# An asymptotic resolution of a conjecture of Szemerédi and Petruska 

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#### Abstract

Consider a 3 -uniform hypergraph of order $n$ with clique number $k$ such that the intersection of all its $k$-cliques is empty. Szemerédi and Petruska proved $n \leq$ $8 m^{2}+3 m$, for fixed $m=n-k$, and they conjectured the sharp bound $n \leq\binom{ m+2}{2}$. This problem is known to be equivalent to determining the maximum order of a $\tau$-critical 3 -uniform hypergraph with transversal number $m$ (details may also be found in a companion paper [9]).

The best known bound, $n \leq \frac{3}{4} m^{2}+m+1$, was obtained by Tuza using the machinery of $\tau$-critical hypergraphs. Here we propose an alternative approach, a combination of the iterative decomposition process introduced by Szemerédi and Petruska with the skew version of Bollobás's theorem on set pair systems. The new approach improves the bound to $n \leq\binom{ m+2}{2}+O\left(m^{5 / 3}\right)$, resolving the conjecture asymptotically.


## 1 Introduction

Let $\mathcal{N}=\left\{N_{1}, \ldots, N_{\ell}\right\}$ be a collection of $k$-subsets of $\{1, \ldots, n\}$. Set $V=\bigcup_{i=1}^{\ell} N_{i}$. Assume that $n=|V|, \ell \geq 2$, and $k \geq 3$. Set $m=n-k$; that is, $\left|\overline{N_{i}}\right|=\left|V \backslash N_{i}\right|=m$. We further
assume that $\mathcal{N}$ satisfies the following two properties:
(i) $\bigcap_{i=1}^{\ell} N_{i}=\varnothing$, but $\bigcap_{j \neq i} N_{j} \neq \varnothing$ for all $i=1, \ldots, \ell$.
(ii) For any $S \subseteq V$ such that $|S| \geq k+1$, there exists a subset $T \subseteq S$ such that $|T|=3$ and $T \nsubseteq N_{i}$, for all $i=1, \ldots, \ell$.

Notice that property ( $i$ ) means that $\mathcal{N}$ is minimal with respect to having empty intersection; and property ( $i i$ ) may be interpreted as stating that, in the 3-uniform hypergraph induced by all 3 -subsets of the $k$-sets in $\mathcal{N}$, the largest cliques have $k$ vertices.

We shall refer to a collection $\mathcal{N}$ satisfying $(i)$ and (ii) as an $(n, m)$-structure. Szemerédi and Petruska [10] conjectured the following:
Conjecture 1. Any $(n, m)$-structure satisfies $n \leq\binom{ m+2}{2}$.
Szemerédi and Petruska give the following construction to show that Conjecture 1, if true, would be sharp. For fixed integers $k, m \geq 3$ such that $k-1=\binom{m+1}{2}$, begin with disjoint sets $A$ and $G$ satisfying $|A|=k-1$ and $|G|=m+1$. Let $A=\left\{a_{1}, \ldots, a_{k-1}\right\}$ and let $\left\{p_{1}, \ldots, p_{k-1}\right\}$ be the set of pairs (2-subsets) of $G$. Define $N_{i}=\left(A \backslash\left\{a_{i}\right\}\right) \cup p_{i}$, for $i=$ $1, \ldots, k-1$. It is easy to check that the collection $\mathcal{N}=\left\{N_{1}, \ldots, N_{k-1}\right\}$ satisfies properties (i) and (ii). In particular, the ( $n, m$ )-structure $\mathcal{N}$ induces a 3 -uniform hypergraph $H$ on vertex set $A \cup G$ with all 3 -subsets included by some $k$-set of $\mathcal{N}$ forming the edge set; furthermore, the order of $H$ is $n=|A|+|G|=k+m=\binom{m+2}{2}$ and $\omega(H)=k$.

For $m=2,3$, there are other extremal ( $n, m$ )-structures; however, it has been conjectured (by us and others) that this construction is the unique extremal structure for $m \geq 4$. For $m=2,3$ and 4 , Conjecture 1 has been verified and all extremal structures have been characterized by Jobson et al. [7].

The best known general bound, $n \leq \frac{3}{4} m^{2}+m+1$, was obtained by Tuza ${ }^{1}$ using the machinery of $\tau$-critical hypergraphs. Here we significantly refine and vindicate an alternative approach first proposed by Jobson et al. [6]. This approach develops the iterative decomposition process introduced by Szemerédi and Petruska and applies the skew version of Bollobás's theorem [2] on the size of cross-intersecting set pair systems. The result (Theorem 23) is an asymptotically tight upper bound:

$$
n \leq\binom{ m+2}{2}+6 m^{5 / 3}+3 m^{4 / 3}+9 m-3
$$

As noted by Gyárfás et al. [4], the Szemerédi and Petruska problem is equivalent to determining the maximum order of a $\tau$-critical 3-uniform hypergraph with transversal number $m$. More generally, they also determined that $O\left(m^{r-1}\right)$ is the correct order of magnitude for the maximum order of a $\tau$-critical $r$-uniform hypergraph with transversal number $m$; the best known bounds were obtained by Tuza [11].

A companion paper [9] shows the details of the mentioned equivalence and presents further remarks on the origin of the Szemerédi and Petruska problem. As an immediate corollary of our main theorem and this equivalence we obtain a new and asymptotically tight bound for the order of a $\tau$-critical 3 -uniform hypergraph.

[^0]Theorem 1. If $H$ is a 3-uniform $\tau$-critical hypergraph with $\tau(H)=t$, then

$$
|V(H)| \leq\binom{ t+2}{2}+O\left(t^{5 / 3}\right)
$$

Section 2 introduces notations and recalls the process, introduced by Szemerédi and Petruska, to decompose an $(n, m)$-structure into stages. Consecutive stages can be viewed as survival times of fewer and fewer $k$-sets of the system. The basic concepts associated with an ( $n, m$ )-structure are the 'kernels' belonging to the surviving $k$-sets at each time and the 'private pairs' selected for the remaining $k$-subsets at each stage (the private pairs are pairs that belong to precisely one $k$-subset of the surviving subsystem at a stage of the decomposition). For example, in the conjectured extremal Szemerédi-Petruska construction described above, there is only one stage; the set $A$ is the kernel of the only stage, and the 2 -subsets of $G$ are the private pairs of the $k$-sets. In Sections 3, 4 and 5 the iterative decomposition process used by Szemerédi and Petruska is extended considerably.

In addition to extending the iterative decomposition process our new approach applies the skew version of Bollobás's theorem [2] on the size of cross-intersecting set pair systems (Theorem 21): If $A_{1}, \ldots, A_{h}$ are $r$-element sets and $B_{1}, \ldots, B_{h}$ are $s$-element sets such that $A_{i} \cap B_{i}=\varnothing$ for $i=1, \ldots, h$, and $A_{i} \cap B_{j} \neq \varnothing$ whenever $1 \leq i<j \leq h$, then $h \leq\binom{ r+s}{r}$. Theorem 21 will be applied in Section 7 with $r=2$ and $s=m$, where the 2 -sets are carefully selected private pairs, and the $m$-sets are derived iteratively from the $k$-sets corresponding to all those private pairs.

Section 3 introduces a recursive procedure, based on the decomposition process, to select special private pairs by 'promoting' the initial pairs as needed. Section 4 describes an additional processing of these special private pairs, known as 'advancement', which sacrifices or deactivates a limited number of private pairs in order to guarantee that others may advance into a protected position before choosing 'free' private pairs in Sections 5 and 6.

Sections 5 and 6 define a large subset of free private pairs chosen from the special advanced pairs obtained in Section 4. A skew cross-intersecting ( $2, m$ )-system ultimately arises from this subset of free private pairs and by appropriately adjusting the complements of the corresponding $k$-sets. This is done by using the recursive process (4) in Section 7, where the proof of the main result, Theorem 23, is concluded.

## 2 The Decomposition Process

We begin by giving definitions and recalling the process, introduced by Szemerédi and Petruska ${ }^{2}$, to decompose $(n, m)$-structures. Much of this section is very similar to their presentation. We assume $\ell \geq 4$ since Szemerédi and Petruska resolve the cases $\ell=2,3$. Let $\mathcal{N}=\left\{N_{1}, \ldots, N_{\ell}\right\}$, be an $(n, m)$-structure. Define a collection of objects in stages, which are also called times, starting with stage 0 . Set $\ell_{0}=\ell, \mathcal{N}^{(0)}=\mathcal{N}$ and $N_{i}^{(0)}=N_{i}$.

[^1]For every $i=1, \ldots, \ell_{0}$, fix a choice of vertex $a_{i}^{(0)} \in \bigcap_{j \neq i} N_{j}$. By definition, $a_{i}^{(0)} \neq a_{j}^{(0)}$, for $i \neq j$. The set $A^{(0)}=\left\{a_{1}^{(0)}, \ldots, a_{\ell_{0}}^{(0)}\right\}$ is called the kernel and $a_{1}^{(0)}, \ldots, a_{\ell_{0}}^{(0)}$ are called the kernel vertices at stage 0 .

Assume that the collection of $k$-subsets $\mathcal{N}^{(j)}=\left\{N_{1}^{(j)}, \ldots, N_{\ell_{j}}^{(j)}\right\}$ and a corresponding kernel $A^{(j)}=\left\{a_{1}^{(j)}, \ldots, a_{\ell_{j}}^{(j)}\right\}$ are defined for all stages $0,1, \ldots, j$. Also assume that $\ell_{j} \geq 4$ and the sets in the 'remainder' structure

$$
R^{(j)}=\left\{N_{1}^{(j)} \backslash \bigcup_{i=0}^{j} A^{(i)}, \ldots, N_{\ell_{j}}^{(j)} \backslash \bigcup_{i=0}^{j} A^{(i)}\right\}
$$

have no common vertex. We refer to $N_{r}^{(j+1)}=N_{r}^{(j)} \backslash \bigcup_{i=0}^{j} A^{(i)}$ as the truncation of $N_{r}^{(j)}$. We now explain the definition of $\mathcal{N}^{(j+1)}$ and $A^{(j+1)}$.

