Stanley's conjectures on the Stern poset

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Abstract. The Stern poset S is a graded infinite poset naturally associated to Stern's triangle, which was defined by Stanley analogously to Pascal's triangle. Let P_n denote the interval of S from the unique element of row 0 of Stern's triangle to the n-th element of row r for sufficiently large r. For $r \ge 1$ let

$$L_n(q) = 2 \cdot \left(\sum_{k=1}^{2^n - 1} A_{P_k}(q)\right) + A_{P_{2^n}}(q),$$

where $A_P(q)$ represents the corresponding P-Eulerian polynomial. For any $n \geq 1$ Stanley conjectured that $L_n(q)$ has only real zeros and $L_{4n+1}(q)$ is divisible by $L_{2n}(q)$. In this paper we obtain a simple recurrence relation satisfied by $L_n(q)$ and affirmatively solve Stanley's conjectures. We also establish the asymptotic normality of the coefficients of $L_n(q)$.

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1 Introduction

Recently, Stanley [9] introduced a class of polynomials $b_n(q)$ by defining $b_1(q) = 1$ and

$$b_{2n}(q) = b_n(q), (1.1)$$

$$b_{4n+1}(q) = qb_{2n}(q) + b_{2n+1}(q), (1.2)$$

$$b_{4n+3}(q) = b_{2n+1}(q) + qb_{2n+2}(q). (1.3)$$

Let

$$L_n(q) = 2 \cdot \left(\sum_{k=1}^{2^n - 1} b_k(q)\right) + b_{2^n}(q). \tag{1.4}$$

The main objective of this paper is to prove Stanley's conjectures on the real-rootedness and divisibility concerning $L_n(q)$.

Let us first review some background. Stanley's conjectures considered here arose in the study of Stern's triangle S, which is an array of numbers similar to Pascal's triangle. We follow Stanley [8] to give a description of Stern's triangle. We number the rows of Stern's triangle by consecutive natural numbers beginning with 0. Row 0 consists of a single 1, row 1 consists of three 1's, and for $r \geq 2$ row r is obtained from row r-1 by inserting between any two consecutive elements their sums and then placing a 1 at the beginning and end. It is clear that row n consists of $2^{r+1}-1$ terms. We number the elements of row r from 0 to $2^{r+1}-2$ and use $\binom{r}{n}$ to denote the (n+1)-th element of row r. Thus, we have the following recurrence relation

where we set $\binom{r}{n} = 0$ for n < 0 or $n > 2^{r+1} - 2$ for convenience. Stanley [8] showed that for any $m \ge 1$ the summation

$$\sum_{n} {r \choose n}^m$$

obeys a homogeneous linear recurrence with constant coefficients, and conjectured the least order of a homogeneous linear recurrence. Speyer [7] proved that the above sum satisfies such a recurrence of the conjectured minimal order. For any $r \geq 1$ Stanley showed that

$$\sum_{n\geq 0} \left\langle {r \atop n} \right\rangle x^n = \prod_{i=0}^{r-1} (1 + x^{2^i} + x^{2 \cdot 2^i}). \tag{1.6}$$

Letting $r \to \infty$ in (1.6), we get

$$\prod_{i\geq 0} (1+x^{2^i}+x^{2\cdot 2^i}) = \sum_{n\geq 1} b_n x^{n-1},$$

where the sequence $\{b_n\}_{n\geq 0}$ with $b_0=0$ is the well-known Stern's diatomic sequence [10]. For more information on Stern's diatomic sequence, see Northshield [6]. It is known that $\{b_n\}_{n\geq 0}$ satisfies the following recurrence relation

$$b_{2n} = b_n, b_{2n+1} = b_n + b_{n+1}.$$
 (1.7)

Comparing the above recurrence relation with (1.1), (1.2) and (1.3), we see that $\{b_n(q)\}_{n\geq 1}$ is a polynomial analogue of Stern's diatomic sequence.

Stanley [9] showed that the polynomials $b_n(q)$ also arise as P-Eulerian polynomials of certain posets P naturally associated to Stern's triangle S. To take S as a poset, we will consider $\binom{r}{n}$ as a symbol instead of a number. According to (1.5), we may impose a partial order \preceq_S on S by letting

$$\binom{r-1}{n} \preceq_{\mathcal{S}} \binom{r}{2n+1}, \qquad \binom{r-1}{n} \preceq_{\mathcal{S}} \binom{r}{2n}, \qquad \binom{r-1}{n} \preceq_{\mathcal{S}} \binom{r}{2n+2}$$

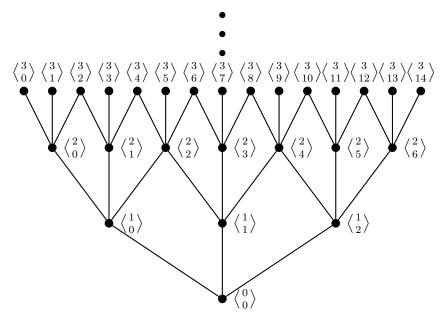


Figure 1.1: The Stern poset \mathcal{S}

for $0 \le n \le 2^r - 2$ and then taking the transitive closure. Following Stanley [9] we call $(S, \le_{\mathcal{S}})$ the Stern poset, denoted by \mathcal{S} . See Figure 1.1 for the first four levels of the Stern poset.

