Approximation and decomposition in lattices of clutters

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Abstract

A clutter on Ω is a family of subsets of Ω that are incomparable under inclusion. Given a collection Σ of clutters on Ω and another clutter Λ on Ω, we study the problem of approximating and decomposing Λ in terms of the clutters in Σ. We want to guarantee when these approximations exist not only for a single element Λ, but for all Λ that belong to a family of clutters $\mathbb{X}$. We use the lattice structure of the collection of clutters to show that the existence of such approximations and decompositions for every clutter Λ is equivalent to the fact that Σ contains a given family of irreducible elements determined by Σ and $\mathbb{X}$. We explicitly compute these irreducible families in some cases and apply our results to clutters arising from discrete objects such as matroids, graphs and secret sharing schemes.

1 Introduction

There are many clutters (that is, families of mutually incomparable sets) that arise from discrete objects; for instance, the clutter of circuits of a matroid or the clutter of maximal independent sets of a graph. This paper is motivated by the following question: given a family Σ of clutters and another clutter $\Lambda \notin \Sigma$, how can one determine which clutters from Σ (if any) are those closest to Λ? And if these exist, is it possible to recover the original clutter Λ from them? This question was considered for matroids in [11, 12] and in a more general setting in [13]. Our goal here is to give a framework for this problem as general as possible, and to give conditions that guarantee a positive answer, and also to apply the results to some particular instances. To do so, we will heavily rely on the fact that the set of clutters on a given ground set has a distributive lattice structure.

To formalize the questions above, we consider four problems transversal to many areas of mathematics: solving equations, giving representations (or realizations), approximating arbitrary elements by elements satisfying some conditions, and factoring an element in terms of elements of some type. Observe that given a map $\Theta : \mathcal{A} \to \mathcal{B}$ between two sets $\mathcal{A}$ and $\mathcal{B}$, and fixing an element $\mathsf{b} \in \mathcal{B}$, solving the equation $\Theta(\mathsf{x}) = \mathsf{b}$

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amounts to determining the preimages of b by Θ. We can also say that the solutions x are realizations of $\mathsf{b} \in \mathcal{B}$ by elements of $\mathcal{A}$. Suppose moreover that $\mathcal{B}$ is endowed with an order or a distance. Then, whenever the equation $\Theta(\mathsf{x}) = \mathsf{b}$ has no solution, or if b is not realizable by elements of $\mathcal{A}$, we can wonder if there are elements $\mathsf{b}_0 \in \mathcal{B}$ close to b and such that the equation $\Theta(\mathsf{x}) = \mathsf{b}_0$ has a solution; that is, we can ask whether there are elements $\mathsf{b}_0 \in \mathcal{B}$ close to b that are realizable by elements of $\mathcal{A}$. Here, "close" refers to the order or the distance we have in $\mathcal{B}$. So one seeks to approximate an element $\mathsf{b} \in \mathcal{B}$ by elements of the image $\mathrm{Im}\ \Theta \subseteq \mathcal{B}$. It can also be that the set $\mathcal{B}$ has an internal operation $*$. Then one can wonder if it is possible to recover the element b if its optimal approximations are known, that is, if the equality $\mathsf{b} = \mathsf{b}_1 * \cdots * \mathsf{b}_r$ where $\mathsf{b}_1, \dots, \mathsf{b}_r \in \mathrm{Im}\ \Theta$ are the optimal approximations of b. All these questions can be considered for a particular element $\mathsf{b} \in \mathcal{B}$, for all elements of the set $\mathcal{B}$, or for all elements of $\mathcal{B}$ that satisfy certain conditions, that is, for a subset, $\mathcal{B}' \subseteq \mathcal{B}$.

The problem that we study in this paper is a concrete instance of this general framework. Given a finite set Ω, the role of $\mathcal{A}$ is played by a collection $\mathfrak{Obj}(\Omega)$ of objects defined on Ω. Among the wide variety of objects that can be considered, in this work we will pay special attention to objects arising in a discrete setting, like matroids with ground set Ω, graphs with vertex set Ω and secret sharing schemes with set of participants Ω (we will introduce the necessary definitions in the sections they are needed).

It is often the case that to an object $\mathcal{X} \in \mathfrak{Obj}(\Omega)$ one can associate an increasing or decreasing family of subsets of Ω, and that this family of subsets is hence determined by its minimal or maximal elements under inclusion. Examples of these minimal or maximal elements are the circuits or the bases of a matroid, minimal vertex covers of a graph or the minimal qualified subsets of participants in a secret sharing scheme. A family of mutually incomparable subsets of Ω is called a *clutter*; that is, a clutter Λ on Ω is a collection of subsets of the powerset 2^Ω such that if $A, B \in \Lambda$ are two different elements of Λ then neither $A \subseteq B$ nor $B \subseteq A$. Clutters are also called *antichains of sets*, *Sperner families* or *simple hypergraphs*. The role of $\mathcal{B}$ above is played by the set of all clutters on Ω, which we denote $\mathrm{Clut}(\Omega)$.

From what has been said so far, we have the following initial situation. On one hand, a collection of discrete objects $\mathfrak{Obj}(\Omega)$ defined on a finite set Ω and, on the other hand, a criterion Θ that associates a clutter $\Theta(\mathcal{X})$ on Ω to any discrete object $\mathcal{X} \in \mathfrak{Obj}(\Omega)$. So we can consider the corresponding map:

$$\begin{aligned} \Theta : \mathfrak{Obj}(\Omega) &\to \mathrm{Clut}(\Omega) \\ \mathcal{X} &\mapsto \Theta(\mathcal{X}) \end{aligned}$$

Let us say that a clutter Δ on Ω is a Θ-*clutter* if there exists a Θ-*realization* for Δ; that is, if there exist $\mathcal{X} \in \mathfrak{Obj}(\Omega)$ such that $\Delta = \Theta(\mathcal{X})$. We denote by $\mathrm{Clut}_\Theta(\Omega)$ the collection of all the Θ-clutters of Ω; that is, $\mathrm{Clut}_\Theta(\Omega) \subseteq \mathrm{Clut}(\Omega)$ is the image of the map Θ. In general the map Θ is not surjective and so $\mathrm{Clut}_\Theta(\Omega) \subsetneq \mathrm{Clut}(\Omega)$, (moreover, in general Θ is not one-to-one and, so there are Θ-clutters with more than one Θ-realization). Therefore a natural question arises at this point: is it possible to approximate and to recover any non Θ-clutter by means of Θ-clutters?

This problem can be posed either for a specific clutter Λ, or for all clutters $\Lambda \in \mathrm{Clut}(\Omega)$, or for all those clutters that verify a certain restriction, that is, for all $\Lambda \in \mathbb{X}$ being $\mathbb{X} \subseteq \mathrm{Clut}(\Omega)$ an specific subset of clutters. For instance, $\mathbb{X}$ can be considered to be the collection of clutters Λ on Ω such that all elements of Ω appear in at least one element of the clutter Λ; or $\mathbb{X}$ can be the collection of those clutters Λ such that no element of Ω appears in all the elements of the clutter Λ, or the collection of clutters whose elements have at least size two. In any case, the fact that we can endow $\mathrm{Clut}(\Omega)$ with a distributive lattice structure allows us to operate with clutters and to characterize when is it possible to approximate and to recover any non Θ-clutter by means of Θ-clutters (and thus, the order and the operation $*$ mentioned in the general framework come from the lattice structure of $\mathrm{Clut}(\Omega)$).

To do this, first in Section 2 we recall some basic concepts about lattices (Subsection 2.1), we endow $\mathrm{Clut}(\Omega)$ with two (different but isomorphic) lattice structures (Subsection 2.2), and we analyze whether the lattice structure is preserved by considering certain constraints (Subsection 2.3). The problems of approximation and decomposition by a subset Σ of a general lattice is considered in Section 3. The main results characterize the existence of approximations, both for the case of approximating a single element or all elements in a family $\mathbb{X}$ (that need not have a lattice structure). A key role is played by the irreducible elements with respect to $\mathbb{X}$, which are needed to guarantee the existence of approximations. We also introduce another useful characterization in terms of what we call "avoidance properties". Next, in Section 4 we provide a complete description of these irreducible elements for some families of clutters, some of which have a lattice structure (Subsection 4.1), and some do not (Subsection 4.2). Finally, by using these descriptions of the irreducible elements (that are summarized in Subsection 4.3), and by combining them with the general results on lattices presented in Theorem 6, in Section 5 we explore if it is possible to approximate and to recover any non Θ-clutter by means of Θ-clutters whenever the combinatorial objects $\mathfrak{Obj}(\Omega)$ under consideration are matroids (Subsection 5.1), graphs (Subsection 5.2) or secret sharing schemes (Subsection 5.3). In each case we also consider the problem when we restrict ourselves to clutters that verify a certain restriction, that is, for all $\Lambda \in \mathbb{X}$ being $\mathbb{X} \subseteq \mathrm{Clut}(\Omega)$ some specific subsets of clutters.

We stress that all results in this paper are about the existence of approximations or decompositions, but we do not give results concerning the actual computation of such approximations. For some clutters related to matroids, this was done in [11, 13], but we do not know of other results in this direction, either for other specific discrete objects or in general.

2 Preliminaries on lattices and clutters

For completeness, we start by recalling some concepts about lattices in general. Then we endow the set of all clutters over a fixed set Ω with a distributive lattice structure (actually, in two different but related ways). We also consider some particular families of clutters and discuss the relation among them and whether they have a lattice structure as well.

2.1 General lattices

Although our main object of interest is the lattice of clutters of a finite set, in some parts of this paper we work with an arbitrary finite lattice $(\mathbb{L}, \leqslant, \vee, \wedge)$. We refer to the books [6, 8] for general definitions about lattices and distributive lattices. For completeness we recall some definitions and notation.

The bottom and top elements of a lattice, if they exist, are denoted by 0 and by 1, respectively. For $A \subseteq \mathbb{L}$, the collection of minimal elements of A is denoted by $\mathrm{minimal}(A, \leqslant)$; if A is a collection of subsets and we want to select those that are minimal under inclusion, we just write $\mathrm{minimal}(A)$; for maximal elements we write $\mathrm{maximal}(A, \leqslant)$ and $\mathrm{maximal}(A)$, respectively.

In a lattice $(\mathbb{L}, \leqslant, \vee, \wedge)$, an element a is *join-irreducible* if $a \neq 0$ and whenever $a = b \vee c$ either $a = b$ or $a = c$. Similarly, an element a is *meet-irreducible* if $a \neq 1$ and whenever $a = b \wedge c$ either $a = b$ or $a = c$. The set of all join-irreducible elements of $\mathbb{L}$ will be denoted $\mathcal{J}(\mathbb{L})$; similary, $\mathcal{M}(\mathbb{L})$ denotes the set of all meet-irreducible elements of $\mathbb{L}$.

It is well-known that if $(\mathbb{L}, \leqslant, \vee, \wedge)$ is a finite distributive lattice, then every element different from 0 has a unique representation as an irredundant join of join-irreducible elements ([8, Cor II.1.13]). Similarly, every element different from 1 has a unique representation as an irredundant meet of meet-irreducible elements.

Let $(\mathbb{L}_1, \leqslant_1, \vee_1, \wedge_1)$ and $(\mathbb{L}_2, \leqslant_2, \vee_1, \wedge_2)$ be two lattices. A map $\psi : \mathbb{L}_1 \to \mathbb{L}_2$ is said to be a *lattice homomorphism* if it is both a join-homomorphism and a meet-homomorphism; that is, if $\psi(a \vee_1 b) = \psi(a) \vee_2 \psi(b)$ and $\psi(a \wedge_1 b) = \psi(a) \wedge_2 \psi(b)$ for any $a, b \in \mathbb{L}_1$. Any lattice homomorphism ψ is necessarily monotone with respect to the associated ordering relation; that is, if $a \leqslant_1 b$ then $\psi(a) \leqslant_2 \psi(b)$.

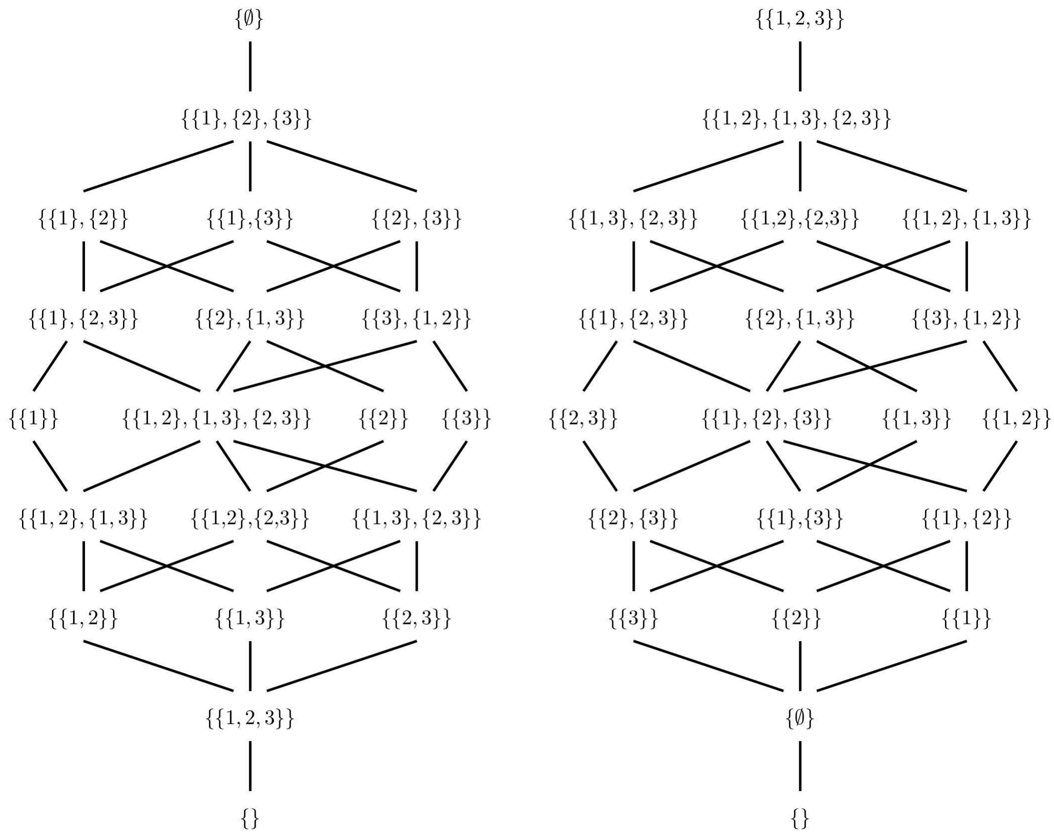
A *lattice isomorphism* is a one-to-one and onto lattice homomorphism. If $\psi : \mathbb{L}_1 \to \mathbb{L}_2$ is a lattice isomorphism then $\psi(\mathcal{J}(\mathbb{L}_1)) = \mathcal{J}(\mathbb{L}_2)$, $\psi(\mathcal{M}(\mathbb{L}_1)) = \mathcal{M}(\mathbb{L}_2)$ and ψ preserves the irredundant representation as a join of join-irreducible elements and the irredundant representation as a meet of meet-irreducible elements.

The *dual* of a lattice $\mathbb{L} = (\mathbb{L}, \leqslant, \vee, \wedge)$ is the lattice $\mathbb{L}^d = (\mathbb{L}, \geqslant, \wedge, \vee)$. So, $\mathcal{J}(\mathbb{L}^d) = \mathcal{M}(\mathbb{L})$ and $\mathcal{M}(\mathbb{L}^d) = \mathcal{J}(\mathbb{L})$.

An *anti-isomorphism* ψ of a lattice $(\mathbb{L}, \leqslant, \vee, \wedge)$ is a lattice isomorphism ψ between the lattice and its dual. If ψ is an anti-isomorphism of $\mathbb{L}$ then $\psi(\mathcal{M}(\mathbb{L})) = \mathcal{J}(\mathbb{L})$ and $\psi(\mathcal{J}(\mathbb{L})) = \mathcal{M}(\mathbb{L})$.

2.2 Two lattices of clutters

Let Ω be a non-empty finite set. By a *clutter* on Ω we mean a collection $\Lambda \subseteq \mathcal{P}(\Omega)$ such that the elements of Λ are pairwise incomparable under inclusion. Note that $\emptyset = \{\}$ and $\{\emptyset\}$ are clutters. The collection of all cutters of Ω is denoted by $\mathrm{Clut}(\Omega)$. We next introduce two natural partial orders on $\mathrm{Clut}(\Omega)$ (we refer to [13] for more details and motivation).



$(\mathrm{Clut}(\{1,2,3\}), \leqslant^+)$ $(\mathrm{Clut}(\{1,2,3\}), \leqslant^-)$

Figure 1: The posets $(\mathrm{Clut}(\Omega), \leqslant^+)$ and $(\mathrm{Clut}(\Omega), \leqslant^-)$ for a $\Omega = \{1, 2, 3\}$.

Given $\Lambda \in \mathrm{Clut}(\Omega)$, we denote by Λ^+ the monotone increasing family of subsets of Ω that has Λ as its minimal elements; that is, $\Lambda^+ = \{A \subseteq \Omega : \ B \subseteq A \text{ for some } B \in \Lambda\}$. This induces a partial order $\leqslant^+$ on $\mathrm{Clut}(\Omega)$ by setting $\Lambda_1 \leqslant^+ \Lambda_2$ if and only if $\Lambda_1^+ \subseteq \Lambda_2^+$. It is straightforward to check that $\Lambda_1 \leqslant^+ \Lambda_2$ if and only if for all $A \in \Lambda_1$ there is $B \in \Lambda_2$ such that $B \subseteq A$.

Similarly, we define $\Lambda^- = \{A \subseteq \Omega : \ A \subseteq B \text{ for some } B \in \Lambda\}$ and $\Lambda_1 \leqslant^- \Lambda_2$ if and only if $\Lambda_1^- \subseteq \Lambda_2^-$. The fact that $\Lambda_1 \leqslant^- \Lambda_2$ is equivalent to the fact that for each $A \in \Lambda_1$ there is $B \in \Lambda_2$ with $A \subseteq B$.

Figure 1 shows the Hasse diagram of both posets $(\mathrm{Clut}(\Omega), \leqslant^+)$ and $(\mathrm{Clut}(\Omega), \leqslant^-)$ when Ω is a set of three elements. Note that the bottom and top elements of the poset $(\mathrm{Clut}(\Omega), \leqslant^+)$ are $\{\} = \emptyset$ and $\{\emptyset\}$, respectively, and those of $(\mathrm{Clut}(\Omega), \leqslant^-)$ are $\{\}$ and $\{\Omega\}$.

The following operations endow the posets $(\mathrm{Clut}(\Omega), \leqslant^+)$ and $(\mathrm{Clut}(\Omega), \leqslant^-)$ with a lattice structure. Let Λ_1, Λ_2 be clutters on Ω. We define

$$\begin{aligned}
\Lambda_1 \sqcap^+ \Lambda_2 &= \text{minimal}\big(\Lambda_1^+ \cap \Lambda_2^+\big) = \text{minimal}\big(\{A_1 \cup A_2 \,:\, A_i \in \Lambda_i\}\big),\\
\Lambda_1 \sqcup^+ \Lambda_2 &= \text{minimal}\big(\Lambda_1^+ \cup \Lambda_2^+\big) = \text{minimal}\big(\Lambda_1 \cup \Lambda_2\big),\\
\Lambda_1 \sqcap^- \Lambda_2 &= \text{maximal}\big(\Lambda_1^- \cap \Lambda_2^-\big) = \text{maximal}\big(\{A_1 \cap A_2 \,:\, A_i \in \Lambda_i\}\big),\\
\Lambda_1 \sqcup^- \Lambda_2 &= \text{maximal}\big(\Lambda_1^- \cup \Lambda_2^-\big) = \text{maximal}\big(\Lambda_1 \cup \Lambda_2\big).
\end{aligned}$$

It is routine to check that indeed these operations give a distributive lattice structure. (We note that $\mathrm{Clut}^+(\Omega)$ is isomorphic to the free distributive lattice on $|\Omega|$ generators, although we will not use this fact in the sequel.)