Because the truncations of the $N_{r}^{(j)}$ 's in $R^{(j)}$ have no common vertex, there exist substructures of $R^{(j)}$ satisfying property ( $i$ ). Stop if $R^{(j)}$ contains such substructure(s) only with two or three sets. Otherwise, let

$$
\mathcal{N}^{(j+1)}=\left\{N_{1}^{(j+1)}, \ldots, N_{\ell_{j+1}}^{(j+1)}\right\} \subset R^{(j)}
$$

be a substructure satisfying $(i)$ and $\ell_{j+1} \geq 4$.
For $i=1, \ldots, \ell_{j+1}$, fix a choice of vertex $a_{i}^{(j+1)} \in \bigcap_{\substack{r=1 \\ r \neq i}}^{\ell_{j+1}} N_{r}^{(j+1)}$ and let $A^{(j+1)}=$ $\left\{a_{1}^{(j+1)}, \ldots, a_{\ell_{j+1}}^{(j+1)}\right\}$ be the kernel at time $j+1$. Observe that the sets in the remainder structure

$$
R^{(j+1)}=\left\{N_{1}^{(j+1)} \backslash \bigcup_{i=0}^{j+1} A^{(i)}, \ldots, N_{\ell_{j+1}}^{(j+1)} \backslash \bigcup_{i=0}^{j+1} A^{(i)}\right\}
$$

have empty intersection.
This process terminates at some time $t$, and defines $\ell_{j}, \mathcal{N}^{(j)}$, and $A^{(j)}$ for $j=0,1, \ldots, t$. The only substructures of the terminal remainder structure, $R^{(t)}$, that satisfy property (i) have 2 or 3 sets. Because $\mathcal{N}=\left\{N_{1}, \ldots, N_{\ell_{0}}\right\}$ is an arbitrary enumeration of $\mathcal{N}$, we may assume that

$$
\mathcal{N}^{(j)}=\left\{N_{1}, \ldots, N_{\ell_{j}}\right\}, \text { for } j=0, \ldots, t .
$$

That is, we linearly order the $k$-sets in $\mathcal{N}$ according to how long their truncations appear in the sequence of chosen substructures. The longer that its truncations appear, the earlier a set appears in this linear ordering.

Define, for $i=1, \ldots, \ell_{0}$, the last time (or stage), denoted $t_{i}$, that truncations of $N_{i}$ appear in a substructure of this decomposition process; equivalently, set

$$
t_{i}=\max \left\{j: N_{i} \in \mathcal{N}^{(j)}\right\}
$$

By definition, $\ell=\ell_{0} \geq \cdots \geq \ell_{t} \geq 4$ and $t=t_{1} \geq \cdots \geq t_{\ell_{0}} \geq 0$. Observe that $x \in\left\{1, \ldots, \ell_{j}\right\}$ implies $j \leq t_{x}$.

Define $A_{j}=\bigcup_{s=0}^{j} A^{(s)}$ and let $A=A_{t}$ be the set of all kernel vertices. Let $G=V \backslash A$ denote the garbage vertices; that is, the vertices remaining after the decomposition process terminates. The selected linear ordering of $\mathcal{N}$ together with the decomposition process induces a natural ordering of $V$ : vertices within a kernel $A^{(s)}$ are ordered according to their subscripts $\left(a_{i}^{(s)}<a_{j}^{(s)}\right.$ if and only if $i<j$ ), kernels are ordered according to stage $\left(A^{(r)}<A^{(s)}\right.$ if and only if $r<s$ ), and garbage vertices are ordered (linearly) last (in no particular order).

The next lemma begins a list of useful properties of the decomposition process. The results mentioned in Lemma 2 were proved in Lemma 5 and in Lemma 6 of [10]. We reprove these results here for completeness.

## Lemma 2.

(a) $t<m \leq|G|$
(b) $|G| \leq 3(m-t-1)$.

Proof. (a) Since $N_{1} \in \mathcal{N}^{(t)}$, we have $\left|\overline{N_{1}} \cap A\right|=t+1$; and because $\left|\overline{N_{1}}\right|=m$, it follows that $t<m$. Next suppose, on the contrary, that $|G| \leq m-1$. Observe that $|A|=$ $|V|-|G| \geq n-m+1=k+1$. By property (ii), there exists a set $T \subset A,|T|=3$, such that $T \nsubseteq N_{i}$, for all $1 \leq i \leq \ell$. Each element of $T$ is missed by at most one $N_{i}, 1 \leq i \leq \ell_{t}$. It follows that because $\ell_{t} \geq 4$, one of these sets contains $T$, a contradiction.
(b) When the decomposition process terminates any subsystem of

$$
R^{(t)}=\left\{N_{1} \backslash A, \ldots, N_{\ell_{t}} \backslash A\right\}
$$

satisfying ( $i$ ) has at most three sets; we may assume that these are $\left\{N_{1} \backslash A, N_{2} \backslash A, N_{3} \backslash A\right\}$ or $\left\{N_{1} \backslash A, N_{2} \backslash A\right\}$. Each set $N_{i}, i=1,2$ or 3 , 'survives' until stage $t$, hence

$$
r=\left|N_{i} \cap G\right|=\left|N_{i} \backslash A\right|=k-\left(\ell_{0}-1\right)-\cdots-\left(\ell_{t}-1\right)=k-|A|+t+1 .
$$

Thus, the subsystem consists of two or three $r$-sets that have empty intersection; so the complements of these sets in $G$ must satisfy $\left|\overline{N_{i}} \cap G\right| \geq \frac{r}{2}$. Consequently,

$$
|G|=\left|N_{1} \cap G\right|+\left|\overline{N_{1}} \cap G\right| \geq r+\frac{r}{2}=\frac{3}{2}(k-|A|+t+1) .
$$

By substituting $k=n-m$ and $n-|A|=|G|$, we obtain

$$
|G| \geq \frac{3}{2}(k-|A|+t+1)=\frac{3}{2}(-m+|G|+t+1),
$$

thus $|G| \leq 3(m-t-1)$ follows.

## 3 Selection of private pairs

For $j, 0 \leq j \leq t_{i}$, a pair of vertices $p \subset N_{i}$ is single-covered with respect to $\mathcal{N}^{(j)}$ if $N_{i}$ is the only set in $\mathcal{N}^{(j)}$ that contains $p$ as a subset. A pair that is contained in at least two sets in $\mathcal{N}^{(j)}$ is called a double-covered pair (at time $j$ ). If $p \subset N_{i}$ is single-covered with respect to $\mathcal{N}^{(j)}$, then it is called a private pair for $N_{i}$ at time $j$, or simply a private pair of $N_{i}$. We also simply say that " $p$ is private for $N_{i}^{(j)}$ " to mean " $p$ is a private pair for $N_{i}$ at time $j "$. Observe that, if a pair is private for $N_{i}^{(j)}$, then it remains a private pair for $N_{i}$ until (and including) time $t_{i}$. Similar terminology is used for private elements. For example, a vertex $v$ is private for $N_{i}^{(j)}$ if $N_{i}$ is the only set in $\mathcal{N}^{(j)}$ that contains $v$.

The following lemma is a rephrasing of Lemma 7(a) proven by Szemerédi and Petruska [10]. Recall the notation $A_{j}=\bigcup_{s=0}^{j} A^{(s)}$.

## Lemma 3.

(c) For all $j=0, \ldots, t$, every pair $p \subset A_{j}$ is double-covered at time $j$.

Proof. (c) Suppose that $p=\left\{a_{i_{1}}^{\left(j_{1}\right)}, a_{i_{2}}^{\left(j_{2}\right)}\right\} \subset A_{j}$, for some $j \in\{0, \ldots, t\}$. Necessarily $j_{1}, j_{2} \leq j$. Because $\ell_{j} \geq 4$, there exists $\left\{i_{3}, i_{4}\right\} \subseteq\left\{1, \ldots, \ell_{j}\right\} \backslash\left\{i_{1}, i_{2}\right\}$. By definition of the kernel, $p \subseteq N_{i_{3}}^{(j)} \cap N_{i_{3}}^{(j)}$ so $p$ is double-covered at time $j$.

The next three lemmas introduce new and powerful tools; they are a crucial refinement of the proof of Szemerédi and Petruska's Lemma 7(b) [10]. In particular, Lemma 4 introduces the novel notion of a 3 -cross.

## Lemma 4.

Suppose that $N_{i} \in \mathcal{N}^{(j)}, Y \subseteq N_{i}$ and $|Y| \leq j$.
(d) There exists a 3 -set $C_{i}^{(j)}(Y) \subseteq\left(N_{i} \cup\left\{a_{i}^{(0)}, \ldots, a_{i}^{(j)}\right\}\right) \backslash Y$ such that $C_{i}^{(j)}(Y) \cap \overline{N_{r}} \neq \varnothing$, for all $r=1, \ldots, \ell$.

Proof. (d) Observe that $S=\left(N_{i} \cup\left\{a_{i}^{(0)}, \ldots, a_{i}^{(j)}\right\}\right) \backslash Y$ has cardinality

$$
|S|=k+(j+1)-|Y| \geq k+1
$$

Applying property (ii) to $S$ produces a desired 3 -set $T=C_{i}^{(j)}(Y)$.
A 3 -set $C_{i}^{(j)}(Y)$, whose existence is established in Lemma $4(\mathrm{~d})$, is called a 3-cross of $N_{i}^{(j)}$ with respect to $Y$. If $C_{i}^{(j)}(Y)$ is a 3 -cross, then it is understood that $N_{i} \in \mathcal{N}^{(j)}$, $Y \subseteq N_{i},|Y| \leq j$ and $C_{i}^{(j)}(Y) \subseteq\left(N_{i} \cup\left\{a_{i}^{(0)}, \ldots, a_{i}^{(j)}\right\}\right) \backslash Y$.

Now we enumerate several important properties of 3 -crosses.

## Lemma 5.

Suppose that $C_{i}^{(j)}(Y)$ is a 3-cross. Let $p=C_{i}^{(j)}(Y) \cap N_{i}$.
(e) $|p|=1$ or $|p|=2$.
(f) If $|p|=1$, then the vertex in $p$ is in $\left(N_{i} \backslash Y\right) \cap G$ and private to $N_{i}^{(j)}$.
(g) If $|p|=2$, then $p$ is a 2 -set from $N_{i} \backslash Y$ and $p$ is a private pair for $N_{i}^{(j)}$.

Proof. (e) Because $\left|C_{i}^{(j)}(Y)\right|=3$ and $C_{i}^{(j)}(Y) \cap \overline{N_{i}} \neq \varnothing$, it follows that $\left|C_{i}^{(j)}(Y) \cap N_{i}\right| \leq 2$. If $\left|C_{i}^{(j)}(Y) \cap N_{i}\right|=0$, then $C_{i}^{(j)}(Y) \subseteq\left\{a_{i}^{(0)}, \ldots, a_{i}^{(j)}\right\}$ which implies that, because $\ell_{j} \geq 3$, there exists some $r$ such that $C_{i}^{(j)}(Y) \subseteq N_{r}$, a contradiction. Therefore, $1 \leq|p| \leq 2$.
(f) Because $|p|=1$ and $C_{i}^{(j)}(Y) \subseteq\left(N_{i} \cup\left\{a_{i}^{(0)}, \ldots, a_{i}^{(j)}\right\}\right) \backslash Y$, it follows that

$$
C_{i}^{(j)}(Y)=\left\{a_{i}^{(\alpha)}, a_{i}^{(\beta)}, v\right\}
$$

for some $v \in N_{i}$ and $0 \leq \alpha<\beta \leq j$. Note $p=\{v\}$ and $v \in N_{i} \backslash Y$.
Assume, to the contrary, that $v$ is not private to $N_{i}^{(j)}$. This means there exists some $r \neq i$ such that $v \in N_{r}^{(j)}$. Because $r \neq i$, the definition of the kernel implies $\left\{a_{i}^{(\alpha)}, a_{i}^{(\beta)}\right\} \subset$ $N_{r}$; therefore, $C_{i}^{(j)}(Y) \subseteq N_{r}$. However, this implies $C_{i}^{(j)}(Y) \cap \overline{N_{r}}=\varnothing$, contradicting that $C_{i}^{(j)}(Y)$ is a 3-cross (Lemma 4(d)). So $v$ is private to $N_{i}^{(j)}$. Because vertices in the kernel are double-covered (Lemma 3(c)), it follows that $v \in G$.
(g) We appropriately modify the argument given to establish (f). Because $|p|=2$ and $C_{i}^{(j)}(Y) \subseteq\left(N_{i} \cup\left\{a_{i}^{(0)}, \ldots, a_{i}^{(j)}\right\}\right) \backslash Y$, it follows that

$$
C_{i}^{(j)}(Y)=\left\{a_{i}^{(\alpha)}, u, v\right\}
$$

for some $u, v \in N_{i}$ and $0 \leq \alpha \leq j$. Note $p=\{u, v\}$ and $p \subseteq N_{i} \backslash Y$.
Assume, to the contrary, that $p$ is not a private pair for $N_{i}^{(j)}$. This means there exists some $r \neq i$ such that $p \subset N_{r}^{(j)}$. Because $r \neq i$, the definition of the kernel implies $a_{i}^{(\alpha)} \in N_{r}$; therefore, $C_{i}^{(j)}(Y) \subseteq N_{r}$. However, this implies $C_{i}^{(j)}(Y) \cap \overline{N_{r}}=\varnothing$, contradicting that $C_{i}^{(j)}(Y)$ is a 3 -cross (Lemma $4(\mathrm{~d})$ ). So $p$ is private pair for $N_{i}^{(j)}$.

