# ON DELANNOY PATHS WITHOUT PEAKS AND VALLEYS 

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#### Abstract

A lattice path is called Delannoy if each of its steps belongs to $\{N, E, D\}$, where $N=(0,1), E=(1,0)$, and $D=(1,1)$ steps. Peak, valley, and deep valley are denoted by $N E, E N$, and $E E N N$ on the lattice path, respectively

In this paper, we find a bijection between $\mathcal{P}_{n, m}(N E, E N)$ and a specific subset of $\mathcal{P}_{n, m}(D, E E N N)$, where $\mathcal{P}_{n, m}(N E, E N)$ is the set of Delannoy paths from the origin to $(n, m)$ without peaks and valleys, and $\mathcal{P}_{n, m}(D, E E N N)$ is the set of Delannoy lattice paths from the origin to ( $n, m$ ) without diagonal steps and deep valleys. We also enumerate the number of Delannoy paths without peaks and valleys on the restricted region $\left\{(x, y) \in \mathbb{Z}^{2}: y \geq k x\right\}$ for a positive integer $k$.


## 1. Introduction

For two lattice points $A$ and $B$, a lattice path from $A$ to $B$ is a sequence of lattice points

$$
\left(v_{1}, v_{2}, \ldots, v_{k}\right)
$$

of $\mathbb{Z}^{2}$ with $v_{1}=A$ and $v_{k}=B$. For each $i=1, \ldots, k-1$, a consecutive difference $s_{i}=$ $v_{i+1}-v_{i}$ of $\left(v_{1}, v_{2}, \ldots, v_{k}\right)$ is called a step of the lattice path. Conventionally, a lattice path can be represented as a word $s_{1} s_{2} \ldots s_{k-1}$ with starting point $v_{1}$.

A $(0,1)$ step is called a north step, denoted by $N$; a $(1,0)$ step is called an east step, denoted by $E$; a $(1,1)$ step is called a diagonal step, denoted by $D$. A lattice path is called Delannoy if each of its steps belongs to $\{N, E, D\}$.

Let $n$ and $m$ be nonnegative integers. Let $\mathcal{P}_{n, m}$ be the set of Delannoy paths from the origin to $(n, m)$. It is well-known [Com74, p. 81] that

$$
\# \mathcal{P}_{n, m}=\sum_{d=0}^{n}\binom{n+m-d}{n-d, m-d, d}=\sum_{j=0}^{n}\binom{n}{j}\binom{m}{j} 2^{j} .
$$

If a pair $N E$ (resp. $E N$ ) appears consecutively on the lattice path, it is called a peak (resp. valley). If a quadruple $E E N N$ appears consecutively on the lattice path, it is called a deep valley.

The collection of Delannoy paths that avoid all specific patterns $\omega_{1}, \ldots, \omega_{k}$ is denoted as

$$
\mathcal{P}_{n, m}\left(\omega_{1}, \ldots, \omega_{k}\right)=\left\{P \in \mathcal{P}_{n, m}: \text { There are no patterns } \omega_{1}, \ldots, \omega_{k} \text { in } P\right\} .
$$

Let $\mathcal{P}_{n, m}(N E, E N)$ be the set of Delannoy paths without peaks and valleys in $\mathcal{P}_{n, m}$. To calculate the number of reduced alignments between two DNA sequences in Bioinformatics, Andrade et al. AANT14 found that the cardinality of $\mathcal{P}_{n, m}(N E, E N)$ is

$$
\begin{equation*}
\sum_{i \geq 0}(-1)^{i}\left[\binom{n+m-3 i}{n-2 i, m-2 i, i}-\binom{(n-1)+(m-1)-3 i}{(n-1)-2 i,(m-1)-2 i, i}\right] \tag{1}
\end{equation*}
$$

which is given by entry A047080 in the OEIS [Slo18.
In this paper, given a lattice path $P=\left(v_{1}, v_{2}, \ldots, v_{k}\right)$, with $v_{0}$ and $v_{k+1}$ satisfying

$$
v_{1}-v_{0}=E \quad \text { and } \quad v_{k+1}-v_{k}=N,
$$

$P^{+}=\left(v_{0}, v_{1}, \ldots, v_{k+1}\right)$ is called an augmented path of $P$, and $P^{-}=\left(v_{2}, \ldots, v_{k-1}\right)$ is called a diminished path of $P$. Given a set $S$ of lattice paths, $S^{+}$and $S^{-}$are defined by

$$
S^{+}=\left\{P^{+}: P \in S\right\} \quad \text { and } \quad S^{-}=\left\{P^{-}: P \in S\right\}
$$

Let $\mathcal{P}_{n, m}{ }^{+}(D, E E N N)^{-}$be the set of Delannoy paths from the origin to ( $n, m$ ) whose augmented path does not include diagonal steps or deep valleys.

