Dimensions of nonbinary antiprimitive BCH codes and some conjectures

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Abstract

Bose-Chaudhuri-Hocquenghem (BCH) codes have been intensively investigated. Even so, there is only a little known about primitive BCH codes, let alone non-primitive ones. In this paper, let q>2 be a prime power, the dimension of a family of non-primitive BCH codes of length $n=q^m+1$ (also called antiprimitive) is studied. These codes are also linear codes with complementary duals (called LCD codes). Through some approaches such as iterative algorithm, partition and scaling, all coset leaders of C_x modulo n with $q^{\lceil \frac{m}{2} \rceil} < x \le 2q^{\lceil \frac{m}{2} \rceil} + 2$ are given for $m \ge 4$. And for odd m the first several largest coset leaders modulo n are determined. Furthermore, a new kind of sequences is introduced to determine the second largest coset leader modulo n with m even and q odd. Also, for even m some conjectures about the first several coset leaders modulo n are proposed, whose complete verification would wipe out the difficult problem to determine the first several coset leaders of antiprimitive BCH codes. After deriving the cardinalities of the coset leaders, we shall calculate exact dimensions of many antiprimitive LCD BCH codes.

Index terms: BCH code, cyclotomic coset, coset leader, LCD code, dimension

1 Introduction

Binary BCH codes were introduced by Hocquenghem in 1959 [1], and independently by Bose and Ray-Chaudhuri in 1960 [2, 3]. Immediately afterwards they were generalized by Gorenstein and Zierler to general finite fields [4]. Since then, BCH codes have been deeply studied and widely developed because of their advantages of good error-correcting capability and efficient encoding and decoding algorithms.

However, as pointed out by Charpin [5] and Ding [6], it is a hard problem to determine the dimension and minimum distance of BCH codes. Therefore, their parameters are determined for only some special classes of code lengths.

Actually, the research of the dimension of BCH codes began at the nearly same time when they were discovered [7]. For narrow-sense primitive BCH codes, many important conclusions on the dimension have been obtained [8–17], while only some sparse results for the nonprimitive case [16–18, 21, 22]. Thereinto, Liu et al. [16] and Li et al. [17] successively studied a kind of BCH codes of length $n=q^m+1$. They acquired some important achievements about the parameters of BCH codes of length n for designed distance $\delta \leq q^{\left \lfloor \frac{m-1}{2} \right \rfloor} + 3$ in [16] and $q^{\left \lfloor \frac{m-1}{2} \right \rfloor} + 3 < \delta \leq q^{\left \lceil \frac{m}{2} \right \rceil}$ in [17], respectively. Also, they proved that this kind of BCH codes is LCD codes. LCD codes were initially named as reversible codes in [19] and introduced in [20] by Massey. Recently, it was found that they can improve information security, especially against side-channel attacks and fault

noninvasive attacks in cryptography [23]. Ding pointed out that it is important but not easy to find the second and third largest coset leaders δ_2 and δ_3 modulo $n=q^m+1$ [6], which is quite helpful to obtain both the dimension of BCH codes and the Bose distance (i.e., the maximal designed distance). In [22], we introduced some new techniques to find out the coset leaders modulo $n=2^m+1$ with $m\not\equiv 0$ mod 8. And we determined the first five largest coset leaders modulo n, as well as all coset leaders for

$$x \le \begin{cases} 2^{t+2} + 7 & \text{if } m = 2t + 1; \\ 2^{2t+2} + 2^{2t+1} + 3 & \text{if } m = 4t + 2; \\ 2^{4t+3} + 2^{4t+2} + 2^{4t+1} + 1 & \text{if } m = 8t + 4. \end{cases}$$

Motivated by the cryptographic importance of LCD codes [23] and the valuable report of Ding in [6], this article aims to study the dimension of nonbinary LCD BCH codes of length $n=q^m+1$ based on Refs. [16, 17, 22]. It extends our previous work on binary antiprimitive LCD BCH codes [22], simplifies some of the proofs and generalizes many of the results to the nonbinary case. The main results are listed as follows.