## Lemma 6.

Suppose that $N_{i} \in \mathcal{N}^{(j)}, Y \subseteq N_{i}$ and $|Y| \leq j$.
(h) There exists a private pair, $p \subseteq N_{i} \backslash Y$, for $N_{i}^{(j)}$ such that either $p$ is a subset of a 3 -cross of $N_{i}^{(j)}$ with respect to $Y$, or $p$ does contain a private vertex for $N_{i}^{(j)}$.

Proof. Lemma 4(d) guarantees there exists a 3 -cross $C_{i}^{(j)}(Y)$. Let $p=C_{i}^{(j)}(Y) \cap N_{i}$; so $p \subset N_{i} \backslash Y$. Lemma 4(e) gives $|p|=1$ or $|p|=2$.

If $|p|=2$, then Lemma $5(\mathrm{~g})$ shows that $p$ is the desired private pair for $N_{i}^{(j)}$.

If $|p|=1$, then Lemma $5(\mathrm{f})$ shows that $p$ contains a private vertex for $N_{i}^{(j)}$. So it suffices now to note that a private pair for $N_{i}^{(j)}$ can be formed by adding any vertex from $N_{i} \backslash Y$ to $p$.

Next we describe an inductive process to select a collection of private pairs for every $N_{i}$. For each $i \in\{1, \ldots, \ell\}$, we define, by induction on time, a set $P_{i}=\left\{p_{i}^{(j)}: 0 \leq j \leq t_{i}\right\}$ of $t_{i}+1$ private pairs for $N_{i}$. The pair $p_{i}^{(j)}$ will be chosen from among the private pairs for $N_{i}$ existing at time $j$.

Notice that, by property Lemma 3(c), any private pair contains at least one garbage vertex. Therefore, $p_{i}^{(j)}$ contains a vertex $g_{i}^{(j)} \in p_{i}^{(j)} \cap G$; call it the anchor of $p_{i}^{(j)}$. The other vertex of $p_{i}^{(j)}$ is the non-anchor; it is denoted $u_{i}^{(j)}$. It is possible that $\left\{g_{i}^{(j)}, u_{i}^{(j)}\right\} \subset G$, but we shall still distinguish one vertex, $g_{i}^{(j)}$, as the anchor of $p_{i}^{(j)}$.

We also define auxiliary sets $P_{i}^{(j)}=\left\{p_{i}^{(s)}: 0 \leq s \leq j\right\}$ and $G_{i}^{(j)}=\left\{g_{i}^{(s)}: 0 \leq s \leq j\right\}$. They are, respectively, the initial segments of the private pairs and the initial segments of the anchors for the private pairs selected for $N_{i}$ up to time $j$.

Recall that every $k$-set $N_{i}$ has a final time, $t_{i}$, associated with it. Now we associate a final time with every $g \in G$.

Definition 1. For $g \in G$, define the critical time for $g$, denoted $t_{g}$, to be the last stage at which there exists a $k$-set that contains $g$. In other words, $t_{g}=\max \left\{t_{i}: g \in N_{i}\right\}$.

We also need the following definition.
Definition 2. For $g \in G$, define the critical index set for $g$, denoted $I_{g}$, to be the set of indices of the $k$-sets that contain $g$ at stage $t_{g}$. In other words,

$$
I_{g}=\left\{i: g \in N_{i}, t_{i}=t_{g}\right\} .
$$

Observe that, by definition, $I_{g} \neq \varnothing$, for all $g \in G$. As we shall see in Lemma 9, the garbage vertices $g \in G$ with $\left|I_{g}\right|=1$ are particularly significant. The private pair selection process utilizes this information, so this motivates the following definition.

Definition 3. For $x \in\{1, \ldots, \ell\}$, define the set of critical garbage vertices for $N_{x}$, denoted $\Gamma_{x}$, to be the set of $g \in N_{x} \cap G$ that have only $N_{x}$ in their critical index set. In other words,

$$
\Gamma_{x}=\left\{g \in G: I_{g}=\{x\}\right\}
$$

It is possible that $\Gamma_{x}=\varnothing$. If $x_{1} \neq x_{2}$, then by definition $\Gamma_{x_{1}} \cap \Gamma_{x_{2}}=\varnothing$.
Now we are ready to describe the private pair selection process. Initially, for $i \in$ $\{1, \ldots, \ell\}$, let $p_{i}^{(0)}=\left\{g_{i}^{(0)}, u_{i}^{(0)}\right\}$ be a private pair for $N_{i}$ at time zero. Such a private pair exists because of property Lemma $6(\mathrm{~h})$ with $Y=\varnothing$. Set $G_{i}^{(0)}=\left\{g_{i}^{(0)}\right\}$ and $P_{i}^{(0)}=\left\{p_{i}^{(0)}\right\}$.

For $j>0$ and $i \in\{1, \ldots, \ell\}$, assume that the sets $P_{i}^{(j-1)}$ and $G_{i}^{(j-1)}$ have already been defined. Also assume that a private pair $p_{i}^{(j)}=\left\{g_{i}^{(j)}, u_{i}^{(j)}\right\}$ has already been chosen for each $N_{i}$ with $j \leq t_{i}$. Now define

$$
G_{i}^{(j)}= \begin{cases}G_{i}^{(j-1)} \cup\left\{g_{i}^{(j)}\right\} & \text { if } j \leq t_{i} \\ G_{i}^{(j-1)} & \text { if } j>t_{i}\end{cases}
$$

and similarly define,

$$
P_{i}^{(j)}= \begin{cases}P_{i}^{(j-1)} \cup\left\{p_{i}^{(j)}\right\} & \text { if } j \leq t_{i} \\ P_{i}^{(j-1)} & \text { if } j>t_{i}\end{cases}
$$

This definition yields $P_{i}^{(j)}=\left\{p_{i}^{(0)}, \ldots, p_{i}^{(j)}\right\}$ and $G_{i}^{(j)}=\left\{g_{i}^{(0)}, \ldots, g_{i}^{(j)}\right\}$; note that $\left|P_{i}^{(j)}\right|=\left|G_{i}^{(j)}\right|=j+1$, for each $0 \leq j \leq t_{i}$.

To complete the iterative process, it remains to describe how to select a private pair $p_{i}^{(j)}$, for each $i \in\left\{1, \ldots, \ell_{j}\right\}$.

Firstly, and very importantly, we prioritize the selection of a private pair for $N_{i}^{(j)}$ that contains a private vertex. Note that a private vertex of any $k$-set belongs to $G$. If $N_{i}$ has a critical garbage vertex in $\Gamma_{i} \backslash G_{i}^{(j-1)}$ that is private to $N_{i}$ at time $j$, then we select it. Otherwise, if $\Gamma_{i} \backslash G_{i}^{(j-1)}$ is empty, but there is a private vertex in $N_{i} \backslash G_{i}^{(j-1)}$ that is private to $N_{i}$ at time $j$, we select it. If either of these types of private vertex exists, then necessarily it is in $\left(N_{i} \cap G\right) \backslash G_{i}^{(j-1)}$; call it $g_{i}^{(j)}$. Complete a private pair containing $g_{i}^{(j)}$ by adding a vertex $a_{x}^{(j)}$, where $x \in\left\{1, \ldots, \ell_{j}\right\} \backslash\{i\}$. Note that $a_{x}^{(j)} \in N_{i}$ by definition of the kernel. So, in this case, the pair selected is $p_{i}^{(j)}=\left\{g_{i}^{(j)}, a_{x}^{(j)}\right\}$, and it is a subset of $N_{i}$. The fact that $p_{i}^{(j)}$ is a private pair for $N_{i}^{(j)}$ is due to $g_{i}^{(j)}$ being a private vertex for $N_{i}$ at time $j$.

If there is no vertex of $N_{i} \backslash G_{i}^{(j-1)}$ that is private to $N_{i}$ at time $j$, apply Lemma $6(\mathrm{~h})$ with $Y=G_{i}^{(j-1)}$ to produce a pair $\left\{g_{i}^{(j)}, u\right\}$ single-covered by $\mathcal{N}^{(j)}$ such that $g_{i}^{(j)} \in$ $G \backslash\left\{g_{i}^{(0)}, \ldots, g_{i}^{(j-1)}\right\}$ and $u \in A \cup G$. If $u \in G$, then set $u_{i}^{(j)}=u$ which completes the private pair selection in this case. Otherwise, in anticipation of the need for well-behaved private pairs later, we adjust $\left\{g_{i}^{(j)}, g\right\}$. In this case $u \notin G$, so we may assume that $u=a_{r}^{(s)}$ for some $1 \leq r \leq \ell$ and $0 \leq s \leq t_{r}$. Note that $r \neq i$ because $a_{r}^{(s)} \in N_{i}$. We now 'promote' $a_{r}^{(s)}$, which means setting the non-anchor vertex $u_{i}^{(j)}$ of the private pair to the latest possible kernel vertex that substitutes for $a_{r}^{(s)}$ (preserving the private pair property). This is accomplished as follows:

$$
u_{i}^{(j)}= \begin{cases}a_{r}^{(s)} & \text { if } \quad j<s \leq t_{r}  \tag{1}\\ a_{r}^{(y)} \quad \text { if } \quad s \leq j \leq y, \text { where } y=\min \left\{t_{i}, t_{r}\right\} \\ a_{x}^{\left(t_{i}\right)} \quad \text { if } \quad s \leq t_{r}<j \leq t_{i}, \quad \text { with any } x \in\left\{1, \ldots, \ell_{t_{i}}\right\} \backslash\{i\}\end{cases}
$$

Notice that promotion is not applied if $g_{i}^{(j)}$ is a private vertex. Furthermore, (1) changes the non-anchor (that is $u_{i}^{(j)} \neq u$ ) only if $s \leq j$. Also observe that if $u_{i}^{(j)}=a_{r}^{(y)}$ with $y \notin\left\{t_{i}, t_{r}\right\}$, then the non-anchor is unchanged (that is, $u_{i}^{(j)}=u$ under these conditions). We next prove that this promotion process in fact produces a private pair for $N_{i}$ at time $j$, furthermore, anchor vertices are not repeated for the private pairs in $N_{i}$.
Lemma 7. The pair $p_{i}^{(j)}=\left\{g_{i}^{(j)}, u_{i}^{(j)}\right\}$ as defined by (1) is a private pair for $N_{i}$ at time j. Furthermore, $g_{i}^{(j)} \in G \backslash\left\{g_{i}^{(0)}, \ldots, g_{i}^{(j-1)}\right\}, j=1, \ldots, t_{i}$, and if $u_{i}^{(j)}$ is a kernel vertex, then $u_{i}^{(j)}=a_{x}^{(y)}$, where $j \leq y$ and $x \leq \ell$.