Fixing a positive integer n, suppose that

$$2^k \le n - 1 < 2^{k+1}$$

for some $k \geq 0$. A little thought shows that for $r \geq k+1$ the interval $\begin{bmatrix} {n \choose 0}, {r \choose n-1} \end{bmatrix}$ of $\mathcal S$ is the ordinal sum of the chain $\begin{bmatrix} {n \choose 0}, {r-k-1 \choose 0} \end{bmatrix}$ and the interval $\begin{bmatrix} {r-k-1 \choose 0}, {r \choose n-1} \end{bmatrix}$, the latter being isomorphic to the interval $\begin{bmatrix} {n \choose 0}, {k+1 \choose n-1} \end{bmatrix}$ of $\mathcal S$ for any $r \geq k+1$. Thus for sufficiently large r we may associate to the n-th element of row r in Stern's triangle the poset $\begin{bmatrix} {n \choose 0}, {k+1 \choose n-1} \end{bmatrix}$, denoted by P_n . Stanley [9] obtained the following result.

Theorem 1.1 ([9]) For any $n \ge 1$ let $b_n(q)$ be defined as in (1.1), (1.2) and (1.3). Then $b_n(q)$ is equal to the P_n -Eulerian polynomial, namely

$$b_n(q) = \sum_{\sigma \in \mathcal{L}(P_n)} q^{\text{des}(\sigma)},$$

where $\mathcal{L}(P_n)$ denotes the set of linear extensions of the poset P_n , provided that P_n is naturally labeled.

Therefore, the polynomials $L_n(q)$ defined in (1.4) can be considered as Eulerian row sums of Stern's triangle. Stanley [9] proposed the following interesting conjectures.

Conjecture 1.2 ([9]) For any $n \ge 1$ the polynomial $L_n(q)$ has only real zeros.

Conjecture 1.3 ([9]) For any $n \ge 1$ the polynomial $L_{4n+1}(q)$ is divisible by $L_{2n}(q)$.

The aim of this paper is to prove the above conjectures. Motivated by Conjecture 1.2, we also study the asymptotically normality of $L_n(q)$ since the coefficients of many real-rooted polynomials follow asymptotically normal limit laws, see [1, 3, 4, 12] for instance. Recall that, for a polynomial sequence $\{F_n(q)\}_{n\geq 0}$ given by

$$F_n(q) = \sum_{k=0}^{n} a_{n,k} q^k \tag{1.8}$$

with $a_{n,k}$ all nonnegative, the coefficients $a_{n,k}$ are said to be asymptotically normal by a central limit theorem if

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \sum_{k \le \mu_n + x\sigma_n} p(n, k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt \right| = 0, \tag{1.9}$$

where

$$p(n,k) = \frac{a_{n,k}}{\sum_{j=0}^{n} a_{n,j}}$$

and μ_n, σ_n^2 are respectively the mean and variance of $a_{n,k}$. We say that $a_{n,k}$ are asymptotically normal by a local limit theorem on \mathbb{R} if

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \sigma_n p(n, \lfloor \mu_n + x \sigma_n \rfloor) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0.$$
 (1.10)

The rest of this paper is organized as follows. In Section 2 we will first derive a recurrence relation satisfied by $L_n(q)$ and then give a proof of Conjecture 1.2. In Section 3 we will prove Conjecture 1.3 based on the recurrence relations satisfied by $L_{2n}(q)$, $L_{2n+1}(q)$ and $L_{4n+1}(q)$. In Section 4 we will show that the coefficients of $L_n(q)$ are asymptotically normal by central and local limit theorems.

2 Real zeros

The aim of this section is to prove Conjecture 1.2. Our proof is based on the following recurrence relation satisfied by $L_n(q)$.

Theorem 2.1 For any $n \ge 2$ we have

$$L_{n+1}(q) = 3L_n(q) + 2(q-1)L_{n-1}(q). (2.1)$$

Proof. Substituting (1.4) into (2.1) we obtain that

$$2 \cdot \left(\sum_{k=1}^{2^{n+1}-1} b_k(q)\right) + b_{2^{n+1}}(q) = 3 \cdot \left(2\sum_{k=1}^{2^n-1} b_k(q) + b_{2^n}(q)\right) + 2(q-1) \cdot \left(2 \cdot \left(\sum_{k=1}^{2^{n-1}-1} b_k(q)\right) + b_{2^{n-1}}(q)\right).$$

By using (1.1) and (1.3), we find that

$$b_{2^n}(q) = b_2(q) = b_1(q) = 1$$
 and $b_3(q) = 1 + q$.

