On the regular 2-connected 2-path Hamiltonian graphs

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Abstract: A graph G is l-path Hamiltonian if every path of length not exceeding l is contained in a Hamiltonian cycle. It is well known that a 2-connected, k-regular graph G on at most $3k-1$ vertices is edge-Hamiltonian if for every edge uv of G, $\{u, v\}$ is not a cut-set. Thus G is 1-path Hamiltonian if $G \setminus \{u, v\}$ is connected for every edge uv of G. Let $P = uvz$ be a 2-path of a 2-connected, k-regular graph G on at most $2k$ vertices. In this paper, we show that there is a Hamiltonian cycle containing the 2-path P if $G \setminus V(P)$ is connected. Therefore, the work implies a condition for a 2-connected, k-regular graph to be 2-path Hamiltonian. An example shows that the $2k$ is almost sharp, i.e., the number is at most $2k+1$.

Keywords: Hamiltonian cycle; l-path Hamiltonian; k-regular graph; edge-Hamiltonian

1 Introduction

All graphs mentioned in this paper are finite simple graphs. Standard graph theory notation and terminology not explained in this paper, we refer the reader to [1]. A *Hamiltonian cycle* in a graph G is a cycle containing all the vertices of G, and a graph

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with a Hamiltonian cycle is called *Hamiltonian*. Dirac's Theorem [2] states that every n-vertex graph with minimum degree at least $\frac{n}{2}$ is Hamiltonian.

One particular classic subarea on Hamiltonian graph theory is about Hamiltonian cycles containing specified elements of a graph. One of these directions is the study of l-path Hamiltonian. A graph G on n vertices is said to be *l-path Hamiltonian* if every path of length not exceeding l, $1 \leqslant l \leqslant n-2$, is contained in a Hamiltonian cycle (i.e., a Hamiltonian graph is 0-path Hamiltonian). A graph G is said to be *edge-Hamiltonian, or 1-path Hamiltonian* if every edge of G is contained in a Hamiltonian cycle. Kronk in [4] considered the l-path Hamiltonian.

Theorem 1 ([4]). *Let G be a graph on n vertices, if $d(a)+d(b) \geqslant n+l$ for every pair of non-adjacent vertices a and b, then G is l-path Hamiltonian.*

It is not difficult to see that Kronk's work is sharp. Due to the theorem above, we try to explore such problems on k-regular graphs.

Many problems and conjectures on Hamiltonian regular graphs have been investigated by various authors. The problem of determining the values of k for which all 2-connected, k-regular graphs on n vertices are Hamiltonian was first suggested by Szekeres (see [3]). Jackson in [3] showed that every 2-connected, k-regular graph on at most $3k$ vertices is Hamiltonian. The strongest result of these works given by Li in [5] is that all 2-connected, k-regular graphs, $k \geqslant 14$, on at most $3k+4$ vertices are Hamiltonian except two kinds of well defined families of graphs.

Li in [6] showed the following result that under almost the same conditions in [3], the graphs are edge-Hamiltonian.

Theorem 2 ([6]). *Let G be a 2-connected, k-regular graph on $n \leqslant 3k-1$ vertices, and let $e_0 = uv$ be any edge of G such that $\{u, v\}$ is not a cut-set, then G has a Hamiltonian cycle containing e_0.*

In other words, if G is a 2-connected, k-regular graph on at most $3k-1$ vertices, and $G \setminus V(P)$ is connected for every path P of length 1, then G is 1-path Hamiltonian.

By Theorem 1, we have that 2-connected, k-regular graphs on at most $2k-2$ vertices are 2-path Hamiltonian. Naturally, what else can we say about the 2-path Hamiltonian regular graphs? In this paper, we are going to prove the following.

Theorem 3. *Let G be a 2-connected, k-regular graph on $n \leqslant 2k$ vertices, and let $P = uvz$ be any path of G such that $\{u, v, z\}$ is not a cut-set, then G has a Hamiltonian cycle containing P.*

The following corollary follows from Theorem 2 and Theorem 3.