Proof. We may assume that no vertex of $N_{i} \backslash G_{i}^{(j-1)}$ is private to $N_{i}$ at time $j$, since otherwise, the private pair selected has the form $p_{i}^{(j)}=\left\{g_{i}^{(j)}, a_{x}^{(j)}\right\}$ satisfying the claim with $j=y$.

Notice that the anchor of $p_{i}^{(j)}$ is not affected by the promotion process, so $g_{i}^{(j)} \in$ $G \backslash\left\{g_{i}^{(0)}, \ldots, g_{i}^{(j-1)}\right\}$ follows from the generation of $\left\{g_{i}^{(j)}, u\right\}$ via Lemma 6(h). Assuming that the non-anchor is a kernel vertex, the formula (1) yields either $u_{i}^{(j)}=a_{r}^{(y)}$, where $y=s$ and $j<s$, or $u_{i}^{(j)}=a_{x}^{(y)}$, where $y \in\left\{t_{i}, t_{r}\right\}$; in each case $j \leq y$. Thus, the second part of the claim follows.

It remains to show that $p_{i}^{(j)}=\left\{g_{i}^{(j)}, u_{i}^{(j)}\right\}$, as defined by (1), is a private pair for $N_{i}^{(j)}$. First we verify that $p_{i}^{(j)} \subset N_{i}$. Clearly, $g_{i}^{(j)} \in N_{i}$, so it suffices to show that promotion produces $u_{i}^{(j)} \in N_{i}$. If promotion does not change the non-anchor (line 1 of (1)), then $a_{r}^{(s)} \in N_{i}$ follows from the generation of $\left\{g_{i}^{(j)}, u\right\}$ via Lemma $6(\mathrm{~h})$. So we may assume that promotion does change $u$. Because $i \neq r$, if $t_{r}<t_{i}$, then $a_{r}^{\left(t_{r}\right)} \in N_{i}$; whereas if $t_{i} \leq t_{r}$, then $a_{r}^{\left(t_{i}\right)} \in N_{i}$ (line 2 of (1)). Similar reasoning shows that $x \neq i$ implies $a_{x}^{\left(t_{i}\right)} \in N_{i}$ (line 3 of (1)) because $x \in\left\{1, \ldots, \ell_{t_{i}}\right\} \backslash\{i\}$ means $t_{i} \leq t_{x}$.

Now we verify the privacy of $p_{i}^{(j)}$. If $u_{i}^{(j)}$ is set to $a_{r}^{(s)}$ (that is, $j<s$ corresponding to line 1 of (1)), then promotion does not change the non-anchor which means $p_{i}^{(j)}$ retains the privacy granted from the application of Lemma $6(\mathrm{~h})$ which generated it. So we may assume that promotion does change the non-anchor; that is, $s \leq j$. Because $\left\{g_{i}^{(j)}, a_{r}^{(s)}\right\}$ is single-covered by $N_{i}$ at time $j$ and $s \leq j$, the anchor vertex $g_{i}^{(j)}$ does not belong to any $N_{x} \in \mathcal{N}^{(s)} \backslash\left\{N_{r}\right\}$ such that $t_{x} \geq j$. In particular, if $\left\{g_{i}^{(j)}, a_{x}^{\left(t_{i}\right)}\right\} \subset N_{q}$, and $N_{q} \in \mathcal{N}^{(j)}$, then $q \in\{i, r\}$. Hence the pair $\left\{g_{i}^{(j)}, a_{x}^{\left(t_{i}\right)}\right\}$ is single-covered by $N_{i}$ at time $j$, if either $x=r$ or $t_{r}<j$. These cases correspond to the second and third line of (1).

In the exceptional case when $j \leq t_{r}<t_{i}$ the second line of (1) yields $p_{i}^{(j)}=\left\{g_{i}^{(j)}, a_{r}^{\left(t_{r}\right)}\right\}$. This is a single-covered pair, since $s \leq t_{r}$ and $\left\{g_{i}^{(j)}, a_{r}^{(s)},\right\}$ is single-covered at time $j$.

The next lemma establishes a property of the selected private pairs that is essential to building small transversal sets for arcs of the digraph $D$ constructed in Section 5 .

Lemma 8. Suppose that $p_{i}^{(j)}=\left\{g_{i}^{(j)}, u_{i}^{(j)}\right\}$ is a selected private pair for $N_{i}^{(j)}$. If $u_{i}^{(j)}=a_{x}^{(y)}$ where $y \notin\left\{t_{i}, t_{x}\right\}$ and $g_{i}^{(j)}$ is not a private vertex for $N_{i}$ at time $j$, then $p_{i}^{(j)}$ is a subset of a 3 -cross of $N_{i}^{(j)}$ with respect to $G_{i}^{(j-1)}$.

Proof. Because the private pair selection process prioritizes selecting private pairs with private vertices, the hypothesis that $g_{i}^{(j)}$ is not a private vertex for $N_{i}$ at time $j$ means $N_{i} \backslash G_{i}^{(j-1)}$ has no private vertices at time $j$. Consequently, the process to select $p_{i}^{(j)}$ must first have applied Lemma $6(\mathrm{~h})$ to generate a pair $\left\{g_{i}^{(j)}, u\right\}$ in which $u$ is a kernel vertex (since the final private pair has a non-anchor in the kernel). Lemma 6(h) guarantees a 3$\operatorname{cross} C_{i}^{(j)}\left(G_{i}^{(j-1)}\right)$ containing $\left\{g_{i}^{(j)}, u\right\}$. The promotion process must not have changed the non-anchor $u$ because the final non-anchor is $u_{i}^{(j)}=a_{x}^{(y)}$ with $y \notin\left\{t_{i}, t_{x}\right\}$ (see observation prior to Lemma 7). Therefore, $p_{i}^{(j)}=\left\{g_{i}^{(j)}, u\right\} \subseteq C_{i}^{(j)}\left(G_{i}^{(j-1)}\right)$, as desired.

Let $P^{(j)}=\left\{p_{i}^{(j)}: 1 \leq i \leq \ell_{j}\right\}$ denote the private pairs defined by this process at time $j$ and let $P_{i}=\left\{p_{i}^{(j)}: 0 \leq j \leq t_{i}\right\}$ be the set of $t_{i}+1$ private pairs defined for $N_{i}$ by this process. The collection of all selected pairs is defined as

$$
P=\bigcup_{j=0}^{t} P^{(j)}=\bigcup_{i=0}^{\ell} P_{i}
$$

Next we list for reference obvious properties of the private pairs in $P$ which are summarized in Lemma 8 of [10]:

- $P^{\left(j_{1}\right)} \cap P^{\left(j_{2}\right)}=\varnothing$, for $0 \leq j_{1}<j_{2} \leq t ;$
- any pair in $\bigcup_{s=0}^{j} P^{(s)}$ is at most single-covered by $\mathcal{N}^{(j)}$, for $j=0, \ldots, t$;
- $\left|P^{(j)}\right|=\ell_{j}$ and every $\left|P_{i} \cap P^{(j)}\right|=1$, for all $0 \leq j \leq t$ and $1 \leq i \leq \ell_{j}$.


## 4 Advancement

In this section we describe a technical modification to the private pairs, called advancement, that enables our final tight asymptotic bounds. This modification concerns rare, but troublesome private pairs that we alter and reorder. During this reordering, troublesome private pairs are given improved non-anchors and advanced in time (hence the
process name) to protect their anchors, guaranteeing that the anchors are swapped via the recursive process (4) defined later in Section 7. Advancement preserves the number of pairs in $P$, but a few pairs will lose their private nature, sacrificing themselves to advance vital troublesome private pairs. To limit the size of the required sacrifices and identify these vital troublesome pairs, we first partition the $k$-sets into three types : a $k$-set $N_{x}$ is weightless if $\left|\Gamma_{x}\right|=0$; it is light if $0<\left|\Gamma_{x}\right|<m^{1 / 3}$ and heavy if $m^{1 / 3} \leq\left|\Gamma_{x}\right|$.

The justification for this otherwise arbitrary appearing trichotomy is that it optimizes the bounds in Theorem 23. As we shall see, weightless $k$-sets are innocuous, so there will be no need to address them further. Heavy sets require too many sacrifices to correct via advancement; their challenges will be settled efficiently by observing that there are few heavy $k$-sets (see Lemma 13) so a crude solution is affordable (definition of $T_{H}$ in Section 6). This leaves the correction of light $k$-sets that define the 'troublesome' pairs (see Definition 5).

Before defining troublesome pairs, we define and examine this precursor to them:
Definition 4. A private pair $p_{i}^{(j)} \in P$ is problematic if it has the form $p_{i}^{(j)}=\left\{g, a_{x}^{(y)}\right\}$ with $j \leq y<t_{x}$ and $g \in N_{x}$.

The following lemmas present important properties of problematic private pairs.
Lemma 9. If $p_{i}^{(j)}=\left\{g, a_{x}^{(y)}\right\}$ is a problematic private pair, then
(i) $i \neq x$,
(ii) for any $z \in\{1, \ldots, \ell\}$, if $g \in N_{z}$ and $y \leq t_{z}$, then $z \in\{i, x\}$,
(iii) $I_{g} \subseteq\{i, x\}$,

Proof. (i) Because $a_{x}^{(y)} \in p_{i}^{(j)} \subseteq N_{i}$ and $a_{x}^{(y)} \notin N_{x}$, it follows that $x \neq i$.
(ii) Assume, to the contrary, that there exists $z$ such that $g \in N_{z}, t_{z} \geq y$, and $z \notin\{i, x\}$. Because $y \leq t_{z}$ and $z \neq x$, it follows that $a_{x}^{(y)} \in N_{z}$. Because $j \leq y$, we have $p_{i}^{(j)}=\left\{g, a_{x}^{(y)}\right\} \subseteq N_{i}^{(j)} \cap N_{z}^{(j)}$, contradicting that $p_{i}^{(j)}$ is private for $N_{i}$ at stage $j$.
(iii) Consider $z \in I_{g}$; so $g \in N_{z}$ and $t_{z}=t_{g}$. Because $g \in N_{x}$, we have $t_{x} \leq t_{g}$. Hence, $y<t_{x} \leq t_{z}=t_{g}$. Now (ii) implies that $z \in\{i, x\}$. Therefore, $I_{g} \subseteq\{i, x\}$.

