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A minimum semi-degree sufficient condition for one-to-many disjoint path covers in semicomplete digraphs

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Abstract

Let D be a digraph. We define the minimum semi-degree of D as $\delta^0(D) := \min\{\delta^+(D), \delta^-(D)\}$. Let k be a positive integer, and let $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$ be any two disjoint subsets of $V(D)$. A set of k internally disjoint paths joining source set S and sink set T that cover all vertices D are called a one-to-many k-disjoint directed path cover (k-DDPC for short) of D. A digraph D is semicomplete if for every pair x, y of vertices of it, there is at least one arc between x and y.

In this paper, we prove that every semicomplete digraph D of sufficiently large order n with $\delta^0(D) \geq \lceil (n + k - 1)/2 \rceil$ has a one-to-many k-DDPC joining any disjoint source set S and sink set T, where $S = \{s\}, T = \{t_1, \ldots, t_k\}$.

Keywords: Semicomplete digraph; Minimun semi-degree; Disjoint path cover

AMS subject classification (2020): 05C07, 05C20, 05C35, 05C70.

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1. Introduction

For terminology and notation not defined here, we refer to [1] and [2]. In this paper, a path always means a directed path. An *x-y path* is a directed path which is from x to y for two vertices $x, y \in V(D)$. For two vertices x and y on a path P satisfying x precedes y, let xPy denote the subpath of P from x to y. A digraph D is *strongly connected*, or *strong* for short, if there exists an x-y path and a y-x path, for every pair x, y of distinct vertices in D.

Let $D = (V(D), A(D))$ be a digraph, and let D_1 and D_2 be two disjoint subdigraphs of D satisfying $V(D_1) \cup V(D_2) = V(D)$. For $x \in V(D_i)$, let $N^+_{D_i}(x) := \{y \in V(D_i) \setminus x \colon xy \in A(D)\}$. $N^-_{D_i}(x)$ is defined similarly. Let $d^+_{D_i}(x) := |N^+_{D_i}(x)|$, $d^-_{D_i}(x) := |N^-_{D_i}(x)|$ and $N_{D_i}(x) := N^+_{D_i}(x) \cup N^-_{D_i}(x)$.

A *k-disjoint path cover* of an undirected graph G is a set of k internally disjoint paths connecting given disjoint source set and sink set such that all vertices of G is covered by the path set. The k-disjoint path cover problem (k-DPC for short) has been studied by many researchers, see [7, 8, 10, 11, 12, 13, 14, 16, 17]. The problem of k-disjoint path cover can be classified into three types according to the number of elements in the source set and the sink set: one-to-one, one-to-many and many-to-many. The one-to-one type considers disjoint path covers joining a single pair of source s and sink t [4, 16, 17], and the one-to-many type is about disjoint path covers which join a single source s and a set of k distinct sinks $t_1, t_2, \ldots, t_k$ [12, 18]. The many-to-many type considers disjoint path covers between a set of k sources $s_1, s_2, \ldots, s_k$ and another set of k sinks $t_1, t_2, \ldots, t_k$ [7, 8, 11, 13, 14].

Let k be a positive integer, let $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$ be two disjoint subsets of $V(D)$. A set of disjoint path $\{P_1, \ldots, P_k\}$ of D is a *k-disjoint directed path cover* (*one-to-many k-DDPC* for short) of D, if $\bigcup_{i=1}^{k} V(P_i) = V(D)$ and $V(P_i) \cap V(P_j) = \{s\}$ for all $i \neq j$, where P_i is a path from s to t_i.

A digraph D is *semicomplete* if for every pair x, y of vertices of it, there is at least one arc between x and y. In this paper, we study the problem of one-to-many k-DDPC in semicomplete digraphs and prove the following main result. Note that our argument is inspired by that of [9],

Theorem 1. *Let D be a semicomplete digraph of order $n \geq (9k)^5$, where k (≥ 2) is an integer. If $\delta^0(D) \geq \lceil (n+k-1)/2 \rceil$, then D has a one-to-many k-DDPC for any disjoint source set S and sink set T, where $S = \{s\}, T = \{t_1, \ldots, t_k\}$.*

2. Preliminaries

We now introduce two results concerning the existence of a Hamiltonian path (cycle) in semicomplete digraphs.

Theorem 2. *[15] Every semicomplete digraph contains a Hamiltonian path.*

Theorem 3. *[3] Every strong semicomplete digraph on $n \geq 3$ vertices has a Hamiltonian cycle.*

The following is the definition of *H-subdivision*.

Definition 1. *[5] Let H be a (multi)digraph and D be a digraph. Let $\mathcal{P}(D)$ denote the set of paths in D. An H-*subdivision *in D is a pair of mappings $f\colon V(H) \to V(D)$ and $g\colon A(H) \to \mathcal{P}(D)$ such that:*

(a) $f(u) \neq f(v)$ for all disjoint $u, v \in V(H)$ and

(b) for every $uv \in A(H)$, $g(uv)$ is an $f(u)$-$f(v)$ path in D, and disjoint arcs map into internally disjoint paths in D.

A digraph D is *H-linked* if every injective mapping $f\colon V(H) \to V(D)$ can be extended to an H-subdivision in D. A digraph D is *one-to-many k-linked* if it has a set of k paths $\{P_1, \ldots, P_k\}$ for any disjoint vertex subsets $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$ such that every path P_i is from s to t_i and $V(P_i) \cap V(P_j) = \{s\}$ for $i \neq j$.

Ferrara, Jacobson and Pfender [6] gave a sufficient condition for a digraph D to be H-linked. By this result, Zhou [18] found a specific digraph H and got the following corollary.

Corollary 4. *[18] Let k (≥ 2) be an integer, and let D be a digraph with order $n \geq 80k$. If $\delta^0(D) \geq \lceil (n+k-1)/2 \rceil$, then D is one-to-many k-linked.*

3. Proof of Theorem 1

In the rest of this paper, let D be a semicomplete digraph which satisfies the assumption of Theorem 1. Let $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$ be any two disjoint subsets of D. An *S-T path* in D is a set of k-disjoint paths $P_1, \ldots, P_k$, where every path P_i is from s to t_i and $V(P_i) \cap V(P_j) = \{s\}$ for all $i \neq j$. By Corollary 4, D contains at least one S-T path. Let L be a maximum S-T path, that is, it covers the most vertices of D among all S-T paths. Suppose that L is not a one-to-many k-DDPC (this implies that $\bigcup_{i=1}^{k} V(P_i) \subsetneqq V(D)$). We aim to obtain a larger S-T path from L and hence this produces a contradiction.