Corollary 4. *Let G be a 2-connected, k-regular graph on at most $2k$ vertices, if $G \setminus V(P)$ is connected for every path P of length at most 2, then G is 2-path Hamiltonian.*

We shall present an example which shows that the best bound of Theorem 4 is at most $2k+1$. Let H_i, $i = 1, 2$, be a graph which is obtained from K_{k+1} by deleting one edge $e_i = a_i b_i$. We can construct a 2-connected, k-regular graph G on 2k+2 vertices from two disjoint copies H_1 and H_2 by adding a_1a_2 and b_1b_2. There is a 2-path in G that is not contained in any Hamiltonian cycle of G. Thus, the problems on regular 2-connected l-path Hamiltonian graphs with n vertices are interesting in $2k - l \leqslant n \leqslant 2k + 1$.

2 Proof of Theorem 3

The proof of Theorem 3 is divided into two cases. We first consider the case of $k \geqslant 5$ and we prove it by using the classic hopping lemma ([7], Lemma 12.3). In the end, we consider the cases of $k = 3$ and $k = 4$.

We fist assume $k \geqslant 5$. Let G be a 2-connected, k-regular graph on $n \leqslant 2k$ vertices, and let $P = uvz$ be a path of G such that $\{u, v, z\}$ is not a cut-set. We define a new graph G_1 by inserting two vertices w_1 and w_2 on the edges $e_1 = uv$ and $e_2 = vz$ of P respectively. Then we have $G_1 = (G - \{e_1, e_2\}) \cup \{w_1, w_2\} \cup \{uw_1, w_1v, vw_2, w_2z\}$, $P_1 = uw_1vw_2z$ and $|V(G_1)| = n_1 \leqslant 2k + 2$. Clearly, it is sufficient to prove that G_1 is Hamiltonian.

Suppose that G_1 is not Hamiltonian. Let $C_1 = c_1, c_2, \cdots, c_{n_1-r_1}$ be a longest cycle of G_1 containing w_1 and w_2 (Note that $G_1 - V(P_1)$ is connected.), such that the number of components of $R_1 = G_1 - C_1$ is as small as possible. Let $r_1 = |R_1|$, R_1' be the largest component of R_1 and $r_1' = |R_1'|$. The subscripts of c_i will be reduced modulo $n_1 - r_1$ throughout. Obviously, we have $|V(C_1)| = n_1 - r_1 \geqslant 6$.

For any $A, B \subseteq V(G_1)$, let

$$e(A, B) = |\{uv \in E(G_1) : u \in A, v \in B\}|$$

$$e(A) = |\{uv \in E(G_1) : u, v \in A\}|.$$

For any $D \subseteq V(C_1)$, let

$$D^+ = \{c_{i+1} : c_i \in D\} \text{ and } D^- = \{c_{i-1} : c_i \in D\}.$$

Case 1. R_1 contains an isolated vertex v_0.

Define that $Y_0 = \emptyset$, and for any $j \geqslant 1$,

$$X_j = N(Y_{j-1} \cup \{v_0\})$$

$$Y_j = \{c_i \in C_1 : c_{i-1}, c_{i+1} \in X_j\}$$

and

$$X = \bigcup_{i=1}^{\infty} X_j, \quad Y = \bigcup_{i=0}^{\infty} Y_j, \quad x = |X| \geqslant k \quad \textit{and} \quad y = |Y|.$$

By the hopping lemma, we have $X \subset V(C_1)$, $X \cap Y = \emptyset$ and X dose not contain two consecutive vertices of C_1.

Let $S_1, S_2, \cdots, S_x$ be the sets of vertices contained in the open segments of C_1 between vertices of X. Put $\phi = \{S_i : |S_i| \geqslant 2, 1 \leqslant i \leqslant x\}$. Then $S_i = \{c_l, c_{l+1}, \cdots, c_m\} \in \phi$ is said to be ψ-connected to $S_j = \{c_q, c_{q+1}, \cdots, c_z\} \in \phi$ if $|S_i|$ is odd and c_q and c_z are both joined to c_{l+e} for all odd e, $1 \leqslant e \leqslant m - l - 1$. Now, $c_{l+1}, c_{l+3}, \cdots, c_{m-1}$ are called

P-vertices of S_i. Set $P = \{c_i \in V(C_1) : c_i$ is a P-vertex of some S_j which is ψ-connected to some S_t of $\phi\}$, and $p = |P|$.