Lemma 9(i) shows that the anchor of a problematic private pair is not a private vertex, thus the pair is obtained through promotion (1). The next lemma highlights the size of a critical index set of the anchor vertex, in particular, distinguishing whether it is one or two.

Lemma 10. Suppose $p_{i}^{(j)}=\left\{g, a_{x}^{(y)}\right\}$ is a problematic private pair.
(i) $\left|I_{g}\right|=1$ or $\left|I_{g}\right|=2$.
(ii) If $\left|I_{g}\right|=2$, then $I_{g}=\{i, x\}$ and there are at most two problematic private pairs with anchor $g$.
(iii) For $\left|I_{g}\right|=1$, if $I_{g}=\{i\}$ then $p_{i}^{(j)}$ is the only problematic private pair with anchor $g$; otherwise, $I_{g}=\{x\}$ and every problematic private pair with anchor $g$ has a non-anchor from $\overline{N_{x}}$.

Proof. (i) Because $I_{g} \neq \varnothing$, this claim is just a rephrasing of Lemma 9(iii).
(ii) Because $\left|I_{g}\right|=2$ and Lemma 9(iii), it follows that $I_{g}=\{i, x\}$. Consider an arbitrary problematic private pair $p_{u}^{(\alpha)}$ with anchor $g$. Lemma 9(iii) implies $u \in\{i, x\}$. Now simply observe that $N_{i}$ and $N_{x}$ each may have at most one private pair containing $g$ because anchors are never repeated in the private pair selection process.
(iii) By Lemma 9 (iii), $\{z\}=I_{g} \subset\{i, x\}$. If $z=i$, then $p_{i}^{(j)}$ is the unique private pair for $N_{i}$ with anchor $g$; otherwise, $z=x$. Consequently, every problematic private pair with anchor $g$ that is not a private pair for $N_{x}$ has non-anchor from $\overline{N_{x}}$.

Now we are ready to define troublesome private pairs.
Definition 5. A private pair $p_{i}^{(j)}=\left\{g, a_{x}^{(y)}\right\}$ is troublesome if $p_{i}^{(j)}$ is problematic, $N_{i}$ is light, and $g \in \Gamma_{i}$.

These pairs are a concern because they cause particularly thorny conflicts that must be removed to obtain a skew $(2, m)$-system that appears in Theorem 22. The next lemma introduces a possible replacement pair for a troublesome pair. The replacement pair for $p_{i}^{(j)}$ is a new pair $\hat{p}_{i}^{(j)}$ with the same anchor but improved non-anchor, $a_{x}^{\left(t_{x}\right)}$, that is immune to swaps (see the recursive process (4) in Section 7) because $t_{x}$ is a terminal stage for $N_{x}$. This immunity ensures that, in the ultimate skew $(2, m)$-system considered, the replacement pair will continue to intersect $m$-sets that contain $a_{x}^{\left(t_{x}\right)}$. Note however that, in contrast to the original pair, the replacement pair may not be private for $N_{i}$ at time $j$, though it is private for $N_{i}$ at later stages, as the next lemma specifies.

Lemma 11. If $p_{i}^{(j)}=\left\{g, a_{x}^{(y)}\right\}$ is a troublesome private pair, then the replacement pair $\hat{p}_{i}^{(j)}=\left\{g, a_{x}^{\left(t_{x}\right)}\right\}$ is private for $N_{i}^{(\alpha)}$, for any $\alpha \geq y$.

Proof. Lemma 9(i) says $x \neq i$, and $t_{x}<t_{i}$ because $g \in N_{x}$ and $I_{g}=\{i\}$; therefore, $a_{x}^{\left(t_{x}\right)} \in N_{i}$. This means that the pair $\left\{g, a_{x}^{\left(t_{x}\right)}\right\}$ is a subset of $N_{i}$. Indeed, Lemma 9(ii) implies that this pair is private for $N_{i}^{(y)}$. Consequently, this pair is private for $N_{i}$ at any time $\alpha$ satisfying $\alpha \geq y$.

Because a replacement private pair is not necessarily private at the same time as the original, to use this new pair we must advance the replacement pair in time at the expense of destroying another private pair; this is "advancement". The advancement expense is justified because these replacement pairs contain important anchors whose swap must be protected to prevent many other conflicts. Only light $k$-sets are considered so that the expense of advancing these replacement pairs is limited. In some cases it may not be possible to advance a given troublesome pair because no pair remains to be sacrificed. The process takes this possibility into account (see 'neutralized' pairs below).

Advancement: Here is the formal description of the advancement process. For each light $k$-set $N_{i}$, we view the collection $P_{i}$ of private pairs as a list: $p_{i}^{(0)}, \ldots, p_{i}^{\left(t_{i}\right)}$. The process treats troublesome private pairs in this list according to the order in which they appear earlier troublesome pairs are processed before later ones. Suppose that $p_{i}^{(j)}=\left\{g_{i}^{(j)}, a_{x}^{(y)}\right\}$ is the next troublesome pair to consider for $N_{i}$. By definition, this means $g_{i}^{(j)} \in \Gamma_{i}$, that is $g_{i}^{(j)}$ is a critical garbage vertex in $N_{i}$. Because $p_{i}^{(j)}$ is problematic, $j \leq y<t_{x}$.

If there exists a private pair $p_{i}^{(\alpha)}=\left\{g_{i}^{(\alpha)}, u_{i}^{(\alpha)}\right\}$ for $N_{i}$ with $\alpha \geq y$ and $g_{i}^{(\alpha)} \notin \Gamma_{i}$, then set $\left\{g_{i}^{(j)}, a_{x}^{\left(t_{x}\right)}\right\}$ as the "replacement pair" to be the new private pair for $N_{i}$ at time $\alpha$, and move the pair $\left\{g_{i}^{(\alpha)}, u_{i}^{(\alpha)}\right\}$ to time $j$ to take the place of the displaced original troublesome pair. In this case we refer to the replacement pair $\left\{g_{i}^{(j)}, a_{x}^{\left(t_{x}\right)}\right\}$, which is the new $p_{i}^{(\alpha)}$, as advanced; the moved pair $\left\{g_{i}^{(\alpha)}, u_{i}^{(\alpha)}\right\}$ is labeled deactivated as it may no longer be private to $N_{i}$ at the stage at which it now appears, time $j$. If there is no private pair $p_{i}^{(\alpha)}=\left\{g_{i}^{(\alpha)}, u_{i}^{(\alpha)}\right\}$ for $N_{i}$ with $\alpha \geq y$ and $g_{i}^{(\alpha)} \notin \Gamma_{i}$, then we simply label the troublesome pair $p_{i}^{(j)}=\left\{g_{i}^{(j)}, a_{x}^{(y)}\right\}$ neutralized.

After all troublesome pairs for $N_{i}$ are processed, some pairs are deactivated, and the remaining pairs are all private pairs for $N_{i}$ at the times at which they appear since Lemma 11 guarantees that replacement pairs are private. The only remaining troublesome pairs are neutralized. Also note that this advancement process does not alter the set of anchors. In particular, distinct private pairs for $N_{i}$ still have different anchors.

To clearly indicate post-advancement definitions, we employ math bold face font for these new sets. For example, the new set of pairs for $N_{i}$ is denoted $\mathbb{P}_{i}$. The union of all new pairs for all $N_{i}$ 's is denoted $\mathbb{P}$. Similarly, $\mathbb{G}_{i}^{(j)}$ are anchors for the pairs in $\mathbb{P}_{i}$ for $N_{i}$ up to time $j$. A pair in $\mathbb{P}$ that is not deactivated is active. Observe that active pairs are private pairs.

## 5 A digraph

In this section we introduce an auxiliary digraph $D$ on the kernel $A$ as its vertex set, where a vertex $a_{x}^{(y)}$ represents the private pair $p_{x}^{(y)} \in \mathbb{P}$. The arcs of $D$ will be labeled with garbage vertices and will serve as the code of 'conflicting' private pairs. Thus an independent set in $D$ will determine a special subset of non-conflicting private pairs in $\mathbb{P}$. These pairs represented by the vertices of a large enough independent set of $D$ will be used in Section 7 to define a large skew cross-intersecting $(2, m)$-system.

The final skew cross-intersecting $(2, m)$-system will be generated by the recursive process (4) in Section 7. To protect critical garbage vertices during this process we introduce the following definition.

Definition 6. For $x \in\{1, \ldots, \ell\}$, define the set of protected private anchors for $N_{x}$ at time $y$, denoted $\Lambda_{x}^{(y)}$, to be the set:

$$
\Lambda_{x}^{(y)}=\left(\mathbb{G}_{x}^{(y-1)} \cap \Gamma_{x}\right) \backslash\left\{g \in G: g \text { is the anchor of a neutralized private pair of } N_{x}\right\} .
$$

This set, $\Lambda_{x}^{(y)}$, contains all anchors for non-neutralized private pairs for $N_{x}$ up to time $y$. In the special case in which $N_{x}$ is light, these anchors will get swapped via the recursive procedure (4) to produce $M_{x}^{(y)}$ (defined later in Section 7); this explains condition (2.4) below.

With these definitions complete, we are now ready to define an arc-labeled digraph $D$ on the vertex set $A$. Recall that $A$ is the set of kernel vertices; and every element of $A$ has the form $a_{x}^{(y)}$, where $1 \leq x \leq \ell$ and $0 \leq y \leq t_{x}$. The arcs of the digraph $D$ are defined as follows.

$$
\begin{equation*}
a_{i}^{(j)} \xrightarrow{g} a_{x}^{(y)} \text { is an arc of } D \tag{2}
\end{equation*}
$$

if and only if all the following are true:
(2.1) $p_{i}^{(j)}=\left\{g, a_{x}^{(y)}\right\} \in \mathbb{P}_{i}$ and $p_{i}^{(j)}$ is active,
(2.2) $g \in N_{x} \cap N_{i}$,
(2.3) $j \leq y<t_{x}$, and if $j=y$ then $i<x$,
(2.4) if $N_{x}$ is light, then $g \notin \Lambda_{x}^{(y)}$.

Together, conditions $(2.1),(2.2),(2.3)$ imply that $p_{i}^{(j)}$ is a problematic pair. Because conditions (2.1) requires $p_{i}^{(j)}$ is active, the pair is also a private pair for $N_{i}$. Condition (2.3) specifies that $y<t_{x}$ which means that $p_{i}^{(j)}$ is not an advanced pair.

We refer to the vertex $a_{i}^{(j)}$ as the tail of the arc $a_{i}^{(j)} \xrightarrow{g} a_{x}^{(y)}$; naturally, $a_{x}^{(y)}$ is the head. Observe that (2.1) implies $g \in G$ because the private pair $p_{i}^{(j)}$ must intersect $G$. It also implies the digraph $D$ has out-degree at most one because an out-going arc from vertex $a_{i}^{(j)}$, if there is one, is determined by $p_{i}^{(j)}$. Condition (2.2) guarantees that $g$ is not a private vertex to $N_{i}$ at time $j$; that is, the production of the private pair $p_{i}^{(j)}=\left\{g, a_{x}^{(y)}\right\}$ must have invoked Lemma 6(h).