We construct two bijections $\delta$ from $\mathcal{P}_{n, m}(N E, E N)$ to $\mathcal{P}_{n, m}{ }^{+}(D, E E N N)^{-}$and $\pi$ from $\mathcal{P}_{n, m}{ }^{+}(D, E E N N)^{-}$to $\mathcal{P}_{n, m}(N E, E N)$.


Let $k$ be a positive integer. Let $\mathcal{P}_{n, k n}^{(k)}$ be the set of Delannoy paths $P \in \mathcal{P}_{n, k n}$ on the half-plane $y \geq k x$. Song [Son05] introduced a $k$-Schröder path $P$ of size $n$ if $P \in \mathcal{P}_{n, k n}^{(k)}$ for nonnegative integers $n$, where the cardinality of $\mathcal{P}_{n, k n}^{(k)}$ equals

$$
\# \mathcal{P}_{n, k n}^{(k)}=\sum_{d=0}^{n} \frac{1}{k n-d+1}\binom{k n+n-d}{k n-d, n-d, d}=\frac{1}{n} \sum_{j=1}^{n}\binom{k n}{j-1}\binom{n}{j} 2^{j} .
$$

It is known [Slo18, A086581] that the number of Delannoy paths without diagonal steps and deep valleys from $(0,0)$ to $(n, n)$ on the half-plane $y \geq x$ is as follows:

$$
\# \mathcal{P}_{n, n}^{(1)}(D, E E N N)=\sum_{m \geq 0} \frac{1}{m+1}\binom{2 m}{m}\binom{n+m}{3 m}
$$

In this paper, we prove that $\mathcal{P}_{n, k n}^{(k)}(N E, E N)$ and $\mathcal{P}_{n, k n}^{(k)}(D, E E N N)$ are equinumerous via two bijections $\delta$ and $\pi$.


For $k=1$ or 2 , we find the numbers of Delannoy paths without peaks and valleys from $(0,0)$ to $(n, k n)$ on the half-plane $y \geq k x$ as follows:

$$
\begin{aligned}
\# \mathcal{P}_{n, n}^{(1)}(N E, E N) & =\sum_{m \geq 0} \frac{1}{m+1}\binom{2 m}{m}\binom{n+m}{3 m}, \\
\# \mathcal{P}_{n, 2 n}^{(2)}(N E, E N) & =\sum_{m \geq 0} \frac{(-1)^{n-m}}{m+1}\binom{3 m+1}{m}\binom{m+1}{n-m} .
\end{aligned}
$$

The remainder of this paper is organized as follows. In Section 2, we review previous studies and analyze the number of Delannoy paths without peaks and valleys. In Section 3, we construct two bijections $\delta$ and $\pi$ between $\mathcal{P}_{n, m}(N E, E N)$ and $\mathcal{P}_{n, m}{ }^{+}(D, E E N N)^{-}$. In Section [4, we address Delannoy paths without peaks and valleys on the half-plane $y \geq k x$. In Section 5, we enumerate the cardinalities of $\mathcal{P}_{n, n}^{(1)}(N E, E N)$ and $\mathcal{P}_{n, 2 n}^{(2)}(N E, E N)$. We also present Conjecture 5, which is about Catalan paths avoiding symmetric peaks [Eli21], and Conjecture 7, which is about inversion sequences avoiding the pattern 102 MS15.

## 2. Delannoy Paths

Let $n$ and $m$ be nonnegative integers. Let $h(n, m)$ be the cardinality of $\mathcal{P}_{n, m}(N E, E N)$. Andrade et al. AANT14 mentioned that $h(n, m)$ satisfies the recurrence

$$
\begin{equation*}
h(n, m)=h(n-1, m)+h(n, m-1)-h(n-2, m-2) \tag{2}
\end{equation*}
$$

for $n \geq 2$ and $m \geq 2$. With initial conditions $h(n, 0)=h(0, m)=1$, they deduced the generating function of $h(n, m)$ and found Formula (1) for $h(n, m)$.