Since

$$e(V(G_1) - X, X) = (n_1 - 2 - x)k + 4 - 2e(V(G_1) - X)$$

$$e(X, V(G_1) - X) \leqslant xk \tag{1}$$

we have

$$2e(V(G_1) - X) \geqslant (n_1 - 2 - 2x)k + 4. \tag{2}$$

On the other hand, under the properties of C_1, we can follow the series of the arguments in [3] and finally have the following inequality:

$$\begin{aligned} 2e(V(G_1) - X) \leqslant p(k + n_1 - x - y - p - 1) + (n_1 - 2x - 2p - 1)(n_1 - 2x) \\ -p(x - y - p) - 2(r_1 - 1)(x - y - 1). \end{aligned} \tag{3}$$

Combining (3) with (2), it can be deduced that

$$p + 4 \leqslant (n_1 - 2x - k)(n_1 - 1 - 2x - p) + k - 2(r_1 - 1)(x - y - 1). \tag{4}$$

By the definitions of X and Y, we have $x \geqslant y$. If $x = y$, we have

$$e(Y \cup \{v_0\}, X) = k + (y - 2)k + 4 = xk + 4 - k,$$

contrary to (1) because of $k \geqslant 5$. It follows that $2(r_1 - 1)(x - y - 1) \geqslant 0$. From the definition of P, we have $p \leqslant \frac{n_1 - 1 - 2x}{2}$, which implies $n_1 - 1 - 2x - p \geqslant 2p - p \geqslant 0$. And $k \geqslant n_1 - 2x - p - 1$ by $n_1 \leqslant 2k + 2$ and $x \geqslant k$. So we have

$$p + 4 \leqslant (n_1 - 2x - k + 1)k - 2(r_1 - 1)(x - y - 1). \tag{5}$$

Therefore by (5), we have $n_1 - 2x - k + 1 > 0$, and then $n_1 > 3k - 1$, a contradiction.

The next two cases in this part are both discussed that R_1 contains no isolated vertex.

For a path $Q = q_1, q_2, \cdots, q_g, g \geqslant 2$, in R_1, let $t(Q)$ denote the number of occurrences of ordered pair (c_i, c_j) of the vertices of C_1 such that c_i is joined to one of q_1 and q_g, c_j is joined to the other, and $e(\{q_1, q_g\}, \{c_{i+1}, c_{i+2}, \cdots, c_{j-1}\}) = 0$. We say that Q satisfies the condition $(*)$ if $t(Q) \geqslant 2$, $N_{C_1}(\{q_1, q_g\}) \not\subset \{u, v, z\}$ and there is a ordered pair (c_i, c_j) of the vertices of C_1 such that u, v, z, w_1 *and* $w_2 \notin \{c_{i+1}, c_{i+2}, \cdots, c_{j-1}\}$. Put $A = N_{C_1}(q_1)$ and $B = N_{C_1}(q_g)$.

Case 2. $2 \leqslant |R_1'| \leqslant k - 1$.

Before the proof of this case, we derive some results about the structure of R_1'.

Lemma 5. *There exists a maximal path Q in R_1' such that Q satisfies $(*)$.*

Proof. Since $|R_1'| \leqslant k - 1$, for any $v_i \in V(R_1')$, $i = 1, \cdots, r_1'$, we have $N_{C_1}(v_i) \geqslant 2$. By the assumption of 2-connectivity and $\{u, v, z\}$ is not a cut-set, there exists a path $Q = q_1, q_2, \cdots, q_g$ in R_1', which is chosen as long as possible such that Q satisfies $(*)$.

If Q is not a maximal path of R_1', let $Q' = b_1, b_2, \cdots, b_s, q_1, q_2, \cdots, q_g, q_{g+1}, \cdots, q_e$ be a maximal path in R_1' containing Q. Without loss of generality, we assume $s \geqslant 1$.

From the definition of Q, it is easy to see that $N_{C_1}(b_1) \leqslant 3$. So $N_{R_1'}(b_1) \geqslant k - 3$, and there is at most one vertex in $R_1^{'}$ which is not adjacent to b_1.