Let H be the subdigraph of D induced by $V(D) \setminus V(L)$. Since D is semicomplete, both L and H are semicomplete. Let $F := \{x \in V(L) \colon \exists y \in V(H) \text{ such that } yx \in A(D)\}$, $R := \{x \in V(L) \colon \exists y \in V(H) \text{ such that } xy \in A(D)\}$. Let $F_m := F \setminus R$ and $R_m := R \setminus F$. Since D is semicomplete, for every vertex $h \in V(H)$ and every vertex $x \in R_m$, there is an arc from x to h. Similarly, for every vertex $y \in F_m$, there is an arc from h to y. For a vertex $v \in V(P_i) \setminus \{s\}$, its predecessor is denoted by $v^- \in V(P_i) \setminus \{t_i\}$. Similarly, for a vertex $v \in V(P_i) \setminus \{t_i\}$, its successor is denoted by $v^+ \in V(P_i) \setminus \{s\}$. Let $F^- := \{x^- \colon x \in F \setminus S\}$ and $R^+ := \{x^+ \colon x \in R \setminus T\}$. Similarly, let $F_m^- := \{x^- \colon x \in F_m \setminus S\}$ and $R_m^+ := \{x^+ \colon x \in R_m \setminus T\}$. To prove Theorem 1, we need more preliminary results, including several lemmas and corollaries.

Lemma 5. *The following assertions hold:*

(a) $d_H^-(x) + d_H^+(y) \geq |H| - 1$ *for every pair of vertices* $x, y \in V(H)$.

(b) H *is strong with* $|H| \leq \lfloor \frac{n-k+1}{2} \rfloor - 1$.

Proof. For any vertex $x \in V(H)$, we have $d_H^-(x) + d_H^+(x) \geq |H| - 1$ since H is semicomplete. The case that $|H| = 1$ is trivial, so now we assume that $|H| \geq 2$.

We first consider the case that $|H| = 2$, say $V(H) = \{x, y\}$. There is at least one arc, say $xy \in A(H)$, between x and y since H is semicomplete. We claim that $d_L^-(x) + d_L^+(y) \leq |L| + k$. Suppose that $d_L^-(x) + d_L^+(y) \geq |L| + k + 1$. Let $X = N_L^-(x)$ and $Y = \{v \colon v^+ \in N_L^+(y)\}$. If $s \in N_L^+(y)$, then s^- does not exist. For each path P_i, if $s^+ \in N_L^+(y)$, then there is only one s satisfying $s \in Y$. Therefore, we have $|Y| \geq d_L^+(y) - k$ and

$$|X \cap Y| = |X| + |Y| - |X \cup Y| \geq d_L^-(x) + d_L^+(y) - k - |L| \geq |L| + k + 1 - k - |L| = 1.$$

This implies that there are two vertices $v \in N_L^-(x)$ and $v^+ \in N_L^+(y)$. By replacing the arc vv^+ with the path $vxyv^+$, we obtain a larger S-T path, a contradiction. Therefore, $d_L^-(x) + d_L^+(y) \leq |L| + k$ and

$$d_H^-(x) + d_H^+(y) \geq 2\lceil \frac{n+k-1}{2} \rceil - (|L| + k) \geq |H| - 1 = 1,$$

which satisfies the degree condition in Lemma 5. Hence, $yx \in A(H)$ and so H is strong in this case.

When $|H| \geq 3$, from the argument above, any two vertices $x_1, y_1 \in V(H)$ with $x_1y_1 \in A(D)$ satisfy the degree condition in Lemma 5. If $y_1x_1 \notin A(H)$, then $y_1 \notin N_H^-(x_1)$ and $x_1 \notin N_H^+(y_1)$. Consequently, $|N_H^-(x_1) \cup N_H^+(y_1)| \leq$

$|H|-2$, and so $|N_H^-(x_1)\cap N_H^+(y_1)| = |N_H^-(x_1)|+|N_H^+(y_1)|-|N_H^-(x_1)\cup N_H^+(y_1)| \geq d_H^-(x_1)+d_H^+(y_1)-(|H|-2)\geq 1$. This implies that H contains a y_1-x_1 path of length 2. Therefore, when $x_1y_1\in A(H)$, either $y_1x_1\in A(H)$ or there is a y_1-x_1 path of length 2 in H. For any vertex $x\in V(H)$, we have $|N_H(x)| = |N_H^-(x)\cup N_H^+(x)| = |H|-1$ since H is semicomplete. According to the argument above, there are two paths of both directions between x and every vertex in $N_H(x)$. For any two vertices $x,z\in V(H)$, note that $z\in N_H(x)$ since H is semicomplete, and this implies that there are two paths of both directions between x and z. Hence, H is also strong in this case.

For every pair of vertices $x,y\in V(H)$, there is an x-y path, say P, since H is strong. With a similar argument to that of the case $|H|=2$, we can get $d_L^-(x)+d_L^+(y)\leq |L|+k$, and for every pair of vertices $x,y\in V(H)$,

$$d_H^-(x)+d_H^+(y)\geq |H|-1. \tag{1}$$

In particular, when $V(H)=\{h\}$, $d_H^-(h)+d_H^+(h)=|H|-1=0$ (satisfying the above degree condition).

We now prove the upper bound for $|H|$. Note that $R^+\cap F=\emptyset$. If $v\in R^+\cap F$, then $v^-\in R$. There are two vertices $u_1,u_2\in V(H)$ such that $v^-u_1, u_2v\in A(D)$. Since H is strong, H contains a u_1-u_2 path P. By replacing the arc v^-v with the path v^-Pv, we obtain a larger S-T path, a contradiction. This implies that there is a vertex $v\in V(L)$ such that $N_D^+(v)\subseteq L$ or $N_D^-(v)\subseteq L$. Thus $|L|-1\geq \delta^0(D)\geq \lceil (n+k-1)/2\rceil$, and then $|H|\leq \lfloor (n-k+1)/2\rfloor -1$. □

According to the proof above, we have $R^+\cap F=\emptyset$. Similarly, $R\cap F^-=\emptyset$, so the following corollary holds.