Lemma 1. *The lattices* $\mathrm{Clut}^+(\Omega) = (\mathrm{Clut}(\Omega), \leqslant^+, \sqcup^+, \sqcap^+)$ *and* $\mathrm{Clut}^-(\Omega) = (\mathrm{Clut}(\Omega), \leqslant^-, \sqcup^-, \sqcap^-)$ *are distributive.*

We next introduce two operations on clutters that allow us to relate the lattice structures of $\mathrm{Clut}^+(\Omega)$ and $\mathrm{Clut}^-(\Omega)$ (see Lemma 2). In fact, one of these operations provides a lattice isomorphism between $\mathrm{Clut}^-(\Omega)$ and $\mathrm{Clut}^+(\Omega)$.

The *complementary clutter* of a clutter Λ is the clutter $\Lambda^c = \{\Omega \setminus A \,:\, A \in \Lambda\}$. Observe that it is an involutive operation, that is, $((\Lambda)^c)^c = \Lambda$. Note that $\{\}^c = \{\ \}$ and $\{\emptyset\} = \{\Omega\}$.

The *blocker* (or transversal) of a clutter Λ is the clutter $b(\Lambda) = \text{minimal}(\{B \,:\, B \cap A \neq \emptyset \text{ for all } A \in \Lambda\})$. The blocker operation is also involutive on clutters, that is, $b(b(\Lambda)) = \Lambda$ (see [7] for more details). Note that $b(\{\}) = \{\emptyset\}$ and that $b(\{\Omega\}) = \{\{w\} \,:\, w \in \Omega\}$.

The following lemma summarizes the behaviour of the lattice structure of clutters with respect to the blocker and complementary operations (it is a routine to prove its statements).

Lemma 2. *Let Ω be a finite set. Then:*

1. *The complementary operation c is an involutive isomorphism between the lattice* $\mathrm{Clut}^+(\Omega)$ *and the lattice* $\mathrm{Clut}^-(\Omega)$.

2. *The blocker operation b is an involutive anti-isomorphism of the lattice* $\mathrm{Clut}^+(\Omega)$.

3. *The composition $c \circ b \circ c$ is an involutive anti-isomorphism of the lattice* $\mathrm{Clut}^-(\Omega)$.

2.3 Clutters with additional properties

In this section we introduce some families of clutters that are of interest from the combinatorial point of view.

For a clutter $\Lambda \in \mathrm{Clut}(\Omega)$, its *support* is defined as $\mathrm{supp}(\Lambda) = \bigcup_{A \in \Lambda} A$. We say that Λ has *full support* if $\mathrm{supp}(\Lambda) = \Omega$; that is, all elements of Ω appear in at least one element of the clutter Λ. The collection of all clutters on Ω with full support is denoted by $\mathrm{Clut}_0(\Omega)$.

The *intersection* of Λ is defined as $\mathrm{int}(\Lambda) = \bigcap_{A \in \Lambda} A$. We say that Λ has *empty intersection* if $\mathrm{int}(\Lambda) = \emptyset$; that is, if no element of Ω appears in all the elements of the clutter Λ. The collection of all clutters on Ω with empty intersection is denoted by $\mathrm{Clut}_\emptyset(\Omega)$. Note that $\emptyset \notin \mathrm{Clut}_\emptyset(\Omega)$ since an empty intersection equals Ω.

For a clutter $\Lambda \neq \emptyset$, we define the *rank*, respectively the *corank*, of Λ as the maximum, respectively the minimum, of all the cardinalities of the elements of Λ; that is, $\mathrm{rk}(\Lambda) = \max\{|A| : A \in \Lambda\}$ and $\mathrm{crk}(\Lambda) = \min\{|A| : A \in \Lambda\}$. The collection $\mathrm{Clut}_{c\geq 2}(\Omega)$ contains all clutters on Ω with corank $\mathrm{crk}(\Lambda) \geq 2$; that is, clutters whose elements have size at least two. For technical reasons that will become clear in Lemma 3 below, we set $\mathrm{rk}(\emptyset) = -\infty$ and $\mathrm{crk}(\emptyset) = \infty$, so that $\emptyset \in \mathrm{Clut}_{c\geq 2}(\Omega)$.

The following lemma summarizes how the above families of clutters behave with respect to the blocker and complementary operations.

Lemma 3. *Let Ω be a finite non-empty set and $\Lambda \in \mathrm{Clut}(\Omega)$. Then*

1. $\Lambda \in \mathrm{Clut}_0(\Omega)$ *if and only if* $b(\Lambda) \in \mathrm{Clut}_0(\Omega)$ *if and only if* $\Lambda^c \in \mathrm{Clut}_\emptyset(\Omega)$.

2. $\Lambda \in \mathrm{Clut}_{c\geq 2}(\Omega)$ *if and only if* $b(\Lambda) \in \mathrm{Clut}_\emptyset(\Omega)$ *if and only if* $b(\Lambda)^c \in \mathrm{Clut}_0(\Omega)$.

3. $\Lambda \in \mathrm{Clut}_\emptyset(\Omega)$ *if and only if* $b(\Lambda) \in \mathrm{Clut}_{c\geq 2}(\Omega)$ *if and only if* $\Lambda^c \in \mathrm{Clut}_0(\Omega)$.

Proof. The first part of statement (1) follows from the equality $\mathrm{supp}(\Lambda) = \mathrm{supp}(b(\Lambda))$ that we next prove. Note that it holds for $\Lambda = \emptyset$. For an arbitrary $\Lambda = \{A_1, \ldots, A_k\}$ with $k \geq 1$, take $a \notin \mathrm{supp}(\Lambda)$. If A is a minimal set such that $A \cap A_i \neq \emptyset$ for all $1 \leq i \leq k$, then $a \notin A$. Thus, $\mathrm{supp}(b(\Lambda)) \subseteq \mathrm{supp}(\Lambda)$. Since the blocker operation is involutive, we have an equality. To complete the proof of (1), note that $\mathrm{int}(\Lambda) = \Omega \setminus \mathrm{supp}(\Lambda^c)$.

Statements (2) and (3) are equivalent, and they follow from the facts that $\mathrm{crk}(\Lambda) \leq 1$ if and only if $\mathrm{int}(b(\Lambda)) \neq \emptyset$ and again that $\mathrm{supp}(\Lambda^c) = \Omega \setminus \mathrm{int}(\Lambda)$. □

In addition to the families $\mathrm{Clut}_0(\Omega)$, $\mathrm{Clut}_\emptyset(\Omega)$ and $\mathrm{Clut}_{c\geq 2}(\Omega)$, we can also consider their intersections, that is,

$$\begin{aligned}
\mathrm{Clut}_{0,\emptyset}(\Omega) &= \mathrm{Clut}_0(\Omega) \cap \mathrm{Clut}_\emptyset(\Omega),\\
\mathrm{Clut}_{0,c\geq 2}(\Omega) &= \mathrm{Clut}_0(\Omega) \cap \mathrm{Clut}_{c\geq 2}(\Omega),\\
\mathrm{Clut}_{c\geq 2,\emptyset}(\Omega) &= \mathrm{Clut}_{c\geq 2}(\Omega) \cap \mathrm{Clut}_\emptyset(\Omega),\\
\mathrm{Clut}_{0,c\geq 2,\emptyset}(\Omega) &= \mathrm{Clut}_0(\Omega) \cap \mathrm{Clut}_{c\geq 2}(\Omega) \cap \mathrm{Clut}_\emptyset(\Omega).
\end{aligned}$$

Each of these seven families gives rise to two posets, one for the order $\leqslant^+$ and one for the order $\leqslant^-$. We next show that only four of these posets have a lattice structure in general. Recall that an *interval* in a lattice $(\mathbb{L}, \leqslant, \vee, \wedge)$ is a set of the form $[a, b] = \{x \in \mathbb{L} : a \leqslant x \leqslant b\}$, and it is a sublattice of $\mathbb{L}$. We call the intervals of the form $[a, 1]$ and $[0, b]$ *upper* and *lower intervals*, respectively.

Lemma 4. *Let Ω be a finite non-empty set of size $|\Omega| = n$. Then:*

1. *The following families have a distributive lattice structure:*

   (a) $(\mathrm{Clut}_{c\geq 2}(\Omega), \leqslant^+, \sqcup^+, \sqcap^+)$, $(\mathrm{Clut}_\emptyset(\Omega), \leqslant^+, \sqcup^+, \sqcap^+)$ *and* $(\mathrm{Clut}_{c\geq 2,\emptyset}(\Omega), \leqslant^+, \sqcup^+, \sqcap^+)$ *are distributive sublattices of* $\mathrm{Clut}^+(\Omega)$*; actually,* $\mathrm{Clut}_{c\geq 2}(\Omega)$ *is a lower interval,* $\mathrm{Clut}_\emptyset(\Omega)$ *is an upper interval and* $\mathrm{Clut}_{c\geq 2,\emptyset}(\Omega)$ *is an interval in* $\mathrm{Clut}^+(\Omega)$.

(b) $(\mathrm{Clut}_0(\Omega), \leqslant^-, \sqcup^-, \sqcap^-)$ *is a distributive sublattice of* $\mathrm{Clut}^-(\Omega)$*; actually,* $\mathrm{Clut}_0(\Omega)$ *is an upper interval of* $\mathrm{Clut}^-(\Omega)$.

2. *The following families do not have a lattice structure:*

   (a) *With respect to the order* $\leqslant^+$*, the families* $\mathrm{Clut}_0(\Omega)$*,* $\mathrm{Clut}_{0,c\geq 2}(\Omega)$ *and* $\mathrm{Clut}_{0,\emptyset}(\Omega)$ *do not have a lattice structure for* $n \geq 4$*, and the family* $\mathrm{Clut}_{0,c\geq 2,\emptyset}(\Omega)$ *does not have a lattice structure for* $n \geq 5$.

   (b) *With respect to the order* $\leqslant^-$*, the families* $\mathrm{Clut}_{c\geq 2}(\Omega)$*,* $\mathrm{Clut}_{\emptyset}(\Omega)$*,* $\mathrm{Clut}_{0,c\geq 2}(\Omega)$*,* $\mathrm{Clut}_{0,\emptyset}(\Omega)$*,* $\mathrm{Clut}_{c\geq 2,\emptyset}(\Omega)$ *and* $\mathrm{Clut}_{0,c\geq 2,\emptyset}(\Omega)$ *do not have a lattice structure for* $n \geq 4$.

Proof. For (1), by Lemmas 2 and 3, it is enough to show that $\mathrm{Clut}_0(\Omega)$ is an upper interval in $\mathrm{Clut}^-(\Omega)$, and this is indeed the case since from the definition of $\leqslant^-$ one can check that $\Lambda \in \mathrm{Clut}_0(\Omega)$ is equivalent to $\{\{a_1\}, \dots, \{a_n\}\} \leqslant^- \Lambda$.

For $n = 4$, the posets $(\mathrm{Clut}_0(\Omega), \leqslant^+)$ and $(\mathrm{Clut}_{0,\emptyset}(\Omega), \leqslant^+)$ are not lattices since the clutters $\Lambda_1 = \{\{a_1\}, \{a_2, a_3\}, \{a_3, a_4\}\}$ and $\Lambda_2 = \{\{a_1\}, \{a_2, a_3\}, \{a_2, a_4\}\}$ have two common maximal lower bounds: $\{\{a_1\}, \{a_2, a_3, a_4\}\}$ and $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_2, a_3\}\}$. Again by Lemmas 2 and 3, we conclude that the posets $(\mathrm{Clut}_{0,c\geq 2}(\Omega), \leqslant^+)$, $(\mathrm{Clut}_{\emptyset}(\Omega), \leqslant^-)$ and $(\mathrm{Clut}_{0,\emptyset}(\Omega), \leqslant^-)$ are not lattices either. To generalize to $n > 4$, it is enough to add the singletons $\{a_5\}, \dots, \{a_n\}$ to the construction above.

The poset $(\mathrm{Clut}_{0,c\geq 2,\emptyset}(\Omega), \leqslant^+)$ for $n = 5$ is not a lattice as the clutters $\Lambda_1 = \{\{a_1, a_2\}, \{a_3, a_4, a_5\}\}$ and $\Lambda_2 = \{\{a_1, a_2, a_3\}, \{a_4, a_5\}\}$ have two common minimal upper bounds: $\{\{a_1, a_2\}, \{a_1, a_3, a_4\}, \{a_4, a_5\}\}$ and $\{\{a_1, a_2\}, \{a_2, a_3, a_4\}, \{a_4, a_5\}\}$. To generalize the construction, it is sufficient to add the sets $\{a_1, a_6\}, \dots, \{a_1, a_n\}$ to all the clutters in the construction.

For $n = 4$, the posets $(\mathrm{Clut}_{c\geq 2}(\Omega), \leqslant^-)$, $(\mathrm{Clut}_{0,c\geq 2}(\Omega), \leqslant^-)$, $(\mathrm{Clut}_{c\geq 2,\emptyset}(\Omega), \leqslant^-)$ and $(\mathrm{Clut}_{0,c\geq 2,\emptyset}(\Omega), \leqslant^-)$ are not lattices since $\Lambda_1 = \{\{a_1, a_2\}, \{a_3, a_4\}\}$ and $\Lambda_2 = \{\{a_1, a_3\}, \{a_2, a_4\}\}$ do not have common lower bounds. To generalize the construction, add the pairs $\{a_1, a_5\}, \dots, \{a_1, a_n\}$ to Λ_1 and the pairs $\{a_2, a_5\}, \dots, \{a_2, a_n\}$ to Λ_2. $\square$

Remark. Lemma 4 above gives some bounds on the size of the ground set for which the given families are not lattices. For smaller values, it is sometimes the case that they are lattices, and in some other cases they are not. Since we are interested in the general behaviour of the family, we omit the details for the small values of $|\Omega|$.

3 Decomposition in lattices

We start by extending the notion of decomposition and irreducibility when restricted to a subset $\mathbb{X}$ of our lattice. As for general meet- and join-irreducibility, the top and bottom elements are sometimes excluded from the definitions. In what follows, we write $x \in \mathbb{X}, x \neq y$ to exclude the element y whenever it belongs to $\mathbb{X}$.

Given a finite distributive lattice $(\mathbb{L}, \leqslant, \vee, \wedge)$ and a subset $\mathbb{X} \subseteq \mathbb{L}$, we say that $a \in \mathbb{X}$, $a \neq 1$, is *meet* $\mathbb{X}$*-irreducible* if whenever $a = b_1 \wedge b_2 \wedge \cdots \wedge b_k$ with $b_i \in \mathbb{X}$ for all i, implies $a = b_i$ for some i. We denote by $\mathcal{M}(\mathbb{X}, \mathbb{L})$ the set of meet $\mathbb{X}$-irreducible elements. Note that if $\mathbb{X}$ is a sublattice of $\mathbb{L}$ we have $\mathcal{M}(\mathbb{X}, \mathbb{L}) = \mathcal{M}(\mathbb{X})$; also, if $\mathbb{X} \subseteq \mathbb{L}_1 \subseteq \mathbb{L}$ and $\mathbb{L}_1$ is a sublattice of $\mathbb{L}$, we have $\mathcal{M}(\mathbb{X}, \mathbb{L}) = \mathcal{M}(\mathbb{X}, \mathbb{L}_1)$. The definitions of *join $\mathbb{X}$-irreducibility* and $\mathcal{J}(\mathbb{X}, \mathbb{L})$ are analogous (here we exclude the bottom element 0 from being join $\mathbb{X}$-irreducible, in case it belongs to $\mathbb{X}$).

Let now Σ be a non-empty subset of $\mathbb{L}$. For an element $x \in \mathbb{L}$, we say that $A \subseteq \Sigma$ is a *Σ-meet decomposition* of x if $x = \bigwedge_{a \in A} a$; note that the case $A = \emptyset$ is not excluded, in which case $x = 1$. The definition of *Σ-join decomposition* is analogous.

The fact that $\mathbb{L}$ is finite gives the following lemma.

Lemma 5. *Let $(\mathbb{L}, \leqslant, \vee, \wedge)$ be a finite distributive lattice and $\mathbb{X} \subseteq \mathbb{L}$. Then every element of $x \in \mathbb{X}$ has a $\mathcal{M}(\mathbb{X}, \mathbb{L})$-meet decomposition and a $\mathcal{J}(\mathbb{X}, \mathbb{L})$-join decomposition.*

The following is a generalisation of a well-known result (see for instance Theorem 2.46 in [6]). The convention that empty meets and joins equal 1 and 0, respectively, allows us to write results like this one in a more compact way. In the proofs we do not address explicitly these extreme cases since they are usually trivial. Also, the results in this section are stated for both meets and joins, but by duality it is enough to prove just one of them, and we do so without further mention.

Theorem 6. *Let $(\mathbb{L}, \leqslant, \vee, \wedge)$ be a finite distributive lattice, let $\mathbb{X} \subseteq \mathbb{L}$ and $\Sigma \subseteq \mathbb{L}$. Every element $x \in \mathbb{X}$ has a Σ-meet decomposition if and only if $\mathcal{M}(\mathbb{X} \cup \Sigma, \mathbb{L}) \subseteq \Sigma$. Analogously, every element $x \in \mathbb{X}$ has a Σ-join decomposition if and only if $\mathcal{J}(\mathbb{X} \cup \Sigma, \mathbb{L}) \subseteq \Sigma$.*

Proof. Suppose that every element in $\mathbb{X}$ has a Σ-meet decomposition and let $x \in \mathcal{M}(\mathbb{X} \cup \Sigma, \mathbb{L})$. If x belongs to Σ there is nothing to prove, so suppose $x \in \mathbb{X}$. As x has a Σ-meet decomposition, we can write $x = y_1 \wedge \cdots \wedge y_k$ for some $k \geq 1$ and $y_1, \ldots, y_k \in \Sigma$. But x is meet $(\mathbb{X} \cup \Sigma)-$irreducible, so $k = 1$ and x belongs to Σ, as needed.

For the converse, let $x \in \mathbb{X}$. If x belongs to $\mathcal{M}(\mathbb{X} \cup \Sigma, \mathbb{L})$, then $x \in \Sigma$ and it trivially has a Σ-meet decomposition. Otherwise, by Lemma 5 the element x has a $\mathcal{M}(\mathbb{X} \cup \Sigma, \mathbb{L})$-decomposition, but since $\mathcal{M}(\mathbb{X} \cup \Sigma, \mathbb{L}) \subseteq \Sigma$, it has a Σ-meet decomposition as well. $\square$

The following lemma sometimes simplifies the computation of meet- or join-irreducible elements. We omit the straightforward proof.

Lemma 7. *Let $(\mathbb{L}, \leqslant, \vee, \wedge)$ be a finite distributive lattice and $\mathbb{X}' \subseteq \mathbb{X} \subseteq \mathbb{L}$. If $\mathbb{X}'$ is an upper interval, then $\mathcal{M}(\mathbb{X}', \mathbb{L}) = \mathcal{M}(\mathbb{X}, \mathbb{L}) \cap \mathbb{X}'$. Analogously, if $\mathbb{X}'$ is a lower interval, then $\mathcal{J}(\mathbb{X}', \mathbb{L}) = \mathcal{J}(\mathbb{X}, \mathbb{L}) \cap \mathbb{X}'$.*

In the following subsections we relate Σ-decompositions to the problem of approximating an element of a lattice with elements from Σ (Subsection 3.1) and we characterize the elements that have a Σ decomposition by means of what we call "avoidance properties" (Subsection 3.2).