Thus it suffices to show that

$$2 \cdot \left(\sum_{k=4}^{2^{n+1}-1} b_k(q)\right) + 2(1+q) + 5 = 3 \cdot \left(2\sum_{k=2}^{2^{n-1}} b_k(q) + 3\right) + 2(q-1) \cdot \left(2 \cdot \left(\sum_{k=1}^{2^{n-1}-1} b_k(q)\right) + 1\right),$$

or equivalently

$$\sum_{k=4}^{2^{n+1}-1} b_k(q) = 3 \cdot \left(\sum_{k=2}^{2^{n}-1} b_k(q)\right) + 2(q-1) \cdot \left(\sum_{k=1}^{2^{n-1}-1} b_k(q)\right). \tag{2.2}$$

By using (1.1), (1.2) and (1.3), it is easy to verify that

$$b_{4m}(q) + b_{4m+1}(q) + b_{4m+2}(q) + b_{4m+3}(q) = (1+q)b_{2m}(q) + 3b_{2m+1}(q) + qb_{2m+2}(q).$$
 (2.3)

Substituting (2.3) into the left hand side of (2.2), we get that for $n \geq 2$

$$\sum_{k=4}^{2^{n+1}-1} b_k(q) = \sum_{m=1}^{2^{n-1}-1} (b_{4m}(q) + b_{4m+1}(q) + b_{4m+2}(q) + b_{4m+3}(q))$$
$$= \sum_{m=1}^{2^{n-1}-1} ((1+q)b_{2m}(q) + 3b_{2m+1}(q) + qb_{2m+2}(q))$$

By rewriting $(1+q)b_{2m}(q)$ as $3b_{2m}(q)+(q-2)b_{2m}(q)$, we find that

$$\sum_{k=4}^{2^{n+1}-1} b_k(q) = \sum_{m=1}^{2^{n-1}-1} (3b_{2m}(q) + 3b_{2m+1}(q) + (q-2)b_{2m}(q) + qb_{2m+2}(q))$$

$$= 3 \cdot \left(\sum_{m=1}^{2^{n-1}-1} (b_{2m}(q) + b_{2m+1}(q))\right) + \sum_{m=1}^{2^{n-1}-1} ((q-2)b_{2m}(q) + qb_{2m+2}(q))$$

By virtue of (1.1), we further obtain that

$$\sum_{k=4}^{2^{n+1}-1} b_k(q) = 3 \cdot \left(\sum_{k=2}^{2^n-1} b_k(q)\right) + \sum_{m=1}^{2^{n-1}-1} ((q-2)b_m(q) + qb_{m+1}(q))$$

$$= 3 \cdot \left(\sum_{k=2}^{2^n-1} b_k(q)\right) + q \cdot \left(\sum_{m=1}^{2^{n-1}-1} b_m(q)\right)$$

$$+ q \cdot \left(\sum_{m=1}^{2^{n-1}-1} b_{m+1}(q)\right) - 2 \cdot \left(\sum_{m=1}^{2^{n-1}-1} b_m(q)\right).$$

Note that, for the third summation on the right hand side, we have

$$\sum_{m=1}^{2^{n-1}-1} b_{m+1}(q) = b_{2^{n-1}}(q) + \sum_{m=1}^{2^{n-1}-2} b_{m+1}(q)$$

$$= b_1(q) + \sum_{m=2}^{2^{n-1}-1} b_m(q)$$

$$= \sum_{m=1}^{2^{n-1}-1} b_m(q)$$

where the second equality holds since $b_{2^{n-1}}(q) = b_1(q) = 1$ by (1.1). Thus

$$\sum_{k=4}^{2^{n+1}-1} b_k(q) = 3 \cdot \left(\sum_{k=2}^{2^n-1} b_k(q)\right) + 2(q-1) \cdot \left(\sum_{k=1}^{2^{n-1}-1} b_k(q)\right),$$

as desired in (2.2). This completes the proof.

In view of (2.1) the polynomial sequence $\{L_n(q)\}_{n\geq 1}$ lies in the framework of polynomial sequence of type (0,1) studied by Gross, Mansour, Tucker and Wang [2], who obtained the following result.

Theorem 2.2 ([2, Theorem 2.6]) Let $\{M_n(q)\}_{n\geq 0}$ be the polynomial sequence defined by the recursion

$$M_{n+1}(q) = aM_n(q) + (bq + c)M_{n-1}(q)$$

with initial values $M_0(q) = 1$ and $M_1(q) = t(q - r)$, where $a, b, t > 0, c, r \in \mathbb{R}$, and $r \neq -c/b$. Then for any $n \geq 1$ the polynomial $M_n(q)$ has only real zeros.

The main result of this section is as follows, which gives an affirmative answer to Conjecture 1.2.

Theorem 2.3 For any $n \ge 1$ the polynomial $L_n(q)$ has only real zeros.