Because their process of finding (1) was algebraic, we want to find a combinatorial interpretation of $h(n, m)$. It is natural to consider the set $\mathcal{P}_{n, m}(D, E E N N)$ of Delannoy paths $P$ in $\mathcal{P}_{n, m}$ without diagonal steps and deep valleys as a combinatorial object satisfying the recurrence (2), letting $b(n, m)$ be the cardinality of $\mathcal{P}_{n, m}(D, E E N N)$. By considering the last pattern of the paths, $b(n, m)$ also satisfies

$$
b(n, m)= \begin{cases}b(n-1, m)+b(n, m-1)-b(n-2, m-2) & \text { if }(n, m) \neq(0,0)  \tag{3}\\ 1 & \text { if }(n, m)=(0,0)\end{cases}
$$

Conventionally, $b(n, m)=0$ for $n<0$ or $m<0$. Using the principle of inclusion and exclusion, we obtain

$$
\begin{equation*}
b(n, m)=\sum_{i \geq 0}(-1)^{i}\binom{n+m-3 i}{n-2 i, m-2 i, i} \tag{4}
\end{equation*}
$$

where $i$ keeps track the number of deep valleys, which is not in the OEIS [Slo18].
Comparing (1) with (4), we observe that

$$
\begin{equation*}
h(n, m)=b(n, m)-b(n-1, m-1) . \tag{5}
\end{equation*}
$$

Now, we introduce another combinatorial object whose cardinality is the right-hand side of (5). Recall that $\mathcal{P}_{n, m}{ }^{+}(D, E E N N)^{-}$is the set of Delannoy paths $P$ in $\mathcal{P}_{n, m}$, where the augmented lattice path $P^{+}$does not have a diagonal step $D$ or a deep valley $E E N N$. Let $a(n, m)$ be the cardinality of $\mathcal{P}_{n, m}{ }^{+}(D, E E N N)^{-}$. We show a combinatorial proof of

$$
\begin{equation*}
a(n, m)=b(n, m)-b(n-1, m-1) . \tag{6}
\end{equation*}
$$

Consider a mapping

$$
\tau: \mathcal{P}_{n, m}(D, E E N N) \backslash \mathcal{P}_{n, m}^{+}(D, E E N N)^{-} \rightarrow \mathcal{P}_{n-1, m-1}(D, E E N N)
$$

with two rules:

$$
\begin{aligned}
& \alpha E E N \mapsto E \alpha, \\
& E N N \beta \mapsto N \beta .
\end{aligned}
$$

According to these rules, for a given $E N N \gamma E E N$ in $\mathcal{P}_{n, m}(D, E E N N) \backslash \mathcal{P}_{n, m}{ }^{+}(D, E E N N)^{-}$, we have two candidates $E E N N \gamma$ and $N \gamma E E N$ as $\tau(E N N \gamma E E N)$. Because the first candidate $E E N N \gamma$ includes a deep valley, it does not belong to $\mathcal{P}_{n-1, m-1}(D, E E N N)$. As a result, $E N N \gamma E E N$ should correspond to $N \gamma E E N$ in $\mathcal{P}_{n-1, m-1}(D, E E N N)$.

In addition, the inverse mapping of $\tau$ is well-defined, and $\tau$ is bijective. Therefore, we immediately obtain the equation

$$
b(n, m)-a(n, m)=b(n-1, m-1)
$$

which is equivalent to (6).
Remark. The generating function $F_{B}(x, y)$ of $b(n, m)$ is induced from (3) as follows:

$$
\begin{equation*}
F_{B}(x, y)=\sum_{n, m \geq 0} b(n, m) x^{n} y^{m}=\frac{1}{1-x-y+x^{2} y^{2}} \tag{7}
\end{equation*}
$$

When the generating function of $a(n, m)$ is defined by

$$
F_{A}(x, y)=\sum_{n, m \geq 0} a(n, m) x^{n} y^{m}
$$

Equation (6) yields

$$
F_{B}(x, y)-F_{A}(x, y)=x y F_{B}(x, y) .
$$



Figure 1. Example under mappings $\pi$ and $\delta$
Thus, we have the generating function of $a(n, m)$

$$
F_{A}(x, y)=\frac{1-x y}{1-x-y+x^{2} y^{2}},
$$

which is also the generating function of $h(n, m)$.

## 3. Two Mappings

A North-East lattice path is a lattice path in $\mathbb{Z}^{2}$ with steps in $\{N, E\}$. Let $\mathcal{P}_{n, m}(D)$ be the set of North-East lattice paths from the origin to $(n, m)$. Let $\mathcal{P}_{n, m}(N E)$ be the set of Delannoy paths without peaks in $\mathcal{P}_{n, m}$.

Given $P \in \mathcal{P}_{n, m}(D)$, a mapping $\pi$ changes each peak of $P$ to a diagonal step. Evidently, $\pi$ is well-defined and $\pi(P) \in \mathcal{P}_{n, m}(N E)$. Considering a reverse of $\pi$, given $Q \in \mathcal{P}_{n, m}(N E)$, a mapping $\delta$ changes each diagonal step of $Q$ to a peak. Evidently, $\delta$ is well-defined and $\delta(Q) \in \mathcal{P}_{n, m}(D)$.