3.1 Decomposition and optimal approximations

Let $(\mathbb{L}, \leqslant, \vee, \wedge)$ be a finite distributive lattice and fix a non-empty subset $\Sigma \subseteq \mathbb{L}$. Given $x \in \mathbb{L}$, we define the set $\Sigma_u(x)$ of Σ-*upper completions* of x as

$\Sigma_u(x) = \{y \in \Sigma : x \leqslant y\}$; analgously, and set $\Sigma_\ell(x)$ of Σ-*lower completions* of x is $\Sigma_\ell(x) = \{y \in \Sigma : y \leqslant x\}$. We also define $\Phi_u(x) = \text{minimal}\,(\Sigma_u(x), \leqslant)$ and $\Phi_\ell(x) = \text{maximal}\,(\Sigma_\ell(x), \leqslant)$. The elements of $\Phi_u(x)$ are said to be the *optimal* Σ-*upper completions* of x, and the elements of $\Phi_\ell(x)$ are the *optimal* Σ-*lower completions* of x.

Observe that if the element x admits a Σ-meet decomposition $\{y_1, \ldots, y_r\}$ with $r \geq 1$, then $\{y_1, \ldots, y_r\} \subseteq \Sigma_u(x)$ and, in particular, $\Sigma_u(x) \neq \emptyset$. The analogous statement holds for Σ-join decompositions and Σ-lower completions. So in such a case, there are optimal approximations. The following proposition highlights the role of optimal approximations for the existence of decompositions.

Proposition 8. *Let $(\mathbb{L}, \leqslant, \vee, \wedge)$ be a finite distributive lattice and let $\Sigma \subseteq \mathbb{L}$ be non-empty. The following statements hold for $x \in \mathbb{L}$:*

1. *There exists a Σ-meet decomposition of x if and only if the set $\Phi_u(x)$ of Σ-upper optimal completions of x is a Σ-meet decomposition of x.*

2. *There exists a Σ-join decomposition of x if and only if the set $\Phi_\ell(x)$ of the Σ-lower optimal completions of x is a Σ-join decomposition of x.*

Proof. Suppose that $\Sigma_u(x)$ is non-empty, as otherwise the statement holds trivially. Let $\Phi_u(x) = \{x_1, \ldots, x_m\}$ and suppose that $\{y_1, \ldots, y_r\}$ is a Σ-meet decomposition of x. We only need to check that $x = x_1 \wedge \ldots \wedge x_m$. The equality $x \leqslant x_1 \wedge \ldots \wedge x_m$ follows from the fact that $x \leqslant x_j$ for each $1 \leq j \leq m$. To show that $x_1 \wedge \ldots \wedge x_m \leqslant x$, note that since $\{y_1, \ldots, y_r\}$ is a Σ-meet decomposition of x, we have $\{y_1, \ldots, y_r\} \subseteq \Sigma_u(x)$, and for each i there is j_i such that $x_{j_i} \leqslant y_i$. We thus conclude $x_{j_1} \wedge \ldots \wedge x_{j_r} \leqslant y_1 \wedge \ldots \wedge y_m = x$, as needed. □

The following two corollaries follow directly from Proposition 8. The first one characterizes when there is a unique optimal Σ-completion, and the second translates equality of elements into equality of the sets of optimal completions.

Corollary 9. *Let $(\mathbb{L}, \leqslant, \vee, \wedge)$ be a finite distributive lattice and let $\Sigma \subseteq \mathbb{L}$ be a non-empty subset. Let $x \in \mathbb{L}$. The following statements hold:*

1. *If there is some Σ-meet decomposition of x, then $\Phi_u(x)$ has a unique element if and only if $\Phi_u(x) = \{x\}$, which happens if and only if $x \in \Sigma$.*

2. *If there is some Σ-join decomposition of x, then $\Phi_\ell(x)$ has a unique element if and only if $\Phi_\ell(x) = \{x\}$, which happens if and only if $x \in \Sigma$.*

Corollary 10. *Let $(\mathbb{L}, \leqslant, \vee, \wedge)$ be a finite distributive lattice and let $\Sigma \subseteq \mathbb{L}$ be a non-empty subset. Let $x_1, x_2 \in \mathbb{L}$. The following statements hold:*

1. *If there exist some Σ-meet decompositions of x_1 and of x_2, then $x_1 = x_2$ if and only if $\Phi_u(x_1) = \Phi_u(x_2)$.*

2. *If there exist some Σ-join decomposition of x_1 and of x_2, then $x_1 = x_2$ if and only if $\Phi_\ell(x_1) = \Phi_\ell(x_2)$.*

3.2 Decomposition and avoidance properties

We next characterize when a particular element x has a Σ-decomposition in terms of what we call avoidance properties (which are similar in spirit to Proposition 2.45 in [6]). We say that $x \in \mathbb{L}$ verifies the Σ-*upper avoidance property* if for all $y \in \mathbb{L}$ with $x \leqslant y$ and $x \neq y$, there is $y_0 \in \Sigma$ such that $x \leqslant y_0$ and $y \nleqslant y_0$. Analogously, the element x verifies the Σ-*lower avoidance property* if for all $y \in \mathbb{L}$ with $y \leqslant x$ and $x \neq y$, there is $y_0 \in \Sigma$ such that $y_0 \leqslant x$ and $y_0 \nleqslant y$.

Note that the top element 1 satisfies the Σ-upper avoidance property vacuously, and similarly the bottom element 0 satisfies the Σ-lower avoidance property. If $x \neq 1$ verifies the Σ-upper avoidance property then $\Sigma_u(x) \neq \emptyset$, and if $x \neq 0$ verifies the Σ-lower avoidance property then $\Sigma_\ell(x) \neq \emptyset$. However, the converse is not true, as in fact avoidance properties are equivalent to the existence of decompositions, as we next show.

Theorem 11. *Let $(\mathbb{L}, \leqslant, \vee, \wedge)$ be a finite distributive lattice and let $\Sigma \subseteq \mathbb{L}$ be a non-empty set. The following statements hold for $x \in \mathbb{L}$.*

1. *There is a Σ-meet decomposition of x if and only if x verifies the Σ-upper avoidance property.*

2. *There is a Σ-join decomposition of x if and only if x verifies the Σ-lower avoidance property.*

Proof. Statement (1) holds trivially for $x = 1$. For $x \neq 1$, let $\{y_1, \dots, y_r\}$ be a Σ-meet decomposition of x. Let $y \in \mathbb{L}$ with $x \leqslant y$ and $x \neq y$. If $y \leqslant y_i$ for all i, then $y \leqslant y_1 \wedge \dots \wedge y_r = x$ and hence $y = x$, a contradiction. Thus there is i_0 such that $y \nleqslant y_{i_0}$. By definition, $x \leqslant y_{i_0}$ and $y_{i_0} \in \Sigma$. Hence we take $y_0 = y_{i_0}$ and thus x verifies the Σ-upper avoidance property.

Reciprocally, if $x \neq 1$ verifies the Σ-upper avoidance property, by taking $y = 1$ in the statement of the Σ-upper avoidance property we conclude that $\Sigma_u(x) \neq \emptyset$. Let $\Phi_u(x) = \{x_1, \dots, x_m\}$ be the minimal elements of $\Sigma_u(x)$. Let us check that $x = x_1 \wedge \cdots \wedge x_m$. Since $x \leqslant x_i$, then $x \leqslant x_1 \wedge \dots \wedge x_m$. If the inequality is strict, by taking $y = x_1 \wedge \dots \wedge x_m$ in the Σ-upper avoidance property we conclude that there is $y_0 \in \Sigma$ such that $x \leqslant y_0$ and $x_1 \wedge \dots \wedge x_m \nleqslant y_0$. But then $y_0 \in \Sigma_u(x)$ and hence there is i_0 such that $x_{i_0} \leqslant y_0$, which is a contradiction since $y = x_1 \wedge \dots \wedge x_m \leqslant x_{i_0} \leqslant y_0$. Hence $x = x_1 \wedge \dots \wedge x_m$, as needed. □

Theorem 11 gives a strategy for showing that a candidate family $\mathcal{F} \subseteq \mathbb{L}$ is indeed $\mathcal{M}(\mathbb{X} \cup \Sigma, \mathbb{L})$. This is the approach we will use later in Subsection 4.2.

Theorem 12. *Let $(\mathbb{L}, \leqslant, \vee, \wedge)$ be a finite distributive lattice, let $\mathbb{X} \subseteq \mathbb{L}$ and $\Sigma \subseteq \mathbb{L}$. The following statements hold:*

1. *Let $\mathcal{F} \subseteq \mathcal{M}(\mathbb{X} \cup \Sigma, \mathbb{L})$ and assume that every element $x \in \mathbb{X} \cup \Sigma$ verifies the $\mathcal{F}$-upper avoidance property. Then, $\mathcal{F} = \mathcal{M}(\mathbb{X} \cup \Sigma, \mathbb{L})$.*

2. *Let $\mathcal{F} \subseteq \mathcal{J}(\mathbb{X} \cup \Sigma, \mathbb{L})$ and assume that every element $x \in \mathbb{X} \cup \Sigma$ verifies the $\mathcal{F}$-lower avoidance property. Then $\mathcal{F} = \mathcal{J}(\mathbb{X} \cup \Sigma, \mathbb{L})$.*

Proof. Let $x \in \mathcal{M}(\mathbb{X} \cup \Sigma, \mathbb{L})$. Since it belongs to $\mathbb{X} \cup \Sigma$, it satisfies the $\mathcal{F}-$upper avoidance property. By Theorem 11, the element x has an $\mathcal{F}-$meet decomposition. Since $\mathcal{F} \subseteq \mathcal{M}(\mathbb{X} \cup \Sigma, \mathbb{L})$ and x is meet $(\mathbb{X} \cup \Sigma)-$irreducible, it must be that $x \in \mathcal{F}$, as needed. □

We end this section with a characterization of avoidance properties that makes them easier to check. In particular, in the sequel we use condition (c) in the lemma below.

Lemma 13. *Let $(\mathbb{L}, \leqslant, \vee, \wedge)$ be a finite distributive lattice and let $\Sigma \subseteq \mathbb{L}$ be a non-empty subset and $x \in \mathbb{L}$.*

1. *The following conditions are equivalent:*
   - *(a) The element x verifies the Σ-upper avoidance property.*
   - *(b) For all $y \in \mathbb{L}$ with $y \nleqslant x$ there is $y_0 \in \Sigma$ such that $x \leqslant y_0$ and $y \nleqslant y_0$.*
   - *(c) For all $y \in \mathcal{J}(\mathbb{L})$ with $y \nleqslant x$ there is $y_0 \in \Sigma$ such that $x \leqslant y_0$ and $y \nleqslant y_0$.*
2. *The following conditions are equivalent:*
   - *(a) The element x verifies the Σ-lower avoidance property.*
   - *(b) For all $y \in \mathbb{L}$ with $x \nleqslant y$ there is $y_0 \in \Sigma$ such that $y_0 \leqslant x$ and $y_0 \nleqslant y$.*
   - *(c) For all $y \in \mathcal{M}(\mathbb{L})$ with $x \nleqslant y$ there is $y_0 \in \Sigma$ such that $y_0 \leqslant x$ and $y_0 \nleqslant y$.*

PROOF. We prove (1). Supose $x \neq 1$, as for $x = 1$ the three conditions hold trivially. For (a) $\Rightarrow$ (b), let $y \in \mathbb{L}$ be such that $y \nleqslant x$ and assume $x \nleqslant y$, as otherwise the Σ-upper avoidance property gives y_0 as needed. Let $y' = x \vee y$. By the Σ-upper avoidance property applied to x and y', we find $y_0 \in \Sigma$ such that $x \leqslant y_0$ and $y' \nleqslant y_0$, which also implies $y \nleqslant y_0$. The implication (b) $\Rightarrow$ (c) is straightforward. For (c) $\Rightarrow$ (a), let $y \in \mathbb{L}$ with $x \leqslant y$. Let $y = y_1 \vee \ldots \vee y_r$ be the irredundant join decomposition of the element y. Since $x \leqslant y$, there is $i_0 \in \{1, \ldots, r\}$ with $y_{i_0} \nleqslant x$. As $y_{i_0} \in \mathcal{J}(\mathbb{L})$, there is $y_0 \in \Sigma$ such that $x \leqslant y_0$ i $y_{i_0} \nleqslant y_0$. Finally, $y \nleqslant y_0$ because $y_{i_0} \leqslant y$. □

Remark. In checking whether x satisfies the Σ-upper avoidance property, it is not sufficient to restrict to $y \in \mathcal{J}(\mathbb{L})$ with $x \leqslant y$ and $x \neq y$. Indeed, for the lattice $\mathrm{Clut}^+(\{1,2,3\})$, taking $x = \{\{1,2\}\}$ and $\Sigma = \{y_0\}$ with $y_0 = \{\{1,2\},\{1,3\},\{2,3\}\}$, one can check that x does not satisfy the Σ-upper avoidance property, but that $x \leqslant y_0$ and $y \nleqslant y_0$ for every join-irreducible element y such that $x \leq y$.

The situation is analogous with respect to the Σ-lower avoidance property and $\mathcal{M}(\mathbb{L})$.

4 Decomposition in families of clutters: description of the meet and join irreducible elements

In this section we particularize the results of Section 3 to the two lattices of clutters $\mathrm{Clut}^+(\Omega)$ and $\mathrm{Clut}^-(\Omega)$. Given a subset $\Sigma \subseteq \mathrm{Clut}(\Omega)$, we seek to determine conditions on Σ that guarantee that every element of a subset $\mathbb{X} \subseteq \mathrm{Clut}(\Omega)$ has a Σ-meet or Σ-join decomposition. So according to Theorem 6 we have to determine the families $\mathcal{M}(\mathbb{X} \cup \Sigma, \mathbb{L})$ and $\mathcal{J}(\mathbb{X} \cup \Sigma, \mathbb{L})$ and, after that, we will have to check whether these families are subsets of Σ. This section deals with the first issue while the second one will be considered in Section 5 for some sets Σ arising from discrete objects.

Specifically, and due to the difficulty of giving an explicit description of all the meet and join irreducible elements of the set $\mathbb{X} \cup \Sigma$, here we will only focus on the case in which we take Σ to be a subset of $\mathbb{X}$, so that we have to determine $\mathcal{M}(\mathbb{X}, \mathbb{L})$ and $\mathcal{J}(\mathbb{X}, \mathbb{L})$. Also, we restrict $\mathbb{X}$ to being either the full lattice of clutters or one of the families of clutters identified in Subsection 2.3.

In Subsection 4.1 we deal with the families $\mathbb{X}$ that have a lattice structure, and in Subsection 4.2 with the families $\mathbb{X}$ that do not have a lattice structure. For ease of reference, the results are summarized in Subsection 4.3.

4.1 The meet and join irreducible elements for some families of clutters with a lattice structure

The following proposition identifies join- and meet-irreducible elements in the lattices of clutters $\mathrm{Clut}^+(\Omega)$ and $\mathrm{Clut}^-(\Omega)$. Given a subset $A \subseteq \Omega$ and a nonnegative integer $k \leq |A|$, the *uniform clutter* $\mathcal{U}_{k,A}$ is the clutter $\{B : B \subseteq A \text{ and } |B| = k\}$. We write $\mathcal{M}^+(\Omega)$ instead of $\mathcal{M}(\mathrm{Clut}^+(\Omega))$ and so on.

We omit the easy proof (by Lemma 2 it is sufficient to prove just one of the four items below). We also note that this proposition together with the results in Section 3 give a more general proof of the results in Section 3 of [13].

Proposition 14. *Let Ω be a finite non-empty set.*

1. *In the distributive lattice* $\mathrm{Clut}^+(\Omega)$ *:*
   (a) $\mathcal{M}^+(\Omega) = \{\cup_{a\in A}\{\{a\}\} : \emptyset \subseteq A \subseteq \Omega\} = \{\mathcal{U}_{1,A} : \emptyset \subseteq A \subseteq \Omega\}$.
   (b) $\mathcal{J}^+(\Omega) = \{\{A\} : \emptyset \subseteq A \subseteq \Omega\} = \{\mathcal{U}_{|A|,A} : \emptyset \subseteq A \subseteq \Omega\}$.
2. *In the distributive lattice* $\mathrm{Clut}^-(\Omega)$*:*
   (a) $\mathcal{M}^-(\Omega) = \{\cup_{a\in A}\{\Omega \setminus \{a\}\} : \emptyset \subseteq A \subseteq \Omega\}$.
   (b) $\mathcal{J}^-(\Omega) = \{\{A\} : \emptyset \subseteq A \subseteq \Omega\} = \{\mathcal{U}_{|A|,A} : \emptyset \subseteq A \subseteq \Omega\}$.

Note that $\{\emptyset\} \in \mathcal{J}^+(\Omega)$ and $\{\} \in \mathcal{M}^+(\Omega) \cap \mathcal{M}^-(\Omega)$

For each of the four lattices in Lemma 4, we next determine their join- and meet-irreducible elements. Thanks to Lemma 3, it is enough to find three of these families to determine the other five. This is summarized in the tables below, where we use the following notation: given $* \in \{+,-\}$ and $\star \subset \{0, \emptyset, c \geq 2\}$, we write $\mathcal{M}^*_\star(\Omega)$ instead of $\mathcal{M}(\mathrm{Clut}^*_\star(\Omega), \mathrm{Clut}^*(\Omega))$, and analogously for join-irreducibility. In each column, we have a clutter $\mathcal{A}$ of some irreducible elements and its transformation under either the blocker or the complementary operations.

$\mathcal{A}$	$\mathcal{M}^+_\emptyset(\Omega)$	$\mathcal{M}^+_{c\geq 2}(\Omega)$	$\mathcal{M}^+_{c\geq 2,\emptyset}(\Omega)$
$b(\mathcal{A})$	$\mathcal{J}^+_{c\geq 2}(\Omega)$	$\mathcal{J}^+_\emptyset(\Omega)$	$\mathcal{J}^+_{c\geq 2,\emptyset}(\Omega)$

$\mathcal{A}$	$\mathcal{M}^+_\emptyset(\Omega)$	$\mathcal{J}^+_\emptyset(\Omega)$
$\mathcal{A}^c$	$\mathcal{M}^-_0(\Omega)$	$\mathcal{J}^-_0(\Omega)$

In Proposition 15 below, we determine $\mathcal{M}^-_0(\Omega)$, $\mathcal{M}_{c\geq 2}(\Omega)$ and $\mathcal{M}^+_{c\geq 2,\emptyset}(\Omega)$. For ease of reference, in Subsection 4.3 we write down explicitly all meet- and join-irreducible elements for the sets $\mathbb{X}$ considered here and in the following section.

To state the result, we need to introduce some graph theory terminology (see [5, 21] for general references). A *graph* G is an ordered pair $(V(G), E(G))$ comprising a finite non-empty set $V(G)$ of *vertices* together with a (possibly empty) set $E(G)$ of *edges*, which are two-element subsets of $V(G)$. Let $\mathrm{Graph}(\Omega)$ be the collection of all graphs with vertex set Ω, and note that if $G \in \mathrm{Graph}(\Omega)$ then $E(G) \in \mathrm{Clut}_{c\geq 2}(\Omega)$.

Proposition 15.

1. $\mathcal{M}^+_{c\geq 2}(\Omega) = \{E(G) : G \in \mathrm{Graph}(\Omega)\}$, *that is, all clutters formed by sets of size two.*

2. $\mathcal{M}^+_{c\geq 2,\emptyset}(\Omega) = \mathcal{M}^+_{c\geq 2}(\Omega) \cap \mathrm{Clut}_\emptyset(\Omega)$*, that is, clutters $E(G)$ such that G has at least two edges and there is no vertex that belongs to all edges.*

3. $\mathcal{M}^-_0(\Omega) = \bigcup_{A\subseteq\Omega,|A|\geq 2}\{\{\Omega \setminus \{a\} : a \in A\}\}$.