We consider the following two cases.

Case (a): $b_1 q_2 \in E(G_1)$.

In this case, there is a longer path $Q^{''} = q_1, b_s, b_{s-1}, \cdots, b_1, q_2, \cdots, q_g$ than Q that satisfies $(*)$, a contradiction of the definition of Q.

Case (b): $b_1 q_2 \notin E(G_1)$.

In this case, if $s \geqslant 2$, there is a longer path $Q^{''} = q_1, b_s, b_{s-1}, \cdots, b_1, q_3, \cdots, q_g$ than Q that satisfies $(*)$. If $s = 1$, we claim that $N_{C_1}(q_2) \leqslant 3$, otherwise, $Q^{''} = q_2, q_1, b_1, q_3, \cdots, q_g$ is a longer path than Q that satisfies $(*)$. Therefore, q_2 is joined to every vertex of R_1' except b_1. There is a longer path $Q^{''} = q_1, b_1, q_3, q_2, q_4, \cdots, q_g$ than Q that satisfies $(*)$, a contradiction.

A similar argument holds if $e > g$. $\square$

Lemma 6. *There exists a maximal path Q in R_1' such that $t(Q) \geqslant 3$.*

Proof. Suppose that Q satisfies the property of Lemma 5 and $t(Q) = 2$. Then we consider the following two cases:

case (a): $A = B = \{c_i, c_j\} \not\subset \{u, v, z\}$, $c_i \neq v$ and $c_j \neq v$;

case (b): $A = \{c_{i_1}, c_{i_2}, \cdots, c_{i_s}\}$ and $B = \{c_{j_1}, c_{j_2}, \cdots, c_{j_l}\}$ such that $s \geqslant 2, l \geqslant 2$ and $\{c_{i_1}, c_{i_2}, \cdots, c_{i_s}\} \cap \{c_{j_1}, c_{j_2}, \cdots, c_{j_l}\} = \emptyset$.

If case (a) occurs, we have $g = r'_1 = k - 1$. Without loss of generality, let $\{w_1, w_2\} \notin \{c_{i+1}, c_{i+2}, \cdots, c_{j-1}\}$, then we have $c_d = \mathrm{c}_i{}^-$ or $c_{d'} = \mathrm{c}_j{}^+$ such that $c_d \notin \{w_1, w_2\}$ or $c_{d'} \notin \{w_1, w_2\}$. Clearly, we have

$$N_{C_1}(c_d) \cap [Q \cup \{c_{j-1}, c_{j-2}, \cdots, c_{j-g}\} \cup c_d] = \emptyset$$

or

$$N_{C_1}(c_{d'}) \cap [Q \cup \{c_{i+1}, c_{i+2}, \cdots, c_{i+g}\} \cup c_{d'}] = \emptyset.$$

And there is at least two of $\{v, w_1, w_2\}$ which can not be adjacent to c_d or $c_{d'}$. It follows that

$$d_{C_1}(c_d) \leqslant 2k + 2 - 2(k - 1) - 3 = 1$$

or

$$d_{C_1}(c_{d'}) \leqslant 2k + 2 - 2(k - 1) - 3 = 1.$$

This is a contradiction.

For case (b), without loss of generality, let $w_1,\ w_2 \notin \{c_{j_1}, c_{j_1+1}, \cdots, c_{j_l}\}$, then there exists either some $c_z \in A^+$ satisfying $N_{C_1}(c_z) \cap \left[Q \cup (\bigcup_{h=1}^{l-1} \{c_{j_h+1}, c_{j_h+2}\}) \cup \{w_1, w_2, c_z\}\right] = \emptyset$, or some $c_f \in A^-$ satisfying $N_{C_1}(c_f) \cap \left[Q \cup (\bigcup_{h=2}^{l} \{c_{j_h-1}, c_{j_h-2}\}) \cup \{w_1, w_2, c_f\}\right] = \emptyset$. Which implies