Proof. By (1.4) it is straightforward to compute that $L_1(q) = 3$ and $L_2(q) = 2q + 7$. For any $n \ge 0$ let

$$\tilde{L}_n(q) = \frac{L_{n+1}(q)}{3}.$$

It is clear that

$$\tilde{L}_{n+1}(q) = 3\tilde{L}_n(q) + 2(q-1)\tilde{L}_{n-1}(q)$$

with $\tilde{L}_1(q) = 1$ and $\tilde{L}_2(q) = \frac{2}{3}q + \frac{7}{3}$. It is easy to verify that $\{\tilde{L}_n(q)\}_{n\geq 0}$ satisfies the conditions of Theorem 2.2. Thus for each $n \geq 0$ the polynomial $\tilde{L}_n(q)$ has only real zeros, so does $L_{n+1}(q)$.

Remark. Based on the recurrence relation (2.1), Theorem 2.3 can also be proved by using a result due to Wang and Yeh, see [11, Theorem 1] or [5, Theorem 1.1].

Theorem 2.1 also has some other consequences. For instance, we can determine the ordinary generating function of $\{L_n(q)\}_{n\geq 1}$. Let

$$\Phi(x) = \sum_{n>1} L_n(q) x^n.$$

By (2.1) it is easy to show that

$$\Phi(x) = \frac{x(2(q-1)x+3)}{1-3x-2(q-1)x^2} = \frac{x(2(q-1)x+3)}{1-3x-2(q-1)x^2} = \frac{1}{1-x(2(q-1)x+3)} - 1.$$

We can also determine the shifted Hankel determinants of $\{L_n(q)\}_{n\geq 1}$. Let

$$\mathcal{H}_n^{(k)} = (L_{i+j+k-1}(q))_{1 \le i,j \le n}, \quad H_n^{(k)} = \det \mathcal{H}_n^{(k)}.$$

We obtain the following result.

Corollary 2.4 For any $k \ge 0$ and $m \ge 3$ we have

$$H_1^{(k)} = L_{k+1}(q), \quad H_2^{(k)} = (-1)^{k+1} 2^{k+2} (q-1)^{k+2}, \quad H_m^{(k)} = 0.$$

Proof. We only need to compute $H_2^{(k)}$ and $H_m^{(k)}$ for $m \geq 3$. It is routine to compute that

$$H_2^{(0)} = \det \begin{pmatrix} L_1(q) & L_2(q) \\ L_2(q) & L_3(q) \end{pmatrix} = \det \begin{pmatrix} 3 & 2q+7 \\ 2q+7 & 12q+15 \end{pmatrix} = -2^2(q-1)^2.$$

For $k \ge 1$ applying (2.1) to $H_2^{(k)}$ gives

$$H_2^{(k)} = \det \begin{pmatrix} L_{k+1}(q) & L_{k+2}(q) \\ L_{k+2}(q) & L_{k+3}(q) \end{pmatrix}$$

$$= \det \begin{pmatrix} L_{k+1}(q) & 3L_{k+1}(q) + 2(q-1)L_k(q) \\ L_{k+2}(q) & 3L_{k+2}(q) + 2(q-1)L_{k+1}(q) \end{pmatrix}$$

$$= \det \begin{pmatrix} L_{k+1}(q) & 2(q-1)L_k(q) \\ L_{k+2}(q) & 2(q-1)L_{k+1}(q) \end{pmatrix}$$

$$= 2(q-1) \cdot \det \begin{pmatrix} L_{k+1}(q) & L_k(q) \\ L_{k+2}(q) & L_{k+1}(q) \end{pmatrix}$$

$$= -2(q-1) \cdot \det \begin{pmatrix} L_k(q) & L_{k+1}(q) \\ L_{k+1}(q) & L_{k+2}(q) \end{pmatrix}$$

$$= -2(q-1)H_2^{(k-1)}. \tag{2.4}$$

By iterating 2.4 we obtain

$$H_2^{(k)} = (-1)^k 2^k (q-1)^k H_2^{(0)} = (-1)^{k+1} 2^{k+2} (q-1)^{k+2}$$

as desired.

We proceed to compute $H_m^{(k)}$ for $m \geq 3$. By definition

$$\mathcal{H}_{m}^{(k)} = \begin{pmatrix} L_{k+1}(q) & L_{k+2}(q) & L_{k+3}(q) & \cdots & L_{k+m}(q) \\ L_{k+2}(q) & L_{k+3}(q) & L_{k+4}(q) & \cdots & L_{k+m+1}(q) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ L_{k+m}(q) & L_{k+m+1}(q) & L_{k+m+2}(q) & \cdots & L_{k+2m-1}(q) \end{pmatrix}.$$

By (2.1) the third column of $\mathcal{H}_m^{(k)}$ is a linear combination of the first column and the second column, and thus the Hankel determinant $H_m^{(k)}$ vanishes. This completes the proof.

3 Divisibility

In this section we aim to prove Conjecture 1.3. The idea of the proof is to introduce a polynomial sequence $\{M_n(q)\}_{n\geq 1}$ as defined below, and then to show that $\{M_n(q)L_{2n}(q)\}_{n\geq 1}$ and $\{L_{4n+1}(q)\}_{n\geq 1}$ satisfy the same recurrence relation with the same initial conditions.