(1): For $m \geq 4$, the coset leaders of C_x modulo $n = q^m + 1$ are determined for

$$q^{\lceil \frac{m}{2} \rceil} < x \leq \left\{ \begin{array}{ll} 2q^{\lceil \frac{m}{2} \rceil} + 2 & \text{if } m = 2t; \\ 2q^{\lceil \frac{m}{2} \rceil} + 2q - 1 & \text{if } m = 2t + 1. \end{array} \right.$$

This doubly extends the corresponding results of Refs. [16,17] (in detail, $x < q^{\lceil \frac{m}{2} \rceil}$) about the cosets leaders modulo n.

(2): A new kind of useful sequences is introduced. Consequently, for even m and odd q the second largest coset leader modulo $n = q^m + 1$ is determined. Also, when m is odd, the first six (resp. five) largest coset leaders modulo $n = q^m + 1$ are determined with q odd (resp. even). These results above shall solve a majority of the significative problem proposed by Ding [6] to determine the second and third largest coset leaders.

Furthermore, some conjectures about the first several largest coset leaders modulo n are proposed for even m. If they could be well verified, the problem about first several largest coset leaders of antiprimitive BCH codes would be completely settled.

(3): Through calculating the cardinalities of relevant cyclotomic cosets, dimensions of some nonbinary antiprimitive BCH codes are precisely obtained. Additionally, their Bose distances are given together.

This article is organized as follows. In Section 2, some basic concepts on cyclotomic cosets, BCH codes and LCD codes are reviewed. In Section 3, the parameters of BCH codes of length $n=q^m+1 (m \geq 4)$ with designed distances for $q^{\lceil \frac{m}{2} \rceil} < \delta \leq 2q^{\lceil \frac{m}{2} \rceil} + 2$. In Section 4, the parameters of BCH codes of length $n=q^{2t+1}+1$ with designed distances $\delta > \delta_5$ (resp. $\delta > \delta_6$) are determined when q is even (resp. odd). Some conclusions and conjectures about the first several coset leaders modulo $n=q^m+1$ with m even are presented in Section 5. The final remarks are drawn in Section 6.

2 Preliminaries

In this section, we recall some basic concepts on cyclotomic cosets, cyclic codes, BCH codes and LCD codes. For more details, one can refer to Refs. [23–25].

Let q be a prime power and \mathbb{F}_q be the finite field with q elements. A linear $[n, k, d]_q$ code \mathcal{C} is denoted as a k-dimensional subspace of \mathbb{F}_q^n with minimum (Hamming) distance d. \mathcal{C} is cyclic if $(c_0, c_1, \dots, c_{n-1}) \in \mathcal{C}$ implies $(c_{n-1}, c_0, c_1, \dots, c_{n-2}) \in \mathcal{C}$. By associating

each vector $(c_0, c_1, \dots, c_{n-1}) \in \mathbb{F}_q^n$ with a polynomial $c(x) = c_0 + c_1 x + \dots + c_{n-1} x^{n-1}$ $\in \mathbb{F}_q[x]/(x^n-1)$, then every cyclic code \mathcal{C} is identified with an ideal of $\mathbb{F}_q[x]/(x^n-1)$. Since every ideal of $\mathbb{F}_q[x]/(x^n-1)$ is principal, each cyclic code \mathcal{C} can be identified with $\mathcal{C} = \langle q(x) \rangle$, where q(x) is monic and has the smallest degree among all the generators of \mathcal{C} . This polynomial g(x) is called the generator polynomial of \mathcal{C} .

Denote $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$. If $\gcd(q, n) = 1$ and $x \in \mathbb{Z}_n$, a q-cyclotomic coset modulo n containing x is defined by

$$C_x = \{xq^i \bmod n | 0 \le i \le l-1\} \subseteq \mathbb{Z}_n,$$

where l is the smallest positive integer such that $q^l x \equiv x \mod n$. The cardinality of C_x is denoted by $|C_x| = l$. The smallest integer in C_x is called the *coset leader* of C_x modulo n.

If ξ is a primitive n-th root of unity in some field containing \mathbb{F}_q , $T = \{i | g(\xi^i) = 0\}$ is called the defining set of $\mathcal{C} = \langle q(x) \rangle$. It is well known that T is the union of some q-cyclotomic cosets modulo n. The dimension k of C is determined by k = n - |T| and the minimum distance d can be evaluated by T.

 \mathcal{C} is called a *BCH code* of designed distance δ if $T = C_b \cup C_{b+1} \cup \cdots \cup C_{b+\delta-2}$. And \mathcal{C} can be denoted by $\mathcal{C}(n,q,\delta,b)$ as given in [16,17]. If $b=1,\mathcal{C}$ is called a narrow-sense BCH code, and non-narrow-sense, otherwise. If $n = q^m - 1$, \mathcal{C} is called *primitive*, and non-primitive, otherwise. Particularly, if $n = q^m + 1$, it is called antiprimitive by Ding in [6].

