means that

$$\begin{aligned} |G| &= |A| + |B| + |C| + |H_1| + |H_2| + |H_3| \\ &\leq (4n-4) + (10n-10) + 3[m(k-2,n)-1] \\ &< m, \end{aligned}$$

a contradiction.

Finally, we consider the subcase $B = \emptyset$. First two claims about A and C.

Claim 8. $|A \cup C| \leq 6n-6$.

Proof. Since H_1 has all blue edges to $A \cup C$, there are at most $n-1$ disjoint blue edges within $A \cup C$. Deleting $2n-2$ vertices from $A \cup C$ produces a new subgraph, say D, with no blue edges. The Gallai partition of G restricted to D must therefore have all small parts and red edges in between the parts so by Lemma 6, $|D| \leq 4n-4$. This, in turn, means that $|A \cup C| \leq (4n-4) + (2n-2) = 6n-6$. $\square$

Claim 9. *$A \neq \emptyset$ and $C \neq \emptyset$.*

Proof. First suppose that both $A = \emptyset$ and $C = \emptyset$. Then $G = H_1 \cup H_2 \cup H_3$. With only 3 parts, this Gallai partition can be reduced down to 2 parts, contradicting the assumptions of this case.

Then suppose, without loss of generality, that $C = \emptyset$ and $A \neq \emptyset$. With exactly 4 parts in the partition, we may apply the same argument as the last part of Case 5. $\square$

We may delete at most $2n-2$ vertices in H_1 and at most $2n-2$ vertices in $H_2 \cup H_3$ and leave behind no red edges within H_1 or within $H_2 \cup H_3$. Also since Claim 9 gives $A \neq \emptyset$ and $C \neq \emptyset$, by deleting at most $2n-2$ vertices in $H_1 \cup H_3$ and at most $2n-2$ vertices in $H_1 \cup H_3$, there must be no blue edges remaining in either $H_1 \cup H_2$ or $H_1 \cup H_3$. This comes to a total of $8n-8$ removed vertices, meaning that

$$\begin{aligned} |G| &= |A| + |C| + |H_1| + |H_2| + |H_3| \\ &\leq (6n-6) + (8n-8) + 3[gr_{k-2}(K_3 : F_n) - 1] \\ &< m, \end{aligned}$$

a contradiction, completing the proof of Lemma 7. $\square$

Finally the lower bound lemma.

Lemma 8. *For $k \geq 2$,*

$$gr_k(K_3; F_n) \geq \begin{cases} 4n \times 5^{\frac{k-2}{2}} + 1, & \text{if } k \textit{ is even,} \\ 2n \times 5^{\frac{k-1}{2}} + 1, & \text{if } k \textit{ is odd.} \end{cases}$$

Proof. We prove this result by inductively constructing a coloring of K_n where

$$n = \begin{cases} 4n \times 5^{\frac{k-2}{2}}, & \text{if } k \textit{ is even,} \\ 2n \times 5^{\frac{k-1}{2}}, & \text{if } k \textit{ is odd,} \end{cases}$$

which contains no rainbow triangle and no monochromatic copy of F_3. Let G_1 be a 1-colored complete graph on $2n$ vertices, most notably too small to contain a copy of F_n. Without loss of generality, suppose this coloring uses color 1.

Suppose we have constructed a coloring of G_{2i-1} where i is a positive integer and $i \geq 2$, with $2i-1 < k$, using the $2i-1$ colors $1, 2, \ldots, 2i-1$ and having order $n_{2i-1} = 2n \times 5^{i-1}$ such that G_{2i-1} contains no rainbow triangle and no monochromatic copy of F_n.

If $k = 2i$, we construct $G_{2i} = G_k$ by making two copies of G_{2i-1} and inserting all edges in between the copies in color k. This coloring clearly contains no rainbow triangle and no monochromatic copy of F_n and has order

$$n = 2 \cdot 2n \cdot 5^{\frac{k-2}{2}} = 4n \times 5^{\frac{k-2}{2}},$$

as claimed.

Otherwise, suppose $k \geq 2i+1$. We construct G_{2i+1} by making five copies of G_{2i-1} and inserting edges of colors $2i$ and $2i+1$ between the copies to form a blow-up of the unique 2-colored K_5 with no monochromatic triangle. This coloring clearly contains no rainbow triangle and there is no monochromatic triangle in either of the two new colors so there can be no monochromatic copy of F_n in G_{2i+1}. With

$$|G_{2i+1}| = 5 \cdot 2n \cdot 5^{i-1} = 2n \times 5^{\frac{k-1}{2}},$$

as claimed, completing the proof. $\square$

References

[1] K. Cameron and J. Edmonds. Lambda composition. *J. Graph Theory*, 26(1):9–16, 1997.

[2] S. Fujita, C. Magnant, and K. Ozeki. Rainbow generalizations of Ramsey theory: a survey. *Graphs Combin.*, 26(1):1–30, 2010.

[3] S. Fujita, C. Magnant, and K. Ozeki. Rainbow generalizations of Ramsey theory - a dynamic survey. *Theo. Appl. Graphs*, 0(1), 2014.

[4] T. Gallai. Transitiv orientierbare Graphen. *Acta Math. Acad. Sci. Hungar*, 18:25–66, 1967.

[5] S. K. Gupta, L. Gupta, and A. Sudan. On Ramsey numbers for fan-fan graphs. *J. Combin. Inform. System Sci.*, 22(2):85–93, 1997.

[6] A. Gyárfás and G. Simonyi. Edge colorings of complete graphs without tricolored triangles. *J. Graph Theory*, 46(3):211–216, 2004.

[7] Y. Li and C. C. Rousseau. Fan-complete graph Ramsey numbers. *J. Graph Theory*, 23(4):413–420, 1996.

[8] Q. Lin and Y. Li. On Ramsey numbers of fans. *Discrete Appl. Math.*, 157(1):191–194, 2009.

[9] H. Liu, C. Magnant, A. Saito, I. Schiermeyer, and Y. Shi. Gallai-Ramsey number for K_4. *Submitted*.

[10] C. Magnant and I. Schiermeyer. Gallai-Ramsey number for K_5. *Submitted*.

[11] S. P. Radziszowski. Small Ramsey numbers. *Electron. J. Combin.*, 1:Dynamic Survey 1, 30 pp. (electronic), 1994.

[12] Z. Wang, Y. Mao, C. Magnant, I. Schiermeyer, and J. Zou. Gallai-Ramsey numbers of odd cycles. *Submitted*.

[13] F. Zhang, Chen Y., and Z. Song. Gallai-ramsey numbers of cycles. *Submitted*.

Appendix A Proof of Lemma 5

Proof. Define the function

$$g(k) = \begin{cases} 14 \times 5^{\frac{k-2}{2}} - 1, & \text{if } k \textit{ is even}; \\ 33 \times 5^{\frac{k-3}{2}}, & \text{if } k = 3, 5; \\ 33 \times 5^{\frac{k-3}{2}} + a \times 5^{\frac{k-5}{2}} - a, & \text{if } k \textit{ is odd},\ k \geq 7, \end{cases}$$

where $a > \frac{3}{4}$.

The goal of this lemma is to show that

$$gr_k(K_3 : F_3) \leq g(k).$$

We prove this upper bound by induction on k. The case $k = 1$ is trivial and the case $k = 2$ is precisely $R(F_3, F_3) = 13$. We therefore suppose $k \geq 3$ and let G be a coloring of K_n where $n = g(k)$.

Since G is a Gallai coloring, it follows from Theorem 1 there is a Gallai partition of $V(G)$. Suppose red and blue are the two colors appearing on edges between parts in the Gallai partition. Let t be the number of parts in the partition and choose such a partition where t is minimized. Since $R(F_3, F_3) = 13$, the reduced graph must have at most 12 vertices so $t \leq 12$. Let r be the number of parts of the Gallai partition with order at least 3. Let H_i be the parts of this Gallai partition and, without loss of generality, suppose that $|H_i| \geq |H_{i+1}|$ for all i. This means that $|H_r| \geq 3$ and $|H_{r+1}| \leq 2$.