Corollary 6. *For any $u\in R$, $u^+\notin F$. Similarly, for any $v\in F$, $v^-\notin R$.*

Lemma 7. *The following assertions hold:*

(a) $|F\cup R| = |L|$ *and* $|F\cap R|\leq k$.

(b) *For any path P_i, if x_1 precedes x_2 in P_i, then there are no distinct vertices $y_1,y_2\in V(H)$ such that $x_1y_1, y_2x_2\in A(D)$.*

Proof. For any vertex $x\in V(L)$ and any vertex $y\in V(H)$, there is at least one arc between x and y since H is semicomplete. Observe that $x\in F$, or $x\in R$ (or both) for any vertex $x\in L$, so $|F\cup R|=|L|$.

For each path P_i, we claim that $|(R\cap F)\cap V(P_i)|\leq 1$. Suppose that $u,v\in R\cap F$ are two distinct vertices in P_i and u precedes v. Let Q_1 denote the set of all vertices between u and v. For any path P_i, if $x \notin F$, then $x \in R$ since $|F \cup R| = |L|$. By Corollary 6, $u^+ \notin F$, for any $u \in R$. Therefore, $u^+ \in R$ and $V(u^+P_it_i) \subseteq R$. As $v^- \in V(u^+P_it_i)$, we get that $v^- \in R$. By the assumption, $v \in F$, so there are two vertices $h_1, h_2 \in V(H)$ such that $v^-h_1, h_2v \in A(D)$. By Lemma 5, H is strong and hence contains an h_1-h_2 path P. By replacing v^-v with v^-Pv, we obtain a larger S-T path, a contradiction. Particularly, when $Q_1 = \emptyset$, $u \in R$ and $v = u^+ \in F$ by the assumption. Similar to the argument above, we still can find a larger S-T path, this also produces a contradiction. Consequently, $|(R \cap F) \cap V(P_i)| \leq 1$ for each path P_i, which implies that $|R \cap F| \leq k$.

Now we prove the assertion (b). Assume that such y_1, y_2 exist. Therefore, $x_1 \in R$ and $x_2 \in F$ in P_i. Similar to the proof above, let Q_2 denote the set of all vertices between x_1 and x_2. For any path P_i, if $x \notin F$, then $x \in R$ since $|F \cup R| = |L|$. By Corollary 6, $u^+ \notin F$, for any $u \in R$. Therefore, $x_1^+ \in R$, $V(x_1^+P_it_i) \subseteq R$ and $x_2^- \in R$. Together with the fact $x_2 \in F$, we can find a larger S-T path, a contradiction. Particularly, if $Q_2 = \emptyset$, then $x_1 \in R$ and $x_2 = x_1^+ \in F$ by the assumption. Similar to the argument above, we still find a larger S-T path, and hence this also produces a contradiction. Consequently, for any path P_i, if x_1 precedes x_2 in P_i, then there are no distinct vertices $y_1, y_2 \in H$ such that $x_1y_1, y_2x_2 \in A(D)$. This implies x_2 always precedes x_1 for any $x_1 \in R, x_2 \in F$ in each path P_i. □

The proof of the following lemma is similar to that of Lemma 13 in [9], but we still need to give the proof below.

Lemma 8. $|F|$, $|R| \geq (n + k + 1)/2 - |H|$. *Furthermore,* $|F_m|$, $|R_m| \geq (n - k + 1)/2 - |H|$ *and* $|F_m \cup R_m| \geq |L| - k$.

Proof. For every vertex $x \in V(H)$, $d_L^-(x) \geq \delta^0(D) - (|H| - 1) \geq (n + k + 1)/2 - |H|$, and so $|R| \geq (n + k + 1)/2 - |H|$. The inequality $|F| \geq (n + k + 1)/2 - |H|$ can be proved similarly. By Lemma 7, $|R \cap F| \leq k$, so $|R_m| = |R| - |R \cap F| \geq (n + k + 1)/2 - |H| - k = (n - k + 1)/2 - |H|$. Similarly, we can prove the inequality $|F_m| \geq (n - k + 1)/2 - |H|$.

By Lemma 7, $|F \cup R| = |L|$ and $|F \cap R| \leq k$. Together with the definition of F_m and R_m, $|F \cup R| = |F_m| + |R_m| + |F \cap R|$, so $|R_m \cup F_m| = |F_m| + |R_m| = |F \cup R| - |F \cap R| \geq |L| - k$. □

We rewrite Theorem 1 and give the proof now:

Theorem 1. *Let D be a semicomplete digraph of order $n \geq (9k)^5$, where k (≥ 2) is an integer. If $\delta^0(D) \geq \lceil (n + k - 1)/2 \rceil$, then D has a one-to-many*

k-DDPC for any disjoint source set S and sink set T, where $S = \{s\}, T = \{t_1, \ldots, t_k\}$.

Proof. Recall that at the beginning of Section 3, we supposed that L is not a one-to-many k-DDPC. It suffices to find a larger S-T path more than L, and then this produces a contradiction and thus we prove Theorem 1. Our argument is divided into the following three cases.

Case 1. $|H| = 1$.

Let $V(H) = \{h\}$. Since $|H| = 1$, we have $d_L^-(h) = |R|$, $d_L^+(h) = |F|$ and $d_L^+(h) + d_L^-(h) = d_D^+(h) + d_D^-(h)$. According to whether $n+k$ is even, we distinguish the following two subcases.

Case 1.1. $n + k$ is even.

When $n + k$ is even, we have $\lceil \frac{n+k-1}{2} \rceil = \frac{n+k}{2}$ and $d_L^+(h) + d_L^-(h) \geq 2\delta^0(D) \geq n + k$. Consequently, $|F| + |R| \geq n + k$. By Lemma 7, $|R \cup F| = |L| = n-1$. Therefore, $|R \cap F| = |R| + |F| - |R \cup F| \geq (n+k) - (n-1) = k+1$, which implies that there exists a path P_i such that $u, v \in R \cap F$. Without loss of generality, assume that u precedes v. Arguing similarly as in the proof of Lemma 7, we can find a larger S-T path, say L^*, which contains h. Consequently, L^* is a one-to-many k-DDPC in D, a contradiction.

Case 1.2. $n + k$ is odd.