Proof. We show that statement (1) follows directly from the definition of the operation $\sqcap^+$. Indeed, if $\Lambda = \Lambda_1 \sqcap^+ \cdots \sqcap^+ \Lambda_k$ and $A \in \Lambda$ has size 2, then $A \in \Lambda_1 \cap \cdots \cap \Lambda_k$, so all edge-sets of graphs belong to $\mathcal{M}^+_{c\geq 2}$. If $\Lambda \in \mathrm{Clut}_{0,c\geq 2}(\Omega)$ contains some set A with $|A| \geq 3$, it has a non-trivial meet-decomposition in $\mathrm{Clut}^+_{c\geq 2}(\Omega)$:

$$\Lambda = (\Lambda \setminus \{A\} \cup \{A \setminus \{x\}\}) \sqcap^+ (\Lambda \setminus \{A\} \cup \{A \setminus \{y\}\}),$$

where x, y are two different elements from A. Thus, statement (1) is proved.

Since $\mathrm{Clut}^+_\emptyset$ and $\mathrm{Clut}^-_0(\Omega)$ are upper intervals, statements (2) and (3) follow respectively from statement (1) and part (2.a) of Proposition 14. □

4.2 The meet and join irreducible elements for some non-lattice families of clutters

Recall that in Lemma 4 we identified ten families in $\mathrm{Clut}^+(\Omega)$ and $\mathrm{Clut}^-(\Omega)$ that are not lattices in general. We will focus on four out of these ten families, the ones with respect to the order $\leqslant^+$ (statement (2.a) in Lemma 4). They seem to us the most interesting ones and they are related to each other with the blocker operation, which greatly reduces the number of cases to deal with. We leave the other families to the interested reader.

The following table is an immediate consequence of the properties of the blocker operation (Lemmas 2 and 3); each column gives a collection $\mathcal{A}$ of clutters and its blocker $b(\mathcal{A}) = \{b(\Lambda) : \Lambda \in \mathcal{A}\}$.

$\mathcal{A}$	$\mathcal{M}_0^+(\Omega)$	$\mathcal{M}_{0,\emptyset}^+(\Omega)$	$\mathcal{M}_{0,c\geq 2}^+(\Omega)$	$\mathcal{M}_{0,c\geq 2,\emptyset}^+(\Omega)$
$b(\mathcal{A})$	$\mathcal{J}_0^+(\Omega)$	$\mathcal{J}_{0,c\geq 2}^+(\Omega)$	$\mathcal{J}_{0,\emptyset}^+(\Omega)$	$\mathcal{J}_{0,c\geq 2,\emptyset}^+(\Omega)$

Next, since $\mathrm{Clut}_\emptyset(\Omega)$ is an upper interval, by Lemma 7 it is enough to determine only $\mathcal{M}_0^+(\Omega)$ and $\mathcal{M}_{0,c\geq 2}^+(\Omega)$. For ease of reference, in Subsection 4.3 one can find the full list of meet- and join-irreducible elements for each of the four families $\mathrm{Clut}_0(\Omega)$, $\mathrm{Clut}_{0,\emptyset}(\Omega)$, $\mathrm{Clut}_{0,c\geq 2}(\Omega)$, $\mathrm{Clut}_{0,c\geq 2,\emptyset}(\Omega)$ in the lattice $\mathrm{Clut}^+(\Omega)$. We next find the sets $\mathcal{M}_0^+(\Omega)$ and $\mathcal{M}_{0,c\geq 2}^+(\Omega)$, from which all others will follow. Recall that $\Omega = \{a_1, \ldots, a_n\}$ and that $\mathfrak{S}_n$ denotes the symmetric group on $[n]$.

Proposition 16. *The family $\mathcal{M}_0^+(\Omega)$ is*

$$\{\{\{a_1\}, \ldots, \{a_n\}\}\} \cup \big\{\{\{a_{\sigma(1)}, a_{\sigma(2)}\}, \ldots, \{a_{\sigma(1)}, a_{\sigma(n)}\}\} : \sigma \in \mathfrak{S}_n\big\} \cup$$
$$\big\{\{\{a_{\sigma(1)}\}, \ldots, \{a_{\sigma(r)}\}, \{a_{\sigma(r+1)}, a_{\sigma(n)}\}, \ldots, \{a_{\sigma(n-1)}, a_{\sigma(n)}\}\} : \sigma \in \mathfrak{S}_n, 1 \leqslant r \leqslant n-2\big\}.$$

Proof. Let us denote by $\mathcal{F}$ the collection of clutters in the statement of the theorem. We apply the characterization in Theorem 12. We first show that $\mathcal{F} \subseteq \mathcal{M}_0^+(\Omega)$.

The clutter $\{\{a_1\}, \ldots, \{a_n\}\}$ is clearly meet $\mathrm{Clut}_0(\Omega)$-irreducible since it is meet-irreducible in the lattice $\mathrm{Clut}^+(\Omega)$.

For the other two families of clutters in $\mathcal{F}$, by symmetry we can suppose that σ is the identity permutation. Thus, let us assume first that

$$\begin{aligned}\{\{a_1, a_2\}, \ldots, \{a_1, a_n\}\} &= \Lambda_1 \sqcap^+ \Lambda_2 \sqcap^+ \cdots \sqcap^+ \Lambda_k \\ &= \min\{A_1 \cup \cdots \cup A_k : A_i \in \Lambda_i, 1 \leq i \leq k\},\end{aligned}$$

where $\Lambda_j \in \mathrm{Clut}_0(\Omega)$ for all $1 \leq j \leq k$.

By definition of the $\sqcap^+$ operation, note that if for each i the clutter Λ_i contained a set that did not contain a_1, then $\Lambda_1 \sqcap^+ \cdots \sqcap^+ \Lambda_k$ would contain a set without a_1, a contradiction. Hence, we can suppose that $\Lambda_1 = \{B_1, \ldots, B_t\}$ with $a_1 \in B_j$ for all $1 \leq j \leq t$.

Again by the definition of $\sqcap^+$, we can conclude that for all j and for all $2 \leq i \leq n$, the clutter Λ_j must contain one of the sets $\{a_1\}$, $\{a_i\}$ or $\{a_1, a_i\}$. Thus, Λ_1 contains the sets $\{a_1, a_i\}$ for all $2 \leq i \leq n$, and hence must be exactly the clutter $\{\{a_1, a_2\}, \ldots, \{a_1, a_n\}\}$, as needed.

Finally, we show that the clutter $\{\{a_1\}, \ldots, \{a_r\}, \{a_{r+1}, a_n\}, \ldots, \{a_{n-1}, a_n\}\}$ is meet $\mathrm{Clut}_0(\Omega)$-irreducible. Assuming again that

$$\{\{a_1\}, \ldots, \{a_r\}, \{a_{r+1}, a_n\}, \ldots, \{a_{n-1}, a_n\}\} = \Lambda_1 \sqcap^+ \Lambda_2 \sqcap^+ \cdots \sqcap^+ \Lambda_k,$$

we deduce that $\{\{a_1\}, \ldots, \{a_r\}\} \subset \Lambda_i$ for all $1 \leq i \leq k$, so we can restrict to the elements $\{a_{r+1}, \ldots, a_n\}$ and argue as in the previous case.

Next, we show that every $\Lambda \in \mathrm{Clut}_0(\Omega)$ satisfies the $\mathcal{F}$-upper avoidance property. By Lemma 13, it is enough to check that if $\Lambda' \in \mathcal{J}^+(\Omega)$ is join-irreducible and $\Lambda' \not\leqslant^+ \Lambda$, then there is $\Lambda_0 \in \mathcal{F}$ such that $\Lambda \leqslant^+ \Lambda_0$ and $\Lambda' \not\leqslant^+ \Lambda_0$.

Recall that a clutter $\Lambda' \in \mathcal{J}^+(\Omega)$ is of the form $\Lambda' = \{A'\}$ for some subset $A' \subseteq \Omega$. If $A' = \emptyset$, we can take $\Lambda_0 = \{\{a_1\}, \ldots, \{a_n\}\}$, since $\{\emptyset\}$ is the top element in $\mathrm{Clut}^+(\Omega)$ and covers $\{\{a_1\}, \ldots, \{a_n\}\}$, which in turn is greater than all other clutters in $\mathrm{Clut}^+(\Omega)$.

Hence, we assume that $A' \neq \emptyset$. The fact that $\Lambda' \not\leqslant^+ \Lambda$ implies that this set A' is such that $A \not\subseteq A'$ for all $A \in \Lambda$, and in particular $A' \neq \Omega$. We can suppose $A' = \{a_{r+1}, \ldots, a_n\}$, for $1 \leq r \leq n-1$.

We next claim that there is $a \in \Omega \setminus A'$ such that $\{a\} \notin \Lambda$. Indeed, if it were not the case, since $\Lambda \in \mathrm{Clut}_0(\Omega)$, it must be of the form $\Lambda = \{\{a_1\}, \ldots, \{a_r\}, A_{r+1}, \ldots, A_m\}$, where $A_{r+1}, \ldots, A_m$ are subsets of A', which contradicts the previous paragraph. So we can assume that $\{a_1\} \notin \Lambda$.

Let us now define a clutter $\Lambda_0 \in \mathcal{F}$ as follows.

$$\Lambda_0 = \begin{cases} \{\{a_1, a_2\}, \ldots, \{a_1, a_n\}\} & \text{if } r = 1, \\ \{\{a_2\}, \ldots, \{a_r\}, \{a_1, a_{r+1}\}, \ldots, \{a_1, a_n\}\} & \text{if } 1 < r \leq n-1. \end{cases}$$

We need to check that $\Lambda \leqslant^+ \Lambda_0$ and $\Lambda' \not\leqslant^+ \Lambda_0$. The latter is clear since $\Lambda' = \{A'\} = \{\{a_{r+1}, \ldots, a_n\}\}$ and the condition $\Lambda' \not\leqslant^+ \Lambda_0$ is equivalent to $A \not\subseteq A'$ for all $A \in \Lambda_0$.

As for $\Lambda \leqslant^+ \Lambda_0$, we need to check that for $A \in \Lambda$ there is $A_0 \in \Lambda_0$ such that $A_0 \subseteq A$. We consider two cases according to whether $A \not\subseteq A' \cup \{a_1\}$ or $A \subseteq A' \cup \{a_1\}$.

First, suppose $A \subseteq A' \cup \{a_1\}$. As $A \not\subseteq A'$, we have $a_1 \in A$, and since $\{a_1\} \notin \Lambda$, it must be $\{a_1\} \subsetneq A$. Thus, there is $a \in A \cap A'$ and we can take $A_0 = \{a_1, a\} \in \Lambda_0$ with $A_0 \subseteq A$.

If $A \not\subseteq A' \cup \{a_1\}$, there is $a \in A$ with $a \notin A' \cup \{a_1\}$. Hence we must be in the case $r \geqslant 2$ and $A_0 = \{a\} \in \Lambda_0$ satisfies $A_0 \subseteq A$. □

Recall from Proposition 15 that the meet irreducible elements in the lattice $\mathrm{Clut}_{c\geq 2}(\Omega)$ correspond to edge-sets of graphs. In the following theorem, this class is further restricted to graphs without isolated vertices, that is, graphs where every vertex belongs to at least one edge.

Proposition 17. *The family* $\mathcal{M}^+_{0,c\geq 2}(\Omega)$ *is*

$$\{E(G) : G \in \mathrm{Graph}(\Omega), G \textit{ does not have isolated vertices}\}.$$

Proof. Again, we denote by $\mathcal{F}$ the collection of clutters in the statement of the theorem and apply Theorem 12. Recall that the meet-irreducible elements in the lattice $\mathrm{Clut}_{c\geq 2}(\Omega)$ are the edge-sets of all graphs, so this implies directly that $\mathcal{F} \subseteq \mathcal{M}^+_{0,c\geq 2}(\Omega)$.

Next, to check that the $\mathcal{F}$-upper avoidance property is satisfied, we take $\Lambda \in \mathrm{Clut}_{0,c\geq 2}(\Omega)$ and $\Lambda' \in \mathcal{J}^+(\Omega)$ with $\Lambda' \not\leqslant^+ \Lambda$; we know that $\Lambda' = \{A'\}$ for some subset $A' \subseteq \Omega$, and as in the proof of Theorem 16 $A' \neq \Omega$. We must find $\Lambda_0 \in \mathcal{F}$ such that

$\Lambda \leqslant^+ \Lambda_0$ and $\Lambda' \nleqslant^+ \Lambda_0$. Recall that $\Lambda' \nleqslant^+ \Lambda$ is equivalent to the fact that no set of Λ is contained in A'. In particular, $A' \neq \Omega$

If $A' = \emptyset$, since $\{\emptyset\}$ is the top element in $\mathrm{Clut}^+(\Omega)$, it is enough to find $\Lambda_0 \in \mathcal{F}$ such that $\Lambda \leqslant^+ \Lambda_0$, and this follows since the maximal clutters in $\mathrm{Clut}_{0,c\geq 2}(\Omega)$ belong to $\mathcal{F}$. Indeed, for any $\Lambda_1 \in \mathrm{Clut}_{0,c\geq 2}(\Omega)$, the clutter $\Lambda_2 = \bigcup_{A\in\Lambda_1}\{\{a,b\} : a,b \in A, a \neq b\}$ is such that $\Lambda_1 \leqslant^+ \Lambda_2$ and Λ_2 is the clutter of edges of some graph. Moreover, if this graph has no isolated vertices, the clutter of edges has full support and thus belongs to $\mathrm{Clut}_{0,c\geq 2}(\Omega)$.

Otherwise, let $A' = \{a_r, \ldots, a_n\}$ with $2 \leq r \leq n$. Let G be the complete r-partite graph with parts $\{a_1\}, \{a_2\}, \ldots, \{a_{r-1}\}, \{a_r, \ldots, a_n\}$ and let $\Lambda_0 = E(G)$; that is, edges of G are all sets of the form $\{a, b\}$ where a, b belong to different parts. Clearly $\Lambda' = \{A'\} \nleqslant^+ \Lambda_0$, since no element of Λ_0 is a subset of A'. To check that $\Lambda \leqslant^+ \Lambda_0$, take $A \in \Lambda$; by assumption, A is not contained in A', so at least it contains some element a_i with $1 \leq i < r$. Since Λ has corank at least two, the set A contains another element a_j with $j \neq i$, and so the set $\{a_i, a_j\}$ of Λ_0 is contained in A, as needed. □

4.3 Summary of irreducible clutters

In Tables 1–4 below we present the collections of meet- and join-irreducible elements of the families of clutters considered in the previous two sections. In order to make these tables clear, we group the clutters that appear as meet- or join-irreducible into twelve families. As before, $\Omega = \{a_1, \ldots, a_n\}$.

On one hand, for $1 \leq m \leq n$ let us consider the clutters $\Delta_{1,m}$, $\Delta_{2,m}$ and $\Delta_{3,m}$ defined as follows (we stress that actually $\Delta_{2,m} = \Delta^c_{1,m}$ and $\Delta_{3,m} = b(\Delta_{1,m})$):

$$\begin{aligned}
\Delta_{1,m} &= \{\{a_1\}, \ldots, \{a_m\}\}\\
\Delta_{2,m} &= \{\Omega \setminus \{a_1\}, \ldots, \Omega \setminus \{a_m\}\}\\
\Delta_{3,m} &= \{\{a_1, \ldots, a_m\}\}
\end{aligned}$$

On the other hand, for $2 \leq m \leq n$ let us consider the clutters $\Delta_{4,m}$, $\Delta_{5,m}$ and $\Delta_{6,m}$ defined as follows (we underline that actually $\Delta_{5,m} = \Delta^c_{4,m}$ and that $\Delta_{6,m} = b(\Delta_{4,m})$):

$$\begin{aligned}
\Delta_{4,m} &= \{\{a_1, a_2\}, \ldots, \{a_1, a_m\}\}\\
\Delta_{5,m} &= \{\Omega \setminus \{a_1, a_2\}, \ldots, \Omega \setminus \{a_1, a_m\}\}\\
\Delta_{6,m} &= \{\{a_1\}, \{a_2, \ldots, a_m\}\}
\end{aligned}$$

In addition, for $1 \leq r \leq n-2$ let us consider the clutters $\Delta_{7,r}$, $\Delta_{8,r}$ and $\Delta_{9,r}$ defined as follows (here $\Delta_{8,r} = \Delta^c_{7,r}$ and $\Delta_{9,r} = b(\Delta_{7,r})$):

$$\begin{aligned}
\Delta_{7,r} &= \{\{a_1\}, \ldots, \{a_r\}, \{a_{r+1}, a_n\}, \ldots, \{a_{n-1}, a_n\}\}\\
\Delta_{8,r} &= \{\Omega \setminus \{a_1\}, \ldots, \Omega \setminus \{a_r\}, \Omega \setminus \{a_{r+1}, a_n\}, \ldots, \Omega \setminus \{a_{n-1}, a_n\}\}\\
\Delta_{9,r} &= \{\{a_1, \ldots, a_r, a_n\}, \{a_1, \ldots, a_r, a_{r+1}, \ldots, a_{n-1}\}\}
\end{aligned}$$

The last three families $\Delta_{10,G}$, $\Delta_{11,G}$ and $\Delta_{12,G}$ are indexed by graphs, and they satisfy $\Delta_{11,G} = \Delta^c_{10,G}$ and $\Delta_{12,G} = b(\Delta_{10,G})$. To properly define them, we need to introduce some more concepts. An *independent set* of a graph G is a set of vertices $I \subseteq V(G)$ such that $|e \cap I| \leq 1$ for every edge $e \in E(G)$. A *vertex cover* of G is a set of vertices $C \subseteq V(G)$ such that $|e \cap C| \geq 1$. Let $\mathrm{Cover}(G)$ be the set of all vertex covers of G, and $\mathrm{Ind}(G)$ be the set of all independent sets. It is clear that I is an independent set if and only if $V(G) \setminus I$ is a vertex cover, and that $b(E(G)) = \mathrm{minimal}(\mathrm{Cover}(G))$.

$$\begin{aligned}
\Delta_{10,G} &= \mathrm{minimal}(\mathrm{Cover}(G)) \\
\Delta_{11,G} &= \mathrm{maximal}(\mathrm{Ind}(G)) \\
\Delta_{12,G} &= E(G)
\end{aligned}$$

The next four tables summarize the results in Subsections 4.1 and 4.2. To further simplify the writing, for two clutters $\Lambda_1, \Lambda_2 \in \mathrm{Clut}(\Omega)$, we write $\Lambda_1 \cong \Lambda_2$ if both clutters differ only by a permutation of Ω. Also, all graphs G that appear in the tables have vertex-set $V(G) = \Omega$. A graph is a *star* if there is a vertex $u \in V(G)$ such that $E(G) = \{uv : v \in V(G), v \neq u\}$.

$\mathcal{M}^+(\Omega)$	$\Delta = \{\,\}$ or $\Delta \cong \Delta_{1,m}$ where $1 \leq m \leq n$
$\mathcal{M}^+_0(\Omega)$	$\Delta = \Delta_{1,n}$ or $\Delta \cong \Delta_{4,n}$ or $\Delta \cong \Delta_{7,r}$ where $1 \leq r \leq n-2$
$\mathcal{M}^+_\emptyset(\Omega)$	$\Delta \cong \Delta_{1,m}$ where $2 \leq m \leq n$
$\mathcal{M}^+_{c\geq 2}(\Omega)$	$\Delta \cong \Delta_{12,G}$ where G is a graph with vertex set $V(G) = \Omega$
$\mathcal{M}^+_{0,\emptyset}(\Omega)$	$\Delta = \Delta_{1,n}$ or $\Delta \cong \Delta_{7,r}$ where $1 \leq r \leq n-2$
$\mathcal{M}^+_{0,c\geq 2}(\Omega)$	$\Delta \cong \Delta_{12,G}$ where G is a graph without isolated vertices
$\mathcal{M}^+_{c\geq 2,\emptyset}(\Omega)$	$\Delta \cong \Delta_{12,G}$ where $G \in \mathrm{Graph}(\Omega)$ has no vertex belonging to all edges and $\lvert E(G)\rvert \geq 2$
$\mathcal{M}^+_{0,c\geq 2,\emptyset}(\Omega)$	$\Delta \cong \Delta_{12,G}$ where G has no isolated vertices, is not a star and $\lvert E(G)\rvert \geq 2$.