First an easy fact that will be used throughout the proof.

Fact 4. *If X and Y are two (non-empty) parts of a Gallai partition, say with all red edges in between them, then the subgraph of Y (and similarly X) containing precisely the red edges contains no monochromatic copy of $3K_2$. This means that the removal of at most 4 vertices from Y yields a subgraph with no red edges.*

Indeed, otherwise there would be a red copy of F_3 centered in X.

We first consider the case $k = 3$, so $n = 33$. If $2 \leq t \leq 3$, then by the minimality of t, we may assume $t = 2$, say with corresponding parts H_1 and H_2. Without loss of generality, suppose all edges between H_1 and H_2 are blue. Since $n \geq 33$, we must have $|H_1| \geq 17$. By Fact 4, the subgraph of H_1 containing precisely the blue edges contains no copy of $3K_2$. We may therefore delete at most 4 vertices from H_1 so that H_1 no longer contains any blue edges. This yields a 2-colored K_{13}, but since $R(F_3, F_3) = 13$, it follows that there is a monochromatic F_3 within H_1, a contradiction. This implies that $t \geq 4$.

If $r \geq 5$ and $t \geq 6$, then any choice of 6 parts containing the 5 parts $\mathscr{H} = \{H_1, \ldots, H_5\}$ will contain a monochromatic triangle in the corresponding reduced graph. Such a triangle must contain at least 2 parts from $\mathscr{H}$. The corresponding subgraph of G must therefore contain a monochromatic copy of F_3, a contradiction. Thus, we may assume that either $4 \leq t \leq 5$ or $r \leq 4$. Furthermore, we have the following easy tools.

Claim 10. *If $t \geq 9$, then there are at most 7 parts of order at least 2.*

Proof. Suppose that there are at least 8 parts of order at least 2, say $\mathscr{H} = \{H_1, H_2, \ldots, H_8\}$. Then any choice 9 parts containing $\mathscr{H}$ will contain a monochromatic copy of F_2 in the reduced graph. Note that such a copy of F_2 must contain at least 4 vertices corresponding to parts from $\mathscr{H}$. This means that the corresponding subgraph of G must contain a monochromatic copy of F_3, a contradiction. □

Claim 11. *If X and Y are two parts of a Gallai partition of a graph with no monochromatic copy of F_3, say with all red edges in between them, and $|X| \geq 3$, then the subgraph of Y containing precisely the red edges is a subgraph of C_4, C_5, or $2K_3$.*

Proof. Then in order to avoid creating a red copy of F_3 centered in Y using the red edges to X as in Figure 5, the subgraph of Y containing precisely the red edges has maximum degree at most 2. By Fact 4, this red subgraph of Y also contains no copy of $3K_2$. Thus, the subgraph induced by the red edges within Y must be a subgraph of C_4, C_5, or $2K_3$. □

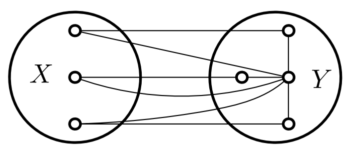

Figure 5: A red copy of F_3

This also leads to another related claim.

Claim 12. *In any Gallai 3-colored K_9 using colors $1, 2, 3$ in which the subgraph containing precisely those edges of color 1 and the subgraph containing precisely those edges of color 2 each are subgraphs of C_4, C_5, or $2K_3$, there must be a monochromatic copy of F_3 in color 3.*

Proof. Let G be a 3-coloring of K_9, say using red (color 1), blue (color 2), and green (color 3). Let G_R, G_B, and G_G be the subgraphs of G containing precisely the red, blue, and green edges respectively and suppose each of G_R and G_B are subgraphs of C_4, C_5, or $2K_3$. Then since $\Delta(G_R), \Delta(G_B) \leq 2$, we have $\delta(G_G) \geq 4$.

First suppose $|G_R \cup G_B| \leq 8$ so there is at least one vertex $w \in G$ with no incident red or blue edges and let $H = G_G \setminus w$. If H is 2-connected, then the circumference of H is at least $\min\{2\delta(H), |H|\} \geq 6$ so H contains a copy of $3K_2$. This along with w forms a green copy of F_3 for a contradiction, so H is not 2-connected. Then G_R and G_B cannot be subgraphs of C_4, C_5, or $2K_3$.

Next suppose $|G_R \cup G_B| = 9$ so $\Delta(G_G) \leq 7$. If there is a vertex v with degree 2 in one of red or blue (say blue) and degree at least 1 in the other color in {red, blue} (so red), then to avoid a rainbow triangle, either red and blue must be a subgraph of the unique 2-coloring of K_5 with no monochromatic triangle or G_B must be a C_4 and a chord of this C_4 must be red. The former case contradicts $|G_R \cup G_B| = 9$ so G_B must be a copy of C_4. With one red edge as a chord of the blue C_4 and the remaining red edges disconnected from this red edge, the remaining red edges must induce a graph on at most 3 vertices, again contradicting the assumption that $|G_R \cup G_B| = 9$. This means there can be no vertex with red (or blue) degree 2 and blue (respectively red) degree 1.

If there is a vertex v' with one incident edge in each of red and blue, then to avoid a rainbow triangle, the edge between those neighbors must be either red or blue. Since the maximum degree of red and blue is at most 2 and to avoid a vertex v as above, these three vertices can have no more incident red or blue edges. This means that $G_R \cup G_B$ must be disconnected. With so many restrictions, the only way for $|G_R \cup G_B| = 9$ is if the red (or blue) graph is a spanning subgraph of C_4 and the blue (respectively red) graph is a spanning subgraph of C_5 and these are disjoint. Such a coloring of K_9 clearly contains a green copy of F_3 to complete the proof of Claim 12. $\square$

When only two colors are present, we get even more by a similar argument.

Fact 5. *In any Gallai* 2*-colored* K_7 *using colors* $1, 2$ *in which the subgraph containing precisely those edges of color* 1 *is a subgraph of* C_4*,* C_5*, or* $2K_3$*, there must be a monochromatic copy of* F_3 *in color* 2*.*

We consider cases based on the value of r. If $r = 0$, then since $n = 33$, there are at least $17 > R(F_3, F_3)$ parts, a contradiction. If $r = 1$, then by Claim 10, there are at most 6 parts of order 2. With a total of $t \leq 12$ parts, there can be at most 11 parts of order at most 2. Since $n = 33$, we have $|H_1| \geq 16$ and H_1 has incident edges to other parts in both red and blue. Thus, by Fact 4, H_1 contains no $3K_2$ in blue or red. By deleting at most 8 vertices from H_1, what remains of H_1 contains no blue edges and no red edges. This yields a 1-colored copy of K_8, which contains a monochromatic F_3, a contradiction.

We may therefore assume that $2 \leq r \leq 5$. We distinguish the following cases to complete the proof.

Case 1. $r = 2$.

Suppose that blue is the color of the edges between H_1 and H_2. By Claim 10, there cannot be many vertices in small parts, so $|H_1| + |H_2| \geq 18$. To avoid a blue F_3, there is no part outside H_1 and H_2 with blue edges to $H_1 \cup H_2$.

If there is a part in $G \setminus (H_1 \cup H_2)$ with all red edges to $H_1 \cup H_2$, then the subgraph induced by red edges in $H_1 \cup H_2$ contains no $3K_2$. Since $|H_1| + |H_2| \geq 18$, deleting at most 4 vertices in $H_1 \cup H_2$ results in a 2-colored of order at least 14. This contains a monochromatic copy of F_3 since $R(F_3, F_3) = 13$, a contradiction. Thus, each part other than H_1 and H_2 has both red and blue edges to $H_1 \cup H_2$.

Let A be the set of parts with red edges to H_1 and blue edges to H_2 and let B be the set of parts with blue edges to H_1 and red edges to H_2. By the minimality of t, we must have $A \neq \emptyset$ and $B \neq \emptyset$. Since $|H_1| + |H_2| \geq 18$, we must have $|H_1| \geq 9$. Next suppose that $|A| \geq 3$. Then by Claims 11 and 12, H_1 contains a monochromatic copy of F_3, a contradiction. We may therefore assume that $|A| \leq 2$.