When $n + k$ is odd, we have $\lceil (n + k - 1)/2 \rceil = (n + k - 1)/2$. Therefore, $d_L^+(h) = d_D^+(h) \geq (n + k - 1)/2$, $d_L^-(h) = d_D^-(h) \geq (n + k - 1)/2$, and $d_L^+(h) + d_L^-(h) \geq 2\delta^0(D) \geq n + k - 1$. Consequently, $|R| + |F| \geq n + k - 1$. Since $|L| = n - 1$, $|R \cap F| = |R| + |F| - |R \cup F| \geq (n + k - 1) - (n - 1) = k$. By Lemma 7, $|R \cap F| \leq k$, so $|R \cap F| = k$. According to the argument in Lemma 5, $d_L^+(h) + d_L^-(h) \leq |L| + k = n + k - 1$. Therefore, $d_L^+(h) + d_L^-(h) = n + k - 1$, and then $d_L^+(h) = d_L^-(h) = (n + k - 1)/2$. Thus we get $|F| = d_L^+(h) = (n + k - 1)/2$, $|R| = d_L^-(h) = (n + k - 1)/2$ and $|R_m| = |F_m| = (n + k - 1)/2 - k = (n - k - 1)/2$ by Lemma 8.

Note that each $y \in F_m$ satisfies $d_{R_m}^+(y) \geq \delta^0(D) - (|F| - 1) = 1$, so there is at least one vertex $x_1 \in R_m$ such that $yx_1 \in A(D)$. By Lemma 7, we have $s \in F_m$, so there exists a vertex $x_1 \in R_m$ in P_i such that $sx_1 \in A(D)$. As $|(R \cap F) \cap V(P_i)| \leq 1$ and $|R \cap F| = k$, there is a vertex $s_j^+ \in F$ in P_j. Together with the fact that $x_1^- \in R$, there exists an x_1^--s_j^+ path whose inner vertex is h. Let $P_i^* := sx_1P_it_i$ and $P_j^* := sP_ix_1^-hs_j^+P_jt_j$. Therefore, we obtain a larger S-T path, say L^*, which contains h. Consequently, L^* is a one-to-many k-DDPC in D, this produces a contradiction (see Figure 1).

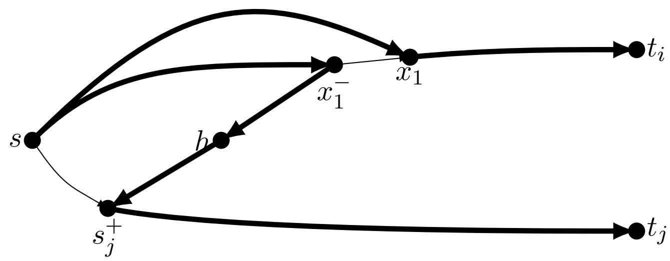

Figure 1

Case 2. $2 \le |H| \le n/2 - n/(50k)$.

By Lemma 8, $|R_m| \ge (n-k+1)/2 - |H| \ge n/(53k)$, so there exists a path P_j which contains at least $n/(53k^2)$ vertices from R_m. Without loss of generality, assume that $j = 1$. Let A be the set of vertices belonging to R_m in P_1. We use A_1 (A_2, respectively) to denote the subpath of A which contains the first (last, respectively) $n/(110k^2)$ inner vertices of A.

The first vertex of A is denoted by t. For any vertex $a \in V(t^+P_1t_1)$, Lemma 7 implies that $a \in R$ and $N_D^-(a) \subseteq V(L)$. By Lemma 8, $|F| \ge (n+k+1)/2 - |H|$ and so

$$d_L^-(a) \ge \delta^0(D) \ge (n+k-1)/2 \ge n+k-|H|-|F| = |L|-|F|+k. \quad (2)$$

Case 2.1. There are two vertices $a_1 \in A_1$ and $a_2 \in A_2$ such that $a_1a_2 \in A(D)$.

By inequality (2), we have $d_L^-(a_1^+) \ge |L| - |F| + k$ and

$$\begin{aligned} |N_L^-(a_1^+) \cap F^-| &= |N_L^-(a_1^+)| + |F^-| - |N_L^-(a_1^+) \cup F^-| \\ &\ge d_L^-(a_1^+) + (|F|-k) - (|L|-1) \\ &\ge |L|-|F|+k+(|F|-k)-(|L|-1) = 1 \end{aligned}$$

(By the definition of F^-, we have $F^- \cap T = \emptyset$. Corollary 6 implies that $F^- \cap R = \emptyset$ and $a_1^+ \notin F^-$, thus $|N_L^-(a_1^+) \cup F^-| \le |L| - 1$. If $s \in F$, then s^- does not exist. For each P_i, if $s_i^+ \in F$, then there is only one s satisfying $s \in F^-$. Therefore, $|F^-| \ge |F| - k$). This implies that there are two vertices $w \in N_L^-(a_1^+) \cap F^-$ and $w^+ \in F$. Lemma 7 implies that $F \cap V(t^+P_1t_1) = \emptyset$, so $w^+ \in V(s^+P_1t)$ or $w^+ \in V(P_i \setminus \{s\})$ $(i \ne 1)$. Therefore, $w \in V(sP_it^-)$ or $w \in V(p_i \setminus \{t_i\})$ $(i \ne 1)$. By Lemma 5, H is strong. When $|H| \ge 3$, by Theorem 3, we get that H contains a Hamiltonian cycle, say C. For a vertex x on C, its predecessor on C is denoted by x^- and its successor on C is denoted by x^+. When $|H| \geq 3$, there exists a vertex $u \in V(H)$ such that $uw^+ \in A(D)$ since $w^+ \in F$. As $a_2^- \in R_m$, there is an arc from a_2^- to u^+. Note that there is a Hamiltonian path from u^+ to u in H. When $|H| = 2$, let $V(H) = \{u, v\}$. There exists a vertex, say $v \in V(H)$, such that $vw^+ \in A(D)$ since $w^+ \in F$. As $a_2^- \in R_m$, there is an arc from a_2^- to u. Note that $uv, vu \in A(H)$ since H is strong.

If $w \in V(sP_1t^-)$, then according to the argument above, there exists a path $a_2^-Q_1w^+$, where Q_1 contains all vertices in H. Let $P_1^* := sP_1wa_1^+P_1a_2^-Q_1w^+P_1a_1a_2P_1t_1$. Now we obtain a larger S-T path L^* which is a one-to-many k-DDPC in D. This produces a contradiction (see Figure 2).