Lemma 10 shows that if there are more than two $\operatorname{arcs}$ in $D$ with label $g$, then $\left|I_{g}\right|=1$. This explains the focus on such vertices given in Definition 3.

The idea is that a labeled arc in $D$ is a code for conflicting elements of an initial $(2, m)$ system developed in Section 7. By eliminating the conflicts exposed by these arcs, this $(2, m)$-system will eventually be reduced to a valid skew cross-intersecting $(2, m)$-system (Theorem 22). The first three arc-defining conditions naturally encode the conflicts, but condition (2.4) is artificial and technical, arising from the special manner in which we treat light $k$-sets in Section 6. Conflict elimination is achieved via a large independent set in $D$, so we now turn our attention to guaranteeing such a set.

Given an independent set $F$ of $D$, call a pair $p_{i}^{(j)} \in \mathbb{P}$ free if $a_{i}^{(j)} \in F$. The size of $F$ determines this size of the final skew cross-intersecting $(2, m)$-system. The complement $T=A \backslash F$ is a transversal (a vertex cover) of the arcs in $D$. We strive for an upper bound on $T$, which yields a lower bound on $F$ and thus on the number of free pairs. In the next section we shall determine a small transversal. The selection of this transversal will apply the aforementioned trichotomy of $k$-sets, a concept originating from the Lemma 10.

## 6 A small transversal set

Our goal in this section is to find a large independent set in digraph $D$ defined in Section 5 , or equivalently, a small transversal set (vertex cover) of the arcs of $D$. More specifically, we seek a transversal set $T \subset A$ such that $|T|=O\left(m^{5 / 2}\right)$. This will guarantee a set of free pairs, $F=A \backslash T$, that has cardinality $|F| \geq|A|-O\left(m^{5 / 2}\right)$ which will play an essential role in the proof of Theorem 23. We build $T$ as the union of four subsets $T_{W}, T_{H}, T_{S}$, and $T_{L}$ that we describe below. Each of these subsets targets arcs of $D$ that arise from different circumstances.

Since the tail of an arc from $D$ represents a problematic private pair, Lemma 9 shows that if an arc in $D$ has the label $g$, then $\left|I_{g}\right|=2$ or $\left|I_{g}\right|=1$. All arcs arising from the case $\left|I_{g}\right|=2$ will be covered by the set $T_{W}$ of their tails:

$$
T_{W}=\left\{a_{i}^{(j)}: a_{i}^{(j)} \xrightarrow{g} a_{x}^{(y)} \text { is an arc of } D \text { and }\left|I_{g}\right|=2\right\} .
$$

The next lemma proves that $T_{W}$ is not too large.
Lemma 12. $\left|T_{W}\right| \leq 6 m$
Proof. Lemma 9 shows that if $\left|I_{g}\right|=2$, then there are at most two arcs in $D$ labeled $g$. Because $|G| \leq 3 m$ (Lemma 2(b)), it follows that there are at most $2 \cdot 3 m$ arcs of this type.

We next turn our attention to covering arcs in $D$ that have a label $g$ such that $\left|I_{g}\right|=1$. Such arcs have a head vertex in a light or heavy $k$-set. We address the latter kind first. We form a small transversal for these arcs using their head vertices (see definition of $T_{H}$ below). First we prove that there are not many heavy $k$-sets.
Lemma 13. The number of heavy $k$-sets is at most $3 m^{2 / 3}$.
Proof. Let $h$ denote the number of heavy $k$-sets. By definition, a $k$-set $N_{x}$ is heavy if $m^{1 / 3} \leq\left|\Gamma_{x}\right|$. Because different heavy sets have disjoint sets of critical garbage vertices, we have

$$
|G| \geq\left|\bigcup_{N_{x} \text { heavy }} \Gamma_{x}\right|=\sum_{N_{x} \text { heavy }}\left|\Gamma_{x}\right| \geq h \cdot m^{1 / 3}
$$

Now $h \leq 3 m^{2 / 3}$ follows from $|G| \leq 3 m$ that is a consequence of Lemma 2(b).
Now define the transversal vertices contributed by the heavy $k$-sets:

$$
T_{H}=\left\{a_{x}^{(y)}: a_{i}^{(j)} \xrightarrow{g} a_{x}^{(y)} \text { is an arc of } D \text { and } N_{x} \text { is heavy }\right\} .
$$

The next lemma proves that this set is not too large.
Lemma 14. $\left|T_{H}\right| \leq 3 m^{5 / 3}$
Proof. A heavy $k$-set $N_{x}$ contributes at most $t_{x} \leq t$ elements to $T_{H}$. Because $t<m$ (Lemma 2(a)) and the number of heavy $k$-sets is at most $3 m^{2 / 3}$ (Lemma 13), we conclude that $\left|T_{H}\right| \leq m \cdot 3 m^{2 / 3}$.

Finally, we address the remaining arcs in $D$; these have a label $g$ such that $\left|I_{g}\right|=1$ and their head vertex is in a light $k$-set. Completing the final construction of our small transversal set $T$ is a very important property we seek to ensure, Property (L):

If $N_{x}$ is a light $k$-set, $g$ is the anchor for $p_{x}^{(y)}$, and $g \in \Gamma_{x}$,
then either $p_{x}^{(y)}$ is neutralized or $a_{x}^{(y)} \notin T$.
The conclusion $a_{x}^{(y)} \notin T$ is equivalent to $a_{x}^{(y)} \in F$ or $p_{x}^{(y)}$ is free. Property (L) guarantees that, if $p_{x}^{(y)}$ is not neutralized, then $g$ gets swapped into $M_{x}^{(y+1)}$ in the recursive process (4) generating the final large skew cross-intersecting $(2, m)$-system in Section 7, thereby avoiding many conflicts present in the initial system. Guaranteeing Property (L) for heavy $k$-sets seems to require a large transversal set, frustrating the primary small transversal objective. This explains the name of Property (L); it only concerns light $k$-sets.

Observe that $T_{H}$ contains the head vertices of all arcs in $D$ that fall into a heavy $k$-set. In contrast, we use tail vertices to cover arcs in $D$ whose label $g$ satisfies $\left|I_{g}\right|=1$ and whose head falls into a light $k$-set. We do this to satisfy Property (L). In particular, define these two sets

$$
\begin{array}{cc}
T_{S}=\left\{a_{i}^{(j)} \in A \backslash T_{W}:\right. & p_{i}^{(j)} \text { is deactivated, or } \\
\left.a_{i}^{(j)} \xrightarrow{g} a_{x}^{(y)} \text { is an arc of } D \text { and } p_{i}^{(j)} \text { is neutralized }\right\},
\end{array}
$$

and

$$
T_{L}=\left\{a_{i}^{(j)} \in A \backslash\left(T_{W} \cup T_{S}\right): a_{i}^{(j)} \xrightarrow{g} a_{x}^{(y)} \text { is an arc of } D \text { and } N_{x} \text { is light }\right\} .
$$

Observe that, in both definitions, the condition $a_{i}^{(j)} \in A \backslash T_{W}$ guarantees that label $g$ satisfies $\left|I_{g}\right|=1$.

The final choice of transversal set is:

$$
T=T_{W} \cup T_{H} \cup T_{S} \cup T_{L} .
$$

Clearly $T$ is a transversal for the $\operatorname{arcs}$ of $D$. We now turn to proving that $T$ satisfies Property (L) and $|T|=O\left(m^{5 / 3}\right)$. We consider Property (L) first.

Lemma 15. T satisfies Property (L)
Proof. Suppose $N_{i}$ is a light $k$-set, $g$ is the anchor for $p_{i}^{(j)}$, and $g \in \Gamma_{i}$. Assume that $p_{i}^{(j)}$ is not neutralized. We must prove that $a_{i}^{(j)} \notin T$. First observe that $a_{i}^{(j)} \notin T_{W} \cup T_{H}$ by definition of $T_{W}$ and $T_{H}$, because $\left|I_{g}\right| \neq 2$ and $N_{i}$ is a light $k$-set.

Next we claim there is no arc of $D$ in which $a_{i}^{(j)}$ is the tail. Suppose, to the contrary, that $a_{i}^{(j)} \xrightarrow{g} a_{x}^{(y)}$ is such an arc of $D$. So, by condition (2.1), we have $p_{i}^{(j)}=\left\{g, a_{x}^{(y)}\right\}$. Because $p_{i}^{(j)} \subseteq N_{i}$, clearly $x \neq i$ since otherwise $a_{i}^{(y)}=a_{x}^{(y)} \in N_{i}$, a contradiction. Recall that, if $a_{i}^{(j)} \xrightarrow{g} a_{x}^{(y)}$ is an arc of $D$, then $p_{i}^{(j)}$ is a problematic pair. Since we additionally assume that $g \in \Gamma_{i}$ and $N_{i}$ is light, it follows that $p_{i}^{(j)}$ is a troublesome pair. However, we
have also assumed that $p_{i}^{(j)}$ is not neutralized. Therefore, $p_{i}^{(j)}$ is an advanced troublesome pair; that is, $p_{i}^{(j)}=\left\{g, a_{x}^{\left(t_{x}\right)}\right\}$. In particular, $y=t_{x}$ which contradicts condition (2.3) that $a_{i}^{(j)} \xrightarrow{g} a_{x}^{(y)}$ is an arc of $D$.

Because $p_{i}^{(j)}$ is not neutralized, $a_{i}^{(j)} \notin T_{S}$. We noted earlier that $a_{i}^{(j)} \notin T_{W} \cup T_{H}$. The last paragraph shows that $p_{i}^{(j)}$ not neutralized implies there is no arc of $D$ that has $a_{i}^{(j)}$ as a tail. It follows that $a_{i}^{(j)} \notin T_{L}$ since these sets are defined taking only tails of arcs. We conclude that $a_{i}^{(j)} \notin T$, as claimed.

To prove $|T|=O\left(m^{5 / 3}\right)$, Lemma 12 and Lemma 14 imply that it suffices to prove that $\left|T_{S} \cup T_{L}\right|=O\left(m^{5 / 3}\right)$. First we bound the number of light $k$-sets.

Lemma 16. The number of light $k$-sets is at most $3 m$.
Proof. Let $N_{\beta_{1}}, \ldots, N_{\beta_{q}}$ be an enumeration of all the light $k$-sets in $\mathcal{N}$. Because $\varnothing \neq$ $\Gamma_{\beta_{i}} \subseteq G$ for all $1 \leq i \leq q$ and these sets of critical garbage vertices are disjoint, we conclude that

$$
q \leq \sum_{i=1}^{q}\left|\Gamma_{\beta_{i}}\right|=\left|\bigcup_{i=1}^{q} \Gamma_{\beta_{i}}\right| \leq|G| .
$$

Therefore $q \leq 3 m$ because Lemma 2(b) implies $|G| \leq 3 m$.
Next we prove that $\left|T_{S}\right|$ is small.
Lemma 17. $\left|T_{S}\right| \leq 3 m^{4 / 3}$.
Proof. Observe that $\left|T_{S}\right|$ is at most the number of deactivated or neutralized pairs in $\mathbb{P}$; we bound this latter number. Every deactivated or neutralized pair for a light $k$-set $N_{i}$ has a unique anchor from $\Gamma_{i}$, thus the number of these pairs is at most $m^{1 / 3}$. Since Lemma 16 proves that the number of light $k$-sets is at most $3 m$, it follows that $\left|T_{S}\right| \leq 3 m^{4 / 3}$.