Table 1: The meet-irreducible elements of some families of clutters with respect to the order $\leqslant^+$.

5 Decomposition in families of clutters: clutters associated to discrete objects

Let us recall from the introduction that given a map $\Theta : \mathfrak{Obj}(\Omega) \to \mathrm{Clut}(\Omega)$ that assigns clutters to combinatorial objects, we consider $\mathrm{Clut}_\Theta(\Omega)$ as the set of those clutters that are in the image of Θ; we also say refer to these clutters as the ones that have Θ-realizations, and as Θ-clutters. The goal of this section is to explore if is it possible to

$\mathcal{M}^-(\Omega)$	$\Delta = \{\}$ or $\Delta \cong \Delta_{2,m}$ where $1 \le m \le n$
$\mathcal{M}_0^-(\Omega)$	$\Delta \cong \Delta_{2,m}$ where $2 \le m \le n$
$\mathcal{M}_\emptyset^-(\Omega)$	$\Delta = \Delta_{2,n}$ or $\Delta \cong \Delta_{5,n}$ or $\Delta \cong \Delta_{8,r}$ where $1 \le r \le n-2$
$\mathcal{M}_{0,\emptyset}^-(\Omega)$	$\Delta = \Delta_{2,n}$ or $\Delta \cong \Delta_{8,r}$ where $1 \le r \le n-2$

Table 2: The meet-irreducible elements of some families of clutters with respect to the order $\leqslant^-$.

$\mathcal{J}^+(\Omega)$	$\Delta = \{\emptyset\}$ or $\Delta \cong \Delta_{3,m}$ where $1 \le m \le n$.
$\mathcal{J}_0^+(\Omega)$	$\Delta = \Delta_{3,n}$ or $\Delta \cong \Delta_{6,n}$ or $\Delta \cong \Delta_{9,r}$ where $1 \le r \le n-2$.
$\mathcal{J}_\emptyset^+(\Omega)$	$\Delta \cong \Delta_{10,G}$
$\mathcal{J}_{c\ge 2}^+(\Omega)$	$\Delta \cong \Delta_{3,m}$ where $2 \le m \le n$.
$\mathcal{J}_{0,\emptyset}^+(\Omega)$	$\Delta \cong \Delta_{10,G}$ where G has no isolated vertices
$\mathcal{J}_{0,c\ge 2}^+(\Omega)$	$\Delta = \Delta_{3,n}$ or $\Delta \cong \Delta_{9,r}$ where $1 \le r \le n-2$
$\mathcal{J}_{c\ge 2,\emptyset}^+(\Omega)$	$\Delta \cong \Delta_{10,G}$ where where G has no vertex belonging to all edges and $\lvert E(G)\rvert \ge 2$
$\mathcal{J}_{0,c\ge 2,\emptyset}^+(\Omega)$	$\Delta \cong \Delta_{10,G}$ where G is not a star, has no isolated vertices and $\lvert E(G)\rvert \ge 2$

Table 3: The join-irreducible elements of some families of clutters with respect to the order $\leqslant^+$.

$\mathcal{J}^-(\Omega)$	$\Delta = \{\emptyset\}$ or $\Delta \cong \Delta_{3,m}$ where $1 \le m \le n$.
$\mathcal{J}_0^-(\Omega)$	$\Delta \cong \Delta_{11,G}$
$\mathcal{J}_\emptyset^-(\Omega)$	$\Delta = \{\emptyset\}$ or $\Delta \cong \Delta_{6,m}$ where $2 \le m \le n$
$\mathcal{J}_{0,\emptyset}^-(\Omega)$	$\Delta \cong \Delta_{11,G}$ where G has no isolated vertices

Table 4: The join-irreducible elements of some families of clutters with respect to the order $\leqslant^-$.

approximate and decompose any clutter by means of Θ-clutters for some special maps Θ related to matroids (Subsection 5.1), to graphs (Subsection 5.2) and to secret sharing schemes (Subsection 5.3). Before doing this, we will first formulate the problem in a precise way and as generally as possible.

Throughout, we write $\mathrm{Clut}_{\mathcal{P}}(\Omega)$ to denote the set of clutters on Ω that satisfy a given restriction $\mathcal{P}$. To help readability, we write $\mathcal{P}_0$ for the restriction of having full support, write $\mathcal{P}_\emptyset$ for the restriction of having empty intersection, and so on. If needed, we identify $\mathrm{Clut}(\Omega)$ with $\mathrm{Clut}_{\mathcal{P}}(\Omega)$ for the empty restriction $\mathcal{P} = \{\}$.

If $\mathcal{P}$ denotes a restriction on clutters, we could be interested in the Θ-clutters that satisfy that restriction. This leads us to the following three definitions:

$$\begin{aligned}
\mathfrak{C}_1(\Omega,\Theta,\mathcal{P}) &= \mathrm{Clut}_\Theta(\Omega)\cap\mathrm{Clut}_\mathcal{P}(\Omega)\\
\mathfrak{C}_2(\Omega,\Theta,\mathcal{P}) &= \Big(\bigcup_{\Omega'\subseteq\Omega}\mathrm{Clut}_\Theta(\Omega')\Big)\cap\mathrm{Clut}_\mathcal{P}(\Omega)\\
\mathfrak{C}_3(\Omega,\Theta,\mathcal{P}) &= \bigcup_{\Omega'\subseteq\Omega}\big(\mathrm{Clut}_\Theta(\Omega')\cap\mathrm{Clut}_\mathcal{P}(\Omega')\big)
\end{aligned}$$

Before continuing, we want to highlight the relationship between these three definitions. Specifically that we have the trivial inclusions $\mathfrak{C}_1(\Omega,\Theta,\mathcal{P})\subseteq\mathfrak{C}_2(\Omega,\Theta,\mathcal{P})\subseteq\mathrm{Clut}_\mathcal{P}(\Omega)\subseteq\mathrm{Clut}(\Omega)$ and $\mathfrak{C}_1(\Omega,\Theta,\mathcal{P})\subseteq\mathfrak{C}_3(\Omega,\Theta,\mathcal{P})\subseteq\mathrm{Clut}(\Omega)$, but in general no inclusion holds between $\mathfrak{C}_2(\Omega,\Theta,\mathcal{P})$ and $\mathrm{Clut}_\mathcal{P}(\Omega)$, or between $\mathfrak{C}_2(\Omega,\Theta,\mathcal{P})$ and $\mathfrak{C}_3(\Omega,\Theta,\mathcal{P})$. The reason is that, in general, the inclusion $\Omega'\subseteq\Omega$ does not imply $\mathrm{Clut}_\mathcal{P}(\Omega')\subseteq\mathrm{Clut}_\mathcal{P}(\Omega)$ (for instance, for the property of having full support we have $\mathrm{Clut}_0(\Omega')\not\subseteq\mathrm{Clut}_0(\Omega)$ whenever $\Omega'\subset\Omega$, whereas $\mathrm{Clut}_\emptyset(\Omega')\subseteq\mathrm{Clut}_\emptyset(\Omega)$ and $\mathrm{Clut}_{c\geq2}(\Omega')\subseteq\mathrm{Clut}_{c\geq2}(\Omega)$).

With these notations, a very general framework would be to consider two restrictions $\mathcal{P}_1$ and $\mathcal{P}_2$ and aiming at decomposing clutters that satisfy $\mathcal{P}_1$ in terms of Θ-clutters that satisfy $\mathcal{P}_2$. Therefore, in order to answer this question, we would apply our results with

$$\begin{aligned}
&\mathbb{L}=\mathrm{Clut}^+(\Omega)\ \text{ or }\ \mathbb{L}=\mathrm{Clut}^-(\Omega),\\
&\mathbb{X}=\mathrm{Clut}_{\mathcal{P}_1}(\Omega),\\
&\Sigma=\mathfrak{C}_1(\Omega,\Theta,\mathcal{P}_2)\ \text{ or }\ \Sigma=\mathfrak{C}_2(\Omega,\Theta,\mathcal{P}_2)\ \text{ or }\ \Sigma=\mathfrak{C}_3(\Omega,\Theta,\mathcal{P}_2).
\end{aligned}$$

Thus, from Theorem 6, to guarantee that every element of $\mathbb{X}$ has a Σ-meet decomposition we need to check the inclusion $\mathcal{M}(\mathbb{X}\cup\Sigma,\mathbb{L})\subseteq\Sigma$, and to guarantee a Σ-join decomposition we need the inclusion $\mathcal{J}(\mathbb{X}\cup\Sigma,\mathbb{L})\subseteq\Sigma$. However, the collections of irreducible elements computed in Section 4 are of the form $\mathcal{J}(\mathbb{X},\mathbb{L})$ and $\mathcal{M}(\mathbb{X},\mathbb{L})$. So in order to apply our results we need that $\Sigma\subseteq\mathbb{X}$.

To guarantee this, in the examples we study in the following subsections we always take $\mathcal{P}_1=\mathcal{P}_2$ and, specifically, we consider the following three situations:

1. First we consider the case without restrictions. We then have $\mathfrak{C}_1(\Omega,\Theta,\mathcal{P}_2)=\mathrm{Clut}_\Theta(\Omega)$ and $\mathfrak{C}_2(\Omega,\Theta,\mathcal{P}_2)=\mathfrak{C}_3(\Omega,\Theta,\mathcal{P}_2)=\bigcup_{\Omega'\subseteq\Omega}\mathrm{Clut}_\Theta(\Omega')$. Thus

$$\begin{aligned}
&\mathbb{L}=\mathrm{Clut}^+(\Omega)\ \text{ o }\ \mathbb{L}=\mathrm{Clut}^-(\Omega)\\
&\mathbb{X}=\mathrm{Clut}(\Omega)\\
&\Sigma=\mathrm{Clut}_\Theta(\Omega)\ \text{ o }\ \Sigma=\bigcup_{\Omega'\subseteq\Omega}\mathrm{Clut}_\Theta(\Omega')
\end{aligned}$$

   and indeed $\Sigma\subseteq\mathbb{X}=\mathbb{L}$, so that $\mathcal{M}(\mathbb{X}\cup\Sigma,\mathbb{L})=\mathcal{M}(\mathbb{L})$ and $\mathcal{J}(\mathbb{X}\cup\Sigma,\mathbb{L})=\mathcal{J}(\mathbb{L})$.

2. Next we consider the case in which $\mathcal{P}_1=\mathcal{P}_2=\mathcal{P}$ where $\mathcal{P}$ is a property satisfying $\mathrm{Clut}_\mathcal{P}(\Omega')\cap\mathrm{Clut}_\mathcal{P}(\Omega)=\emptyset$ for $\Omega'\subsetneq\Omega$. This happens for $\mathcal{P}=\mathcal{P}_0$, $\mathcal{P}_{0,\emptyset}$,

$\mathcal{P}_{0,c\geq 2}$, $\mathcal{P}_{0,\emptyset,c\geq 2}$. We have that $\mathfrak{C}_1(\Omega,\Theta,\mathcal{P}) = \mathfrak{C}_2(\Omega,\Theta,\mathcal{P})$ but we cannot guarantee $\mathfrak{C}_3(\Omega,\Theta,\mathcal{P}) \subseteq \mathrm{Clut}_{\mathcal{P}}(\Omega)$. We thus have

$$\begin{aligned}
&\mathbb{L} = \mathrm{Clut}^+(\Omega) \text{ or } \mathbb{L} = \mathrm{Clut}^-(\Omega)\\
&\mathbb{X} = \mathrm{Clut}_{\mathcal{P}}(\Omega)\\
&\Sigma = \mathrm{Clut}_{\Theta}(\Omega) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega) \text{ or } \Sigma = \bigcup_{\Omega'\subseteq\Omega} \big(\mathrm{Clut}_{\Theta}(\Omega') \cap \mathrm{Clut}_{\mathcal{P}}(\Omega')\big),
\end{aligned}$$

and hence the inclusion $\Sigma \subseteq \mathbb{X}$ holds if and only if $\Sigma = \mathrm{Clut}_{\Theta}(\Omega) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega)$, so we will only consider this collection Σ.

3. Finally, we consider $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}$ where $\mathcal{P}$ satisfies $\mathrm{Clut}_{\mathcal{P}}(\Omega') = \mathrm{Clut}(\Omega') \cap \mathrm{Clut}_{\mathcal{P}}(\Omega)$ for $\Omega' \subsetneq \Omega$. This is the case for $\mathcal{P} = \mathcal{P}_{\emptyset}$, $\mathcal{P}_{c\geq 2}$, $\mathcal{P}_{\emptyset,c\geq 2}$. Here we have $\mathfrak{C}_1(\Omega,\Theta,\mathcal{P}) \subseteq \mathfrak{C}_2(\Omega,\Theta,\mathcal{P}) = \mathfrak{C}_3(\Omega,\Theta,\mathcal{P}) \subseteq \mathrm{Clut}_{\mathcal{P}}(\Omega)$. As before, we take

$$\begin{aligned}
&\mathbb{L} = \mathrm{Clut}^+(\Omega) \text{ o } \mathbb{L} = \mathrm{Clut}^-(\Omega)\\
&\mathbb{X} = \mathrm{Clut}_{\mathcal{P}}(\Omega)\\
&\Sigma = \mathrm{Clut}_{\Theta}(\Omega) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega) \text{ o } \Sigma = \Big(\bigcup_{\Omega'\subseteq\Omega} \mathrm{Clut}_{\Theta}(\Omega')\Big) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega),
\end{aligned}$$

but now $\Sigma \subseteq \mathbb{X}$, so we can consider both families Σ.

In the three following sections we consider different families of combinatorial objects and several maps Θ that associate clutters to them. For each one, we study whereas the inclusion $\mathcal{I}(\mathbb{X},\mathbb{L}) \subseteq \Sigma$ holds in the cases above, where we use $\mathcal{I}$ to denote either $\mathcal{M}$ or $\mathcal{J}$. We will look only at the families $\mathcal{I}(\mathbb{X},\mathbb{L})$ that we have identified in Section 4, so in particular we will not deal with some of the restrictions in the order $\leqslant^-$.

In some of the cases above the set Σ is of the form $\mathrm{Clut}_{\Theta}(\Omega) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega)$, but we only need to check that $\mathcal{I}(\mathbb{X},\mathbb{L}) \subseteq \mathrm{Clut}_{\Theta}(\Omega)$ since $\mathcal{I}(\mathbb{X},\mathbb{L}) \subseteq \mathrm{Clut}_{\mathcal{P}}(\Omega)$ holds trivially.

Note also that in several cases the sets $\mathcal{I}(\mathbb{X},\mathbb{L})$ identified in Section 4 include the clutters $\{\}, \emptyset, \{\Omega\}$, which correspond to the bottom or top elements of $\mathrm{Clut}^+(\Omega)$ and $\mathrm{Clut}^-(\Omega)$. It is easy to check that whenever these three clutters belong to some family $\mathcal{I}(\mathbb{X},\mathbb{L})$ they are only needed to guarantee their own decomposition. Since in general one is not interested in decomposing these particular clutters, in the examples they can be safely omitted from the collections $\mathcal{I}(\mathbb{X},\mathbb{L})$, and we will do so from now on without further mention.

Before moving on to the examples, we finish this general introduction with some open questions related to this section and to the whole paper.

First, we mention that all results in this section given in which cases a clutter can be decomposed in terms of $\Theta-$clutters, but all the results have an existential nature. In the paper [13] we gave some algorithms to compute matroidal completions in the case of the restriction $\mathcal{P}$ being empty, but we do not know if such algorithms behave well when considering other restrictions. We also do not know of any algorithm for other kinds of combinatorial clutters.

For a clutter Λ and a map $\Theta : \mathfrak{Obj}(\Omega) \to \mathrm{Clut}(\Omega)$, having just one optimal Θ-completion is equivalent to Λ being a Θ-clutter (Corollary 9). It might be worth investigating if those elements having exactly two optimal Θ-completions have an interesting characterization, as they should correspond with the clutters that are closest to being Θ-clutters. These could happen for a given choice of order ($\leqslant^+$ or $\leqslant^-$) and type of decomposition (meet or join), or for all of the four choices. More generally, even in the case that there were more than two optimal Θ-completions, it could be that there is a Θ-decomposition using only two of those optimal completions, so it would be also interesting to characterize these clutters.

Finally, the restrictions that we have considered in this paper (like having full support or empty intersection) are very particular to clutters and it is not clear whether they can be expressed in pure lattice theoretic terms. We would be interested in knowing for which arbitrary distributive lattices one can define a notion akin to having full support.

5.1 Matroidal clutters

Matroids are discrete structures that abstract and generalize the notion of linear independence in vector spaces. Namely a *matroid* M is a pair $\mathcal{M} = (\Omega, Ind)$ where Ω is a finite set (called the ground set) and Ind is a non-empty and monotone decreasing family of subsets of Ω (called the *independent sets*) satisfying the following property: if $I_1, I_2 \in Ind$ and $|I_1| > |I_2|$ then there exists $x \in I_1 \setminus I_2$ such that $I_2 \cup \{x\} \in Ind$ (this is sometimes called the augmentation property or the independent set exchange property). We refer to [16] as a general reference in matroid theory. Let $\mathrm{Mat}(\Omega)$ denote the collection of all matroids with ground set Ω.

There are several maps $\Theta : \mathrm{Mat}(\Omega) \to \mathrm{Clut}(\Omega)$ from matroids to clutters. Here we will only consider the ones that arise from taking bases, circuits and hyperplanes, which we next define. A *basis* of a matroid is a maximal independent set, a *circuit* is a minimal dependent set and a *hyperplane* is a maximal set not containing any basis. So we can take $\Theta(M)$ to be one of

$$\begin{aligned}
\Theta(M) &= \mathcal{B}(M) = \{B \subseteq \Omega \,:\, B \text{ is a basis of } M\},\\
\Theta(M) &= \mathcal{C}(M) = \{C \subseteq \Omega \,:\, C \text{ is a circuit of } M\},\\
\Theta(M) &= \mathcal{H}(M) = \{H \subseteq \Omega \,:\, H \text{ is a hyperplane of } \mathcal{M}\}.
\end{aligned}$$

We recall the following characterizations:

- *Basis exchange property* ([16, Cor. 1.2.5]: a clutter $\Delta \neq \{\}$ is the clutter of bases of a matroid if and only if whenever B_1, B_2 are elements of Δ and $x \in B_1 \setminus B_2$, then there is $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \Delta$.

- *Circuit elimination property* ([16, Cor. 1.1.5]: a clutter $\Delta \neq \{\emptyset\}$ is the clutter of circuits of a matroid if and only if whenever C_1 and C_2 are distinct members of Δ and $x \in C_1 \cap C_2$, then there is some member C_3 of Δ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{x\}$.

- *Hyperplane characterization* ([16, Prop. 2.1.18]): a clutter $\Delta \neq \{\Omega\}$ is a hyperplane clutter if and only if $\Delta \neq \{\Omega\}$ and if H_1, H_2 are distinct members of Δ and $x \in \Omega \setminus (H_1 \cup H_2)$, then there is $H_3 \in \Delta$ such that $H_3 \supseteq (H_1 \cap H_2) \cup \{x\}$.