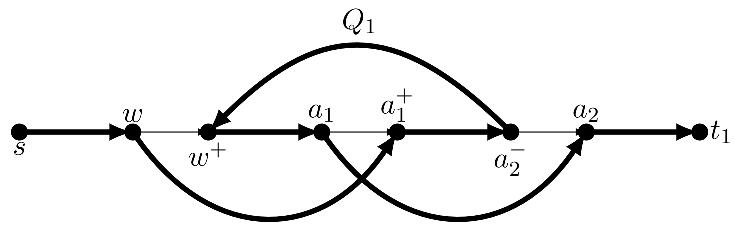

Figure 2

If $w \in V(P_i \setminus \{t_i\})$ $(i \neq 1)$, then according to the argument above, we can also find a path $a_2^-Q_2w^+$, where Q_2 contains all vertices in H. Let $P_1^* := sP_1a_1a_2P_1t_1$ and $P_i^* := sP_iwa_1^+P_1a_2^-Q_2w^+P_it_i$. Therefore, we obtain a larger S-T path L^* which is a one-to-many k-DDPC in D. This produces a contradiction (see Figure 3).

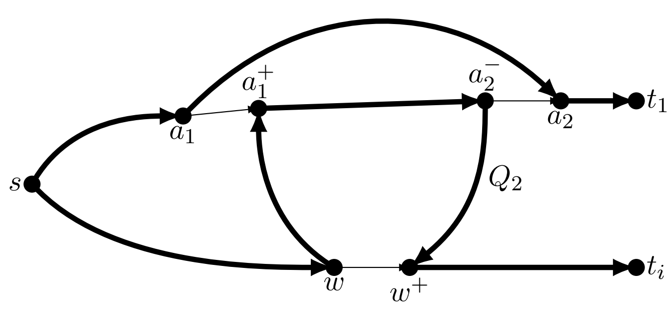

Figure 3

Case 2.2. Case 2.1 does not hold.

By the definition of F_m^-, $F_m^- \cap V(tP_1t_1) = \emptyset$. For any vertex $a \in A_2$, Lemma 7 implies that $N_D^-(a) \subseteq V(L)$. By the assumption, $N_D^-(a) \subseteq V(L) \setminus V(A_1)$. By Lemma 8, $|F_m| \geq (n-k+1)/2 - |H|$. Similar to the argument in (2), we get that $d_{L-A_1}^-(a) \geq \delta^0(D) \geq (n+k-1)/2 \geq n - |H| - |F_m| \geq |L-A_1| + |A_1| - |F_m^-| - k$ (recall that $|F^-| \geq |F| - k$, so $|F| \leq |F^-| + k$ and $|F_m| \leq |F_m^-| + k$). By the definition of F_m^-, $F_m^- \cap V(A_1) = \emptyset$, and so

$$\begin{aligned} |N_{L-A_1}^-(a) \cap F_m^-| &\geq |L-A_1| + |A_1| - |F_m^-| - k + |F_m^-| - |L-A_1| \\ &= |A_1| - k = (n - 110k^3)/(110k^2). \end{aligned} \tag{3}$$

Let $I_1 := sP_1t$ and $I_i := P_i \setminus \{t_i\}$ $(i \neq 1)$. We use G_i to denote the auxiliary bipartite graph whose vertex sets are $V(A_2)$ and $V(I_i) \cap F_m^-$ $(i = 1, \ldots, k)$. For any $a \in V(A_2)$ and $w \in V(I_i) \cap F_m^-$, if $wa \in A(D)$, then there is an edge between a and w in each G_i. As $F_m^- \cap V(tP_1t_1) = \emptyset$, $F_m^- \subseteq V(I_1) \cup \cdots \cup V(I_k)$ and the edges of $G_1 \cup \cdots \cup G_k$ are equivalent to the arcs which are from F_m^- to A_2 in D. Since $|N_{L-A_1}^-(a) \cap F_m^-| \geq (n-110k^3)/(110k^2)$, there exists a G_i satisfying

$$e(G_i) \geq \frac{|A_2|(n-110k^3)}{110k^3} \geq \frac{n(n-110k^3)}{12100k^5} \geq 3n \geq 3|G_i|.$$

This implies that G_i is not planar, so there are vertices a_1, $a_2 \in V(A_2)$ and w_1, $w_2 \in V(I_i) \cap F_m^-$ such that the edges w_1a_1, w_2a_2 cross in G_i.

We first consider the case that $i = 1$, and then w_1, $w_2 \in V(sP_1t) \cap F_m^-$ and w_1^+, $w_2^+ \in F_m$. Together with the fact that a_1^-, $a_2^- \in R_m$, we can find disjoint paths $a_j^-Q_jw_j^+$ $(j = 1, 2)$, where the vertices of Q_j lie in H and $|Q_1 \cup Q_2| \geq 2$. Particularly, when $|H| = 2$, say $V(H) = \{u, v\}$, we have $Q_1 = u$ and $Q_2 = v$, or $Q_1 = v$ and $Q_2 = u$. Let $P_1^* := sP_1w_1a_1P_1a_2^-Q_2w_2^+P_1a_1^-Q_1w_1^+P_1w_2a_2P_1t_1$. Thus we obtain a larger S-T path L^* which contains at least $|L| + 2$ vertices, this produces a contradiction (see Figure 4).