$$d_{C_1}(c_z) \leqslant 2k + 2 - [g + 2(l - 1) + 3] \leqslant k - 2$$

or

$$d_{C_1}(c_f) \leqslant 2k + 2 - [g + 2(l - 1) + 3] \leqslant k - 2$$

a contradiction. □

Corollary 7. *If $t(Q) \geqslant 3$, then $g \leqslant k-2$.*

Proof. Suppose $g \geqslant k-1$ and $t(Q) \geqslant 3$. Then $|A \cup B| \geqslant 3$, we have

$$\begin{aligned} 2k+2 \geqslant |V(G_1)| &= |R_1| + |V(C_1)| \\ &\geqslant r_1 + |A \cup B| + (t(Q)-2)g + 2 \\ &\geqslant 2(k-1) + 3 + 2 = 2k+3 \end{aligned}$$

a contradiction. □

Lemma 8. *There exists a maximal path Q in R_1' such that $t(Q) \geqslant 3$. Then $A = B$.*

Proof. By contradiction. Suppose $B \neq A$ and $|B-A| \geqslant 1$, without loss of generality, $|B| \geqslant |A|$. We have

$$\begin{aligned} V(C_1) = n_1 - r_1 &\geqslant |A \cup B| + |A^+ \cup A^- \cup B^+ \cup B^-| + (\mathrm{t}-2)(\mathrm{g}-2) \\ &= |A| + |B{-}A| + |A^+ \cup B^-| + |(A^- \cup B^+) - (A^+ \cup B^-)| + (\mathrm{t}-2)(\mathrm{g}-2). \end{aligned}$$

Since $g \geqslant 2$, if $c_d \in A^+ \cap B^-$, we have $c_d = w_1$ or $c_d = w_2$. Let $t(Q) = t$, $\sigma = |B{-}A| + |(A^- \cup B^+) - (A^+ \cup B^-)|$, and

$$\theta = \begin{cases} -2 & if\ u \in A,\ v \in A \cap B,\ z \in B \\ -1 & if\ u \in A,\ v \in B \backslash A\ \ or\ \ v \in A \backslash B,\ z \in B \\ 0 & otherwise. \end{cases}$$

So we have

$$n_1 - r_1 \geqslant |A| + \left|A^+\right| + \left|B^-\right| + \theta + \sigma + (t-2)(g-2).$$

By the maximality of Q, $|A| \geqslant k-g+1$ and $|B| \geqslant k-g+1$. Therefore we have

$$\begin{aligned} 2k+2-r_1 &\geqslant |A| + |A^+| + |B^-| + \theta + \sigma + (t-2)(g-2) \\ 2k+2-r_1 &\geqslant 3(k-g+1) + \theta + \sigma + (t-2)(g-2) \\ 1-\theta &\geqslant r_1 - g + k - g + \sigma + (t-3)(g-2). \end{aligned}$$

By Corollary 7, we have

$$-1-\theta \geqslant r_1 - g + \sigma + (t-3)(g-2).$$

Since $t(Q) \geqslant 3$, $g \geqslant 2$, $r_1 \geqslant g$ and $\sigma \geqslant 1$, we have a contradiction when $\theta = 0$ or $\theta = -1$. If $\theta = -2$, we have $(A^- \cup B^+) - (A^+ \cup B^-) = \emptyset$ which implies $\theta = 0$, a contradiction. In fact, let $c_i \in B - A$ be the vertex such that the next vertex of $A \cup B$ after c_i belongs to A. Since $c_{i+1} \notin (A^- \cup B^+) - (A^+ \cup B^-)$, we have c_{i+1} is in B^-, which implies $c_{i+2} \in B \cap A$ and then $c_i = u \in B - A$ and $c_{i+2} = v \in B \cap A$, or $c_i = v \in B - A$ and $c_{i+2} = z \in B \cap A$. According to the definition of θ, we see $\theta = 0$. □

Lemma 9. *There exists a maximal path Q in R_1' such that $t(Q) \geqslant 3$. Then $g = k - t + 1$.*

Proof. Clearly, $g \geqslant k - t + 1$. If $g \geqslant k - t + 2$, by Lemma 8 and Corollary 7, we have $2 \leqslant g \leqslant k - 2$. Thus,

$$2k+2 \geqslant V(C_1) + g \geqslant g(t-2) + 2 + g + t \geqslant (g+1)(t-1) + 3 \geqslant (g+1)(k-g+1) + 3.$$

But since $f(g) = (g+1)(k-g+1)+3$ is a concave function of g and $f(2) = f(k-2) = 3k > 2k+2$, we have $f(g) > 2k+2$, a contradiction. □

Now, let $Q = q_1, q_2, \cdots, q_g$ be a maximal path in R_1' such that $t(Q) \geqslant 3$ and $A = B$. We write $X' = A = B = \{x_1', x_2', \cdots, x_t'\}$.