It is easy to see from these characterizations that all bases have the same number of elements, or, in other words, that if Δ is a clutter of basis then $\mathrm{rk}(\Delta) = \mathrm{crk}(\Delta)$. Also, clutters of circuits belong to $\mathrm{Clut}_{\emptyset}(\Omega)$, unless they have just one element, and clutters of hyperplanes belong to $\mathrm{Clut}_0(\Omega)$, again unless they have just one element. Also, the clutter $\mathcal{B}(M)$ belongs to $\mathrm{Clut}_0(\Omega)$ if and only if M has no loops (that is, circuits of size 1), and it belongs to $\mathrm{Clut}_{\emptyset}(\Omega)$ if and only if it has no coloops (which are precisely the elements that belong to all bases). Similarly, $\mathcal{C}(M)$ belongs to $\mathrm{Clut}_0(\Omega)$ if and only if M has no coloops, and it belongs to $\mathrm{Clut}_{0,c\geq 2}(\Omega)$ if and only if it has no loops. Clutters of hyperplanes belong to $\mathrm{Clut}_{\emptyset}(\Omega)$ if and only if the matroid has no loops, whereas they belong to $\mathrm{Clut}_{0,c\geq 2}(\Omega)$ for all matroids of rank at least three.

Let us also recall the well-known relation between these families of clutters and the dual matroid M^* (the first identity is the definition of the dual matroid):

$$(\mathcal{B}(M))^c = \mathcal{B}(M^*), \quad b(\mathcal{B}(M)) = \mathcal{C}(M^*), \quad (\mathcal{C}(M))^c = \mathcal{H}(M^*).$$

Using the setting introduced in this section, we want to study when it is possible to decompose and approximate clutters on Ω that satisfy a restriction $\mathcal{P}$ by means of clutters of basis, circuits or hyperplanes satisfying that same restriction $\mathcal{P}$. Let us note that this problem was partially studied in [11, 12, 13] (see also the references in [13] for other works relating clutters and matroids). All these papers correspond to taking the restriction $\mathcal{P}$ to be empty. In [12] results were developed for matroids representable over a field. Whereas here we consider arbitrary matroids, it is easy to adapt the tables we present below to the representable case, and naturally one could consider any other family of matroids of interest.

For the sake of completeness, we reproduce the case where the restriction $\mathcal{P}$ is empty. Recall that we need to check whether the corresponding meet- or join-irreducible elements belong to the corresponding family Σ of matroidal clutters. We summarize the results in Table 5 (we present the proofs of all tables at the end of this subsection). Part of the results in this table were already given in [13]. Let us note that in that article we also gave algorithms for finding the optimal matroidal completions of a given clutter, a topic that we do not touch in the present article.

As mentioned above, the restrictions of having full support, empty interesection or corank at least two arise naturally in the setting of matroidal clutters. We next study the approximation and decomposition problems with $\mathcal{P} = \mathcal{P}_0$, $\mathcal{P}_{\emptyset}$, $\mathcal{P}_{c\geq 2}$, $\mathcal{P}_{0,\emptyset}$, $\mathcal{P}_{0,c\geq 2}$, $\mathcal{P}_{c\geq 2,\emptyset}$, $\mathcal{P}_{0,c\geq 2,\emptyset}$. The results are gathered in Tables 6 and 7. Note that some cases are not studied with respect to the order $\leqslant^-$ since we do not have the description of the corresponding irreducible elements.

the inclusion $\mathcal{I}(\mathbb{L}) \subseteq \Sigma$ where $\Sigma = \mathrm{Clut}_\Theta(\Omega)$ or $\Sigma = \bigcup_{\Omega' \subseteq \Omega} \mathrm{Clut}_\Theta(\Omega')$			
	$\Theta = \mathcal{B}$	$\Theta = \mathcal{C}$	$\Theta = \mathcal{H}$
$\mathcal{I}(\mathbb{L}) = \mathcal{M}^+$	(✓ , ✓)	(✓ , ✓)	(× , ✓)
$\mathcal{I}(\mathbb{L}) = \mathcal{M}^-$	(✓ , ✓)	(× , ×)	(✓ , ✓)
$\mathcal{I}(\mathbb{L}) = \mathcal{J}^+ = \mathcal{J}^-$	(✓ , ✓)	(✓ , ✓)	(✓ , ✓)

Table 5: Approximation and decomposition of clutters by Θ-matroidal clutters, the general case. In each cell, the pairs in $\{\checkmark, \times\}^2$ indicate if the inclusion $\mathcal{I}(\mathbb{L}) \subseteq \Sigma$ holds (the first component for $\Sigma = \mathrm{Clut}_\Theta(\Omega)$ and the second for $\Sigma = \bigcup_{\Omega' \subseteq \Omega} \mathrm{Clut}_\Theta(\Omega')$).

the inclusion $\mathcal{I} = \mathcal{I}(\mathbb{X}, \mathbb{L}) \subseteq \Sigma$ where $\mathbb{X} = \mathrm{Clut}_\mathcal{P}(\Omega)$ and $\Sigma = \mathrm{Clut}_\Theta(\Omega) \cap \mathrm{Clut}_\mathcal{P}(\Omega)$ and where $\mathcal{P} = \mathcal{P}_0,\ \mathcal{P}_{0,\emptyset},\ \mathcal{P}_{0,c\geq 2},\ \mathcal{P}_{0,c\geq 2,\emptyset}$							
	$\Theta = \mathcal{B}$	$\Theta = \mathcal{C}$	$\Theta = \mathcal{H}$		$\Theta = \mathcal{B}$	$\Theta = \mathcal{C}$	$\Theta = \mathcal{H}$
$\mathcal{I} = \mathcal{M}^+_0$	×	×	×	$\mathcal{I} = \mathcal{J}^+_0$	×	×	×
$\mathcal{I} = \mathcal{M}^+_{0,\emptyset}$	×	×	×	$\mathcal{I} = \mathcal{J}^+_{0,\emptyset}$	×	×	×
$\mathcal{I} = \mathcal{M}^+_{0,c\geq 2}$	×	×	×	$\mathcal{I} = \mathcal{J}^+_{0,c\geq 2}$	×	×	×
$\mathcal{I} = \mathcal{M}^+_{0,c\geq 2,\emptyset}$	×	×	×	$\mathcal{I} = \mathcal{J}^+_{0,c\geq 2,\emptyset}$	×	×	×
$\mathcal{I} = \mathcal{M}^-_0$	✓	×	✓	$\mathcal{I} = \mathcal{J}^-_0$	×	×	×
$\mathcal{I} = \mathcal{M}^-_{0,\emptyset}$	×	×	×	$\mathcal{I} = \mathcal{J}^-_{0,\emptyset}$	×	×	×

Table 6: Approximation and decomposition of clutters by Θ-matroidal clutters. Case of clutters with full support and with empty intersection or corank greater than or equal to two.

Before turning to the proof of Tables 5–7, we remark that the fact that most entries are in the negative does not imply that there are no decompositions, only that they cannot be guaranteed in all cases. In this situation, if one is interested in decomposing

the inclusion $\mathcal{I} = \mathcal{I}(\mathbb{X}, \mathbb{L}) \subseteq \Sigma$ where $\mathbb{X} = \mathrm{Clut}_{\emptyset}(\Omega)$ and $\Sigma = \mathrm{Clut}_{\Theta}(\Omega) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega)$ or $\Sigma = \left(\bigcup_{\Omega' \subseteq \Omega} \mathrm{Clut}_{\Theta}(\Omega')\right) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega)$ for $\mathcal{P} = \mathcal{P}_{\emptyset}, \mathcal{P}_{c\geq 2}, \mathcal{P}_{c\geq 2,\emptyset}$							
	$\Theta = \mathcal{B}$	$\Theta = \mathcal{C}$	$\Theta = \mathcal{H}$		$\Theta = \mathcal{B}$	$\Theta = \mathcal{C}$	$\Theta = \mathcal{H}$
$\mathcal{I} = \mathcal{M}^{+}_{\emptyset}$	$(\checkmark, \checkmark)$	$(\checkmark, \checkmark)$	$(\times, \checkmark)$	$\mathcal{I} = \mathcal{J}^{+}_{\emptyset}$	$(\times, \times)$	$(\times, \times)$	$(\times, \times)$
$\mathcal{I} = \mathcal{M}^{+}_{c\geq 2}$	$(\times, \times)$	$(\times, \times)$	$(\times, \times)$	$\mathcal{I} = \mathcal{J}^{+}_{c\geq 2}$	$(\checkmark, \checkmark)$	$(\checkmark, \checkmark)$	$(\times, \checkmark)$
$\mathcal{I} = \mathcal{M}^{+}_{c\geq 2,\emptyset}$	$(\times, \times)$	$(\times, \times)$	$(\times, \times)$	$\mathcal{I} = \mathcal{J}^{+}_{c\geq 2,\emptyset}$	$(\times, \times)$	$(\times, \times)$	$(\times, \times)$
$\mathcal{I} = \mathcal{M}^{-}_{\emptyset}$	$(\times, \times)$	$(\times, \times)$	$(\times, \times)$	$\mathcal{I} = \mathcal{J}^{-}_{\emptyset}$	$(\times, \times)$	$(\checkmark, \checkmark)$	$(\times, \checkmark)$

Table 7: Approximation and decomposition of clutters by Θ-matroidal clutters. Case of clutters with empty intersection and corank gretaer than or equal to two. In each cell, the pairs in $\{\checkmark, \times\}^2$ indicate if the inclusion $\mathcal{I}(\mathbb{X}, \mathbb{L}) \subseteq \Sigma$ holds (the first component for $\Sigma = \mathrm{Clut}_{\Theta}(\Omega) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega)$ and the second for $\Sigma = (\bigcup_{\Omega' \subseteq \Omega} \mathrm{Clut}_{\Theta}(\Omega')) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega)$.

a given clutter Λ in terms of clutters of Σ one needs to check if the corresponding avoidance property is fulfilled. Just to mention an example on the positive side, the clutter $\Lambda = \{12, 13, 14, 23, 24\}$ (brackets are omitted for ease of reading) is not a clutter of circuits, but it can be written as a join of clutters of circuits with full support and corank at least two: $\Lambda = \{13, 24\} \sqcup^{+} \{23, 14\} \sqcup^{+} \{12, 134, 234\}$. Note that if we drop the restriction of having full support there are other decompositions, for instance, $\Lambda = \{12, 13, 23\} \sqcup^{+} \{12, 13, 24\}$.

Proof of the results in Table 5, Table 6 and Table 7

To prove the results in the tables, we need to check which of the twelve families of clutters listed in Subsection 4.3 admit realizations as clutters of bases, circuits or hyperplanes, and whether this realizations can be achieved with ground set Ω or just a subset of it.

Let us note that if Δ belongs to $\mathrm{Clut}_{\mathcal{B}}(\Omega')$ (that is, if it is a clutter of basis with ground set Ω'), then it also belongs to $\mathrm{Clut}_{\mathcal{B}}(\Omega)$ for any $\Omega \supseteq \Omega'$; the same holds for clutters of circuits, but not for clutters of hyperplanes. Hence, in the columns corresponding to $\mathcal{B}$ and $\mathcal{C}$ in Tables 5 and 7, both components are always equal. Also, as mentioned in the introduction to this section, the clutters $\{\}, \{\emptyset\}$ and $\{\Omega\}$ are not considered.

All the clutters $\Delta_{1,m}$, $\Delta_{2,m}$ i $\Delta_{3,m}$ for $1 \leq m \leq n$ verify the basis exchange property. As for the circuit elimination property, it is satisfied by clutters $\Delta_{1,m}$ for all m, clutters $\Delta_{2,m}$ only for $m = n$ and clutters $\Delta_{3,m}$ for all m. Finally, the hyperplane characterization holds for clutters $\Delta_{1,m}$ only if $m = n$, clutters $\Delta_{2,m}$ for all m and clutters $\Delta_{3,m}$ for $m \neq n$.

Next, let us look at the clutters $\Delta_{4,m}$, $\Delta_{5,m}$ and $\Delta_{6,m}$ with $2 \leq m \leq n$. They basis condition is verified by $\Delta_{4,m}$ for all m, by $\Delta_{5,m}$ for all m and by the clutter $\Delta_{6,2}$. The only ones that are clutters of circuits are $\Delta_{4,2}$, $\Delta_{5,n}$ and $\Delta_{6,m}$ for all m. Finally, the hyperplane condition is satisfied only by $\Delta_{4,n}$, $\Delta_{5,2}$ and $\Delta_{6,n}$.

As for the clutters $\Delta_{7,r}$, $\Delta_{8,r}$ and $\Delta_{9,r}$ with $1 \leq r \leq n-2$ the only one that is a clutter of basis is $\Delta_{9,n-2}$. The ones that are clutters of circuits are $\Delta_{7,n-2}$ and $\Delta_{8,n-2}$, and the ones that are clutters of hyperplanes are $\Delta_{7,n-2}$, $\Delta_{8,n-2}$ and $\Delta_{9,r}$ for all r.

Finally, we obseve that not all clutters of the form $\Delta_{10,G}, \Delta_{11,G}, \Delta_{12,G}$ are Θ-clutters. For instance, let $n \geq 4$ and let us consider the graph G with vertex set $V(G) = \Omega = \{a_1, \ldots, a_n\}$ and with edges $E(G) = \{\{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}\} \cup \{\{a_3, a_4\}, \{a_3, a_5\}, \ldots, \{a_3, a_n\}\}$. Then the clutter of its minimal vertex cover sets is the clutter $\Delta_{10,G} = \text{minimal}(\text{Cover}(G)) = \{\{a_1, a_3\}, \{a_2, a_3\}, \{a_1, a_2, a_4, \ldots, a_n\}\}$ and the clutter of its maximal independent sets is the clutter $\Delta_{11,G} = \text{maximal}(\text{Ind}(G)) = \{\{a_3\}, \{a_1, a_4, \ldots, a_n\}, \{a_2, a_4, \ldots, a_n\}\}$. So, for this graph we have that the clutters $\Delta_{10,G}$, $\Delta_{11,G}$ and $\Delta_{12,G} = E(G)$ are not clutters neither of basis, nor of circuits, nor of hyperplanes. This completes the proof of the results in Table 5, Table 6 and Table 7.

5.2 Graphic clutters: independence, vertex cover and dominating sets

In this subsection we consider as set of combinatorial objects the set $\mathfrak{Obj}(\Omega) = \text{Graph}(\Omega)$ whose elements are all the graphs G with vertex set $V(G) = \Omega$.

There are many different criteria Γ that can be considered in order to obtain a family of subsets of vertices $\Gamma(G)$ associated with a given graph G. Some of these criteria Γ are defined directly from properties of the vertices and edges of the graph. In any case, for a given criterion Γ, the associated family of subsets of vertices $\Gamma(G)$ is, in general, univocally determined by a clutter $\Theta_\Gamma(G)$ on the set of vertices of the graph. So for a given criterion Γ we can consider its corresponding map

$$\begin{aligned} \Theta_\Gamma : \text{Graph}(\Omega) &\to \text{Clut}(\Omega) \\ G &\mapsto \Theta_\Gamma(G) \end{aligned}$$

In this subsection we analyse the problem of approximation and decomposition of clutters by means of Θ_Γ-clutters for some criteria Γ that somehow involve concepts related with the control of the vertices of the graph. Namely we focus our attention on the dominating sets of vertices of a graph, on the independent set of vertices and on the vertex cover subsets (we stress that there are more criteria in this sense, for instance the forcing sets of vertices and for the immune sets of any forcing process acting on the vertices of the graph). Before continuing let us recall the definition of each of these three concepts (two of which were already introduced in Section 4.3, but for completeness we include all the definitions here).

Let Ω be a finite set and let G be a graph with set of vertices $V(G) = \Omega$ and edges $E(G)$. A *dominating set* of G is a subset $D \subseteq \Omega$ such that every vertex not in D is adjacent to at least one member of D; an *independent set* of G is a set of vertices $I \subseteq \Omega$ such that no two of them are adjacent; and a *vertex cover* is set of vertices $C \subseteq \Omega$ such that for every edge $\{x, y\} \in E(G)$, either $x \in C$ or $y \in C$.

Therefore, now we can consider the following three families of subsets $\Gamma(G)$ associated with a graph G:

$$\Gamma(G) = \mathrm{Dom}(G) = \{D \subseteq \Omega \, : \, D \text{ is a dominating set } G\},$$
$$\Gamma(G) = \mathrm{Ind}(G) = \{I \subseteq \Omega \, : \, I \text{ is an independent set of } G\},$$
$$\Gamma(G) = \mathrm{Cover}(G) = \{C \subseteq \Omega \, : \, C \text{ is a vertex cover of } G\},$$

From the definitions it is clear that $\mathrm{Dom}(G)$ and $\mathrm{Cover}(G)$ are monotone increasing families of subsets, while $\mathrm{Ind}(G)$ is a monotone decreasing family of subsets. Therefore the families $\Gamma(G) = \mathrm{Dom}(G)$ and $\Gamma(G) = \mathrm{Cover}(G)$ are univocally determined by the clutter of their inclusion minimal elements $\Theta_\Gamma(G) = \mathrm{minimal}(\Gamma(G))$; the family $\Gamma(G) = \mathrm{Ind}(G)$ is determined by the clutter of its inclusion maximal elements $\Theta_\Gamma(G) = \mathrm{maximal}(\Gamma(G))$. We also note the following fact, that we use sometimes in the proofs: a maximal independent set of vertices is a minimal dominating set of the graph.

Thus, for studying decomposition problems in terms of vertex-cover and dominating sets it is more natural to consider the order $\leqslant^+$, and the order $\leqslant^-$ is more natural for independent sets. However, in the sequel we study both orders for all families.

We first start by considering no restriction on the family of clutters, that is, to take the property $\mathcal{P}$ to be empty. In this case, the answer is given by Table 8 below, whose proof appears at the end of this subsection.

the inclusion $\mathcal{I}(\mathbb{L}) \subseteq \Sigma$ where $\Sigma = \mathrm{Clut}_{\Theta_\Gamma}(\Omega)$ or $\Sigma = \bigcup_{\Omega' \subseteq \Omega} \mathrm{Clut}_{\Theta_\Gamma}(\Omega')$			
	$\Gamma = \mathrm{Dom}$	$\Gamma = \mathrm{Ind}$	$\Gamma = \mathrm{Cover}$
$\mathcal{I}(\mathbb{L}) = \mathcal{M}^+$	($\times$, ✓)	($\times$, ✓)	($\times$, $\times$)
$\mathcal{I}(\mathbb{L}) = \mathcal{M}^-$	($\times$, $\times$)	($\times$, $\times$)	($\times$, $\times$)
$\mathcal{I}(\mathbb{L}) = \mathcal{J}^+ = \mathcal{J}^-$	($\times$, ✓)	($\times$, ✓)	($\times$, $\times$)

Table 8: Approximation and decomposition of clutters by Γ-graphic clutters. The general case. In each cell, the pairs in $\{\checkmark, \times\}^2$ indicate if the inclusion $\mathcal{I}(\mathbb{L}) \subseteq \Sigma$ holds (the first component for $\Sigma = \mathrm{Clut}_{\Theta_\Gamma}(\Omega)$ and the second for $\Sigma = \bigcup_{\Omega' \subseteq \Omega} \mathrm{Clut}_{\Theta_\Gamma}(\Omega')$).

Many clutters arising from graphs naturally satisify the restrictions of having full support or empty intersection. Namely, it is easy to check that $\Theta_\Gamma(G) \in \mathrm{Clut}_0(\Omega)$ for

$\Gamma = \mathrm{Dom}, \mathrm{Ind}$, and that $\Theta_\Gamma(G) \in \mathrm{Clut}_\emptyset(\Omega)$ for $\Gamma = \mathrm{Cover}$. Moreover, if G has no isolated vertex, then $\Theta_\Gamma(G) \in \mathrm{Clut}_0(\Omega)$ for $\Gamma = \mathrm{Cover}$; and $\Theta_\Gamma(G) \in \mathrm{Clut}_\emptyset(\Omega)$ for $\Gamma = \mathrm{Dom}, \mathrm{Ind}$. Finally, we note that for $\Gamma = \mathrm{Dom}, \mathrm{Ind}, \mathrm{Cover}$ the condition of having corank at least two is equivalent to not having vertices adjacent to all other.