For example, as shown in Figure 1, we have

$$
\begin{aligned}
\pi(E N N E E N N N E N E E N) & =E N D E N N D D E N, \\
\delta(E N D E N N D D E N) & =E N N E E N N N E N E E N
\end{aligned}
$$

Theorem 1. Mappings $\pi$ and $\delta$ are bijections between $\mathcal{P}_{n, m}(N E)$ and $\mathcal{P}_{n, m}(D)$.


Proof. $\delta(\pi(P))=P$ holds for $P \in \mathcal{P}_{n, m}(D)$ and $\pi(\delta(Q))=Q$ for $Q \in \mathcal{P}_{n, m}(N E)$.

From the previous section, the two sets $\mathcal{P}_{n, m}{ }^{+}(D, E E N N)^{-}$and $\mathcal{P}_{n, m}(N E, E N)$ are known to be equinumerous, that is,

$$
\begin{equation*}
a(n, m)=h(n, m) . \tag{8}
\end{equation*}
$$

Here, we provide a bijective proof of (8) under $\pi$ and $\delta$.
Theorem 2. Mappings $\pi$ and $\delta$ between $\mathcal{P}_{n, m}{ }^{+}(D, E E N N)^{-}$and $\mathcal{P}_{n, m}(N E, E N)$ are bijections.


Proof. It is sufficient to show that

$$
\begin{aligned}
P \in \mathcal{P}_{n, m}^{+}(D, E E N N)^{-} & \Longrightarrow \pi(P) \in \mathcal{P}_{n, m}(N E, E N) \\
& Q \in \mathcal{P}_{n, m}(N E, E N)
\end{aligned}
$$

(a) For a given $P \in \mathcal{P}_{n, m}{ }^{+}(D, E E N N)^{-}$, we have $P \in \mathcal{P}_{n, m}(D)$ and $\pi(P) \in \mathcal{P}_{n, m}(N E)$. Thus, $\pi(P)$ avoids peaks $N E$.
Because $P \in \mathcal{P}_{n, m}{ }^{+}(D, E E N N)^{-}$and $P^{+}$do not include a deep valley EENN, only two possibilities exist for each valley $E N$ in $P$ :
(i) If a valley $E N$ follows $N$ in $P^{+}$, then $N E N$ should be changed to $D N$.
(ii) If a valley $E N$ preceds $E$ in $P^{+}$, then $E N E$ should be changed to $E D$.

Hence, $\pi(P)$ avoids peaks $E N$, and we have $\pi(P) \in \mathcal{P}_{n, m}(N E, E N)$.
(b) For a given $Q \in \mathcal{P}_{n, m}(N E, E N)$, we have $Q \in \mathcal{P}_{n, m}(N E)$ and $\delta(Q) \in \mathcal{P}_{n, m}(D)$. Thus, $\delta(Q)^{+}$avoids diagonal steps $D$.
Suppose $\delta(Q)^{+}$includes a deep valley EENN. Then, the path $\pi\left(\delta(Q)^{+}\right)^{-}$includes a valley $E N$. Because

$$
\pi\left(\delta(Q)^{+}\right)^{-}=Q
$$

$Q$ includes a valley $E N$, which contradicts $Q \in \mathcal{P}_{n, m}(N E, E N)$. Hence, $\delta(Q)^{+}$avoids deep valleys $E E N N$, and we have $\delta(Q) \in \mathcal{P}_{n, m}{ }^{+}(D, E E N N)^{-}$.

We complete a combinatorial proof of (1) because (4) is proved by the principle of inclusion and exclusion, (6) is proved bijectively by $\tau$, and (8) is proved bijectively by $\pi$ and $\delta$.

## 4. $k$-SCHRÖDER PATHS

Let $k$ be a positive integer. Define a region $\mathbb{Z}^{2}(k)$ by

$$
\mathbb{Z}^{2}(k)=\left\{(x, y) \in \mathbb{Z}^{2}: y \geq k x\right\} .
$$

Let $\mathcal{P}_{n, m}^{(k)}$ be the set of Delannoy paths $P \in \mathcal{P}_{n, m}$, whose vertices lie on the region $\mathbb{Z}^{2}(k)$. Song Son05] introduced a $k$-Schröder path $P$ of size $n$ if $P \in \mathcal{P}_{n, k n}^{(k)}$ for nonnegative integers $n$, where the cardinality of $\mathcal{P}_{n, k n}^{(k)}$ equals

$$
\# \mathcal{P}_{n, k n}^{(k)}=\sum_{d=0}^{n} \frac{1}{k n-d+1}\binom{k n+n-d}{k n-d, n-d, d}=\frac{1}{n} \sum_{j=1}^{n}\binom{k n}{j-1}\binom{n}{j} 2^{j} .
$$