Given two vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{F}_q^n$, their Euclidean inner product is denoted by $(\mathbf{x}, \mathbf{y}) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.

The Euclidean dual code \mathcal{C}^{\perp} of \mathcal{C} is defined by $\mathcal{C}^{\perp} = \{\mathbf{x} \in \mathbb{F}_q^n \mid (\mathbf{x}, \mathbf{y}) = 0, \forall \ \mathbf{y} \in \mathcal{C}\}.$ is called an LCD code if $\mathcal{C}^{\perp} \cap \mathcal{C} = \{\mathbf{0}\}$, which is equivalent to $\mathcal{C}^{\perp} \oplus \mathcal{C} = \mathbb{F}_q^n$.

Below, we pay main attention to nonbinary antiprimitive BCH codes with defining sets $T = \bigcup_{i=1}^{n-1} C_i$ and $T_0 = \{0\} \bigcup_{i=1}^{n} T_i$, which can be denoted by $C(n, q, \delta, 1)$ and $C(n, q, \delta, 1)$ 1,0), respectively. As thus, they have parameters $\mathcal{C}(n,q,\delta,1)=[n,n-|T|,d\geq\delta]_q$ and $\mathcal{C}(n,q,\delta+1,0) = [n,q^m-|T|,d_0 \geq 2\delta]_q$, respectively.

Throughout this paper, let q be a prime power and $n = q^m + 1$. Suppose that $a, b, c, x \in$ \mathbb{Z}_n and $a \leq b$. We denote $\{x | a \leq x \leq b\}$ by [a,b] and define [a,b] + c = [a+c,b+c]. Let " $x \equiv y$ " denote " $x \equiv y \mod n$ " for short unless otherwise noted. The words "modulo n" are omitted when cyclotomic cosets and coset leaders are mentioned. For example, "x is a coset leader" means "x is a coset leader of C_x modulo n". It is obvious that if $x \neq 0$, $x \in \mathbb{Z}_n$ and q|x, then $\frac{x}{q} \in C_x$ and x is not a coset leader. Therefore, to determine coset leaders in T, one only needs to consider x with $q \nmid x$.

Notation 3 in [22] shall be naturally generalized to the following result.

Proposition 1. Let $n = q^m + 1$ and C_x be a cyclotomic coset modulo n containing $x \in \mathbb{Z}_n$. For $0 \le k \le m-1$, define

$$y_{x,k} \equiv q^k x \mod n \text{ with } y_{x,k} \in \mathbb{Z}_n.$$

Then (1) C_x can be denoted by $C_x=\{y_{x,k},n-y_{x,k}|0\leq k\leq m-1\};$ (2) x is the coset leader of C_x if and only if $y_{x,k}-x\geq 0$ and $n-y_{x,k}-x\geq 0$ for $0 \le k \le m - 1$.

The proposition above is quite necessary to this article and will be continually utilized below. For clarity, an example is provided as follows.

Example 1. If $n = 3^2 + 1 = 10$, we shall obtain the following cyclotomic cosets by Proposition 1.

$$C_1 = \{y_{1,0} = 1, y_{1,1} = 3, y_{1,2} = 10 - 1 = 9, y_{1,3} = 10 - 3 = 7\} = \{1, 3, 9, 7\} = C_7,$$

$$C_2 = \{y_{2,0} = 2, y_{2,1} = 6, y_{2,2} = 10 - 2 = 8, y_{2,3} = 10 - 6 = 4\} = \{2, 6, 8, 4\} = C_4 = C_8,$$

$$C_5 = \{y_{5,0} = 5, y_{5,1} = 5, y_{5,2} = 10 - 5 = 5, y_{5,3} = 10 - 5 = 5\} = \{5\},$$

It is easy to know that 1, 2 and 5 are coset leaders, while 4, 7 and 8 are not. Since 3 and 6 are both divisible by 3, they are out of consideration.