Thus, for graphic clutters we will consider the restrictions $\mathcal{P}_0, \mathcal{P}_\emptyset, \mathcal{P}_{0,\emptyset}$. For the problem of approximating and decomposing clutters Λ verifying property $\mathcal{P}$ by Θ_Γ-clutters also satisfying $\mathcal{P}$, we gather in Table 9 the case $\mathcal{P} = \mathcal{P}_0, \mathcal{P}_{0,\emptyset}$, and in Table 10 the case $\mathcal{P} = \mathcal{P}_\emptyset$. The results are stated for Ω large enough to avoid stating exceptions for all small values of $|\Omega|$, so if the reader is interested in values smaller than five they should double-check the case separately.

the inclusion $\mathcal{I}(\mathbb{X}, \mathbb{L}) \subseteq \Sigma$ where $\mathbb{X} = \mathrm{Clut}_\mathcal{P}(\Omega)$ and $\Sigma = \mathrm{Clut}_{\Theta_\Gamma}(\Omega) \cap \mathrm{Clut}_\mathcal{P}(\Omega)$ for $\mathcal{P} = \mathcal{P}_0, \mathcal{P}_{0,\emptyset}$			
	$\Gamma = \mathrm{Dom}$	$\Gamma = \mathrm{Ind}$	$\Gamma = \mathrm{Cover}$
$\mathcal{I}(\mathbb{X}, \mathbb{L}) = \mathcal{M}_0^+$	✓	✓	×
$\mathcal{I}(\mathbb{X}, \mathbb{L}) = \mathcal{M}_{0,\emptyset}^+$	✓	✓	×
$\mathcal{I}(\mathbb{X}, \mathbb{L}) = \mathcal{M}_0^-$	×	×	×
$\mathcal{I}(\mathbb{X}, \mathbb{L}) = \mathcal{M}_{0,\emptyset}^-$	×	×	×
$\mathcal{I}(\mathbb{X}, \mathbb{L}) = \mathcal{J}_0^+$	✓	✓	×
$\mathcal{I}(\mathbb{X}, \mathbb{L}) = \mathcal{J}_{0,\emptyset}^+$	×	×	✓
$\mathcal{I}(\mathbb{X}, \mathbb{L}) = \mathcal{J}_0^-$	×	✓	×
$\mathcal{I}(\mathbb{X}, \mathbb{L}) = \mathcal{J}_{0,\emptyset}^-$	×	✓	×

Table 9: Approximation and decomposition of clutters by Γ-graphic clutters. Case of clutters with full support and empty intersection.

Proof of the results in Table 8, Table 9 and Table 10

Recall that in Table 8 we want to decide in which cases we have the inclusion $\mathcal{I}(\mathbb{L}) \subseteq \Sigma$ where $\mathcal{I}(\mathbb{L}) = \mathcal{M}^+, \mathcal{M}^-, \mathcal{J}^+, \mathcal{J}^-$ and where $\Sigma = \mathrm{Clut}_{\Theta_\Gamma}(\Omega)$ or $\Sigma = \bigcup_{\Omega' \subseteq \Omega} \mathrm{Clut}_{\Theta_\Gamma}(\Omega')$. While in Table 9, and for the properties $\mathcal{P} = \mathcal{P}_0, \mathcal{P}_{0,\emptyset}$, we want to decide in which cases

the inclusion $\mathcal{I}(\mathbb{X},\mathbb{L}) \subseteq \Sigma$ where $\mathbb{X} = \mathrm{Clut}_{\emptyset}(\Omega)$ and $\Sigma = \mathrm{Clut}_{\Theta_\Gamma}(\Omega) \cap \mathrm{Clut}_{\emptyset}(\Omega)$ or $\Sigma = \left(\bigcup_{\Omega' \subseteq \Omega} \mathrm{Clut}_{\Theta_\Gamma}(\Omega')\right) \cap \mathrm{Clut}_{\emptyset}(\Omega)$			
	$\Gamma = \mathrm{Dom}$	$\Gamma = \mathrm{Ind}$	$\Gamma = \mathrm{Cover}$
$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{M}^{+}_{\emptyset}$	$(\times\ ,\ \checkmark)$	$(\times\ ,\ \checkmark)$	$(\times\ ,\ \times)$
$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{M}^{-}_{\emptyset}$	$(\times\ ,\ \times)$	$(\times\ ,\ \times)$	$(\times\ ,\ \times)$
$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{J}^{+}_{\emptyset}$	$(\times\ ,\ \times)$	$(\times\ ,\ \times)$	$(\checkmark\ ,\ \checkmark)$
$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{J}^{-}_{\emptyset}$	$(\times\ ,\ \checkmark)$	$(\times\ ,\ \checkmark)$	$(\checkmark\ ,\ \checkmark)$

Table 10: Approximation and decomposition of clutters by Γ-graphic clutter. Case of clutters with empty intersection. In each cell, the pairs in $\{\checkmark, \times\}^2$ indicate if the inclusion $\mathcal{I}(\mathbb{L}) \subseteq \Sigma$ holds (the first component for $\Sigma = \mathrm{Clut}_{\Theta_\Gamma}(\Omega) \cap \mathrm{Clut}_{\emptyset}(\Omega)$ and the second for $\Sigma = (\bigcup_{\Omega' \subseteq \Omega} \mathrm{Clut}_{\Theta_\Gamma}(\Omega')) \cap \mathrm{Clut}_{\emptyset}(\Omega)$.

we have the inclusion $\mathcal{I}(\mathbb{X},\mathbb{L}) \subseteq \Sigma$ where $\mathbb{X} = \mathrm{Clut}_{\mathcal{P}}(\Omega)$, the family $\mathcal{I}(\mathbb{X},\mathbb{L})$ is any of $\mathcal{M}^{+}_{0}, \mathcal{M}^{+}_{0,\emptyset}, \mathcal{M}^{-}_{0}, \mathcal{M}^{-}_{0,\emptyset}, \mathcal{J}^{+}_{0}, \mathcal{J}^{+}_{0,\emptyset}, \mathcal{J}^{-}_{0}, \mathcal{J}^{-}_{0,\emptyset}$, and $\Sigma = \mathrm{Clut}_{\Theta_\Gamma}(\Omega) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega)$. Whereas in Table 10, and now for the property $\mathcal{P} = \mathcal{P}_{\emptyset}$, we want to decide in which cases we have the inclusion $\mathcal{I}(\mathbb{X},\mathbb{L}) \subseteq \Sigma$ where $\mathbb{X} = \mathrm{Clut}_{\mathcal{P}}(\Omega)$, the family $\mathcal{I}(\mathbb{X},\mathbb{L})$ is any of $\mathcal{M}^{+}_{\emptyset}, \mathcal{M}^{-}_{\emptyset}, \mathcal{J}^{+}_{\emptyset}, \mathcal{J}^{-}_{\emptyset}$, and either $\Sigma = \mathrm{Clut}_{\Theta_\Gamma}(\Omega) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega)$ or $\Sigma = \left(\bigcup_{\Omega' \subseteq \Omega} \mathrm{Clut}_{\Theta_\Gamma}(\Omega')\right) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega)$. At this point observe that in both Table 9 and Table 10 we have that $\mathcal{I}(\mathbb{X},\mathbb{L}) \subseteq \mathrm{Clut}_{\mathcal{P}}(\Omega)$.

Therefore, if we denote by $\mathcal{I}$ any of the above mentioned irreducible sets of elements, we have that in Table 8, Table 9 and Table 10 we must check if for a given $\Delta \in \mathcal{I}$ there exist graphs G with vertex set $V(G) = \Omega$ and such that $\Delta = \Theta_\Gamma(G)$; and, in addition, in Table 8 and Table 10 we also must check if there exist graphs G with vertex set $V(G) = \Omega' \subseteq \Omega$ and such that $\Delta = \Theta_\Gamma(G)$.

First let us consider the case $\Gamma = \mathrm{Cover}$.

Clearly there is nothing to say whenever $\mathcal{I} = \mathcal{J}^{+}_{\emptyset}, \mathcal{J}^{+}_{0,\emptyset}$ because on both cases we have that $\Delta = \Delta_{10,G}$ is actually a Cover-clutter. Moreover, observe that the clutter $\Delta_{6,m}$ is the collection of minimal vertex covers of the star G with vertex set $V(G) = \{a_1, \ldots, a_m\}$ and edges $E(G) = \{\{a_1, a_2\}, \ldots, \{a_1, a_m\}\}$. So the case $\mathcal{I} = \mathcal{J}^{-}_{\emptyset}$ is done.

Therefore we only must prove that in all remaining cases there are clutters $\Delta \in \mathcal{I}$ for which there does not exists a graph G with $\mathrm{minimal}(\mathrm{Cover}(G)) = \Delta$.

Let us consider the clutter $\Delta = \Delta_{1,n} \in \mathcal{M}^{+}, \mathcal{M}^{+}_{0}, \mathcal{M}^{+}_{\emptyset}, \mathcal{M}^{+}_{0,\emptyset}$, the clutter $\Delta = \Delta_{2,m} \in \mathcal{M}^{-}, \mathcal{M}^{-}_{0}$ (with $2 < m \neq n$), the clutter $\Delta = \Delta_{3,n-1} \in \mathcal{J}^{+}, \mathcal{J}^{-}$, the clutter $\Delta_{9,n-2} \in \mathcal{J}^{+}_{0}$, the clutter $\Delta = \Delta_{5,n} \in \mathcal{M}^{-}_{\emptyset}$ and the clutter $\Delta = \Delta_{8,r} \in \mathcal{M}^{-}_{0,\emptyset}$. It is not hard to check that in any case there exists $A \in b(\Delta)$ with $|A| \neq 2$. Hence it follows that there does not exist a graph G with $E(G) = b(\Lambda)$. Since $b(\text{minimal}(\text{Cover}(G))) = E(G)$, we conclude that there does not exists a graph G with $\text{minimal}(\text{Cover}(G)) = b(b(\Delta)) = \Delta$ as needed. Finally, note that the clutter $\Delta_{1.n}$ is a particular case of a clutter $\Delta_{11,G}$ by taking G to be the complete graph with vertex set Ω, so this takes care of the remaining cases $\mathcal{J}_0^-, \mathcal{J}_{0,\emptyset}^-$.

Now we examine the cases $\Gamma = \text{Dom}$ and $\Gamma = \text{Ind}$. Set $\Omega = \{a_1, \ldots, a_n\}$.

First recall that if G is a graph with vertex set Ω then $\Theta_\Gamma(G) \in \text{Clut}_0(\Omega)$; that is, $\text{Clut}_{\Theta_\Gamma}(\Omega) \subseteq \text{Clut}_0(\Omega)$. Therefore, if $m < n$ then neither the clutter $\Delta_{1,m}$ nor the clutters $\Delta_{3,m}, \Delta_{6,m}$ admit Γ-graph realizations by using graphs G with $V(G) = \Omega$ (because $a_n \notin A$ if $A \in \Delta$). The same argument applies to the clutter $\Delta = \Delta_{2,m}$ where $m = 1$, and to the clutter $\Delta = \Delta_{10,G}$ if G is a graph with isolated vertices. So, for $\Gamma = \text{Dom}, \text{Ind}$ we have that the first component of the pairs in Table 8 and Table 10 must be "$\times$" whenever $\mathcal{I} = \mathcal{M}^+, \mathcal{M}^-, \mathcal{J}^+, \mathcal{J}^-, \mathcal{M}_\emptyset^+, \mathcal{J}_\emptyset^-$. Moreover we have that for $\mathcal{I} = \mathcal{J}_\emptyset^+$ both components of the pairs in Table 10 must be "$\times$".

Next let us show that, for $\mathcal{I} = \mathcal{M}^-$, the second component of the pairs in Table 8 must be "$\times$" and that, for $\mathcal{I} = \mathcal{M}_\emptyset^-$, both components of the pairs in Table 10 must be "$\times$". Actually we claim that by means of graphs with vertex set Ω it is not possible to obtain a Γ-graph realization for the clutter $\Delta_{2,m}$ with $m = n$. Let us prove our claim. On one hand, it is clear that there does not exist a graph G with vertex set Ω and whose maximal independent sets of vertices are all the subsets of size $n-1$; in other words, the clutter $\Delta_{2,n}$ does not admit an Ind-graph realization. On the other hand, from [14, Proposition 5] it follows that the uniform clutter $\mathcal{U}_{n-1,n}$ does not admit a Dom-graph realization; since $\Delta_{2,n} = \mathcal{U}_{n-1,n}$, our claim follows.

At this point, the proof of the results in Table 8 and Table 10 will be completed by showing that with graphs G with vertex set $V(G) = \Omega' \subseteq \Omega$ is it possible to obtain Γ-graph realizations for all the clutters $\Delta \in \mathcal{M}^+, \mathcal{J}^+, \mathcal{J}^-, \mathcal{M}_\emptyset^+, \mathcal{J}_\emptyset^-$. Let us prove it. Let $\Delta \in \mathcal{M}^+, \mathcal{J}^+, \mathcal{J}^-, \mathcal{M}_\emptyset^+, \mathcal{J}_\emptyset^-$. Then, either $\Delta = \Delta_{1,m}$ (where $1 \le m \le n$), or $\Delta = \Delta_{3.m}$ (where $1 \le m \le n$), or $\Delta = \Delta_{6.m}$ (where $2 \le m \le n$). It is easy to check that the complete graph $G = K_{\Omega'}$ on the set of vertices $\Omega' = \{a_1, \ldots, a_m\} \subseteq \Omega$ is a Dom-graph realization and an Ind-graph realization for $\Delta_{1,m}$; that the null graph $G = \overline{K_{\Omega'}}$ on the set of vertices $\Omega' = \{a_1, \ldots, a_m\} \subseteq \Omega$ is a Dom-graph realization and an Ind-graph realization for $\Delta_{3,m}$; and that the star G with vertex set $V(G) = \{a_1, \ldots, a_m\}$ and edge set $E(G) = \{\{a_1, a_2\}, \ldots, \{a_1, a_m\}\}$ is a Dom-graph realization and an Ind-graph realization for $\Delta_{6,m}$.

To finish, let us prove the results in Table 9. Clearly there is nothing to say whenever $\Gamma = \text{Ind}$ and $\mathcal{I} = \mathcal{J}_0^-, \mathcal{J}_{0,\emptyset}^-$, as in both cases we have that if $\Delta \in \mathcal{I}$ then $\Delta = \Delta_{11,G}$ which is a Ind-clutter by definition. The cases $\mathcal{M}_0^-, \mathcal{M}_{0,\emptyset}^-$ follow by considering the clutter $\Delta_{2,n}$, which we saw before it is not Γ-realizable. So it only remains to prove three facts: that there are Γ-graph realizations for all the clutters $\Delta \in \mathcal{M}_0^+, \mathcal{M}_{0,\emptyset}^+, \mathcal{J}_0^+$; that there are clutters $\Delta \in \mathcal{J}_{0,\emptyset}^+$ that do not admit a Γ-graph realization; and that by means of graphs with vertex set Ω it is not possible to obtain a Dom-graph realization for all the clutters of the form $\Delta_{11,G}$.

If $\Delta \in \mathcal{M}_0^+, \mathcal{M}_{0,\emptyset}^+, \mathcal{J}_0^+$, then either Δ is one of the clutters $\Delta_{1,n}$, $\Delta_{3,n}$, $\Delta_{4,n}$, or $\Delta_{6,n}$, or Δ is of the form $\Delta_{7,r}$ or $\Delta_{9,r}$ for $1 \leq r \leq n-2$. All these clutters are Γ-clutters. Namely, we have that a realization of $\Delta_{1,n}$ is given by the complete graph $G = K_n$; a realization of $\Delta_{3,n}$ is given by the null graph $G = \overline{K_\Omega}$; a realization of $\Delta_{4,n}$ is given by the union $G = K_{\{a_1\}} \cup K_{\{a_2,\ldots,a_n\}}$; a realization of $\Delta_{6,n}$ is given by the star G with edge set $E(G) = \{\{a_1, a_2\}, \ldots, \{a_1, a_n\}\}$; a realization of $\Delta_{7,r}$ is given by the graph G with set of vertices $V(G) = \Omega$ obtained from the complete graph K_Ω by removing the edges $\{a_{r+1}, a_n\}, \ldots, \{a_{n-1}, a_n\}$; and, finally, the graph G with vertex set $V(G) = \Omega$ and edge set $E(G) = \{\{a_n, a_{r+1}\}, \ldots, \{a_n, a_{n-1}\}\}$ is a realization for $\Delta_{9,r}$.

If $\Delta \in \mathcal{J}^+_{0,\emptyset}$ then $\Delta = \Delta_{10,G} = \mathrm{minimal}(\mathrm{Cover}(G))$ for some graph G with vertex set $V(G) = \Omega$. Observe that if the graph G has isolated vertices then $\Delta_{10,G} \notin \mathrm{Clut}_0(\Omega)$. Therefore, since $\Theta_\Gamma(G') \in \mathrm{Clut}_0(V(G'))$ for both $\Gamma = \mathrm{Ind}, \mathrm{Dom}$, we conclude that if the graph G has isolated vertices then there can be no graph G' with vertex set $V(G') = \Omega$ and with $\mathrm{minimal}(\mathrm{Dom}(G')) = \Delta_{10,G}$ or with $\mathrm{maximal}(\mathrm{Ind}(G')) = \Delta_{10,G}$.

Finally, let $n \geq 4$ and let us consider the graph G with set of vertices $V(G) = \Omega = \{a_1, \ldots, a_n\}$ obtained from the complete graph K_Ω by removing the edges $\{a_1, a_4\}$, $\{a_2, a_3\}$. Then $\Delta_{11,G} = \mathrm{maximal}(\mathrm{Ind}(G)) = \{\{a_1, a_4\}, \{a_2, a_3\}, \{a_5\}, \ldots, \{a_n\}\}$, and it is not difficult to check that there does not exist a graph G' with vertex set $V(G') = \Omega$ and with $\mathrm{minimal}(\mathrm{Dom}(G')) = \Delta_{11,G}$. This completes the proof of the results gathered in Table 8, Table 9 and Table 10.

5.3 Access structures and ideal secret sharing decompositions

Secret sharing was introduced by Blakley [1] and Shamir [18]. A comprehensive introduction to this topic can be found in [19].

A *secret sharing scheme* $\mathbb{S}$ is a method to distribute a secret value $k \in \mathcal{K}$ of a finite set $\mathcal{K}$ among a set of participants Ω. Every participant $p \in \Omega$ receives a *share* $s_p \in \mathcal{S}_p$ in such a way that only some subsets of participants, the *qualified subsets*, are able to reconstruct the secret k from their shares (for each participant p, its corresponding set of shares $\mathcal{S}_p$ is a finite set too). Only *perfect* secret sharing schemes are going to be considered, that is, schemes in which the shares of the participants in a *non-qualified subset* provide absolutely no information about the value of the secret. Besides, we are dealing here with *unconditional security*, that is, we are not making any assumption on the computational power of the participants.

The *access structure* of a secret sharing scheme $\mathbb{S}$ is the family of the qualified subsets, $\Gamma(\mathbb{S}) \subseteq 2^\Omega$. In general, access structures are considered to be *monotone increasing*, that is, any subset of Ω containing a qualified subset is qualified. So, the access structure $\Gamma(\mathbb{S})$ is determined by the family of the *minimal qualified subsets*, $\Delta(\mathbb{S})$, which is called the *basis* of $\Gamma(\mathbb{S})$ (observe that $\Delta(\mathbb{S})$ is a clutter on Ω).

Ito, Saito and Nishizeki [9] proved, in a constructive way, that given a monotone increasing family of subsets $\Gamma \subseteq 2^\Omega$, there exists a secret sharing scheme $\mathbb{S}_\Gamma$ with access structure Γ (that is, with $\Gamma = \Gamma(\mathbb{S}_\Gamma)$). In other words, the result of Ito, Saito and Nishizeki states that given a clutter $\Delta \subseteq 2^\Omega$ there exists a secret sharing scheme $\mathbb{S}_\Delta$ with basis Δ (that is, with $\Delta = \Delta(\mathbb{S}_\Delta)$).