Put $D = \{S_i, 1 \leqslant i \leqslant t\}$, where S_i is the set of vertices contained in the open segment of C_1 between two vertices of X'. Let $D' = \{S_i^*, i = 1, 2\}$ denote the element of D which contains w_1 or w_2 (If w_1 and w_2 is contained in a same segment, let $D' = S^*$). Let $D'' = D - D'$. The structure of D has two cases:

Case (a): w_1 and w_2 is contained in a same segment S^*.

By Lemma 9, we have

$$\begin{aligned}
n_1 &\geqslant |V(C_1)| + |V(R_1)| \\
&\geqslant g(t-1) + 2 + (|S^*| - 2) + \sum_{S_i \in D''} (|S_i| - g) + t + g + (r_1 - g) \\
&\geqslant (g+1)t + 2 + (|S^*| - 2) + \sum_{S_i \in D''} (|S_i| - g) + (r_1 - g) \\
&\geqslant (g+1)(k-g+1) + 2 + (|S^*| - 2) + \sum_{S_i \in D''} (|S_i| - g) + (r_1 - g).
\end{aligned}$$

Put $f'(g) = (g+1)(k-g+1) + 2$. Since $f'(g)$ is a concave function of g with $f'(2) = 3k - 1 = f'(k-2)$, we obtain a contradiction that

$$2k + 2 \geqslant 3k - 1 + (|S^*| - 2) + \sum_{S_i \in D''} (|S_i| - g) + (r_1 - g).$$

Case (b): w_1 is contained in S_1^*, and w_2 is contained in S_2^*.

By Lemma 9, we have

$$\begin{aligned}
n_1 &\geqslant |V(C_1)| + |V(R_1)| \\
&\geqslant g(t-2) + 2 + \sum_{i=1}^{2} (|S_i^*| - 1) + \sum_{S_i \in D''} (|S_i| - g) + t + g + (r_1 - g) \\
&\geqslant (g+1)(t-1) + 3 + \sum_{i=1}^{2} (|S_i^*| - 1) + \sum_{S_i \in D''} (|S_i| - g) + (r_1 - g) \\
&\geqslant (g+1)(k-g) + 3 + \sum_{i=1}^{2} (|S_i^*| - 1) + \sum_{S_i \in D''} (|S_i| - g) + (r_1 - g).
\end{aligned}$$

Put $f''(g) = (g+1)(k-g) + 3$. When $2 \leqslant g \leqslant k-3$, $f''(g)$ is a concave function of g with $f''(2) = 3k - 3 = f''(k-3)$, we have

$$2k + 2 \geqslant 3k - 3 + \sum_{i=1}^{2} (|S_i^*| - 1) + \sum_{S_i \in D''} (|S_i| - g) + (r_1 - g).$$

There is a contradiction when $k \geqslant 6$ from

$$5 - k \geqslant \sum_{i=1}^{2} (|S_i^*| - 1) + \sum_{S_i \in D''} (|S_i| - g) + (r_1 - g).$$

When $k = 5$, we have $r_1 = g$, $|S_i| = g$ for all $S_i \in D''$ and $|S_i^*| = 1$ for $i = 1, 2$. By Lemma 9, we have $t = k - g + 1 = 6 - g$. For any elements S_i and S_j of D'', we have