3 Dimensions of BCH codes with relatively small distance

In this section, suppose that $n = q^m + 1$ with $m \ge 4$. The coset leaders of C_x are determined for

$$x \le \begin{cases} 2q^t + 2 & \text{if } m = 2t; \\ 2q^{t+1} + 2q - 1 & \text{if } m = 2t + 1. \end{cases}$$

We split into two subsections according to the parity of m. In each subsection, the coset leaders and the cardinalities of the cyclotomic cosets containing them are firstly determined. Then the dimension of many BCH codes with relatively small designed distance shall be naturally calculated.

3.1 BCH codes of length $n = q^m + 1$ for m odd

Throughout this subsection, let $n = q^m + 1$ with $m = 2t + 1 \ge 5$.

Theorem 1. If $x \not\equiv 0 \mod q$, then the following statements hold:

- (1) If $1 \le x \le q^{t+1} q 1$, then x is a coset leader (see [16, 17]).
- (2) If $q^{t+1} + q + 1 \le x \le q^{t+1} + q^t 2$, then x is a coset leader. (3) Given an integer $\alpha \in [1, q 2]$. If $q^{t+1} + \alpha q^t + 2 \le x \le q^{t+1} + (\alpha + 1)q^t 2$, then
 - (4) If $q^{t+1} + (q-1)q^t + 2 \le x \le 2q^{t+1} 2q 1$, then x is a coset leader.
 - (5) Given three integers $\alpha \in [1, q-1]$, $\beta = 1$ or 2 and $1 \le \gamma \le \beta q 1$. If

$$x = q^{t+1} + \alpha q^t \pm 1$$
 or $x = \beta q^{t+1} \pm \gamma$,

then x is not a coset leader.

Proof. See Appendix A.1.

Lemma 2. If $1 \le x \le 2q^{t+1} - 2q - 1$, then $|C_x| = 2m$.

Proof. Seeking a contradiction, suppose $|C_x| = k$ with k < 2m. It follows that $x(q^k - 1) \equiv k$ 0. Since k|2m and m is odd, we have $k=m,\frac{2m}{3}$ or $k\leq \frac{2m}{5}$. (1) If k=m, then $(q^k-1,n)=(q^m-1,q^m+1)=2$. From $1\leq x\leq q^{t+1}-2q-1<\frac{n}{2}$,

- one has $x(q^k-1) \not\equiv 0$, a contradiction.
- (2) If $m \equiv 0 \mod 3$ and $k = \frac{2m}{3}$, then $(q^k 1, n = q^m + 1) = q^{\frac{m}{3}} + 1$. Since $1 \le x \le q^{t+1} 2q 1 < \frac{n}{q^{\frac{m}{3}} + 1} = q^{\frac{2m}{3}} q^{\frac{m}{3}} + 1$, so there holds $x(q^k 1) \not\equiv 0$.
- (3) If $k \leq \frac{2m}{5}$, then $1 \leq x(q^k 1) \leq (2q^{t+1} 2q 1)(q^{\frac{2m}{5}} 1) < n$ and $x(q^k 1) \not\equiv 0$. Collecting all previous discussions, we have $x(q^k 1) \not\equiv 0$, this yields a contradiction. Hence, $|C_x| = 2m$ for $1 \le x \le 2q^{t+1} - 2q - 1$.

The previous results on the coset leaders and cardinalities are sufficient to give the following conclusion.

Theorem 3. Suppose that $n = q^m + 1$ with $m = 2t + 1 \ge 5$. If $\alpha \in [1, q - 2]$, then the following statements hold:

(1) The narrow-sense BCH codes $C(n, q, \delta, 1)$ have parameters

$$\begin{cases} \left[n, n - 2m \lceil (\delta - 1)(1 - 1/q) \rceil + 4m(q - 1), d \geq \delta \right]_q & \text{if } q^{t+1} + q + 1 \leq \delta \leq q^{t+1} + q^t - 2; \\ \left[n, n - 2m \lceil (\delta - 1)(1 - 1/q) \rceil + 4m(\alpha + q - 1), d \geq \delta \right]_q & \text{if } q^{t+1} + \alpha q^t + 2 \leq \delta \leq q^{t+1} + (\alpha + 1)q^t - 2; \\ \left[n, n - 2m \lceil (\delta - 1)(1 - 1/q) \rceil + 8m(q - 1), d \geq \delta \right]_q & \text{if } q^{t+1} + (q - 1)q^t + 2 \leq \delta \leq 2q^{t+1} - 2q - 1; \\ \left[n, n - 2m(2q^{t+1} - 2q^t - 6q + 7), d \geq 2q^{t+1} + 2q \right]_q & \text{if } 2q^{t+1} - 2q + 1 \leq \delta \leq 2q^{t+1} + 2q. \end{cases}$$