Next suppose that $|B| \leq 10$, so $|A| + |B| \leq 12$ and $|H_1| + |H_2| \geq 21$. Since the subgraph induced by the red edges within each of H_1 and H_2 contains no copy of $3K_2$, we may delete at most 8 vertices from $H_1 \cup H_2$ (at most 4 vertices from each of H_1 and H_2) so that what remains of $H_1 \cup H_2$ contains no red edges. This yields a 2-colored K_{13}, which contains a monochromatic copy of F_3 since $R(F_3, F_3) = 13$. We may therefore assume that $|B| \geq 11$.

Furthermore, with $|B| \geq 11$, by the same argument as above (applying Claim 12), we must have $|H_2| \leq 8$. This, in turn, means that $|H_1| \geq 10$ so $|H_1 \cup B| \geq 21$. Since $A \neq \emptyset$, we may remove at most 8 vertices from $H_1 \cup B$ (at most 4 from each of H_1 and B) to obtain a subgraph of order at least 13 containing no red edges. Since $R(F_3, F_3) = 13$, this subgraph contains a monochromatic copy of F_3 for a contradiction, completing the proof of Case 1.

Case 2. $r = 3$.

Disregarding the relative orders of the parts H_1, H_2, and H_3 for this case, we may suppose without loss of generality, that the edges from H_2 to H_3 are red and all edges from H_1 to $H_2 \cup H_3$ are blue since a monochromatic triangle among these large parts would produce a monochromatic copy of F_3. We first claim that there is no part outside H_1, H_2, and H_3 with blue edges to H_1 so suppose, to the contrary, that there is such a part, say H', with blue edges to H_1. To avoid a blue triangle in the reduced graph, all edges from H' to $H_2 \cup H_3$ must be red. Then H' together with H_2 and H_3 yields a red triangle in the reduced graph, yielding a red F_3 in G, a contradiction. There can therefore be no such part H' with blue edges to H_1.

Thus all vertices outside $H_1 \cup H_2 \cup H_3$ have red edges to H_1. Let A be the set of parts with blue edges to H_2 and with red edges to H_3. Let B be the set of parts with red edges to H_2 and with blue edges to H_3. Let C be the set of parts with blue edges to $H_2 \cup H_3$. Suppose, for a contradiction, that $|H_2 \cup H_3| \geq 17$. Since the subgraph induced by blue edges within $H_2 \cup H_3$ contains no $3K_2$, by deleting at most 4 vertices from $H_2 \cup H_3$ we obtain a subgraph of $H_2 \cup H_3$ containing no blue edges. This yields a 2-colored copy of K_{13}, a contradiction since $R(F_3, F_3) = 13$. This means $|H_2 \cup H_3| \leq 16$.

First we consider the case when $C \neq \emptyset$. We claim that each vertex of C has at most 2 incident edges in each of red and blue to $A \cup B$. Otherwise suppose that there is a vertex, say $u \in C$, with 3 incident edges in either red or blue, say in red, to $A \cup B$. Then these 3 edges along with H_1 yields a red copy of F_3 centered at u, a contradiction. This means that $|A| + |B| \leq 4$, and symmetrically, that $|C| \leq 4$. Then $|H_1| + |H_2| + |H_3| \geq 25$. Since $|H_2| + |H_3| \leq 16$, we have $|H_1| \geq 9$. If $|A| + |B| + |C| \geq 3$, then by the same arguments used above, the subgraphs of H_1 induced by red and blue edges must be subgraphs of C_5 so by Claim 12, there exists a monochromatic copy of F_3. On the other hand, if $|A| + |B| + |C| \leq 2$, then $|H_1| \geq 15$. Since the subgraphs of H_1 induced by red edges and blue edges each contain no monochromatic copy $2K_2$, by deleting at most 8 vertices from H_1 to remove all red and blue edges, we obtain a monochromatic K_7 and so a monochromatic copy of F_3, a contradiction.

We may therefore suppose $C = \emptyset$. By minimality of t, we also have $A \neq \emptyset$ and $B \neq \emptyset$. We first prove a claim.

Claim 13. $|A| \leq 5$ *and* $|B| \leq 5$.

Proof. We focus on A but the same argument holds for B. Suppose, for a contradiction, that $|A| \geq 6$ and let v be a vertex in a smallest part H_A^0 within A. Since A consists only of parts of order 1 or 2, there are at least 3 parts within A. If $|A| \geq 7$, then there are at least 5 vertices in $A \setminus H_A^0$ so v has at least 3 incident edges in either red or blue. Then using the red or blue edges to H_1 or respectively to H_2, v is the center of a red or blue copy of F_3. This means that we may assume that $|A| = 6$ and that A consists of exactly 3 parts each of order 2. At least one of these three parts, say H_A^1, has all one color, red or blue, to the other two parts by the definition of the Gallai partition. Then for any vertex $v \in H_A^1$, using the red or blue edges to H_1 or respectively to H_2, v is again the center of a red or blue copy of F_3, for a contradiction. □

Next suppose $|A \cup B| \leq 4$. Since $n = 33$, we have $|H_1| + |H_2| + |H_3| \geq 29$. Additionally since $|H_2| + |H_3| \leq 16$, we also have $|H_1| \geq 13$. Since $A \cup B \neq \emptyset$, by Fact 4, the subgraph of H_1 containing precisely the red edges contains no $3K_2$ and by Claim 11, the subgraph of H_1 containing precisely the blue edges has order at most 5 and maximum degree at most 2. We can therefore delete at most 4 vertices from H_1 such that what remains of H_1 contains no red edges. By Claim 12, the resulting 2-colored copy of K_9 must contain a monochromatic F_3, a contradiction.

Thus we may assume that $|A \cup B| \geq 5$. Then at least one of A or B has order at least 3, say $|A| \geq 3$. To avoid a red or blue copy of F_3, by Claim 11, for each i with $1 \leq i \leq 3$, the subgraph of H_i containing precisely the red (or similarly blue) edges is a subgraph of C_4, C_5, or $2K_3$. By Claim 12, we know that $|H_i| \leq 8$ so $|H_1| + |H_2| + |H_3| \leq 24$, meaning that $|A \cup B| \geq 9$ so one of A or B has order 5. On the other hand, since $n = 33$ and $|A \cup B| \leq 10$, we must have $|H_1 \cup H_2 \cup H_3| \geq 23$ so $7 \leq |H_i| \leq 8$ for all i. Without loss of generality, suppose $|A| = 5$ so A consists of at least 3 parts of the Gallai partition, each of order at most 2. By Claim 11, there is no vertex in A with red or blue degree at least 3 so the only possible configuration is for A to be the unique 2-colored K_5 with no monochromatic K_3 using red and blue. To avoid creating a blue copy of F_3 centered in H_2, H_1 must have no blue edges and by Claim 11, any red edges in H_1 must be a subgraph of C_4, C_5, or $2K_2$. Since H_1 is a 2-colored complete graph of order at least 7 where the subgraph containing precisely the red edges is a subgraph of C_4, C_5, or $2K_3$, we see that H_1 contains a green copy of F_3, completing the proof of Case 2.

Case 3. $r = 4$.

For the proof of this case, we disregard the relative ordersof the parts $|H_i|$ for $i \leq 4$. In order to avoid a monochromatic triangle within the reduced graph restricted to the 4 largest parts, up to symmetry, we may assume that either

1. all edges from $H_1 \cup H_2$ to $H_3 \cup H_4$ are red with all remaining edges between the parts being blue, or

2. all edges from H_i to H_{i+1} are red for $1 \leq i \leq 3$ and all remaining edges between the parts are blue.

In either coloring, by Claims 11 and 12, we have $|H_i| \leq 8$ for all i. Let A be the set of vertices outside $\cup_i H_i$.