We now consider the case that $i \neq 1$, without loss of generality, assume that $i = 2$. Consequently, w_1, $w_2 \in V(P_2 \setminus \{t_2\}) \cap F_m^-$ and w_1^+, $w_2^+ \in F_m$. According to the argument above, w_1a_1 and w_2a_2 cross in G_2. Without loss of generality, assume that a_1 precedes a_2 in P_1 and w_2 precedes w_1 in P_2. By Theorem 2, H has a Hamiltonian path. Suppose that there is a Hamiltonian path from u to v in H. Since $a_2^- \in R_m$ and $w_2^+ \in F_m$, we have a_2^-u, $vw_2^+ \in A(D)$, which implies that there exists a path $a_2^-Qw_2^+$, where Q contains all vertices in H. Let $P_1^* := sP_2w_2a_2P_1t_1$ and $P_2^* :=$

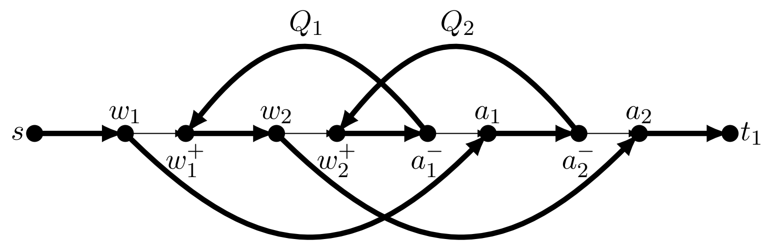

Figure 4

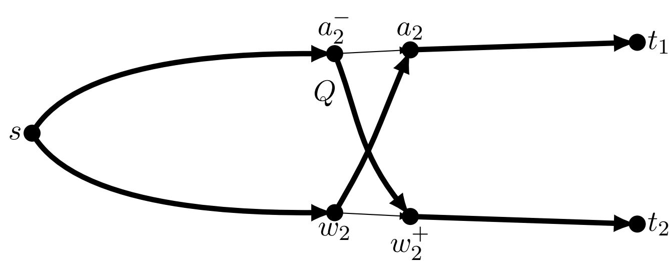

Figure 5

$sP_1a_2^-Qw_2^+P_2t_2$. Therefore, we obtain a larger S-T path L^* which is a one-to-many k-DDPC in D. This produces a contradiction (see Figure 5).

Case 3. $n/2 - n/(50k) \le |H| \le \lceil (n-k)/2 \rceil - 1 = \lfloor (n-k+1)/2 \rfloor - 1$.

By Lemma 8, $|R_m| \ge (n-k+1)/2 - |H| \ge 1$. Similarly, $|F_m| \ge 1$. By Lemma 7, $|R \cup F| = |L|$, $|R \cap F| \le k$ and $|L| - k \le |R_m \cup F_m| = |R_m| + |F_m| \le |L|$. Since $\lceil (n+k-1)/2 \rceil + 1 \le |L| \le n/2 + n/(50k)$, we deduce that $\lceil (n-k+1)/2 \rceil \le |R_m| + |F_m| \le n/2 + n/(50k)$.

Note that each $h \in V(H)$ satisfies

$$\begin{aligned} d_L^-(h) \ge \delta^-(D) - (|H|-1) &\ge \lceil (n+k-1)/2 \rceil - \lceil (n-k)/2 \rceil + 1 + 1 \\ &\ge (n+k-1)/2 - (n-k+1)/2 + 2 = k+1 \end{aligned}$$

and so $|R| \ge k+1$. Similarly, $d_L^+(h) \ge \delta^+(D) - (|H|-1) \ge k+1$, so $|F| \ge k+1$. For any vertex $x \in R_m$, Lemma 7 implies that $N_D^-(x) \subseteq V(L)$. Furthermore,

$$|L| - \delta^-(D) \le n/2 + n/(50k) - (n+k-1)/2 < n/(50k),$$

so for every vertex $x \in R_m$ and a vertex set $Z_1 \subseteq V(L)$ satisfying $|Z_1| \ge n/(50k)$, there exists a vertex $a_1 \in Z_1$ such that $a_1x \in A(D)$. Similarly, for every vertex $y \in F_m$ and a vertex set $Z_2 \subseteq V(L)$ satisfying $|Z_2| \geq n/(50k)$, there exists a vertex $a_2 \in Z_2$ such that $ya_2 \in A(D)$.

Case 3.1. $1 \leq |R_m| \leq k$.

By Lemma 7, $|(R \cap F) \cap V(P_i)| \leq 1$. Together with the fact that $|R| \geq k+1$, we deduce that there exists a path P_i such that $x_1 \in R$ and $x_1^+ \in R_m$. Observe that $|F_m| \geq \lceil (n-k+1)/2 \rceil - k \geq n/3$ since $|R_m| \leq k$, and so one of the k paths of L, say P_j, contains at least $n/(3k)$ vertices from F_m. Thus there is a subpath $y_1 P_j y_2$ on path P_j such that $V(y_1 P_j y_2) \subseteq F_m$ and $|y_1 P_j y_2| = n/(20k)$.

We first consider the case that $j \neq i$, without loss of generality, assume that $i = 1$ and $j = 2$. There is a vertex $a_1 \in V(y_1 P_2 y_2^-)$ such that $a_1 x_1^+ \in A(D)$ since $x_1^+ \in R_m$ and $|y_1 P_2 y_2^-| \geq n/(50k)$. As $x_1 \in R$, there exists a vertex $h_1 \in V(H)$ such that $x_1 h_1 \in A(D)$. According to the argument in Case 2.1, we deduce that H has a Hamiltonian cycle. As $a_1^+ \in F_m$, $h_1^- a_1^+ \in A(D)$. Note that there is a Hamiltonian path from h_1 to h_1^- in H. This implies that there exists a path $x_1 Q_1 a_1^+$, where Q_1 contains all vertices in H. Let $P_1^* := sP_2 a_1 x_1^+ P_1 t_1$ and $P_2^* := sP_1 x_1 Q_1 a_1^+ P_2 t_2$. Now we obtain a larger S-T path L^* which is a one-to-many k-DDPC in D. This produces a contradiction (see Figure 6).