We first give recurrence relations which are satisfied by $\{L_{2n}(q)\}_{n\geq 1}$, $\{L_{2n-1}(q)\}_{n\geq 1}$ and $\{L_{4n+1}(q)\}_{n\geq 1}$ respectively. For notational convenience let

$$E_n(q) = L_{2n}(q), \qquad O_n(q) = L_{2n-1}(q), \qquad J_n(q) = L_{4n+1}(q)$$
 (3.1)

for any $n \ge 1$. The recurrence relations are as follows.

Theorem 3.1 Let

$$a = (5+4q), b = -4(q-1)^2, h = 2O_3(q).$$

Then for any $n \geq 2$ we have

$$E_{n+1}(q) = aE_n(q) + bE_{n-1}(q), (3.2)$$

$$O_{n+1}(q) = aO_n(q) + bO_{n-1}(q), (3.3)$$

$$J_{n+1}(q) = ab^{n-1}h + a^2J_n(q) + b^2J_{n-1}(q) + \sum_{k=2}^{n-1} 2a^2b^{n-k}J_k(q).$$
 (3.4)

Proof. Let us first prove (3.2). From (2.1) it follows that

$$E_{n+1}(q) = L_{2(n+1)}(q) = 3L_{2n+1}(q) + 2(q-1)L_{2n}$$

$$= 3(3L_{2n}(q) + 2(q-1)L_{2n-1}(q))$$

$$+ 2(q-1)(3L_{2n-1}(q) + 2(q-1)L_{2(n-1)}(q)).$$

By simplification we obtain

$$E_{n+1}(q) = 9L_{2n}(q) + 12(q-1)L_{2n-1}(q) + 4(q-1)^2L_{2(n-1)}(q),$$

which could be rewritten as

$$E_{n+1}(q) = (5+4q)L_{2n}(q) - 4(q-1)^2 L_{2(n-1)}(q)$$

$$+ 4(1-q) \left(L_{2n}(q) - 3L_{2n-1}(q) - 2(q-1)L_{2(n-1)}(q) \right).$$

Again by (2.1) we obtain

$$E_{n+1}(q) = (5+4q)E_n(q) - 4(q-1)^2 E_{n-1}(q),$$

as desired.

Similarly, we compute

$$O_{n+1}(q) = L_{2(n+1)+1}(q)$$

$$= 3L_{2(n+1)}(q) + 2(q-1)L_{2n+1}$$

$$= 3(3L_{2n+1}(q) + 2(q-1)L_{2n}(q)) + 2(q-1)(3L_{2n}(q) + 2(q-1)L_{2n-1}(q))$$

$$= 9L_{2n+1}(q) + 12(q-1)L_{2n}(q) + 4(q-1)^{2}L_{2n-1}(q)$$

$$= (5+4q)L_{2n+1}(q) - 4(q-1)^{2}L_{2(n-1)+1}(q)$$

$$+ 4(1-q)(L_{2n+1}(q) - 3L_{2n}(q) - 2(q-1)L_{2n-1}(q))$$

$$= (5+4q)O_{n}(q) - 4(q-1)^{2}O_{n-1}(q),$$

where the last equality follows from (2.1). This completes the proof of (3.3).

We proceed to prove (3.4). Note that

$$J_{n+1}(q) = L_{4(n+1)+1}(q) = O_{2(n+1)}(q).$$

By virtue of (3.3), there holds

$$J_{n+1}(q) = aO_{2n+1}(q) + bO_{2n}(q)$$

$$= a \left(aO_{2n}(q) + bO_{2n-1}(q)\right) + b \left(aO_{2n-1}(q) + bO_{2n-2}(q)\right)$$

$$= a^2 J_n(q) + b^2 J_{n-1}(q) + 2abO_{2n-1}(q). \tag{3.5}$$

On the other hand, for any $m \geq 2$ we have

$$O_{2m-1}(q) = aO_{2m-2}(q) + bO_{2m-3}(q) = aJ_{m-1}(q) + bO_{2m-3}(q).$$

Iterating the above process leads to

$$O_{2n-1}(q) = aJ_{n-1}(q) + bO_{2n-3}(q)$$

$$= aJ_{n-1}(q) + b (aJ_{n-2}(q) + bO_{2n-5}(q))$$

$$= aJ_{n-1}(q) + b (aJ_{n-2}(q) + b (aJ_{n-3}(q) + bO_{2n-7}(q)))$$

$$= \cdots$$

$$= b^{n-2}O_3(q) + \sum_{k=2}^{n-1} ab^{n-k-1}J_k(q).$$
(3.6)

Substituting (3.6) into (3.5), we get

$$J_{n+1}(q) = a^2 J_n(q) + b^2 J_{n-1}(q) + 2ab \left(b^{n-2} O_3(q) + \sum_{k=2}^{n-1} ab^{n-k-1} J_k(q) \right)$$
$$= 2ab^{n-1} O_3(q) + a^2 J_n(q) + b^2 J_{n-1}(q) + \sum_{k=2}^{n-1} 2a^2 b^{n-k} J_k(q),$$

as desired. The proof is complete.