For the first coloring, any vertex of A must form a monochromatic triangle with at least one pair of parts H_i and H_j, producing a monochromatic copy of F_3, so A must be empty. Then

$$|G| = \sum_{i=1}^{4} |H_i| \leq 4 \cdot 8 = 32,$$

a contradiction.

For the second coloring, every vertex of A must have red edges to H_1 and H_4 and blue edges to H_2 and H_3. By minimality of t, the set A must be a single part of the Gallai partition with $|A| \leq 2$ and by the same calculation as above, $A \neq \emptyset$, meaning that

$$|A| + \sum_{i=1}^{4} |H_i| \leq 2 + 4 \cdot 8 = 34.$$

This implies that $1 \leq |A| \leq 2$ and $|H_i| = 8$ for all i except at most one, for which $|H_i| = 7$.

By Facts 1 and 2, if H_1 contains two blue edges, then $H_3 \cup H_4$ must contain no blue edges. Then $H_3 \cup H_4$ is a 2-colored copy of a complete graph on at least 15 vertices, which must contain a monochromatic copy of F_3 for a contradiction. We may therefore assume that H_1 and similarly H_4 each contain at most one blue edge and symmetrically, H_2 and H_3 each contain at most one red edge.

Suppose H_1 contains a blue edge. If H_4 also contains a blue edge, then by Fact 2, H_2 and H_3 each contain at most one blue edge. Then there exist two vertices in $H_2 \cup H_3$ whose removal yields a 2-colored complete graph on at least $15 - 2 = 13$ vertices, which must contain a monochromatic copy of F_3, for a contradiction. This means H_4 contains no blue edge. Then there exists a vertex in $H_3 \cup H_4$ (more specifically in H_3) whose removal yields a 2-colored complete graph on at least $15 - 1 = 14$ vertices, which again contains a monochromatic copy of F_3. This means that H_1, and similarly H_4, contains no blue edges and symmetrically, H_2 and H_3 each contain no red edges.

Finally since H_1 contains no blue edges and, by Claim 11, the subgraph of H_1 containing precisely the red edges is contained in C_4, C_5, or $2K_3$, H_1 contains a green copy of F_3, a contradiction to complete the proof of Case 3.

Case 4. $r = 5$.

Certainly $t = 5$ and to avoid a monochromatic triangle in the reduced graph, and the reduced graph must be the unique 2-colored K_5 consisting of a blue cycle say $H_1H_2H_3H_4H_5H_1$ and a complementary red cycle. By Claims 11 and 12, we have $|H_i| \leq 8$ for all i. If H_1 contains a blue edge, then $H_2 \cup H_5$ contains at most one blue edge by Facts 1 and 2. Then by deleting one vertex from $H_2 \cup H_5$, we can obtain a 2-colored complete graph, meaning that $|H_2 \cup H_5| \leq 13$ to avoid making a monochromatic copy of F_3. By symmetry, this same fact holds for other parts and for red as well.

Suppose first that H_1 contains at least one edge in both red and blue. Then $H_2 \cup H_5$ contains at most one blue edge and $|H_2 \cup H_5| \leq 13$. Similarly $H_3 \cup H_4$ contains at most one red edge and $|H_3 \cup H_4|$ so $|G| = \sum |H_i| \leq 8 + 2 \cdot 13 = 34$. By Claim 11, the subgraph of H_i consisting of the red (respectively blue) edges is a subgraph of C_4, C_5, or $2K_3$ for all i with $1 \leq i \leq 5$.

If H_1 contains both a red $2K_2$ and a blue $2K_2$, then each H_i (for $2 \leq i \leq 5$) is missing either red or blue edges, so by Fact 5, $|H_i| \leq 6$. Then $|G| = |H_1| + \sum_{i=2}^{5} |H_i| = 8 + 6 \cdot 4 = 32$, a contradiction. We may therefore assume that no part H_i contains both a red $2K_2$ and a blue $2K_2$. This means that for every part H_i, either the red or the blue edges are a subgraph of P_3. Since every 3-coloring of K_8 in which the subgraph containing the edges of one color is a subgraph of P_3 and the subgraph containing edges of a second color is a subgraph of C_4, C_5, or $2K_3$ must contain a monochromatic copy of F_3, this means that $|H_i| \leq 7$ for all i.

Then we get the following claim.

Claim 14. *We have $|H_i| = 7$ for at most 2 values of i with $1 \leq i \leq 5$.*

Proof. Assume, to the contrary, that there are three such values of i. In particular, note that by Fact 5, each such part H_i must contain at least one red and at least one blue edge. Up to symmetry, there are two possible cases:

(i) $|H_1| = |H_2| = |H_3| = 7$, and

(ii) $|H_1| = |H_2| = |H_4| = 7$.

First suppose $|H_1| = |H_2| = |H_3| = 7$. Then considering H_1 and H_3 as one part with all blue edges to H_2, Fact 2 yields a monochromatic (blue) copy of F_3.

Thus suppose $|H_1| = |H_2| = |H_4| = 7$. Then considering H_1 and H_2 as one part with all red edges to H_4, Fact 2 again yields a monochromatic (red) copy of F_3. □

From Claim 14, we have $|G| \leq 3 \cdot 6 + 2 \cdot 7 = 32$, a contradiction, completing the proof of Case 4 and the case when $k = 3$.

Before getting into the case where $k \geq 4$, we prove a useful claim.

Claim 15. *In any Gallai 4-colored copy of K_{33} using colors $1,2,3,4$ in which the subgraph containing precisely those edges of color 4 is a subgraph of K_3, there is a monochromatic copy of F_3.*

Proof. Since G is a Gallai coloring, it follows from Theorem 1 there is a Gallai partition of $V(G)$. Suppose colors 1 and 2 are the two colors appearing on edges between parts in the Gallai partition. Let t be the number of parts in the partition and choose such a partition where t is minimized. Since $R(F_3,F_3) = 13$, the reduced graph must have at most 12 vertices so $t \leq 12$. Let r be the number of "large" parts of the Gallai partition with order at least 3. Let H_i be the parts of this Gallai partition and, without loss of generality, suppose that $|H_i| \geq |H_{i+1}|$ for all i. This means that $|H_r| \geq 3$ and $|H_{r+1}| \leq 2$.

If $2 \leq t \leq 3$, then by the minimality of t, we may assume $t = 2$, say with corresponding parts H_1 and H_2. Without loss of generality, suppose all edges between H_1 and H_2 are color 1. Since $n = 33$, we must have $|H_1| \geq 17$. If $|H_2| \geq 13$, then H_2 contains at least one edge that is not color 2 or 3, so either an edge with color 1 or a subgraph of a triangle with color 4. If H_2 contains an edge with color 1, then H_1 contains a subgraph of a triangle with color 4 and does not contain a copy of $2K_2$ with color 1. Then by deleting at most 4 vertices, we can remove all edges of color 1 and 4 from H_1, leaving behind a 2-colored subgraph of order at least 13, and hence there is a monochromatic F_3. If H_2 contains at least one edge with color 4, then H_1 contains no edge of color 4 and no copy of $3K_2$ in color 1. Then, by deleting at most 4 vertices from H_1, we can obtain a subgraph of order at least 13 containing no edges of color 1 or 4, and hence there is a monochromatic F_3. If $|H_2| \leq 12$, then $|H_1| \geq 21$. Since $H_2 \neq \emptyset$, H_1 does not contain a copy of $3K_2$ with color 1 and also H_1 contains at most a triangle with color 4. Thus, by removing at most 6 vertices, we can produce a subgraph of H_1 of order at least 15 with no edges of 1 and 4, and hence there is a monochromatic F_3. We may therefore assume that $t \geq 4$.

If $r \geq 5$ and $t \geq 6$, then any choice of 6 parts containing the 5 parts $\mathscr{H} = \{H_1, \dots, H_5\}$ will contain a monochromatic triangle in the corresponding reduced graph. Such a triangle must contain at least 2 parts from $\mathscr{H}$. The corresponding subgraph of G must therefore contain a monochromatic copy of F_3, a contradiction. Thus, we may assume that either $4 \leq t \leq 5$ or $r \leq 4$.