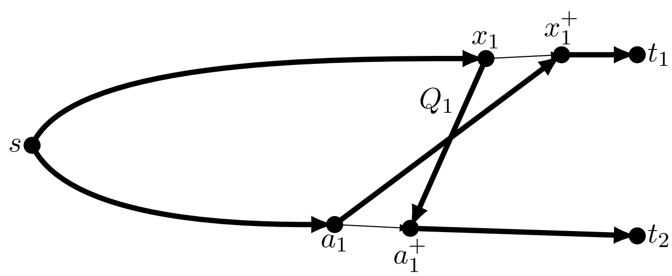

Figure 6

We next consider the case that $j = i$, without loss of generality, assume that $j = i = 1$. Let $I_1 := y_1 P_1 y_3$, $I_2 := y_3^+ P_1 y_4$, $I_3 := y_4^+ P_1 y_5$, and $I_4 := y_5^+ P_1 y_2$, where $|I_1| = n/(240k)$, $|I_2| = n/(240k)$, $|I_3| = n/(48k)$, and $|I_4| = n/(48k)$. According to the argument at the beginning of Case 3, there is a vertex $a_3 \in V(I_3)$ such that $a_3 x_1^+ \in A(D)$ since $x_1^+ \in R_m$ and $|I_3| > n/(50k)$. Similarly, there are two vertices $a_1 \in I_1$ and $a_4 \in I_4$ such that $a_1 a_4 \in A(D)$ (since $a_1 \in F_m$ and $|I_4| > n/(50k)$). As $x_1 \in R$, there exists a vertex $h_1 \in V(H)$ such that $x_1 h_1 \in A(D)$. For a vertex $a_2 \in I_2$, we have $a_2 \in F_m$, so $h_1^- a_2 \in A(D)$. According to the argument in Case 2.1, we deduce that H contains a Hamiltonian cycle, and so there is a Hamiltonian path from h_1 to h_1^- in H. This implies that there exists a path $x_1Q_2a_2$, where Q_2 contains all vertices in H. Let $P_1^* := sP_1a_1a_4P_1x_1Q_2a_2P_1a_3x_1^+P_1t_1$. Thus we obtain a larger S-T path L^* which contains at least $|H| - |I_1| - |I_2| - |I_3| - |I_4| \geq n/2 - n/(50k) - n/(20k) > 0$ vertices more than L. This produces a contradiction (see Figure 7).

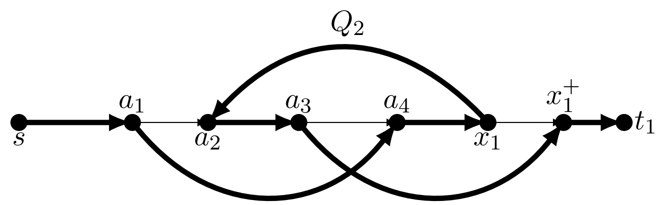

Figure 7

Case 3.2. $|F_m| = 1$.

In this case, $n-k$ is odd, $|H| = (n-k+1)/2-1$ and $|L| = (n+k-1)/2+1$ by Lemma 8. Recall that $|F| \geq k+1$ at the beginning of Case 3. By Lemma 7, $|R \cap F| \leq k$ and $|(R \cap F) \cap V(P_j)| \leq 1$. Therefore, $s \in F_m$ and there exists exactly a vertex s_j^+ in each P_j such that $s_j^+ \in R \cap F$ by Lemma 7. As $|R_m| + |F_m| \geq \lceil (n-k+1)/2 \rceil$, $|R_m| \geq \lceil (n-k+1)/2 \rceil - 1 \geq n/3$. Consequently, one of the k paths of L, say P_i, contains at least $n/(3k)$ vertices from R_m. There is a subpath $x_1P_ix_2$ on P_i such that $V(x_1P_ix_2) \subseteq R_m$ and $|x_1P_ix_2| = n/(20k)$.

Without loss of generality, assume that $i = 1$. There are two vertices $s \in F_m$ and $s_2^+ \in R \cap F$ in P_2. According to the argument at the beginning of Case 3, there is a vertex $a_1 \in V(x_1^+P_1x_2)$ such that $sa_1 \in A(D)$ since $s \in F_m$ and $|x_1^+P_1x_2| > n/(50k)$. As $s_2^+ \in F$, there exists a vertex $h_1 \in V(H)$ such that $h_1s_2^+ \in A(D)$. By the assumption, $a_1^- \in R_m$, so there is an arc from a_1^- to h_1^+. According to the argument in Case 2.1, we deduce that H contains a Hamiltonian cycle. Note that there is a Hamiltonian path from h_1^+ to h_1 in H. This implies that there exists a path $a_1^-Qs_2^+$, where Q contains all vertices in H. Let $P_1^* := sa_1P_1t_1$ and $P_2^* := sP_1a_1^-Qs_2^+P_2t_2$. Thus we obtain a larger S-T path L^* which is a one-to-many k-DDPC in D. This produces a contradiction (see Figure 8).

Case 3.3. $|F_m| \geq 2$ and $|R_m| \geq k+1$.

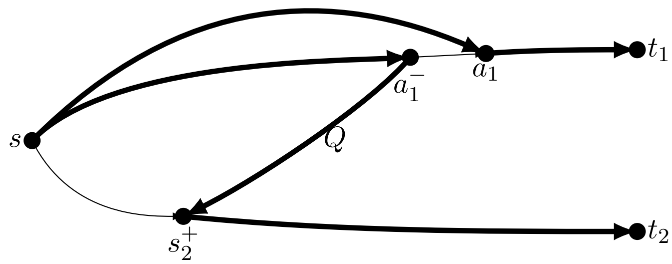

Figure 8

Since $|R_m| \geq k+1$, there is a path P_i such that $x_1, x_2 \in R_m$, and x_1 precedes x_2. Similarly, as $|F_m| \geq 2$, there is a path P_j such that $y_1, y_2 \in F_m$, and y_1 precedes y_2.

Case 3.3.1. There exists a subpath $x_1P_ix_2$ on P_i such that $|x_1P_ix_2| \geq n/(20k)$.

We first consider the case that $j \neq i$, without loss of generality, assume that $i = 1$, $j = 2$, and $y_1 = y_2^-$. Lemma 7 implies that $V(x_1P_ix_2) \subseteq R_m$. Let $A_1 := x_1P_1x_3$ and $A_2 := x_3^+P_1x_4$, where $|A_1| = n/(48k)$ and $|A_2| = n/(48k)$. According to the argument at the beginning of Case 3, there is a vertex $a_1 \in A_1$ such that $a_1x_2 \in A(D)$ since $x_2 \in R_m$ and $|A_1| > n/(50k)$. Similarly, as $y_2^- \in F_m$ and $|A_2| > n/(50k)$, there exists a vertex $a_2 \in A_2$ such that $y_2^-a_2 \in A(D)$. By Theorem 2, H has a Hamiltonian path. Suppose that there is a Hamiltonian path from u to v in H. Since $x_2^- \in R_m$ and $y_2 \in F_m$, x_2^-u, $vy_2 \in A(D)$. This implies that there exists a path $x_2^-Q_1y_2$, where Q_1 contains all vertices in H. Let $P_1^* := sP_1a_1x_2P_1t_1$ and $P_2^* := sP_2y_2^-a_2P_1x_2^-Q_1y_2P_2t_2$. Thus we obtain a larger S-T path L^* which contains at least $|H| - |V(a_1P_1a_2)| \geq n/2 - n/(50k) - n/(24k) > 0$ vertices more than L. This produces a contradiction (see Figure 9).