Recently, Yang and Jiang [YJ21] showed that the cardinality of $\mathcal{P}_{n, m}^{(k)}$ equals

$$
\# \mathcal{P}_{n, m}^{(k)}=\sum_{d=0}^{n} \frac{m-k n+1}{m-d+1}\binom{m+n-d}{m-d, n-d, d}=\frac{m-k n+1}{n} \sum_{j=1}^{n}\binom{m}{j-1}\binom{n}{j} 2^{j}
$$

with $\# \mathcal{P}_{n, m}^{(k)}=0$ if $m<k n$.
Let $\mathcal{P}_{n, m}^{(k)+}(D, E E N N)^{-}$be the set of North-East lattice paths $P$ in $\mathcal{P}_{n, m}^{(k)}$, where the augmented lattice path $P^{+}$does not have a deep valley $E E N N$, and let $\mathcal{P}_{n, m}^{(k)}(N E, E N)$ be the set of Delannoy paths without peaks and valleys in $\mathcal{P}_{n, m}^{(k)}$.

Corollary 3. Mappings $\pi$ from $\mathcal{P}_{n, m}^{(k)}{ }^{+}(D, E E N N)^{-}$to $\mathcal{P}_{n, m}^{(k)}(N E, E N)$ and $\delta$ from $\mathcal{P}_{n, m}^{(k)}(N E, E N)$ to $\mathcal{P}_{n, m}^{(k)}{ }^{+}(D, E E N N)^{-}$are bijections.

$$
\begin{aligned}
& \mathcal{P}_{n, m}{ }^{+}(D, E E N N)^{-} \underset{\delta}{\stackrel{\pi}{\rightleftarrows}} \mathcal{P}_{n, m}(N E, E N) \\
& \cup \cup \cup \\
& \mathcal{P}_{n, m}^{(k)+}(D, E E N N)-\stackrel{\pi}{\rightleftarrows} \mathcal{P}_{n, m}^{(k)}(N E, E N)
\end{aligned}
$$

Proof. Because, for any lattice point $(x, y)$, if $(x, y) \in \mathbb{Z}^{2}(k)$ and $(x+1, y+1) \in \mathbb{Z}^{2}(k)$ then $(x, y+1) \in \mathbb{Z}^{2}(k)$, we know that

$$
\begin{aligned}
P \in \mathcal{P}_{n, m}^{(k)^{+}}(D, E E N N)^{-} & \Longrightarrow \pi(P) \in \mathcal{P}_{n, m}^{(k)}(N E, E N) \\
Q \in \mathcal{P}_{n, m}^{(k)}(N E, E N) & \Longrightarrow \delta(Q) \in \mathcal{P}_{n, m}^{(k)+}(D, E E N N)^{-}
\end{aligned}
$$

which completes the proof.

## 5. $k$-SChröder paths without peaks and valleys

Because every $k$-Schröder path should begin with $N$ and end with $E$, for a $k$-Schröder path $P$ without deep valleys, $P^{+}$does not have deep valleys. Thus, we have

$$
\mathcal{P}_{n, k n}^{(k)}(D, E E N N)=\mathcal{P}_{n, k n}^{(k)+}(D, E E N N)^{-}
$$

From Corollary 3, we obtain

$$
\# \mathcal{P}_{n, k n}^{(k)}(D, E E N N)=\# \mathcal{P}_{n, k n}^{(k)}(N E, E N)
$$

Let us divide $\mathcal{P}_{n, k n}^{(k)}(N E, E N)$ by the last step. Define $D \mathcal{P}_{n, k n}^{(k)}(N E, E N)$ as the set of $P \in \mathcal{P}_{n, k n}^{(k)}(N E, E N)$ ending with diagonal step $D$. In addition, define $E \mathcal{P}_{n, k n}^{(k)}(N E, E N)$ as the set of $P \in \mathcal{P}_{n, k n}^{(k)}(N E, E N)$ ending with east step $E$. By definition, it holds that

$$
\# \mathcal{P}_{n, k n}^{(k)}(N E, E N)=\delta_{n, 0}+\# D \mathcal{P}_{n, k n}^{(k)}(N E, E N)+\# E \mathcal{P}_{n, k n}^{(k)}(N E, E N)
$$