If $r = 0$, then there are at least 17 small parts. Since $R(F_3, F_3) = 13$, it follows that there is a monochromatic F_3.

If $r = 1$, then let A be the set of parts with edges with color 1 to H_1 and B be the set of parts with edges with color 2 to H_1. Without loss of generality, suppose $|A| \geq |B|$. If $|A| \geq 11$, then A contains no copy of $3K_2$ in color 2 so by deleting at most 4 vertices, we can obtain a subgraph of order at least 7 within A in which there is no edge with color 2. Since all parts within A have order at most 2 and the edges in between the parts in A all have color 3, it follows that there is a monochromatic copy of F_3 in color 3 within A. This means that $|B| \leq |A| \leq 10$ and so $|H_1| \geq 13$. By Fact 4, there is no copy of $3K_2$ in either color 1 or color 2 within H_1. This means that if $|H_1| \geq 15$, then removing 8 vertices from H_1 to destroy all edges of colors 1 and 2 would yield a subgraph colored entirely in color 3 except for a subgraph of a triangle colored in color 4, clearly containing a copy of F_3. This means that $|H_1| \leq 14$ so $|A| = 10$ and $9 \leq |B| \leq 10$. Furthermore, if we could remove all edges of colors 1 and 2 from H_1 by deleting at most 6 vertices, what remains would easily contain a copy of F_3 in color 3, we must remove at least 7 vertices from H_1 to destroy all edges of colors 1 and 2 from H_1. There must therefore be a copy of $2K_2$ in one of color 1 or 2 and at least an edge in the other color within H_1. In particular, there is at least one edge in color 1 within H_1. This means that there can be at most one edge in color 1 within A. Since A is made up of parts of the Gallai partition of order at most 2, this means that the edges of color 2 within A form a complete graph minus a matching and with $|A| = 10$, there is a copy of F_3 in color 2 within A.

If $r = 5$, then $t = 5$ and to avoid a monochromatic triangle in the reduced graph, and the reduced graph must be the unique 2-colored K_5 consisting of a cycle say $H_1H_2H_3H_4H_5H_1$ with color 1 and a complementary cycle with color 2. Without loss of generality, suppose that if there is any edge of color 4, it appears within H_1, meaning that H_i contains no edge of color 4 for $2 \leq i \leq 5$. By Claim 11, within each part H_i the edges of colors 1 and 2 are each subgraphs of either C_4, C_5, or $2K_3$. By Claim 12, we have $|H_i| \leq 8$ for $2 \leq i \leq 5$. Since $n = 33$, either there are three parts of order at least 7 or one part of order 8 and another part of order at least 7. First suppose there is a part of order 8, say H_1, and another part of order at least 7, say H_2. By Fact 5, there can be no vertex $v \in H_1$ such that $H_1 \setminus \{v\}$ has no edges of color i where i is either of 1 or 2. This means that H_1 contains either a triangle or a copy of $2K_2$ in each of colors 1 and 2. By Facts 1 and 2, H_2 must have no edges of color 1, and so by Fact 5, H_2 contains a monochromatic copy of F_3. We may therefore assume there is no part of order 8, so there are at least 3 parts of order 7. Finally suppose three parts have order 7, say $|H_1| = |H_2| = |H_3| = 7$. By Fact 5, each part H_i contains at least one edge of color 1 and one edge of color 2. By Facts 1 and 2, since H_2 has at least one edge in color 1, each of H_1 and H_3 must have at most one edge in color 1, meaning that they each have exactly one edge in color 1. Similarly, since each of H_1 and H_3 has at least one edge of color 2, they must each also have at most one edge of color 2, so H_1 and H_3 must each have exactly one edge of color 1 and one edge of color 2. Merging these two edges into a single color and applying Fact 5, we obtain a monochromatic copy of F_3 for a contradiction.

If $r = 4$, then in order to avoid a monochromatic triangle within the reduced graph restricted to the 4 largest parts, up to symmetry, we may assume that all edges from H_i to H_{i+1} have color 1 for $1 \leq i \leq 3$ and all remaining edges between the parts have color 2. Since $n = 33$ and $|H_i| \leq 8$ (by Claim 12) for $1 \leq i \leq 4$, it follows that $t \geq 5$. Let A be the set of vertices in $G \setminus (H_1 \cup H_2 \cup H_3 \cup H_4)$. In order to avoid creating a monochromatic triangle in the reduced graph using two large parts, all vertices in A have all edges in color 1 to H_1 and H_4 and all vertices in A have all edges in color 2 to H_2 and H_3. If $|A| \leq 8$, then we may treat A as one part of the Gallai partition and apply the arguments in the above case when $r = 5$. We may therefore assume $|A| \geq 9$. Since A is made up entirely of parts from the Gallai partition of order at most 2, each vertex has at least $\frac{|A|-2}{2} \geq 3$ incident edges in either color 1 or 2. Then again treating A as a single part of the Gallai partition, Fact 1 yields a monochromatic copy of F_3.

If $r = 3$, then we assume that the edges from H_1 to $H_2 \cup H_3$ have color 1 and the edges from H_2 to H_3 have color 2. By Claim 12, $|H_2|, |H_3| \leq 8$. If there is a part H_i with $i \geq 4$ with edges of color 1 to H_1, then to avoid creating a copy of F_3 in color 1, all edges from H_i to $H_2 \cup H_3$ must have color 2, making a copy of F_3 in color 2. This means that all parts H_i with $i \geq 4$ must have edges of color 2 to H_1. If $|H_2| \geq 7$ and $|H_3| \geq 7$, then by Fact 5, each of H_2 and H_3 contains at least one edge of color 1 and one edge of color 2. To avoid creating a copy of F_3 in color 1 (centered at a vertex in H_1), each of H_2 and H_3 contains exactly one edge in color 1. By Fact 1, each of H_2 and H_3 contains exactly one edge in color 2. Then as in the case $r = 5$ above, we may merge colors 1 anr 2 into a single color and apply Fact 5 to obtain a monochromatic copy of F_3. Then $|H_2| \leq 6$ or $|H_3| \leq 6$ so without loss of generality, suppose $|H_2| \leq 6$. Let A be the set of vertices not in large parts, so $A = \{H_i \,|\, 4 \leq i \leq t\}$. If $|A| \geq 3$, then $|H_1| \leq 8$ (by Claim 12) and so $|A| \geq 11$. By deleting at most 4 vertices from A, we can remove all edges of color 2, leaving behind a subgraph of A of order at least 7 consisting of parts of the Gallai partition each of order at most 2. Since all edges between these parts have color 1, there is a copy of F_3 in color 1. We may therefore assume that $|A| \leq 2$ so $|H_1| \geq 17$. Since H_1 contains no copy of $3K_2$ in either color 1 or color 2, we may remove at most 8 vertices from H_1 to leave behind a subgraph with no edges of colors 1 or 2. This is a copy of K_9 in which color 4 is a subgraph of a triangle and all remaining edges have color 3, clearly producing a copy of F_3 in color 3.