We continue to prove Conjecture 1.3. Now define the sequence $\{M_n(q)\}_{n\geq 1}$ by letting $M_1(q) = \frac{J_1(q)}{E_1(q)}$ and $M_2(q) = \frac{J_2(q)}{E_2(q)}$ and

$$M_{n+1}(q) = aM_n(q) + bM_{n-1}(q), (3.7)$$

where a = (5 + 4q) and $b = -4(q - 1)^2$ as in Theorem 3.1. The main result of this section is as follows, which gives an affirmative answer to Conjecture 1.3.

Theorem 3.2 For any $n \ge 1$ let $E_n(q), J_n(q)$ be given by (3.1), let $M_n(q)$ be given by (3.7), and let $H_n(q)$ be defined by

$$H_n(q) = E_n(q)M_n(q). (3.8)$$

Then $H_n(q) = J_n(q)$, and hence $L_{4n+1}(q)$ is divisible by $L_{2n}(q)$.

Proof. Since $H_1(q) = J_1(q)$ and $H_2(q) = J_2(q)$, it suffices to show that the sequence $\{H_n(q)\}_{n\geq 1}$ satisfies the same recurrence relation as $\{J_n(q)\}_{n\geq 1}$, namely

$$H_{n+1}(q) = ab^{n-1}h + a^2H_n(q) + b^2H_{n-1}(q) + \sum_{k=2}^{n-1} 2a^2b^{n-k}H_k(q),$$
 (3.9)

where

$$a = (5+4q), b = -4(q-1)^2, h = 2O_3(q).$$

By (3.2) and (3.7) we have

$$H_{n+1}(q) = E_{n+1}(q)M_{n+1}(q)$$

$$= (aE_n(q) + bE_{n-1}(q)) (aM_n(q) + bM_{n-1}(q))$$

$$= a^2H_n(q) + b^2H_{n-1}(q) + ab (E_n(q)M_{n-1}(q) + E_{n-1}(q)M_n(q)).$$

For notational convenience, let

$$T_n(q) = E_n(q)M_{n-1}(q) + E_{n-1}(q)M_n(q).$$
(3.10)

Then by (3.2) and (3.7)

$$T_n(q) = E_n(q)M_{n-1}(q) + E_{n-1}(q)M_n(q)$$

$$= (aE_{n-1}(q) + bE_{n-2}(q))M_{n-1}(q) + E_{n-1}(q)(aM_{n-1}(q) + bM_{n-2}(q))$$

$$= 2aH_{n-1}(q) + bT_{n-1}(q),$$

where the last equality is due to (3.8) and (3.10).

Iterating the above process, we get

$$H_{n+1}(q) = a^{2}H_{n}(q) + b^{2}H_{n-1}(q) + abT_{n}(q)$$

$$= a^{2}H_{n}(q) + b^{2}H_{n-1}(q) + ab(2aH_{n-1}(q) + bT_{n-1}(q))$$

$$= a^{2}H_{n}(q) + b^{2}H_{n-1}(q) + 2a^{2}bH_{n-1}(q) + ab^{2}T_{n-1}(q)$$

$$= a^{2}H_{n}(q) + b^{2}H_{n-1}(q) + 2a^{2}bH_{n-1}(q) + ab^{2}(2aH_{n-2}(q) + bT_{n-2}(q))$$

$$= a^{2}H_{n}(q) + b^{2}H_{n-1}(q) + 2a^{2}bH_{n-1}(q) + 2a^{2}b^{2}H_{n-2}(q) + ab^{3}T_{n-2}(q)$$

$$= \cdots$$

$$= a^{2}H_{n}(q) + b^{2}H_{n-1}(q) + 2a^{2}bH_{n-1}(q) + 2a^{2}b^{2}H_{n-2}(q) + \cdots$$

$$+ 2a^{2}b^{n-2}H_{2}(q) + ab^{n-1}T_{2}(q)$$

$$= a^{2}H_{n}(q) + b^{2}H_{n-1}(q) + \sum_{k=2}^{n-1} 2a^{2}b^{n-k}H_{k} + ab^{n-1}T_{2}(q).$$

It is straightforward to verify that

$$T_2(q) = 2O_3(q).$$

Thus $\{H_n(q)\}_{n\geq 1}$ and $\{J_n(q)\}_{n\geq 1}$ satisfy the same recurrence relation with the same initial conditions. This completes the proof.

4 Asymptotic normality

In this section we will show that the coefficients of $L_n(q)$ are asymptotically normal by central and local limit theorems. For a polynomial sequence $\{F_n(q)\}_{n\geq 0}$ as in (1.8), it is not hard to show that if the coefficients $a_{n,k}$ are asymptotically normal by a local limit theorem on \mathbb{R} then these numbers are also asymptotically normal by a central limit theorem. Thus, we will focus on proving that the coefficients of $L_n(q)$ are asymptotically normal by a local limit theorem.