(2) The BCH codes $C(n, q, \delta + 1, 0)$ have parameters

$$\begin{cases} [n,q^m-2m\lceil(\delta-1)(1-1/q)\rceil+4m(q-1),d\geq 2\delta]_q & \text{if } q^{t+1}+q+1\leq \delta\leq q^{t+1}+q^t-2;\\ [n,q^m-2m\lceil(\delta-1)(1-1/q)\rceil+4m(\alpha+q-1),d\geq 2\delta]_q & \text{if } q^{t+1}+\alpha q^t+2\leq \delta\leq q^{t+1}+(\alpha+1)q^t-2;\\ [n,q^m-2m\lceil(\delta-1)(1-1/q)\rceil+8m(q-1),d\geq 2\delta]_q & \text{if } q^{t+1}+(q-1)q^t+2\leq \delta\leq 2q^{t+1}-2q-1;\\ [n,q^m-2m(q^{t+1}-2q^t-6q+7),d\geq 4q^{t+1}+4q]_q & \text{if } 2q^{t+1}-2q+1\leq \delta\leq 2q^{t+1}+2q. \end{cases}$$

Proof. With the conclusions of the coset leaders and cardinalities in hand, it is natural to obtain the dimension of BCH codes for given designed distance. Since the proof is very similar to that of Theorem 3.7 in [22], we have it omitted here. П

Remark: The following theorems about giving the dimension can be also verified in the similar way to that of Theorem 3.7 in [22], then they will be omitted, too.

BCH codes of length $n = q^m + 1$ for m even

In this subsection, suppose that $n = q^m + 1$ with m = 2t > 4.

Theorem 4. If $x \not\equiv 0 \mod q$, then the following statements hold:

- (1) If $1 \le x \le q^t 1$, then x is a coset leader(see [16, 17]).
- (2) If $q^t + 2 \le x \le 2q^t 2$, then x is a coset leader.
- (3) If $x = q^t + 1, 2q^t 1, 2q^t + 1$ or $2q^t + 2$, then x is not a coset leader.

Proof. See Appendix A.2.

To calculate the actual dimension of BCH codes, it still needs to get the cardinalities of the cosets containing the coset leaders below.

Lemma 5. If
$$1 \le x \le 2q^t - 2$$
, then $|C_x| = 2m$.

Proof. Seeking a contradiction, suppose that $|C_x| = k$ with k < 2m, it follows that

- $x(q^k-1) \equiv 0$. Since k|2m and m is even, we have $k=m, \frac{m}{2}, k=\frac{2m}{3}$ or $k \leq \frac{2m}{5}$. (1) If $k=m, \frac{m}{2}$, then $(q^k-1, n)=(q^m-1, q^m+1)=1$ or 2. From $1 \leq x \leq 2q^t-2 < \frac{n}{2}$, one has $x(q^k-1) \not\equiv 0$, a contradiction.
- (2) If $m \equiv 0 \mod 3$ and $k = \frac{2m}{3}$, then $(q^k 1, n = q^m + 1) = q^{\frac{m}{3}} + 1$. Since $1 \le x \le 2q^t 2 < \frac{n}{q^{\frac{m}{3}} + 1} = q^{\frac{2m}{3}} q^{\frac{m}{3}} + 1$, we get $x(q^k 1) \not\equiv 0$.
- (3) If $k \leq \frac{2m}{5}$, it can be derived that $1 \leq x(q^k-1) \leq (2q^t-2)(q^{\frac{2m}{5}}-1) < n$ and $x(q^k - 1) \not\equiv 0.$

All the three cases contradict $x(q^k-1) \equiv 0$. Hence, one can easily know that $|C_x|=2m$ for every $1 \le x \le 2q^{t+1} - 2q - 1$.

Based on these results above, the dimension of some BCH codes can be obtained.