If $r = 2$, then we assume that the edges from H_1 to H_2 have color 1. First suppose $|H_1| \geq 14$. Then by removing at most 4 vertices, we obtain a subgraph of H_1 with no edges of color 2 and by removing at most an additional 2 vertices, we obtain a subgraph of H_1 in which the edges of color 1 induce a subgraph of $2K_2$. In this remaining subgraph of H_1, a colored copy of K_8, any edges of color 1 induce a subgraph of $2K_2$ and any edges of color 4 induce a subgraph of K_3 and all remaining edges have color 3, yielding a copy of F_3 in color 3. We may therefore assume that $|H_1| \leq 13$ and similarly $|H_2| \leq 13$. Let A be the set of vertices not in large parts, so $A = \{H_i \,|\, 3 \leq i \leq t\}$. Note that $|A| \geq 7$ since $n = 33$. Let B be the set of vertices in A with edges of color 1 to H_1 and let C be the set of vertices in A with edges of color 2 to H_1 so $C = A \setminus B$. Note that all edges from B to H_2 must have color 2 to avoid creating a copy of F_3 in color 1. If $|C| \geq 3$, then by Claim 12, $|H_1| \leq 8$ and similarly if $|B| \geq 3$, then $|H_2| \leq 8$. Since $|A| \geq 7$, at least one of B or C has order at least 3 so at least one of H_1 or H_2 has order at most 8. First suppose $|C| \leq 2$ so $5 \leq |B| \leq 8$, and $3 \leq |H_2| \leq 8$. With $|H_1| \leq 13$, this means $|G| \leq 31$ which is a contradiction. Next suppose $|B| \leq 2$ so $|C| \geq 5$ and $3 \leq |H_1| \leq 8$. With $|H_2| \leq 13$, this means $|C| \geq 10$. Furthermore, by removing at most 2 vertices from H_2, we obtain a subgraph of H_2 with no edges of color 4 and in which colors 1 and 2 both satisfy Claim 11. By Claim 12, this means that $|H_2| \leq 8 + 2 = 10$. This implies that $|C| \geq 13$.

By Claim 11, we may remove at most 4 vertices from C to obtain a subgraph of C of order at least 9 with no edges of color 2. Since C is made up of only parts of the Gallai partition of order at most 2 and all edges between these parts in the aforementioned subgraph of C have color 2, this clearly produces a copy of F_3 in color 2. We may therefore assume that $|B|, |C| \geq 3$ so $|H_1|, |H_2| \leq 8$. As in the proof of the case $r = 3$, one of B or H_2 has at most 6 vertices so this means $|C| \geq 11$. By Claim 11, the removal of at most 2 vertices from C leaves a subgraph in which the edges of color 2 form a subgraph of $2K_2$. This subgraph is a complete graph of order at least 9 where all except a matching has color 1, which contains a copy of F_3 in color 1, to complete the proof of Claim 15. $\square$

For the remainder of this proof, we suppose $k \geq 4$. Since G is a Gallai coloring, it follows from Theorem 1 that there is a Gallai partition of $V(G)$. Suppose that the two colors appearing on edges in between the parts of the Gallai partition are red and blue. Let t be the number of parts in this partition and choose such a partition where t is minimized. Let $H_1, H_2, \dots, H_t$ be the parts of this partition, say with $|H_1| \geq |H_2| \geq \cdots \geq |H_t|$.

If $2 \leq t \leq 3$, then by the minimality of t, we may assume that $t = 2$. Let H_1 and H_2 be the corresponding parts and suppose all edges from H_1 to H_2 are red. If $|H_2| \leq 2$, then by Fact 4, H_1 contains no red $3K_2$ and by deleting 4 vertices, we can obtain a subgraph of H_1 with no red edges. This means that

$$|G| = |H_1| + |H_2| \leq [g(k-1) - 1] + 4 + 2 < n,$$

a contradiction. If $|H_1| \geq 3$ (and $|H_2| \geq 3$), then by deleting at most 8 vertices in total, we can obtain subgraphs of H_1 and H_2 with no red edges inside. This means that

$$|G| = |H_1| + |H_2| \leq 2[g(k-1) - 1] + 8 < n,$$

a contradiction when $k \geq 5$. Hence, we may assume that $k = 4$ so $n = 69$. By Claim 11, we can remove at most 4 vertices from each part H_i for $i = 1, 2$ to obtain subgraphs in which there are no red edges. This means that $|H_i| \leq [g(3) - 1] + 4 = 36$, implying that $|H_i| \geq 69 - 36 = 33$ for $i = 1, 2$. Suppose now that by deleting q_i vertices from H_i, we can obtain a subgraph in which there are no red edges for $i = 1, 2$. If $q_1 + q_2 \leq 4$, then

$$69 = |H_1| + |H_2| \leq 2 \cdot 32 + 4 = 68 < 69,$$

a contradiction. Hence, we may assume that $q_1 + q_2 \geq 5$, say with $3 \leq q_1 \leq 4$, also meaning that $1 \leq q_2 \leq 4$. Hence, H_1 contains a red copy of $2K_2$ and H_2 contains a red edge. Since $|H_1| \geq 33 > 5$, there is a red copy of F_3 using these red edges and the red edges between H_1 and H_2, a contradiction. We may therefore assume that $t \geq 4$.

Since $R(F_3, F_3) = 13$, it follows that $4 \leq t \leq 12$. Let r be the number of "large" parts of the Gallai partition with order at least 3. As before, we disregard the relative orders of the parts H_i for $i \leq r$. By Fact 4, we can remove at most 8 vertices from each part H_i for $i \leq r$ to obtain subgraphs with no red or blue edges. If $0 \leq r \leq 4$, then we get

$$\begin{aligned} |G| &= \sum_{i=1}^{r} |H_i| + \sum_{i=r+1}^{t} |H_i| \\ &\leq r[g(k-2)-1] + 8r + 2(t-r) \\ &< g(k), \end{aligned}$$

a contradiction when $k \geq 5$ or $k = 4$ and $0 \leq r \leq 2$. Hence, we assume, for a moment, that $k = 4$ and $3 \leq r \leq 4$.

In order to avoid a monochromatic triangle within the reduced graph restricted to the r large parts, we may assume that each large part is adjacent in red to another large part. By Claim 11, the red edges in each of these parts form a subgraph of C_4, C_5, or $2K_2$. By Fact 4 (and the minimality of t which guarantees that each part has blue edges to some other part), we may remove at most 4 vertices from each large part to obtain subgraphs with no blue edges. By Claim 12 applied within these subgraphs, we see that $|H_i| \leq 8 + 4 = 12$ for each i with $1 \leq i \leq r$. This means that

$$g(4) = 69 = \sum_{i=1}^{r} |H_i| + \sum_{i=r+1}^{t} |H_i| \leq r \cdot 12 + 2(t-r) \leq 64,$$

a contradiction.

In order to avoid a monochromatic triangle within the reduced graph restricted to the r large parts, we must have $r \leq 5$. Since the cases with $r \leq 4$ have already been considered, we may therefore assume, for the remainder of the proof, that $r = 5$.

Certainly $t = 5$ and to avoid a monochromatic triangle in the reduced graph restricted to the 5 large parts, the reduced graph must be the unique 2-colored copy of K_5 consisting of a blue cycle say $H_1H_2H_3H_4H_5H_1$ and a complementary red cycle. First some helpful claims.

Claim 16. *If one part, say H_1, contains a blue (or red) copy of $2K_2$, then there are no blue (respectively red) edges in $H_2 \cup H_3 \cup H_4 \cup H_5$.*

Proof. Suppose H_1 contains a blue copy of $2K_2$. By Fact 2, the parts H_2 and H_5 each contain no blue edges. Then treating $H_1 \cup H_3$ (or symmetrically $H_1 \cup H_4$) as one part, if H_3 (respectively H_4) contains a blue edge, then H_2 along with $H_1 \cup H_3$ (respectively H_5 along with $H_1 \cup H_4$) violates Fact 4. □

Claim 17. *There is no blue (or red) copy of $4K_2$ in $H_1 \cup H_2 \cup H_3 \cup H_4 \cup H_5$ as a disjoint union of subgraphs, not including the edges between the parts.*

Before proving this claim, we would like to note that it is possible for $H_1 \cup H_2 \cup H_3 \cup H_4 \cup H_5$ to contain a blue (or symmetrically red) $3K_2$. Indeed, placing one blue edge in each of H_1, H_2, and H_4 does not produce a blue copy of F_3.

Proof. If there is a red copy of $4K_2$ in $H_1 \cup H_2 \cup H_3 \cup H_4 \cup H_5$ as a disjoint union of subgraphs, then there are only three possible cases that do not immediately violate Fact 4 or Claim 16. In each of these cases, there is only one part that does not have a blue edge so there must be three parts in a row, say H_1, H_2, and H_3 that each contain a blue edge. Then by considering $H_1 \cup H_3$ as a single part, this structure violates Fact 2. □

Finally we claim that the red and blue edges can be completely destroyed from within all parts H_i by the removal of a total of at most 8 vertices.