The main tool we use here is the following criterion, which goes back to Harper [3].

Theorem 4.1 ([1, Theorem 2]) Suppose that $\{F_n(q)\}_{n\geq 0}$ is a real-rooted polynomial sequence with nonnegative coefficients as in (1.8). Let

$$\mu_n = \frac{F_n'(1)}{F_n(1)},\tag{4.1}$$

$$\sigma_n^2 = \frac{F_n''(1)}{F_n(1)} + \mu_n - \mu_n^2. \tag{4.2}$$

If $\sigma_n^2 \to +\infty$ when $n \to +\infty$, then the coefficients of $F_n(q)$ are asymptotically normal by a local limit theorem with mean μ_n and variance σ_n^2 .

In order to compute the mean and variance corresponding to $L_n(q)$, we next derive Binet's formula for these polynomials by solving (2.1). Note that the characteristic equation of this recurrence relation is

$$x^2 - 3x - 2(q - 1) = 0$$

with roots

$$r(q) = \frac{3 + \sqrt{8q + 1}}{2}$$
 and $s(q) = \frac{3 - \sqrt{8q + 1}}{2}$

where r(q) + s(q) = 3 and r(q)s(q) = -2(q-1). So the general solution of (2.1) is

$$L_n(q) = C(r(q))^{n-1} + D(s(q))^{n-1},$$

where the coefficients C and D are to be determined.

The initial conditions $L_1(q) = 3$ and $L_2(q) = 7 + 2q = 3r(q) + 3s(q) - r(q)s(q)$ yield the following system:

$$C + D = 3,$$

$$Cr(q) + Ds(q) = 2q + 7.$$

Solving this, we get

$$C = \frac{2q+7-3s(q)}{r(q)-s(q)}$$
 and $D = -\frac{2q+7-3r(q)}{r(q)-s(q)}$.

Thus

$$L_n(q) = \frac{2q + 7 - 3s(q)}{r(q) - s(q)} \cdot (r(q))^{n-1} - \frac{2q + 7 - 3r(q)}{r(q) - s(q)} \cdot (s(q))^{n-1}$$

$$= \frac{3 - s(q)}{r(q) - s(q)} \cdot (r(q))^n - \frac{3 - r(q)}{r(q) - s(q)} \cdot (s(q))^n$$

$$= 3 \cdot \frac{(r(q))^n - (s(q))^n}{r(q) - s(q)} - r(q)s(q) \cdot \frac{(r(q))^{n-1} - (s(q))^{n-1}}{r(q) - s(q)}. \tag{4.3}$$

The main result of this section is as follows.

Theorem 4.2 Let the sequence $\{L_n(q)\}_{n\geq 1}$ be defined as in (1.4). Then the coefficients of $L_n(q)$ are asymptotically normal by a local limit theorem.

Proof. Let us first compute μ_n and σ_n^2 . It is easy to show that

$$r(q) = \frac{3 + \sqrt{8q + 1}}{2},$$

$$r'(q) = \frac{2}{\sqrt{8q + 1}},$$

$$r''(q) = -\frac{8}{(8q + 1)\sqrt{8q + 1}},$$

$$s(q) = \frac{3 - \sqrt{8q + 1}}{2},$$

$$s'(q) = -\frac{2}{\sqrt{8q + 1}},$$

$$s''(q) = \frac{8}{(8q+1)\sqrt{8q+1}}.$$

Thus

$$r(1) = 3, \quad r'(1) = \frac{2}{3}, \quad r''(1) = -\frac{8}{27},$$
 (4.4)

$$s(1) = 0, \quad s'(1) = -\frac{2}{3}, \quad s''(1) = \frac{8}{27}.$$
 (4.5)

Let

$$F_n(q) = \frac{(r(q))^n - (s(q))^n}{r(q) - s(q)}.$$

It is straightforward to verify that

$$F'_n(q) = \frac{n[((r(q))^{n-1}r'(q) - (s(q))^{n-1}s'(q))(r(q) - s(q))]}{(r(q) - s(q))^2} - \frac{[((r(q))^n - (s(q))^n)(r'(q) - s'(q))]}{(r(q) - s(q))^2}$$

and

$$F_n''(q) = \frac{G_n'(q)(r(q) - s(q))^2 - 2G_n(q)(r(q) - s(q))(r'(q) - s'(q))}{(r(q) - s(q))^4}.$$

with

$$G_n(q) = (n-1)((r(q))^n r'(q) + (s(q))^n s'(q))$$

$$+ ((s(q))^{n-1} - n(r(q))^{n-1})r'(q)s(q) + ((r(q))^{n-1} - n(s(q))^{n-1})s'(q)r(q)$$

and

$$\begin{split} G_n'(q) = &(n-1)((r(q))^n r''(q) + n(r(q))^{n-1}(r'(q))^2 + (s(q))^n s''(q) + n(s(q))^{n-1}(s'(q))^2) \\ &+ ((s(q))^{n-1} - n(r(q))^{n-1})(r''(q)s(q) + r'(q)s'(q)) \\ &+ ((n-1)(s(q))^{n-2}s'(q) - n(n-1)(r(q))^{n-2}r'(q))r'(q)s(q) \\ &+ ((r(q))^{n-1} - n(s(q))^{n-1})(s''(q)r(q) + s'(q)r'(q)) \\ &+ ((n-1)(r(q))^{n-2}r'(q) - n(n-1)(s(q))^{n-2}s'(q))s'(q)r(q). \end{split}$$