Define three generating functions $F, F_{D}$, and $F_{E}$ as

$$
\begin{aligned}
& F=F^{(k)}(x) \\
&=\sum_{n \geq 0} \# \mathcal{P}_{n, k n}^{(k)}(N E, E N) x^{n}, \\
& F_{D}=F_{D}^{(k)}(x) \\
& F_{E}=\sum_{n \geq 1} \# D \mathcal{P}_{n, k n}^{(k)}(N E, E N) x^{n}, \\
&(x)=\sum_{n \geq 1} \# E \mathcal{P}_{n, k n}^{(k)}(N E, E N) x^{n} .
\end{aligned}
$$

These satisfy the identity

$$
F=1+F_{D}+F_{E} .
$$

On the left of Figure 2, every path $P \in D \mathcal{P}_{n, k n}^{(k)}(N E, E N)$ can be decomposed into $k$ subpaths as follows. For $j=1, \ldots, k-1$, let $N_{j}$ be the last north steps between lines $y=k x+(j-1)$ and $y=k x+j$. Consider $k$ subpaths $P_{1}, \ldots, P_{k}$ by removing $(k-1)$ north steps $N_{1}, \ldots, N_{k-1}$ and the last diagonal step $D$. Because $P$ avoids peaks and valleys, subpaths $P_{1}, \ldots, P_{k-1}$ are unable to end with an east step $E$, but $P_{k}$ does not have this restriction. This decomposition yields the identity

$$
\begin{equation*}
F_{D}=\left(1+F_{D}\right)^{k-1} F x . \tag{9}
\end{equation*}
$$

On the right in Figure 2, every path $P \in E \mathcal{P}_{n, k n}^{(k)}(N E, E N)$ can be decomposed into $(k+1)$ parts as follows. For $j=1, \ldots, k$, let $N_{j}$ be the last north steps between lines $y=k x+(j-1)$ and $y=k x+j$. Consider $k$ subpaths $P_{1}, \ldots, P_{k+1}$ by removing $k$ north steps $N_{1}, \ldots, N_{k}$ and the last diagonal step $E$. Because $P$ avoids peaks and valleys, subpaths $P_{1}, \ldots, P_{k}$ are unable to end with an east step $E$, but $P_{k+1}$ should not be a path of length 0 . This decomposition yields the identity

$$
\begin{equation*}
F_{E}=\left(1+F_{D}\right)^{k}(F-1) x \tag{10}
\end{equation*}
$$

Mutiplying (9) by $\left(1+F_{D}\right)$ and subtracting this by (10), $F_{E}$ is expressed by $F_{D}$ as follows:

$$
\begin{equation*}
F_{E}=\left(1+F_{D}\right) F_{D}-x\left(1+F_{D}\right)^{k} \tag{11}
\end{equation*}
$$

Substituting $F_{E}$ in (10) by (11) and simplifying, we have

$$
\begin{equation*}
F_{D}=x\left(1+F_{D}\right)^{k+1}-x^{2}\left(1+F_{D}\right)^{2 k-1} \tag{12}
\end{equation*}
$$

Because the right-hand sides of (9) and (12) are the same, $F$ is expressed by $F_{D}$ as follows:

$$
\begin{equation*}
F=\left(1+F_{D}\right)^{2}-x\left(1+F_{D}\right)^{k} . \tag{13}
\end{equation*}
$$



Figure 2. Decomposition of 3-Schröder paths
To obtain all generating functions, calculating $F_{D}$ from (12) is sufficient.
Enumerations of $\mathcal{P}_{n, n}^{(1)}(D, E E N N)$ and $\mathcal{P}_{n, n}^{(1)}(N E, E N)$. For $k=1$, we have

$$
\begin{aligned}
F^{(1)}(x) & =\frac{(1-x)^{2}-\sqrt{(1-x)^{4}-4 x^{2}(1-x)}}{2 x^{2}} \\
& =1+x+2 x^{2}+5 x^{3}+13 x^{4}+35 x^{5}+97 x^{6}+275 x^{7}+\cdots, \\
F_{D}^{(1)}(x) & =\frac{(1-x)^{2}-\sqrt{(1-x)^{4}-4 x^{2}(1-x)}}{2 x} \\
& =x+x^{2}+2 x^{3}+5 x^{4}+13 x^{5}+35 x^{6}+97 x^{7}+\cdots, \\
F_{E}^{(1)}(x) & =\frac{(1-x)^{3}-(1-x) \sqrt{(1-x)^{4}-4 x^{2}(1-x)}}{2 x^{2}}-1 \\
& =x^{2}+3 x^{3}+8 x^{4}+22 x^{5}+62 x^{6}+178 x^{7}+\cdots,
\end{aligned}
$$

where the sequence of the coefficients in $F^{(1)}(x)$ and $F_{E}^{(1)}(x)$ are given by entries A086581 and A188464 in the OEIS Slo18.