Claim 18. *There exists a set of at most 4 vertices V_0 such that $H_i \setminus (V_0 \cap H_i)$ contains no blue (or similarly red) edges for all i with $1 \leq i \leq 5$.*

Proof. If all blue edges within parts H_i are disjoint, then the claim follows by Claim 17 (in fact with only 3 vertices) so suppose there is a pair of adjacent blue edges, say in H_1. Also if H_1 contains a blue copy of $2K_2$, then the claim follows from Claim 16 so by Claim 11, we may assume that the blue edges in H_1 are contained in a triangle. Then by Fact 1, H_2 and H_5 contain no blue edges. Similarly, if H_3 (or H_4) contains two adjacent blue edges, then by Fact 1, H_4 (respectively H_3) contains no blue edges. Otherwise H_3 and H_4 each contain at most one blue edge. In either case, the removal of at most 4 vertices (two from H_1 and either two from H_3 or one from each of H_3 and H_4) destroys all blue edges within the parts. □

By Claim 18, the removal of at most 8 vertices destroys all red and blue edges within the parts H_i. This means that, to avoid a monochromatic copy of F_3, we have

$$|G| = \sum_{i=1}^{5} |H_i| \leq 5[g(k-2) - 1] + 8.$$

If k is even, this means that

$$\begin{aligned} |G| &\leq 5[g(k-2) - 1] + 8 \\ &= 5\left[14 \times 5^{\frac{k-4}{2}} - 2\right] + 8 \\ &= 14 \times 5^{\frac{k-2}{2}} - 2 \\ &< 14 \times 5^{\frac{k-2}{2}} - 1 = |G|, \end{aligned}$$

a contradiction.

If k is odd and $k \geq 7$, this means that

$$\begin{aligned} |G| &\leq 5[g(k-2) - 1] + 8 \\ &= 5\left[33 \times 5^{\frac{k-5}{2}} + a \times 5^{\frac{k-7}{2}} - a - 1\right] + 8 \\ &< 33 \times 5^{\frac{k-3}{2}} + a \times 5^{\frac{k-5}{2}} - a = |G|, \end{aligned}$$

a contradiction.

We may therefore assume that $k = 5$ and so $n = 33 \cdot 5 = 165$, say with red and blue being colors 4 and 5 respectively. Then by Claim 11, the subgraphs induced by red or blue within each part $\{H_1, H_2, H_3, H_4, H_5\}$ are contained in C_5 or C_4 or $2C_3$.

Without loss of generality, suppose that $|H_1| \geq |H_2| \geq |H_3| \geq |H_4| \geq |H_5|$. If $|H_1| \geq |H_2| \geq |H_3| \geq 33$, then it follows from Claim 15, that for each i with $i \in \{1, 2, 3\}$, H_i contains $2K_2$ in either red or blue. Then there is a pair of parts in $\{H_1, H_2, H_3\}$ that violates either Fact 2 or 4.

Next suppose that $|H_2| \geq 33$ but $|H_3| \leq 32$. Then $|H_1| \geq 35$ and say red is the color of the edges between H_1 and H_2. Since the reduced graph is the unique 2-coloring of K_5 with no monochromatic triangle, there is another part, say H_3, with all blue edges to $H_1 \cup H_2$. By Claim 11, the blue subgraph of $H_1 \cup H_2$ is contained in C_4, C_5, or $2K_3$. Similarly using Facts 2 or 4, for $i \in \{1, 2\}$, if H_i contains a vertex with red degree 2, then H_{3-i} contains no red edges. If H_2 contains either no red edges and a subgraph of K_3 in blue or no blue edges and a subgraph of K_3 in red, then we may apply Claim 15 within H_2 to complete the proof. This means that H_2 contains two independent edges (a copy of $2K_2$) that are either both red, or both blue, or one red and one blue. If both edges are red, then H_1 contains no red edges. If both edges are blue, then H_1 contains no blue edges. If H_2 contains at least one red and at least one blue edge, then the red and blue subgraphs of H_1 are both subgraphs of K_3. In any of these cases, the removal of at most two vertices from H_1 leaves behind a subgraph of H_1 of order at least 33 in which one of red or blue is absent and the other is a subgraph of K_3. It is on this subgraph that we may apply Claim 15 to complete the proof in this case.

Finally, we may assume that $|H_2| \leq 32$ so since $n = 165$, we have $|H_1| \geq 37$. Since we have shown that $gr_3(K_3 : F_3) = 33$, it is safe to assume that for $2 \leq i \leq 5$, the parts H_i all have order exactly 32 and contain no red and no blue edges, so we assume, for the remainder of the proof, that $|H_1| = 37$.

As above, if there is a subgraph of H_1 of order at least 33 in which one of red or blue is absent and the other is a subgraph of K_3, then we may apply Claim 15 to complete the proof, so suppose this is not the case. By Claim 11, the red and blue subgraphs of H_1 are each a subgraph of C_4, C_5, or $2K_3$, so the only possible remaining possible cases for red and blue subgraphs of H_1 are precisely as follows:

- a red copy of C_5 and a blue copy of C_5,
- a red copy of C_5 and a blue copy of $2K_3$ (or symmetrically a red copy of $2K_3$ and a blue copy of C_5), or
- a red copy of $2K_3$ and a blue copy of $2K_3$ (possibly missing at most one edge from exactly copy).

Since H_1 contains no rainbow triangle, Theorem 1 gives a partition of $V(H_1)$, say into parts $X_1, X_2, \ldots, X_a$. Suppose green and purple (colors 2 and 3) are the colors that appear between parts of this partition and choose such a partition so that a is minimized. Since $R(F_3, F_3) = 13$, we see that $a \leq 12$. Let b be the number of "large" parts X_i of order at least 3 in this partition and call all other parts "small".

If $2 \leq a \leq 3$, then by the minimality of a, we may assume $a = 2$. Let X_1 and X_2 be the two parts of this partition and suppose all edges from X_1 to X_2 have color 3. If $|X_1| \geq 25$, then by deleting at most 12 vertices, we can obtain a subgraph of X_1 in which there are no edges of colors $3, 4, 5$. This subgraph is a 2-coloring of a complete graph of order at least 13, which must contain a monochromatic copy of F_3, a contradiction. If $|X_1| = 25 - i$ for some i with $1 \leq i \leq 6$, then $|X_2| = 12 + i$. In order to avoid creating a monochromatic copy of F_3, X_2 contains at least i disjoint edges with colors in $\{3, 4, 5\}$. By Fact 2 and the assumptions on H_1, each of these edges in X_2 precludes one such edge from appearing within X_1. The result is that we can remove at most $12 - 2i$ vertices from X_1 to obtain a subgraph of X_1 in which there are no edges with any color in $\{3, 4, 5\}$. Such a subgraph is a 2-colored complete graph of order at least $25 - i - (12 - 2i) = 13 + i$, so this subgraph contains a monochromatic copy of F_3, completing the proof in the case $a \leq 3$.

Therefore, suppose $4 \leq a \leq 12$. In order to avoid a monochromatic triangle within the reduced graph restricted to the large parts, we have $b \leq 5$. Since we know that H_1 satisfies one of the cases listed above concerning the presence of red and blue edges, it is clear that $b \geq 1$. We consider cases based on the value of b.

First a small claim that will be used within the cases.

Claim 19. $|X_1| \leq 12$.

Proof. If $|X_1| \geq 13$, then applying Claim 11, by deleting at most 4 vertices, there is a subgraph of X_1 in which no edges with colors 2 or 3 appear. Since $|X_1| - 4 \geq 9$, it follows from Claim 12 that this subgraph contains a monochromatic copy of F_3, a contradiction. □

Case 1. $b = 1$.