From the above formulas we deduce that

$$F_n(1) = 3^{n-1}, \quad F'_n(1) = 2(n-2)3^{n-3}, \quad G_n(1) = 2(n-2)3^{n-1},$$

 $G'_n(1) = 4(n^2 - 3n + 4)3^{n-3}, \quad F''_n(1) = 4(n^2 - 7n + 12)3^{n-5}.$

By Binet's formula for $L_n(q)$, we obtain

$$L_n(q) = 3F_n(q) - r(q)s(q)F_{n-1}(q)$$

and hence

$$L'_{n}(q) = 3F'_{n}(q) - r'(q)s(q)F_{n-1}(q) - r(q)s'(q)F_{n-1}(q) - r(q)s(q)F'_{n-1}(q)$$

$$L''_{n}(q) = 3F''_{n}(q) - r''(q)s(q)F_{n-1}(q) - r'(q)s'(q)F_{n-1}(q) - r'(q)s(q)F'_{n-1}(q)$$

$$- r'(q)s'(q)F_{n-1}(q) - r(q)s''(q)F'_{n-1}(q) - r(q)s'(q)F'_{n-1}(q)$$

$$- r'(q)s(q)F'_{n-1}(q) - r(q)s'(q)F'_{n-1}(q) - r(q)s(q)F''_{n-1}(q).$$

So

$$L_n(1) = 3^n$$
, $L'_n(1) = 2(n-1) \cdot 3^{n-2}$, $L''_n(1) = 4(n-2)(n-3) \cdot 3^{n-4}$.

From (4.1) it follows that

$$\mu_n = \frac{L'_n(1)}{L_n(1)} = \frac{2(n-1)}{9}$$

and

$$\sigma_n^2 = \frac{L_n''(1)}{L_n(1)} + \mu_n - \mu_n^2$$

$$= \frac{4(n-2)(n-3)}{81} + \frac{2(n-1)}{9} - \frac{4(n-1)^2}{81}$$

$$= \frac{2(n-1)(2n+7)}{81},$$

which tends to $+\infty$ as n approaches infinity. Then combining Theorem 2.3 and Theorem 4.1, we complete the proof.

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References

- [1] E.A. Bender, Central and local limit theorems applied to asymptotic enumeration, J. Combin. Theory Ser. A 15 (1973) 91–111.
- [2] J.L. GROSS, T. MANSOUR, T.W. TUCKER AND D.G.L. WANG, Root geometry of polynomial sequences I: type (0,1), J. Math. Anal. Appl. 433 (2016) 1261–1289.

- [3] L.H. Harper, Stirling behavior is asymptotically normal, Ann. Math. Statist. 38 (1967) 410–414.
- [4] H.-K. HWANG, H.-H. CHERN AND G.-H. DUH, An asymptotic distribution theory for Eulerian recurrences with applications, Adv. in Appl. Math. 112 (2020) 101960.
- [5] L.L. LIU AND Y. WANG, A unified approach to polynomial sequences with only real zeros, Adv. in Appl. Math. 38 (2007) 542–560.
- [6] S. NORTHSHIELD, Sterns diatomic sequence 0, 1, 1, 2, 1, 3, 2, 3, 1, 4, ..., Amer. Math. Monthly, **117** (2010) 581–598.
- [7] D.E. Speyer, *Proof of a conjecture of Stanley about Sterns array*, arxiv.org/abs/1901.06301.
- [8] R.P. Stanley, Some linear recurrences motivated by Sterns diatomic array, Amer. Math. Monthly, 127 (2020) 99–111.
- [9] R.P. Stanley, From Stern's triangle to upper homogeneous posets, Institut Mittag-Leffler Algebraic Combinatorics Online Workshop (ACOW), 20 April 2020.
- [10] M. Stern, Ueber eine zahlentheoretische Funktion, J. Reine Angew. Math., 55 (1858) 193–220.
- [11] Y. Wang and Y.-N. Yeh, Polynomials with real zeros and Pólya frequency sequences, J. Combin. Theory Ser. A 109 (2005) 63–74.
- [12] Y. Wang, H.-X. Zhang and B.-X. Zhu, Asymptotic normality of Laplacian coefficients of graphs, J. Math. Anal. Appl. 455 (2017) 2030–2037.