We next consider the case that $j = i$, without loss of generality, assume that $j = i = 1$ and $y_1 = y_2^-$. The argument for the case that $j = i$ is similar to that of the case $j \neq i$, so there exists a path $x_2^-Q_2y_2$, where Q_2 contains all vertices in H. Let $P_1^* := sP_1y_2^-a_2P_1x_2^-Q_2y_2P_1a_1x_2P_1t_1$. Thus we obtain a larger S-T path L^* which contains at least $|H| - |V(a_1P_1a_2)| \geq n/2 - n/(50k) - n/(24k) > 0$ vertices more than L. This produces a contradiction (see Figure 10).

Case 3.3.2. Case 3.3.1 does not hold.

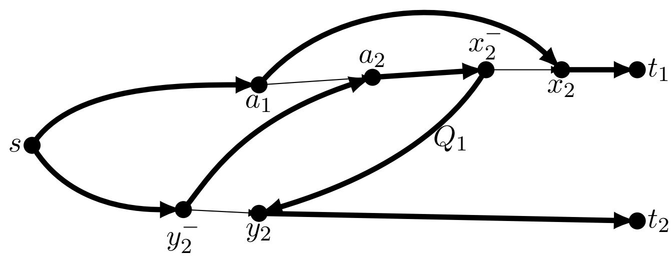

Figure 9

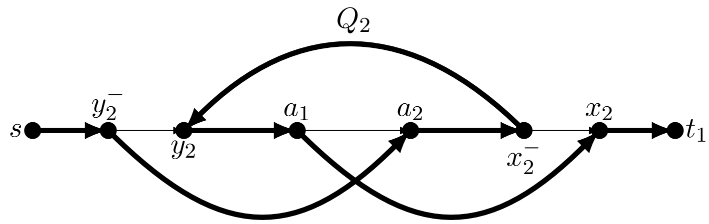

Figure 10

There is not a subpath $x_1P_ix_2$ in each path P_i such that $|x_1P_ix_2| \geq n/(20k)$, so $|R_m| < n/(20)$. Recall that $|R_m|+|F_m| \geq \lceil(n-k+1)/2\rceil$ at the beginning of Case 3, so $|F_m| \geq \lceil(n-k+1)/2\rceil - n/(20) \geq n/3$. One of the k paths of L, say P_j, contains at least $n/(3k)$ vertices from F_m. Without loss of generality, assume that $j = 2$. We can find a subpath $y_1P_2y_2$ such that $V(y_1P_2y_2) \subseteq F_m$ and $|V(y_1P_2y_2)| \geq n/(20k)$. There is a path P_i such that $x_1, x_2 \in R_m$ and x_1 precedes x_2 since $|R_m| \geq k+1$.

We first consider the case that $j \neq i$, without loss of generality, assume that $i = 1$, $j = 2$, and $x_2 = x_1^+$. Let $A_2 := y_3P_2y_2$ and $A_1 := y_4P_2y_3^-$, where $|A_2| = n/(48k)$ and $|A_1| = n/(48k)$. According to the argument at the beginning of Case 3, there is a vertex $a_2 \in A_2$ such that $y_1a_2 \in A(D)$ since $y_1 \in F_m$ and $|A_2| > n/(50k)$. Similarly, as $x_1^+ \in R_m$ and $|A_1| > n/(50k)$, there exists a vertex $a_1 \in A_1$ such that $a_1x_1^+ \in A(D)$. By Theorem 2, H has a Hamiltonian path. Assume that there is a Hamiltonian path from u to v in H. Since $x_1 \in R_m$ and $y_1^+ \in F_m$, x_1u, $vy_1^+ \in A(D)$. This implies that there exists a path $x_1Q_1y_1^+$, where Q_1 contains all vertices in H. Let $P_1^* := sP_1x_1Q_1y_1^+P_2a_1x_1^+P_1t_1$ and $P_2^* := sP_2y_1a_2P_2t_2$. Thus we obtain a larger S-T path L^* which contains at least $|H| - |V(a_1P_2a_2)| \geq n/2 - n/(50k) - n/(24k) > 0$ vertices more than L. This produces a contradiction (see Figure 11).

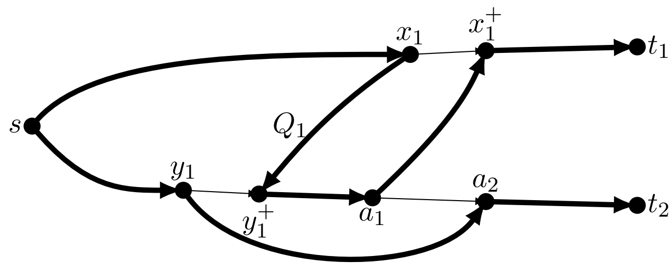

Figure 11

We next consider the case that $i = j = 2$, without loss of generality, assume that $x_2 = x_1^+$. Arguing similarly as that of the case $i \neq j$, we get that there exists a path $x_1 Q_2 y_1^+$, where Q_2 contains all vertices in H. Let $P_2^* := sP_2 y_1 a_2 P_2 x_1 Q_2 y_1^+ P_2 a_1 x_1^+ P_2 t_2$. Thus we obtain a larger S-T path L^* which contains at least $|H| - |V(a_1 P_2 a_2)| \geq n/2 - n/(50k) - n/(24k) > 0$ vertices more than L. This produces a contradiction (see Figure 12).

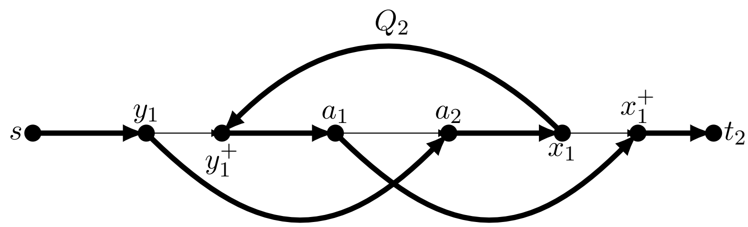

Figure 12

□

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