Theorem 4. For $k=1$, the coefficients of $x^{n}$ of $F, F_{D}$, and $F_{E}$ are as follows:

$$
\begin{aligned}
& {\left[x^{n}\right] F^{(1)}(x)=\sum_{m \geq 0} \frac{1}{m+1}\binom{2 m}{m}\binom{n+m}{3 m},} \\
& {\left[x^{n}\right] F_{D}^{(1)}(x)=\sum_{m \geq 0} \frac{1}{m+1}\binom{2 m}{m}\binom{n+m-1}{3 m},} \\
& {\left[x^{n}\right] F_{E}^{(1)}(x)=\sum_{m \geq 1} \frac{1}{m+1}\binom{2 m}{m}\binom{n+m-1}{3 m-1} .}
\end{aligned}
$$

Proof. Let

$$
C(x)=\frac{1-\sqrt{1-4 x}}{2 x}=\sum_{n \geq 0} \frac{1}{n+1}\binom{2 n}{n} .
$$

It is well-known that $C(x)$ satisfies

$$
C(x)=1+x C(x)^{2} .
$$

Suppose $H(x)$ satisfies the equation

$$
a(x) H(x)=1+b(x) H(x)^{2},
$$

where $a(0) \neq 0$ and $b(0)=0$. Evidently, $H(x)$ is expressed with $C(x), a(x)$, and $b(x)$ by

$$
\begin{equation*}
H(x)=\frac{1}{a(x)} C\left(\frac{b(x)}{a(x)^{2}}\right) . \tag{14}
\end{equation*}
$$

From (9) and (12), we obtain

$$
F_{D}=x\left(1+F_{D}\right)^{2}-x^{2}\left(1+F_{D}\right) \quad \text { and } \quad F_{D}=x F
$$

and, eliminating $F_{D}$,

$$
(1-x) F=1+\frac{x^{2}}{1-x} F^{2} .
$$

According to (14), $F=F^{(1)}(x)$ is expressed by

$$
F^{(1)}(x)=\frac{1}{1-x} C\left(\frac{x^{2}}{(1-x)^{3}}\right)=\sum_{m \geq 0} \frac{1}{m+1}\binom{2 m}{m} \frac{x^{2 m}}{(1-x)^{3 m+1}}
$$

and the two identities

$$
F_{D}^{(1)}(x)=x F^{(1)}(x) \quad \text { and } \quad F_{E}^{(1)}(x)=(1-x) F^{(1)}(x)-1,
$$

completing the proof.

The concept of symmetric peaks was presented by Flórez and Rodríguez [FR20 as follows. Every peak can be extended to a unique maximal subsequence of the form $N^{i} E^{j}$ for $i, j \geq 1$, which we call the maximal mountain of the peak. A peak is symmetric if its maximal mountain $N^{i} E^{j}$ satisfies $i=j$, and it is asymmetric otherwise. Elizalde Eli21, Theorem 2.1] found the generating function $C_{s p, a p}(t, r, z)$ for Dyck paths with respect to the number of symmetric peaks and asymmetric peaks. We found that the equation

$$
C_{s p, a p}(0,1, x)=1+x F_{E}^{(1)}(x)
$$

holds true.
Conjecture 5. For a positive integer n, there exists a bijection between the following two sets:
(i) the set $E \mathcal{P}_{n, n}^{(1)}(N E, E N)$ of Delannoy paths from $(0,0)$ to $(n, n)$ consisting of northsteps $N=(0,1)$, east-steps $E=(1,0)$, and diagonal-steps $D=(1,1)$ that avoid patterns $N E$ and $E N$, end with east step $E$, and are not below $y=x$
(ii) the set of Catalan paths of size $n+1$ avoiding symmetric peaks.

Enumerations of $\mathcal{P}_{n, 2 n}^{(2)}(D, E E N N)$ and $\mathcal{P}_{n, 2 n}^{(2)}(N E, E N)$. For $k=2$, from the next theorem, the series $F, F_{D}$, and $F_{E}$ begins with

$$
\begin{aligned}
& F^{(2)}(x)=1+x+3 x^{2}+11 x^{3}+44 x^{4}+186 x^{5}+818 x^{6}+3706 x^{7}+\cdots \\
& F_{D}^{(2)}(x)=x+2 x^{2}+6 x^{3}+22 x^{4}+89 x^{5}+381 x^{6}+1694 x^{7}+\cdots \\
& F_{E}^{(2)}(x)=x^{2}+5 x^{3}+22 x^{4}+97 x^{5}+437 x^{6}+2012 x^{7}+\cdots
\end{aligned}
$$