To prove $\left|T_{L}\right|=O\left(m^{5 / 3}\right)$, we first develop the next tool which finally applies the leverage the 3 -crosses provide via Lemma 8 .

Lemma 18. For any $0 \leq x \leq \ell$, any $g \in G$, and any $y$ satisfying $0 \leq y \leq t_{x}$, there is at most one problematic pair with anchor $g$ and non-anchor $a_{x}^{(y)}$.

Proof. Suppose, to the contrary, that there are two problematic private pairs, $p_{i_{1}}^{\left(j_{1}\right)}=$ $\left\{g, a_{x}^{(y)}\right\}$ and $p_{i_{2}}^{\left(j_{2}\right)}=\left\{g, a_{x}^{(y)}\right\}$. This means, by definition, that $j_{1}, j_{2} \leq y<t_{x}$. Because each $k$-set has at most one private pair with anchor $g$, it follows that $i_{1} \neq i_{2}$. Without loss of generality, $i_{1}<i_{2}$. This means $t_{i_{1}} \geq t_{i_{2}}$, so $g$ is not a private vertex of $N_{i_{2}}^{\left(j_{2}\right)}$. Since $p_{i_{1}}^{\left(j_{1}\right)}$ is private for $N_{i_{1}}^{\left(j_{1}\right)}$, it follows that $t_{i_{2}}<j_{1} \leq t_{i_{1}}$. Because $j_{1} \leq y<t_{x}$, we conclude that $y \notin\left\{t_{i_{2}}, t_{x}\right\}$.

Now apply Lemma 8 to $p_{i_{2}}^{\left(j_{2}\right)}=\left\{g, a_{x}^{(y)}\right\}$ to obtain a 3 -cross with respect to $G_{i_{2}}^{\left(j_{2}-1\right)}$ containing $p_{i_{2}}^{\left(j_{2}\right)}$. Recall that this 3 -cross $C$ satisfies

$$
p_{i_{2}}^{\left(j_{2}\right)} \subset C \subseteq\left(N_{i_{2}} \cup\left\{a_{i_{2}}^{(0)}, \ldots, a_{i_{2}}^{\left(j_{2}\right)}\right\}\right) \backslash G_{i_{2}}^{\left(j_{2}-1\right)}
$$

Because $C$ is a 3 -cross, it must satisfy $C \cap \overline{N_{i_{2}}} \neq \varnothing$. However, $p_{i_{2}}^{\left(j_{2}\right)} \subset N_{i_{2}}$ and $C$ contains $p_{i_{2}}^{\left(j_{2}\right)}$. Consequently, we conclude that $C=\left\{a_{i_{2}}^{(\alpha)}, g, a_{x}^{(y)}\right\}$, for some $a_{i_{2}}^{(\alpha)} \in\left\{a_{i_{2}}^{(0)}, \ldots, a_{i_{2}}^{\left(j_{2}\right)}\right\}$. Since $t_{i_{1}}>t_{i_{2}}$, it follows that $a_{i_{2}}^{(\alpha)} \in N_{i_{1}}$. Therefore, $C \subset N_{i_{1}}$, contradicting that it is a 3 -cross (which requires $C \cap \overline{N_{i_{1}}} \neq \varnothing$ ).
Lemma 19. $\left|T_{L}\right| \leq 3 m^{5 / 3}$.
Proof. Consider now an arbitrary light $k$-set $N_{x}$. Let $a_{x}^{\left(y_{1}\right)}, \ldots, a_{x}^{\left(y_{r}\right)}$, with $y_{1}<\cdots<y_{r}$, be the collection of all kernel vertices in $\overline{N_{x}}$ that are the head of an arc of $D$ whose tail is in $T_{L}$. We seek to give an upper bound on the number of tails of these arcs since this gives an upper bound for the number of vertices in $T_{L}$ contributed by $N_{x}$, due to Lemma 10(iii) and Lemma 18.

Suppose, for some $1 \leq w \leq r$, that $a_{i}^{(j)} \xrightarrow{g} a_{x}^{\left(y_{w}\right)}$ is an arc of $D$ and $a_{i}^{(j)} \in T_{L}$. As noted right after the definition of $T_{L}$, this means that $\left|I_{g}\right|=1$. Lemma 10(iii) gives $I_{g}=\{i\}$ or $I_{g}=\{x\}$. If $I_{g}=\{i\}$, then $p_{i}^{(j)}=\left\{g, a_{x}^{\left(y_{w}\right)}\right\}$ is a troublesome pair. Since we are assuming that $a_{i}^{(j)} \in T_{L}$, it follows that $a_{i}^{(j)} \notin T_{S}$ so $p_{i}^{(j)}$ is an advanced pair. However, this implies that $y_{w}=t_{x}$ which contradicts condition (2.3) that $a_{i}^{(j)} \xrightarrow{g} a_{x}^{\left(y_{w}\right)}$ is an arc of $D$. So $I_{g}=\{x\}$.

In other words, if $N_{x}$ is a light $k$-set, and for $w$ fixed, $a_{i}^{(j)} \xrightarrow{g} a_{x}^{\left(y_{w}\right)}$ is an arc of $D$ with $a_{i}^{(j)} \in T_{L}$, then $g \in \Gamma_{x}$. Lemma 18 states that there is at most one arc entering $a_{x}^{\left(y_{w}\right)}$ with label $g$. Since $N_{x}$ is light, $\left|\Gamma_{x}\right| \leq m^{1 / 3}$, thus at most $\left|\Gamma_{x}\right| \cdot r \leq m^{1 / 3} \cdot r \operatorname{arcs}$ with tail in $T_{L}$ enter $a_{x}^{\left(y_{w}\right)}$, for $w=1, \ldots, r$. If $r \leq m^{1 / 3}$, then the number of vertices in $T_{L}$ contributed by $N_{x}$ would be at most $m^{2 / 3}$. Therefore, because Lemma 16 shows $q \leq 3 m$, to prove this lemma's conclusion $\left(\left|T_{L}\right| \leq 3 m^{5 / 3}\right)$ it suffices to prove $r \leq m^{1 / 3}$.

Toward this latter end, define, for $1 \leq w \leq r$,

$$
\Delta_{x}(w)=\left\{g: a_{i}^{(j)} \xrightarrow{g} a_{x}^{\left(y_{w}\right)} \text { is an arc of } D \text { and } a_{i}^{(j)} \in T_{L}\right\} .
$$

By the definition of $a_{x}^{\left(y_{1}\right)}, \ldots, a_{x}^{\left(y_{r}\right)}$, we have $\Delta_{x}(w) \neq \varnothing$, for $w=1, \ldots, r$. The prior paragraphs shows $\Delta_{x}(w) \subseteq \Gamma_{x}$, for all $1 \leq w \leq r$. Consider an arbitrary $g \in \Delta_{x}(w)$. By definition this means $D$ contains an arc $a_{i}^{(j)} \xrightarrow{g} a_{x}^{\left(y_{w}\right)}$ and $a_{i}^{(j)} \in T_{L}$. Note that (2.4) guarantees that $g \notin \Lambda_{x}^{\left(y_{w}\right)}$; hence $g \notin G_{x}^{\left(y_{w}-1\right)}$. Therefore, $\Delta_{x}(w) \subseteq \Gamma_{x} \backslash G_{x}^{\left(y_{w}-1\right)}$.

Because $\left|\Gamma_{x}\right| \leq m^{1 / 3}$, to prove $r \leq m^{1 / 3}$ it is enough to prove:

$$
\begin{equation*}
\Gamma_{x} \backslash G_{x}^{\left(y_{1}-1\right)} \supsetneq \Gamma_{x} \backslash G_{x}^{\left(y_{2}-1\right)} \supsetneq \cdots \supsetneq \Gamma_{x} \backslash G_{x}^{\left(y_{r}-1\right)} . \tag{3}
\end{equation*}
$$

Consider again an arbitrary $g \in \Delta_{x}(w) \subseteq \Gamma_{x} \backslash G_{x}^{\left(y_{w}-1\right)}$, for some $w \in\{1, \ldots, r-1\}$, with $\operatorname{arc} a_{i}^{(j)} \xrightarrow{g} a_{x}^{\left(y_{w}\right)}$ in $D$. If $g$ is not private for $N_{x}$ at time $y_{w}$, say $g \in N_{u}^{\left(y_{w}\right)}$ with $u \neq x$, then $a_{x}^{\left(y_{w}\right)} \in N_{u}$ contradicting that $p_{i}^{(j)}=\left\{g, a_{x}^{\left(y_{w}\right)}\right\}$ is a private pair for $N_{i}^{(j)}$. So $g$ is a private vertex for $N_{x}$ at time $y_{w}$ and $g \notin G_{x}^{\left(y_{w}-1\right)}$. In other words, private vertices are available to produce the private pair $p_{x}^{\left(y_{w}\right)}$. Because private pair selection prioritizes vertices from $\Gamma_{x} \backslash G_{x}^{\left(y_{w}-1\right)}$ that are private to $N_{x}$ at time $y_{w}$, some vertex in $\Gamma_{x} \backslash G_{x}^{\left(y_{w}-1\right)}$ was chosen as the anchor for $p_{x}^{\left(y_{w}\right)}$. This private vertex is absent from $\Gamma_{x} \backslash G_{x}^{\left(y_{w}\right)}$ since it is added to $G_{x}^{\left(y_{w}-1\right)}$ to produce $G_{x}^{\left(y_{w}\right)}$. Because $y_{w+1}-1 \geq y_{w}$, we conclude, $\Gamma_{x} \backslash G_{x}^{\left(y_{w}-1\right)} \supsetneq \Gamma_{x} \backslash G_{x}^{\left(y_{w+1}-1\right)}$.

Theorem 20. $|T| \leq 6 m^{5 / 3}+3 m^{4 / 3}+6 m$.
Proof. Simply observe that by definition $T=T_{W} \cup T_{H} \cup T_{S} \cup T_{L}$. Now apply $\left|T_{W}\right| \leq 6 m$ (Lemma 12), $\left|T_{H}\right| \leq 3 m^{5 / 3}$ (Lemma 14), $\left|T_{S}\right| \leq 3 m^{4 / 3}$ (Lemma 17), and $\left|T_{L}\right| \leq 3 m^{5 / 3}$ (Lemma 19).