$e(S_i, S_j) = 0$ because of the maximality of C_1. Firstly, if there is some q_i *of* $Q - \{q_1\}$ such that $N_{C_1}(q_i) \cap S_i \neq \emptyset$ for some $S_i \in D''$. By Lemma 9, since $q_{i-1}q_g \in E(G_1)$, then $Q' = q_1, q_2, \cdots, q_{i-1}, q_g, q_{g-1}, \cdots, q_i$ is a path satisfying $(*)$ in R_1', which implies $2g+1 \leqslant g$. So we have $N_{C_1}(Q) \subset X'$. Secondly, we have $e(X', V(G_1) - X') \leqslant kt = 5t$. Moreover, we also have

$$kt \geqslant e(V(G_1) - X', X') \geqslant gt + (t-2)g(k-g+1) + 4.$$

By Lemma 6 we deduce that

$$(5-g)(6-g) \geqslant (4-g)g(6-g) + 4.$$

Because $2 \leqslant g \leqslant 3$ and g is an integer, we have

$$(5-g)(6-g) \leqslant (4-g)g(6-g) + 4$$

a contradiction.

If $g = k-2$, we have $t = 3$. So there exists $x_i' \in X'$ such that $x_i^{'-} \notin \{w_1, w_2\}$ or $x_i^{'+} \notin \{w_1, w_2\}$. It is clearly that

$$d_{G_1}(x_i^{'-}) \leqslant 2k + 2 - 2(k-2) - 2 = 4$$

or

$$d_{G_1}(x_i^{'+}) \leqslant 2k + 2 - 2(k-2) - 2 = 4$$

a contradiction.

Case 3. $|R_1'| \geqslant k$.

By the assumption of connectivity and $\{u, v, z\}$ is not a cut-set, there exists $x'' \in N_{C_1}(R_1')$, such that $x^{''-} \notin \{w_1, w_2\}$. It is clearly that $N_{G_1}(x^{''-}) \cap R' = \emptyset$, and at least two of $\{v, w_1, w_2\}$ cannot be adjacent to $x^{''-}$. It follows that

$$d_{G_1}(x^{''-}) \leqslant 2k + 2 - k - 2 - 1 = k - 1$$

a contradiction.

These contradictions complete our proof in this part. We next discuss the cases of $k = 3$ and $k = 4$. Similarly, let C be a longest cycle of G containing P and $R = G - C$. Clearly, $|C| \geqslant 4$. By Theorem 1, we only need to discuss the cases that $2k-1 \leqslant |V(G)| = n \leqslant 2k$.

When $k = 3$, $5 \leqslant n \leqslant 6$. If $n = 5$, we consider the following two cases.

Case (a): $|C| = 5$. Theorem 3 holds.

Case (b): $|C| = 4$. Let $C = uvzx_1$. Then R is an isolated vertex v_0. It is easy to see that there exist two consecutive vertices of $\{u, z, x_1\}$ which are adjacent to v_0. A contradiction of that C is the longest cycle of G containing P.

If $n = 6$, we consider the following three cases.

Case (a): $|C| = 6$. Theorem 3 holds.

Case (b): $|C| = 5$. Let $C = uvzx_1x_2$. Then R is an isolated vertex v_0. By assumption, we have $N_C(v_0) = \{v, z, x_2\}$ or $N_C(v_0) = \{u, v, x_1\}$. By symmetry, we consider the case of $N_C(v_0) = \{v, z, x_2\}$. Since $ux_1 \in E(G)$, there is a Hamiltonian cycle $C' = u, v, z, v_0, x_2, x_1, u$ containing P. Theorem 3 holds.

Case (c): $|C| = 4$. Let $C = uvzx_1$.

Subcase (c1): R contains an isolated vertex v_0. It is similar to that of the case(b) when $k = 3$ and $n = 5$.

Subcase (c2): R contains no isolated vertex. So the vertices of C are adjacent to R. This contradict with the assumption that C is the longest cycle of G containing P.

When $k = 4$, $7 \leqslant n \leqslant 8$. If $n = 7$, we consider the following four cases.

Case (a): $|C| = 7$. Theorem 3 holds.

Case (b): $|C| = 6$. Let $C = uvzx_1x_2x_3$. Then R is an isolated vertex v_0. By assumption, we have $N_C(v_0) = \{u, v, z, x_2\}$. If $x_1x_3 \notin E(G)$, we have $x_1u, x_1v \in E(G)$ which makes $d_G(x_3) \leqslant 3$, a contradiction. So we have $x_1x_3 \in E(G)$. There is a Hamiltonian cycle $C' = u, v, z, v_0, x_2, x_1, x_3, u$ containing P. Theorem 3 holds.