One of the main parameters in secret sharing is the *information rate* $\rho(\mathbb{S})$ *of the scheme* $\mathbb{S}$, which is defined as the ratio between the length (in bits) of the secret and the maximum length of the shares given to the participants. That is, $\rho(\mathbb{S}) = \log|\mathcal{K}|/\max_{p\in\Omega}\log|\mathcal{S}_p|$, where $\mathcal{K}$ is the set of secrets and $\mathcal{S}_p$ is the set of shares given to p. In any perfect secret sharing scheme $0 < \rho(\mathbb{S}) \leq 1$ because the size of the share of any participant is at least the size of the secret [19]. Due to efficiency reasons a high information rate is desirable. A secret sharing scheme is said to be *ideal* if its information rate is equal to one, that is, if all shares have the same size as the secret.

Unfortunately, for a given monotone increasing family of subsets, the schemes constructed by the method in [9] are in general very inefficient because the size of the shares is much larger than the size of the secret and so, in most cases, their information is very small. Thus, when designing a secret sharing scheme for a given monotone increasing family of subsets Γ (or, equivalently, for a clutter Δ), we may try to maximize the information rate. The *optimal information rate* of Γ is defined by $\rho^*(\Gamma) = \sup(\rho(\mathbb{S}))$, where the supremum is taken over all possible secret sharing schemes $\mathbb{S}$ with access structure Γ. The monotone increasing family Γ is said to be *ideal* if there exists an ideal secret sharing scheme $\mathbb{S}$ for Γ (analogously, a clutter Δ is said to be *ideal* if there exists an ideal secret sharing scheme $\mathbb{S}$ for Δ).

Since not all clutters are ideal, to characterize the ideal access structures and to provide bounds on the optimal information rate are two important problems in secret sharing that have received considerable attention.

Our goal is to decide when it is possible to approximate and to recover any non ideal clutter by means of ideal clutters. We stress that from our results it follows that there are different ways to do it. Among all of them we have the one corresponding to the join decomposition with respect the order $\leqslant^+$. It is worth mentioning that this descomposition is the one used in the *decomposition method* introduced by Stinson in [20] to provide lower bounds on the optimal information rate (see for instance [15, Proposition 3.2]). Therefore, the following two open problem arise at this point: is it possible to obtain lower bounds on the optimal information rate by using the join decomposition with respect the order $\leqslant^-$? is it possible to obtain upper bounds on the optimal information rate by using meet decompositions? In this work we are not going to focus our attention on answering these two questions.

Returning to our approximation and decomposition problem, to solve it we will formalize it in our general framework.

Let Ω be a finite set of $n > 5$ participants (the perfect secret sharing schemes on sets of $n \leq 5$ participants has been completely studied in [10]). Let us denote by $\mathrm{IdealSSS}(\Omega)$ the set of all the ideal secret sharing schemes $\mathbb{S}$ defined on the set of participants Ω. Now we consider the map

$$\begin{array}{rcl}\Theta : \mathfrak{Obj}(\Omega) = \mathrm{IdealSSS}(\Omega) & \to & \mathrm{Clut}(\Omega) \\ \mathbb{S} & \mapsto & \Delta(\mathbb{S})\end{array}$$

With this notation, and from the results collected in the following three tables, we obtain the different ways to approximate and decompose a non-ideal access structure that satisfies a property $\mathcal{P}$ by means of ideal access structures fulfilling that same property.

As in the case of graphs, the first table (Table 11) summarizes the results when we do not consider any constraints on the clutters, that is to say, whenever $\mathcal{P} = \{\}$. (We emphasize that unlike the case of graphs, here there is no difference between the value of the two components of the ordered pairs.)

the inclusion $\mathcal{I}(\mathbb{L}) \subseteq \Sigma$ where $\Sigma = \mathrm{Clut}_\Theta(\Omega)$ or $\Sigma = \bigcup_{\Omega' \subseteq \Omega} \mathrm{Clut}_\Theta(\Omega')$	
$\mathcal{I}(\mathbb{L}) = \mathcal{M}^+$	(✓ , ✓)
$\mathcal{I}(\mathbb{L}) = \mathcal{M}^-$	(✓ , ✓)
$\mathcal{I}(\mathbb{L}) = \mathcal{J}^+ = \mathcal{J}^-$	(✓ , ✓)

Table 11: Approximation and decomposition of clutters by ideal access structures. The general case. In each cell, the pairs in $\{\checkmark, \times\}^2$ indicate if the inclusion $\mathcal{I}(\mathbb{L}) \subseteq \Sigma$ holds (the first component for $\Sigma = \mathrm{Clut}_\Theta(\Omega)$ and the second for $\Sigma = \bigcup_{\Omega' \subseteq \Omega} \mathrm{Clut}_\Theta(\Omega')$).

However, in a secret sharing scheme it is natural to require one or more of the following three constraints: that all participants appear in at least one minimal qualified subset; that very minimal qualified subset has at least two participants; and that no participant appears in all the minimal qualified subsets. So it is reasonable to consider clutters with the restriction $\mathcal{P}$ where $\mathcal{P} = \mathcal{P}_0$, $\mathcal{P}_\emptyset$, $\mathcal{P}_{c\geq 2}$, $\mathcal{P}_{0,\emptyset}$, $\mathcal{P}_{0,c\geq 2}$, $\mathcal{P}_{c\geq 2,\emptyset}$, $\mathcal{P}_{0,c\geq 2,\emptyset}$. The answer to the problem of approximation and decomposition of clutters verifying $\mathcal{P}$ by using Θ-clutters that also verify $\mathcal{P}$ follows from the results of Table 12 if $\mathcal{P} = \mathcal{P}_0, \mathcal{P}_{0,\emptyset}, \mathcal{P}_{0,c\geq 2}, \mathcal{P}_{0,c\geq 2,\emptyset}$, and from Table 13 whenever $\mathcal{P} = \mathcal{P}_\emptyset, \mathcal{P}_{c\geq 2}, \mathcal{P}_{c\geq 2,\emptyset}$.

Proof of the results in Table 11, Table 12 and Table 13

From [2] we know that if G is a connected graph then there exists an ideal secret sharing scheme $\mathbb{S}$ with basis $\Delta(\mathbb{S})$ the edges of the graph G if and only if the graph G is a complete multipartite graph. Therefore, in general not all the clutters of the form $\Delta_{12,G}$ admit an ideal secret sharing representation. Next let us prove that the same apply for the clutters $\Delta_{10,G}$ and $\Delta_{11,G}$.

the inclusion $\mathcal{I}(\mathbb{X},\mathbb{L}) \subseteq \Sigma$ where $\mathbb{X} = \mathrm{Clut}_{\mathcal{P}}(\Omega)$ and $\Sigma = \mathrm{Clut}_{\Theta}(\Omega) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega)$ and where $\mathcal{P} = \mathcal{P}_0,\ \mathcal{P}_{0,\emptyset},\ \mathcal{P}_{0,c\geq 2},\ \mathcal{P}_{0,c\geq 2,\emptyset}$			
$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{M}_0^+$	✓	$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{J}_0^+$	✓
$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{M}_{0,\emptyset}^+$	✓	$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{J}_{0,\emptyset}^+$	×
$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{M}_{0,c\geq 2}^+$	×	$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{J}_{0,c\geq 2}^+$	✓
$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{M}_{0,c\geq 2,\emptyset}^+$	×	$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{J}_{0,c\geq 2,\emptyset}^+$	×
$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{M}_0^-$	✓	$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{J}_0^-$	×
$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{M}_{0,\emptyset}^-$	×	$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{J}_{0,\emptyset}^-$	×

Table 12: Approximation and decomposition of clutters by ideal access structures. Case of clutters with full support and with empty intersection or corank greater than or equal to two.

the inclusion $\mathcal{I}(\mathbb{X},\mathbb{L}) \subseteq \Sigma$ where $\mathbb{X} = \mathrm{Clut}_{\mathcal{P}}(\Omega)$ where $\Sigma = \mathrm{Clut}_{\Theta}(\Omega) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega)$ or $\Sigma = \left(\bigcup_{\Omega' \subseteq \Omega} \mathrm{Clut}_{\Theta}(\Omega')\right) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega)$ and where $\mathcal{P} = \mathcal{P}_{\emptyset},\ \mathcal{P}_{c\geq 2},\ \mathcal{P}_{\emptyset,\geq 2}$			
$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{M}_{\emptyset}^+$	(✓ , ✓)	$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{J}_{\emptyset}^+$	(× , ×)
$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{M}_{c\geq 2}^+$	(× , ×)	$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{J}_{c\geq 2}^+$	(✓ , ✓)
$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{M}_{\emptyset,c\geq 2}^+$	(× , ×)	$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{J}_{\emptyset,c\geq 2}^+$	(× , ×)
$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{M}_{\emptyset}^-$	(× , ×)	$\mathcal{I}(\mathbb{X},\mathbb{L}) = \mathcal{J}_{\emptyset}^-$	(✓ , ✓)

Table 13: Approximation and decomposition of clutters by ideal access structures. Case of clutters with empty intersection and corank greater than or equal to two. In each cell, the pairs in $\{\checkmark, \times\}^2$ indicate if the inclusion $\mathcal{I}(\mathbb{L}) \subseteq \Sigma$ holds (the first component for $\Sigma = \mathrm{Clut}_{\Theta_\Gamma}(\Omega) \cap \mathrm{Clut}_{\mathcal{P}}(\Omega)$ and the second for $\Sigma = (\bigcup_{\Omega' \subseteq \Omega} \mathrm{Clut}_{\Theta_\Gamma}(\Omega') \cap \mathrm{Clut}_{\mathcal{P}}(\Omega))$.

First we are going to prove that not all the clutters of the form $\Delta_{10,G}$ admit an ideal secret sharing representation. For instance, let $n \geq 5$ and let us consider the graph G with vertex set $V(G) = \Omega = \{a_1, \ldots, a_n\}$ and with edges $E(G) = \{\{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}\} \cup \{\{a_3, a_4\}, \{a_3, a_5\}, \ldots, \{a_3, a_n\}\}$. Then the clutter of its minimal vertex cover sets is the clutter $\Delta_{10,G} = \text{minimal}(\text{Cover}(G)) = \{\{a_1, a_3\}, \{a_2, a_3\}, \{a_1, a_2, a_4, \dots, a_n\}\}$. This clutter has three elements and so we can apply the results in [15]. In this case, from [15, Proposition 3.2], we get that there does not exists an ideal secret sharing scheme $\mathbb{S}$ with basis $\Delta(\mathbb{S}) = \Delta_{10,G}$. Therefore, the clutter $\Delta_{10,G}$ is not a Θ-clutter.

Now let us show that not all the clutters of the form $\Delta_{11,G}$ admit an ideal secret sharing representation. For instance, let $n \geq 4$ and let G be the graph with vertex set $V(G) = \Omega = \{a_1, \dots, a_n\}$ and with edges $E(G) = \{\{a_1, a_2\}, \{a_2, a_3\}\} \cup \{\{a_3, a_4\}, \dots, \{a_3, a_n\}\}$. For this graph we have that the clutter of the maximal independent sets of vertices is $\Delta_{11,G} = \text{minimal}(\text{Ind}(G)) = \{\{a_1, a_3\}, \{a_1, a_4, a_5, \dots, a_n\}, \{a_2, a_4, \dots, a_n\}\}$. Now, applying again [15, Proposition 3.2], we conclude that there does not exists an ideal secret sharing scheme $\mathbb{S}$ with basis $\Delta(\mathbb{S}) = \Delta_{11,G}$. So the clutter $\Delta_{11,G}$ is not a Θ-clutter.

In addition to the clutters $\Delta_{10,G}$, $\Delta_{11,G}$ i $\Delta_{12,G}$, next we are going to prove that there is another irreducible clutter which is not a Θ-clutter. Namely, let us show that if $n \geq 5$ and if $1 \leq r \leq n-3$, then the clutter $\Delta_{8,r}$ does not admit an ideal secret sharing representation. To prove this fact we use the *independent sequence method* which was introduced by Blundo, De Santis, De Simone and Vaccaro in [3] and was generalized by Padró and Sáez in [17] (here we use the notations gathered in [15, Proposition 2.1]). Let Γ be the monotone increasing family if substes defined by $\Delta_{8,r}$. Let us consider the subsets $B_1 = \Omega \setminus \{a_1, a_{r+1}, a_{r+2}, a_n\}$, $B_2 = \Omega \setminus \{a_1, a_{r+1}, a_n\}$, $B_3 = \Omega \setminus \{a_1, a_{r+1}\}$, and the subsets $X_1 = \{a_1, a_{r+1}\}$, $X_2 = \{a_1\}$ and $X_3 = \{a_{r+1}\}$. Then we have that $X_1 \cup B_1 = \Omega \setminus \{a_{r+2}, a_n\} \in \Gamma$, $X_2 \cup B_1 = \Omega \setminus \{a_{r+1}, a_{r+2}, a_n\} \notin \Gamma$, $X_2 \cup B_2 = \Omega \setminus \{a_{r+1}, a_n\} \in \Gamma$, $X_3 \cup B_2 = \Omega \setminus \{a_1, a_n\} \notin \Gamma$ and that $X_3 \cup B_3 = \Omega \setminus \{a_1\} \in \Gamma$. Therefore we have that the sequence of subsets $\emptyset \neq B_1 \subsetneq B_2 \subsetneq B_3 \notin \Gamma$ is made independent by the subset $A = \{a_1, a_{r+1}\} \notin \Gamma$. So, by applying the independent sequence method, we conclude that $\rho^*(\Gamma) \leq 2/3$. In particular, there does not exists an ideal secret sharing scheme $\mathbb{S}$ with basis $\Delta(\mathbb{S}) = \Delta_{8,r}$, as we wanted to prove.

At this point, the proof will be completed by showing that the clutters $\Delta = \Delta_{1,m}$, $\Delta_{2,m}, \Delta_{3,m}$ (where $1 \leq m \leq n$), the clutters $\Delta = \Delta_{4,m}, \Delta_{5,m}, \Delta_{6,m}$ (where $2 \leq m \leq n$) and the clutters $\Delta = \Delta_{7,r}, \Delta_{9,r}$ (where $1 \leq r \leq n-2$) are all Θ-clutters. Specifically, we are always going to provide θ-realizations over the set Ω and, therefore, we do not have to analyze the case Ω' with $\Omega' \subseteq \Omega$. To do this we will use the vector space structures introduced by Brickell [4].

It is said that a monotone increasing family of subsets Γ is a *vector space access structure* if there exist a vector space E over a finite field $\mathbb{K}$ and a map $\psi : \mathcal{P} \cup \{D\} \longrightarrow E \setminus \{0\}$, where $D \notin \mathcal{P}$ is the *dealer*, such that if $A \subseteq \mathcal{P}$ then, $A \in \Gamma$ if and only if $\psi(D)$ is a linear combination of the vectors in the set $\{\psi(p) : p \in A\}$. In such a case, an ideal secret sharing scheme $\mathbb{S}$ for Γ is obtained in the following way: given a secret value $k \in \mathbb{K}$, the dealer takes at random an element $v \in E$ such that $v \cdot \psi(D) = k$, and gives the share $s_p = v \cdot \psi(p)$ to p. Therefore, the vector space structures are ideal. The map ψ is said to be a *vector space realization* of Γ.

So the proof of our result will be completed by presenting a vector space realization $\psi_{i,j} : \mathcal{P} \cup \{D\} \longrightarrow E \setminus \{0\}$ for the monotone increasing family of subsets $\Gamma_{\Delta_{i,j}}$ associated to the clutter $\Delta_{i,j}$.

From now on, $\{e_1, \dots, e_d\}$ will denote the elements of the canonical basis of the

d-dimensional vector space $E = \mathbb{K}^d$. The vector space realizations $\psi_{i,j}$ are listed next. We consider $\mathbb{K}$ to be as large as needed.

- $\psi_{1,m} : \mathcal{P} \cup \{D\} \longrightarrow \mathbb{K}^2 \setminus \{0\}$ where $\psi_{1,m}(D) = e_1$, $\psi_{1,m}(a_i) = e_1$ if $1 \le i \le m$, and $\psi_{1,m}(a_i) = e_2$ if $i > m$.
- $\psi_{2,m} : \mathcal{P} \cup \{D\} \longrightarrow \mathbb{K}^{n-1} \setminus \{0\}$ where $\psi_{2,m}(D) = e_1 + \cdots + e_{n-1}$, $\psi_{2,m}(a_i) = e_i$ for all $i < m$, $\psi_{2,m}(e_m) = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_{m-1} e_{m-1}$ (being $\alpha_1, \ldots, \alpha_{m-1} \in \mathbb{K} \setminus \{0\}$ different elements of the finite field $\mathbb{K}$), and $\psi_{2,m}(a_i) = e_{i-1}$ for $m < i \le n$.
- $\psi_{3,m} : \mathcal{P} \cup \{D\} \longrightarrow \mathbb{K}^{m+1} \setminus \{0\}$ where $\psi_{3,m}(D) = e_1 + \cdots + e_m$, $\psi_{3,m}(a_i) = e_i$ if $1 \le i \le m$, and $\psi_{3,m}(a_i) = e_{m+1}$ if $i > m$.
- $\psi_{4,m} : \mathcal{P} \cup \{D\} \longrightarrow \mathbb{K}^3 \setminus \{0\}$ where $\psi_{4,m}(D) = e_1 + e_2$, $\psi_{4,m}(a_1) = e_1$, $\psi_{4,m}(a_i) = e_2$ if $2 \le i \le m$ and $\phi_{4,m}(a_i) = e_3$ if $i > m$.
- $\psi_{5,m} : \mathcal{P} \cup \{D\} \longrightarrow \mathbb{K}^{n-1} \setminus \{0\}$ where $\psi_{5,m}(D) = e_2 + \cdots + e_{n-1}$, $\psi_{5,m}(a_i) = e_i$ for all $i < m$ and $\psi_{5,m}(e_m) = \alpha_2 e_2 + \ldots + \alpha_{m-1} e_{m-1}$ (being $\alpha_2, \ldots, \alpha_{m-1} \in \mathbb{K} \setminus \{0\}$ different elements of the finite field $\mathbb{K}$), and $\psi_{5,m}(a_i) = e_{i-1}$ for $m < i \le n$.
- $\psi_{6,m} : \mathcal{P} \cup \{D\} \longrightarrow \mathbb{K}^m \setminus \{0\}$ where $\psi_{6,m}(D) = e_1 + \cdots + e_{m-1}$, $\psi_{6,m}(a_1) = e_1 + \cdots + e_{m-1}$, $\psi_{6,m}(a_i) = e_{i-1}$ if $2 \le i \le m$ and $\psi_{6,m}(a_i) = e_m$ if $m < i \le n$.
- $\psi_{7,r} : \mathcal{P} \cup \{D\} \longrightarrow \mathbb{K}^2 \setminus \{0\}$ where $\psi_{7,r}(D) = e_1 + e_2$, $\psi_{7,r}(a_i) = e_1 + e_2$ if $1 \le i \le r$, $\psi_{7,r}(a_i) = e_1$ if $r + 1 \le i < n$ and $\psi_{7,r}(a_n) = e_2$.
- $\psi_{9,r} : \mathcal{P} \cup \{D\} \longrightarrow \mathbb{K}^n \setminus \{0\}$ where $\psi_{9,r}(D) = e_1 + \cdots + e_r + e_n$, $\psi_{9,r}(a_i) = e_i$ if $i \ne r + 1$, and $\psi_{9,r}(a_{r+1}) = e_{r+2} + \cdots + e_n$.

This completes the proof of the results in Table 11, Table 12 and Table 13.

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