## 7 A skew cross-intersecting system

In this section we apply the following theorem, first proven by Frankl [3] (see also [8]); it is the skew version of a theorem due to Bollobás [2]. This theorem is also presented in the book by Babai and Frankl ([1], pages 94-95).
Theorem 21. (Bollobás's Theorem - Skew Version) If $A_{1}, \ldots, A_{h}$ are r-element sets and $B_{1}, \ldots, B_{h}$ are s-element sets such that
(a) $A_{i}$ and $B_{i}$ are disjoint for $i=1, \ldots, h$,
(b) $A_{i}$ and $B_{j}$ intersect whenever $1 \leq i<j \leq h$
then $h \leq\binom{ r+s}{r}$.
A linearly ordered collection of pairs of sets, $\left\{\left(A_{i}, B_{i}\right)\right\}_{i=1}^{h}$, satisfying the hypotheses of Theorem 21 is called a skew intersecting set pair $(r, s)$-system; abbreviate this to skew $(r, s)$-system.

Theorem 21 will be applied here with $r=2$ and $s=m$ to obtain a skew $(2, m)$-system, where the 2 -sets are members in the set $\mathbb{P}$ of free pairs specified by the set $F \subset V \backslash T$ in Section 6, and the $m$-sets are derived iteratively from the $k$-sets corresponding to all the pairs in $\mathbb{P}$ as follows.

Recall that each $k$-set has a private pair at each stage until the $k$-set survives. First to every $N_{i}$ associate $t_{i}+1 \mathrm{~m}$-sets denoted $M_{i}^{(0)}, \ldots, M_{i}^{\left(t_{i}\right)}$. At stage 0 , set $M_{i}^{(0)}=\overline{N_{i}}$, for all $i=1, \ldots, \ell_{0}$. For $i=1, \ldots, \ell$ and $j=1, \ldots, t_{i}$, recursively define

$$
M_{i}^{(j)}= \begin{cases}\left(M_{i}^{(j-1)} \backslash\left\{a_{i}^{(j-1)}\right\}\right) \cup\left\{g_{i}^{(j-1)}\right\} & \text { if } p_{i}^{(j-1)}=\left\{g_{i}^{(j-1)}, u_{i}^{(j-1)}\right\} \text { is free }  \tag{4}\\ M_{i}^{(j-1)} & \text { if } p_{i}^{(j-1)} \text { is not free }\end{cases}
$$

Note that, because $a_{i}^{(j-1)} \in M_{i}^{(j-1)}, g_{i}^{(j-1)} \notin M_{i}^{(j-1)}$, and $\left|M_{i}^{(0)}\right|=m$, it follows that $\left|M_{i}^{(j)}\right|=m$, for all $i=1, \ldots, \ell$ and $j=1, \ldots, t_{i}$. This recursive process will never remove $a_{i}^{\left(t_{i}\right)}$ from $M_{i}^{(0)}$ because the process halts at stage $j=t_{i}$.

Now define the set-pair system

$$
\mathcal{F}=\left\{\left(p_{i}^{(j)}, M_{i}^{(j)}\right): p_{i}^{(j)} \in \mathbb{P} \text { is free }\right\}
$$

where $\mathcal{F}$ is ordered linearly and chronologically via lexicographical order:

$$
\left(p_{i}^{(j)}, M_{i}^{(j)}\right)<\left(p_{x}^{(y)}, M_{x}^{(y)}\right) \quad \Longleftrightarrow \quad(j<y) \text { or }(j=y \text { and } i<x)
$$

Theorem 22. $\mathcal{F}$ is a skew (2,m)-system.
Proof. Clearly $\left|p_{i}^{(j)}\right|=2$ and $\left|M_{i}^{(j)}\right|=m$, for all $\left(p_{i}^{(j)}, M_{i}^{(j)}\right) \in \mathcal{F}$. Because $p_{i}^{(j)}=$ $\left\{g_{i}^{(j)}, u_{i}^{(j)}\right\}$ is private to $N_{i}$ at time $j$, it follows that $p_{i}^{(j)} \subset N_{i}$; so, $p_{i}^{(j)} \cap M_{i}^{(0)}=\varnothing$. Observe that $g_{i}^{(j)} \notin M_{i}^{(j)}$, because $g_{i}^{(j)}$ is added to $M_{i}^{(j)}$ at time $j+1$ by the recursive process generating the $m$-sets. Moreover, $\left\{g_{i}^{(j)}, u_{i}^{(j)}\right\} \cap\left\{g_{i}^{(0)}, \ldots, g_{i}^{(j-1)}\right\}=\varnothing$, by the iterative choice of private pairs, thus $u_{i}^{(j)} \notin M_{i}^{(0)}$ implying that $u_{i}^{(j)} \notin M_{i}^{(j)}$. Therefore, $p_{i}^{(j)} \cap M_{i}^{(j)}=\varnothing$, showing that hypothesis (a) is satisfied in Theorem 21.

Now we prove the system satisfies hypothesis (b). Suppose $\left(p_{i}^{(j)}, M_{i}^{(j)}\right),\left(p_{x}^{(y)}, M_{x}^{(y)}\right) \in \mathcal{F}$ and $\left(p_{i}^{(j)}, M_{i}^{(j)}\right)<\left(p_{x}^{(y)}, M_{x}^{(y)}\right)$, in particular, $j \leq y$. We must prove $p_{i}^{(j)} \cap M_{x}^{(y)} \neq \varnothing$.

If $i=x$, then $j<y$ so $g_{i}^{(j)} \in M_{i}^{(j+1)}=M_{x}^{(j+1)}$ because $p_{i}^{(j)}$ is free and therefore (4) swaps $g_{i}^{(j)}$ into $M_{i}^{(j+1)}$. In this case, $g_{i}^{(j)} \in M_{x}^{(y)}$. Consequently, we may assume $i \neq x$.

If $g_{i}^{(j)} \in M_{x}^{(y)}$, then $p_{i}^{(j)} \cap M_{x}^{(y)} \neq \varnothing$. So we may assume $g_{i}^{(j)} \notin M_{x}^{(y)}$. Elements from $G$ are only added to $M_{x}^{(0)}$ to get to $M_{x}^{(y)}$, so $g_{i}^{(j)} \notin M_{x}^{(0)}=\overline{N_{x}}$ which implies $g_{i}^{(j)} \in N_{x}$. Since $j \leq y$ and $p_{i}^{(j)}$ is private to $N_{i}$ at time $j$, we conclude that $u_{i}^{(j)} \notin N_{x}$ because $N_{x}$ survives at stage $j$ (that is, $t_{x} \geq y \geq j$ ). So $u_{i}^{(j)} \in \overline{N_{x}}=M_{x}^{(0)}$.

If $u_{i}^{(j)} \in G$, or $u_{i}^{(j)}=a_{x}^{(s)}$ and $s \geq y$, then $a_{x}^{(s)}$ is not removed from $M_{x}^{(0)}$ during the process generating $M_{x}^{(y)}$, meaning $p_{i}^{(j)} \cap M_{x}^{(y)} \neq \varnothing$. So we may assume that $u_{i}^{(j)}=a_{x}^{(s)}$, for some $s \leq y-1$. The private pair selection and the promotion process (1) guarantees that $j \leq s$.

Setting $g=g_{i}^{(j)}$, we find $p_{i}^{(j)}=\left\{g, a_{x}^{(s)}\right\}$ with $g \in N_{i} \cap\left(N_{x} \cap G\right)$ and $j \leq s \leq y-1<$ $y \leq t_{x}$. In other words, according to Definition $4, p_{i}^{(j)}$ is a problematic pair. Lemma 10(i) yields $\left|I_{g}\right|=1$ or $\left|I_{g}\right|=2$. If $\left|I_{g}\right|=2$, then $a_{i}^{(j)} \in T_{W}$ which contradicts the assumption that $p_{i}^{(j)}$ is free. So we may assume $\left|I_{g}\right|=1$. Lemma 9 (iii) implies that $I_{g}=\{i\}$ or $I_{g}=\{x\}$.

If $N_{x}$ is heavy, then the definition of $T_{H}$ means $a_{x}^{(s)} \in T_{H}$, so $p_{x}^{(s)}$ is not free. Accordingly, $a_{x}^{(s)}$ is never removed from $M_{x}^{(0)}$ in the production of $M_{x}^{(y)}$. Hence, $a_{x}^{(s)} \in p_{i}^{(j)} \cap M_{x}^{(y)}$. So we may assume that $N_{x}$ is light.

If $g \in \Lambda_{x}^{(y)}$, then $g$ is the anchor for some non-neutralized private pair $p_{x}^{(\beta)}$, for some $0 \leq \beta \leq y-1$ and $g \in \Gamma_{x}$. In this case, Property (L) guarantees that $a_{x}^{(\beta)} \notin T$. It follows that $p_{x}^{(\beta)}$ is free so (4) swaps $g$ into $M_{x}^{(\beta+1)}$. This means $g \in M_{x}^{(y)}$ yielding $g \in p_{i}^{(j)} \cap M_{x}^{(y)}$. So we may assume that $g \notin \Lambda_{x}^{(y)}$.

Because $p_{i}^{(j)}$ is free, we conclude that $a_{i}^{(j)} \notin T_{S}$ so $p_{i}^{(j)}$ is active.
To summarize, we now have these remaining conditions: $p_{i}^{(j)}=\left\{g, a_{x}^{(s)}\right\}, p_{i}^{(j)}$ is active, $g \in N_{i} \cap N_{x}, j \leq s<y \leq t_{x}, N_{x}$ is light, and $g \notin \Lambda_{x}^{(y)}$. These conditions guarantee $a_{i}^{(j)} \xrightarrow{g} a_{x}^{(s)}$ is an arc of $D$. But this case can not occur because the definition of $T_{L}$ would give $a_{i}^{(j)} \in T_{L}$, implying $a_{i}^{(j)} \notin F$. This contradicts the assumption that $p_{i}^{(j)}$ is free.

Now we may state the main theorem of the paper, a new upper bound on the order of an ( $n, m$ )-structure

Theorem 23. Any ( $n, m$ )-structure satisfies

$$
n \leq\binom{ m+2}{2}+6 m^{5 / 3}+3 m^{4 / 3}+9 m-3
$$

Proof. Recall that $F=A \backslash T$ is a free set of pairs; so $|F|=|A|-|T|$, and the skew Bollobás theorem yields $|F| \leq\binom{ m+2}{2}$. Therefore, $|A| \leq\binom{ m+2}{2}+|T|$. By Lemma 2(b), $|G| \leq 3(m-1)$; and using the bound on $|T| \leq 6 m^{5 / 3}+3 m^{4 / 3}+6 m$ in Theorem 20 we obtain

$$
\begin{aligned}
n & =|G|+|A| \\
& \leq|G|+\binom{m+2}{2}+|T| \\
& \leq 3(m-1)+\binom{m+2}{2}+6 m^{5 / 3}+3 m^{4 / 3}+6 m \\
& \leq\binom{ m+2}{2}+6 m^{5 / 3}+3 m^{4 / 3}+9 m-3 .
\end{aligned}
$$

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[^0]:    ${ }^{1}$ Personal communication (2019).

[^1]:    ${ }^{2}$ We have endeavored to use the same notation introduced by Szemerédi and Petruska. Some important exceptions: we use $a_{i}^{(j)}$ for their $x_{i}^{(j)}$; also $k$ and $\ell$ here refer to their quantities $n$ and $k$, respectively.