Let A be the set of parts with edges with color 2 to X_1 and B be the set of parts with edges with color 3 to X_1. By Claim 19, we have $|X_1| \leq 12$, so $|A| \geq 13$ or $|B| \geq 13$. Without loss of generality, let $|A| \geq 13$. By deleting at most 4 vertices from A, there is a subgraph of A of order at least 9 with no edges of color 2. Since each part within A has order at most 2 and all edges between these parts have color 3, A clearly contains a copy of F_3 in color 3, a contradiction.

Case 2. $b = 2$.

Assume that the edges from X_1 to X_2 have color 2. By Claim 19, we know that $|X_1| \leq 12$ and $|X_2| \leq 12$. Let Y be the set of parts with edges with color 2 to X_1 and Z be the set of parts with edges with color 3 to X_1. Let Y' be the set of parts with edges with color 2 to X_2 and Z' be the set of parts with edges with color 3 to X_2. If $|Z| \geq 11$, then by deleting at most 4 vertices from

Z, we obtain a subgraph of Z in which there are no edges with color 3. Since Z contains only small parts and all edges in between these parts have color 2, Z contains a copy of F_3 with color 2, a contradiction. This means that $|Z| \leq 10$ and symmetrically, $|Y|, |Y'|, |Z'| \leq 10$. Since $|X_1| + |X_2| \leq 24$, this also means that $|Y| \geq 3$, $|Z| \geq 3$, $|Y'| \geq 3$ and $|Z'| \geq 3$.

Claim 20. *If $|Y| \geq 3$, $|Z| \geq 3$, $|Y'| \geq 3$ and $|Z'| \geq 3$, then $|X_1| \leq 10$ and $|X_2| \leq 10$.*

Proof. Assume, to the contrary, that $|X_1| \geq 11$. The subgraph of X_1 induced by the edges with each color i for $2 \leq i \leq 5$ is a subgraph of one of C_5 or C_4 or $2C_3$. Since X_1 contains no rainbow triangle, there is a Gallai partition of the vertices of X_1 in which all edges between the parts have color 1. Recall that the subgraphs of H_1 of colors 4 and 5 are each subgraphs of C_5 or $2K_3$ and by Claim 11, the subgraphs of X_1 of colors 2 and 3 are each subgraphs of either C_5, C_4, or $2K_3$. To avoid a rainbow triangle, these must either share vertices (for example, two complementary copies of K_5) or be vertex disjoint with all edges of color 1 in between. Therefore, each part of this partition has order at most 5, meaning that there are at least 3 parts. Hence, there is a copy of F_3 in color 1, a contradiction. □

We may therefore assume that $|X_1|, |X_2| \leq 10$. Indeed, in the proof above, if $|X_1| \in \{9, 10\}$, then the subgraphs of X_1 in color i with $2 \leq i \leq 5$ are very restricted to avoid having 3 parts in the Gallai partition (with all edges of color 1 in between the parts. This observation is used in the following proof.

Claim 21. *If $|X_1| \geq 9$, then $|X_2| \leq 6$.*

Proof. Assume, to the contrary, that $|X_2| \geq 7$. If $|X_1| = 10$, then by the arguments in the proof of Claim 20, the subgraph of X_1 in each of the colors $2, 3, 4, 5$ must be a copy of C_5, forming two copies of K_5, each consisting of complementary monochromatic copies of C_5 in pairs of these colors. By Claim 11, since there is a part in G with all edges of color 4 to $X_1 \cup X_2$ and a part in G with all edges in color 5 to $X_1 \cup X_2$, there can be no edges of color 4 or 5 within X_2. By Fact 2, there can be no edge with color 2 in X_2. Even if there are edges of color 3 within X_2, by Claim 11, these must form a subgraph of C_4, C_5, or $2K_3$. Then by Fact 5, there is a copy of F_3 in color 1 within X_2, for a contradiction. Similarly, if $|X_1| = 9$, then again using the arguments from the proof of Claim 20, X_1 must contain two (complementary) 5-cycles using two colors from $2, 3, 4, 5$, and a complete graph on 4 vertices colored with the remaining colors from $\{2, 3, 4, 5\}$, and hence there are again no edges with a color in $\{2, 4, 5\}$ in X_2 and the edges of color 3 are restricted to subgraphs of C_4, C_5, and $2K_3$. By Fact 5, there is again a copy of F_3 in color 1 as a subgraph of X_2, a contradiction. □

From Claim 21, we have $|X_1| + |X_2| \leq 16$ so this means that $|Y| \geq 11$ or $|Z| \geq 11$, say $|Y| \geq 11$. By deleting at most 4 vertices from Y, we obtain a subgraph of Y with no edges of color 2. Since Y consists of small parts of the

Gallai partition of H_1 and all edges between the parts in this subgraph of Y have color 3, this subgraph contains a copy of F_3 in color 3, a contradiction.

Case 3. $b = 3$.

In order to avoid a monochromatic triangle in the reduced graph restricted to the large parts, we assume that the edges from X_1 to $X_2 \cup X_3$ have color 2 and the edges from X_2 to X_3 have color 3. In order to avoid a monochromatic triangle in the reduced graph using two large parts, for each part X_i with $4 \leq i \leq a$, the edges from X_i to X_1 have color 3. Let $X = \bigcup_{i=4}^{a} X_i$. By Claim 21 (note that Claims 20 and 21 can be applied to any pair of the parts X_1, X_2, X_3 here), we have $|X_1| + |X_2| + |X_3| \leq 24$. This means $|X| \geq 13$. By Claim 11, we can delete at most 4 vertices from X to obtain a subgraph of X with no edges of color 3. Since X consists entirely of small parts of the partition and the edges between these parts within the aforementioned subgraph are colored entirely with color 2, this subgraph of X contains a copy of F_3 in color 2.

Case 4. $b = 4$.

In order to avoid a monochromatic triangle within the reduced graph restricted to the 4 largest parts, up to symmetry, we may assume that all edges from X_i to X_{i+1} have color 2 for $1 \leq i \leq 3$ and all remaining edges between these large parts have color 3. By Claim 20, we have $|X_i| \leq 10$ for $1 \leq i \leq 4$. In order to avoid creating a monochromatic triangle in the reduced graph using two of the large parts, all small parts must have edges of color 2 to $X_1 \cup X_4$ and edges of color 3 to $X_2 \cup X_3$. Thus, by the minimality of a, we may assume that $4 \leq a \leq 5$ so there is at most one small part. If there exists some X_j such that $|X_j| \geq 9$, then $|X_i| \leq 6$ for $1 \leq i \neq j \leq a$. In this case,

$$|H_1| = \sum_{i=1}^{a} |X_i| \leq 10 + 3 \cdot 6 + 2 = 30 < 37,$$

a contradiction. On the other hand, if $|X_i| \leq 8$ for all $1 \leq i \leq 4$, then

$$|H_1| = \sum_{i=1}^{a} |X_i| \leq 4 \cdot 8 + 2 = 34 < 37,$$

again a contradiction.

Case 5. $b = 5$.

Then $a = 5$ with all large parts and in order to avoid a monochromatic triangle in the reduced graph, the reduced graph must be the unique 2-colored K_5 consisting of a cycle say $X_1X_2X_3X_4X_5X_1$ with color 2 and a complementary cycle with color 3. From Claim 20, $|X_i| \leq 10$ for all $1 \leq i \leq 5$. If there exists some part X_j with $|X_j| \geq 9$, then $|X_i| \leq 6$ for $1 \leq i \neq j \leq 5$, and hence

$$|H_1| = \sum_{i=1}^{a} |X_i| \leq 10 + 4 \cdot 6 = 34 < 37,$$

a contradiction. Thus, we may assume that $|X_i| \le 8$ for $1 \le i \le 5$, say with $|X_1| \ge |X_2| \ge |X_3| \ge |X_4| \ge 7$. For $1 \le i \le 4$, by the arguments leading up to Fact 5, in order to avoid a copy of F_3 in color 1 within X_i, each part X_i contains at least 3 disjoint edges of colors from $\{2, 3, 4, 5\}$ for a total of at least 12 such edges. Since there are at most 8 such edges (two for each of these colors), this is a contradiction, completing the proof of this last case and the proof of Lemma 5. $\square$