Case (c): $|C| = 5$. Let $C = uvzx_1x_2$.

Subcase (c1): R contains an isolated vertex v_0. It is easy to see that there exist two consecutive vertices of $\{u, z, x_1, x_2\}$ which are adjacent to v_0, a contradiction.

Subcase (c2): R contains no isolated vertex. Since G is 4-regular graph, we have $d_C(R) \geqslant 6$ and $N_C(R) \geqslant 3$. When $N_C(R) = 3$, by assumption, we have $N_C(R) = \{v, z, x_2\}$ or $N_C(R) = \{u, v, x_1\}$. By symmetry, we consider the case of $N_C(R) = \{v, z, x_2\}$ in which we have $d_G(u) \leqslant 3$, a contradiction. When $N_C(R) \geqslant 4$, there exist two consecutive vertices of $\{u, z, x_1, x_2\}$ which are adjacent to R, a contradiction.

Case (d): $|C| = 4$. Let $C = uvzx_1$. For every connected component R' of R, $N_C(R') \geqslant 3$. Clearly, there exist two consecutive vertices of $\{u, z, x_1\}$ which are adjacent to R', a contradiction.

If $n = 8$, we consider the following five cases.

Case (a): $|C| = 8$. Theorem 3 holds.

Case (b): $|C| = 7$. Let $C = uvzx_1x_2x_3x_4$. Then R is an isolated vertex v_0. By assumption, we have $N_C(v_0) = \{u, v, z, x_2\}$, $N_C(v_0) = \{u, v, z, x_3\}$, $N_C(v_0) = \{u, v, x_1, x_3\}$ or $N_C(v_0) = \{v, z, x_2, x_4\}$. By the same discussion as for n=7 when k=4, there is a Hamiltonian cycle containing P in all cases.

Case (c): $|C| = 6$. Let $C = uvzx_1x_2x_3$.

Subcase (c1): R contains two isolated vertices v_0 and v_1. By assumption, we have $N_C(v_0) = N_C(v_1) = \{u, v, z, x_2\}$, $d_G(x_1) \leqslant 3$, a contradiction.

Subcase (c2): R is an edge $e = v_0v_1$. Since G is a 4-regular graph, we have $d_C(R) \geqslant 6$ and $N_C(R) \geqslant 3$. When $N_C(R) = 3$, we have $N_C(v_0) = N_C(v_1)$. By assumption, we have $N_C(R) = \{u, v, x_1\}$, $N_C(R) = \{v, z, x_3\}$, $N_C(R) = \{u, v, x_2\}$, $N_C(R) = \{v, z, x_2\}$, $N_C(R) = \{u, z, x_2\}$ or $N_C(R) = \{x_1, x_3, v\}$. In the discussion of all cases, either there is a contradiction of regularity, or there is a Hamiltonian cycle containing P. When $N_C(R) \geqslant 4$, it is clear that there is no consecutive vertices of $\{u, z, x_1, x_2, x_3\}$ which are adjacent to R. So $N_C(R) = \{u, v, z, x_2\}$. We claim $x_1x_3 \in E(G)$, and then there is a Hamiltonian cycle $C' = u, v, z, v_1, v_0, x_2, x_1, x_3, u$ containing P. Otherwise, $ux_1, vx_1 \in E(G)$ which makes $d_G(x_3) \leqslant 3$, a contradiction.

Case (d): $|C| = 5$. It is similar to that of the case(c) when $k = 4$ and $n = 7$.

Case (e): $|C| = 4$. Let $C = uvzx_1$. Obviously, R contains no isolated vertex. Let R' be a connected component of R. If $N_C(R') \geqslant 3$, it is clear that there exist two consecutive vertices of $\{u, z, x_1\}$ which are adjacent to R', a contradiction. If $N_C(R') = 2$, we have $N_C(R) = \{v, x_1\}$, which makes $d_G(u) \leqslant 3$, a contradiction.

Thus, we complete the proof.

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