where the sequence of the coefficients in $F_{D}^{(2)}(x)$ is given by entry A200753 in the OEIS Slo18. However, the sequence

$$
1,1,3,11,44,186,818,3706,17182,81136, \ldots
$$

of the coefficients in $F^{(2)}(x)$ is not in the OEIS Slo18.
Theorem 6. For $k=2$, the coefficients of $x^{n}$ of $F$ and $F_{D}$ are as follows:

$$
\begin{aligned}
& {\left[x^{n}\right] F^{(2)}(x)=\sum_{m \geq 0} \frac{(-1)^{n-m}}{m+1}\binom{3 m+1}{m}\binom{m+1}{n-m},} \\
& {\left[x^{n}\right] F_{D}^{(2)}(x)=\sum_{m \geq 1} \frac{(-1)^{n-m}}{m}\binom{3 m}{m-1}\binom{m}{n-m}}
\end{aligned}
$$

Proof. Setting $k=2$ in (12), we obtain

$$
F_{D}^{(2)}(x)=\left(x-x^{2}\right)\left(1+F_{D}^{(2)}(x)\right)^{3}
$$

and, letting $t=x-x^{2}$,

$$
A(t)=t(1+A(t))^{3}
$$

where the power series $A(t)$ satisfies $A\left(x-x^{2}\right)=F_{D}^{(2)}(x)$. Setting $k=2$ in (9), we obtain

$$
F_{D}^{(2)}(x)=\frac{x F^{(2)}(x)}{1-x F^{(2)}(x)} .
$$

Substituting $F_{D}^{(2)}(x)$ in (12) with it, we obtain

$$
x F^{(2)}(x)=\left(x-x^{2}\right)\left(\frac{1}{1-x F^{(2)}(x)}\right)^{2}
$$

and, letting $t=x-x^{2}$,

$$
B(t)=t\left(\frac{1}{1-B(t)}\right)^{2},
$$

where the power series $B(t)$ satisfies $B\left(x-x^{2}\right)=x F^{(2)}(x)$.
Using the Lagrange inversion formula, we obtain the coefficients of $t^{n}$ in $A(t)$ and $B(t)$. Replacing $t$ with $x-x^{2}$, we obtain the coefficients of $x^{n}$ in $F_{D}^{(2)}(x)$ and $x F^{(2)}(x)$.

The function $F_{D}^{(2)}(x)$ is mentioned in MS15, Theorem 3.7]. It is shown here that $F_{D}^{(2)}(x)$ is the same as the generating function of $\mathcal{I} \mathcal{S}_{n}(102)$. In the remark of [MS15, p.168], they were unable to find other combinatorial interpretations in the literature.

Conjecture 7. For a positive integer n, there exists a bijection between the following two sets:
(i) the set $D \mathcal{P}_{n, 2 n}^{(2)}(N E, E N)$ of Delannoy paths from $(0,0)$ to $(n, 2 n)$ consisting of northsteps $N=(0,1)$, east-steps $E=(1,0)$, and diagonal-steps $D=(1,1)$ that avoid the patterns $N E$ and $E N$, end with diagonal step $D$, and are not below $y=2 x$ and
(ii) the set $\mathcal{I S}_{n}(102)$ of inversion sequences of length $n$ avoiding the pattern 102.

Enumerations of $\mathcal{P}_{n, 3 n}^{(3)}(D, E E N N)$ and $\mathcal{P}_{n, 3 n}^{(3)}(N E, E N)$. From Equation (12) for $k=3$, we obtain

$$
F_{D}=x\left(1+F_{D}\right)^{4}-x^{2}\left(1+F_{D}\right)^{5} .
$$

According to a computer program, we find that the series $F, F_{D}$, and $F_{E}$ begins with

$$
\begin{aligned}
& F^{(3)}(x)=1+x+4 x^{2}+20 x^{3}+111 x^{4}+657 x^{5}+4065 x^{6}+25981 x^{7}+\cdots, \\
& F_{D}^{(3)}(x)=x+3 x^{2}+13 x^{3}+67 x^{4}+380 x^{5}+2288 x^{6}+14351 x^{7}+\cdots, \\
& F_{E}^{(3)}(x)=x^{2}+7 x^{3}+44 x^{4}+277 x^{5}+1777 x^{6}+11630 x^{7}+\cdots,
\end{aligned}
$$

where the sequence of the coefficients in $F_{D}^{(3)}(x)$ is given by entry A200754 in the OEIS [Slo18.
Note that, for $k \geq 3$, we were unable to find a solution for (12).

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