Theorem 6. Suppose that $n = q^m + 1$ with $m = 2t \ge 4$. Then the following statements hold:

(1) The narrow-sense BCH codes $C(n, q, \delta, 1)$ have parameters

$$\left\{ \begin{array}{ll} [n,n-2m\lceil(\delta-1)(1-1/q)\rceil+2m, d \geq \delta]_q & \mbox{if } q^t+2 \leq \delta \leq 2q^t-2; \\ [n,n-4m(q^t-q^{t-1}-1), d \geq 2q^t+3]_q & \mbox{if } 2q^t-1 \leq \delta \leq 2q^t+3. \end{array} \right.$$

(2) The BCH codes $C(n, q, \delta + 1, 0)$ have parameters

$$\left\{ \begin{array}{ll} [n,q^m-2m\lceil(\delta-1)(1-1/q)\rceil+2m, d \geq 2\delta]_q & \mbox{if } q^t+2 \leq \delta \leq 2q^t-2; \\ [n,q^m-4m(q^t-q^{t-1}-1), d \geq 4q^t+6]_q & \mbox{if } 2q^t-1 \leq \delta \leq 2q^t+3. \end{array} \right.$$

4 Dimensions of $C(q^{2t+1}+1,q,\delta,b)$ with relatively large δ

Throughout this section, let $n = q^m + 1$ with $m = 2t + 1 \ge 5$. It will be divided into two subsections by the parity of q. In every subsection, we first determine the first several largest coset leaders and calculate the cardinalities of the cosets containing them. Consequently, it is possible to obtain the exact dimension of some BCH codes in terms of relatively large designed distances.

4.1 BCH codes over \mathbb{F}_q of odd characteristic

In this subsection, let q be an odd prime power, we give the following results.

Lemma 7. Let n and q be given as above. Consider

$$\delta_1 = \frac{n}{2}, \ \delta_2 = \frac{n}{q+1} \cdot \frac{q-1}{2}, \ \delta_3 = \delta_2 - \frac{2\delta_2 + (q-1)^2}{q^2} \ and \ \delta_4 = \delta_3 - (q-1)^2.$$
If $m = 5$, put $\delta_5 = \delta_4 - (q-1)$ and $\delta_6 = \delta_5 - 1$;
if $m \geq 7$, let $\delta_5 = \delta_4 - (q^2 - 1)(q-1)^2$ and $\delta_6 = \delta_5 - (q-1)^2$. Then $\delta_i(i = 1, 2, \dots, 6)$ are all coset leaders.

To derive the conclusion of Lemma 8, we first introduce an **Iterative algorithm**, a quite useful technology introduced in [22] to determine the fist several largest coset leaders modulo $2^m + 1$. Adopting it, we can partition $I^{(t)}$ into 2^{t-2} disjoint subintervals, where

$$I^{(t)} = \begin{cases} [1, q-2] & \text{if } t = 2; \\ [1, (q-1)^2 q^{2t-5}] & \text{if } t \ge 3. \end{cases}$$

This is a key to settling the very intractable problem.

Iterative Algorithm 1 (IA-1, for short):

1) If
$$t=2$$
, let $I_1=I_{2^0}=[1,q-2]=[a_1,b_1];$
2) If $t=3$, let $I_2=I_{2^1}=[q-1,(q-1)^2q^{2\times 1-1}]=[a_2,b_2];$
3) If $t=4$, consider that
$$I_3=I_{2^1+1}=I_1+b_2=[a_1+(q-1)^2q,b_1+(q-1)^2q]=[a_3,b_3],$$
 $I_4=I_{2^2}=[a_2+b_2,(q-1)^2q^{2\times 2-1}]=[a_2+(q-1)^2q,(q-1)^2q^3]=[a_4,b_4];$
4) Let $t\geq 5$. Suppose that a partition of $I^{(t-1)}$ is given by $I^{(t-1)}=[1,(q-1)^2q^{2t-7}]=I_1\bigcup I_2\cdots\bigcup I_{2^{t-3}},$ where $I_j=[a_j,b_j].$ For $u=2^{t-3}+j$ with $1\leq j\leq 2^{t-3}-1$, consider

$$I_u = I_j + b_{2^{t-3}} = [a_j + (q-1)^2 q^{2t-7}, b_j + (q-1)^2 q^{2t-7}].$$

For $u = 2^{t-2}$, let $I_u = [a_{2^{t-3}} + b_{2^{t-3}}, (q-1)^2 q^{2t-5}] = [a_{2^{t-2}}, b_{2^{t-2}}]$. Consequently, a partition of $I^{(t)} = [1, (q-1)^2 q^{2t-5}]$ can be obtained as

$$I^{(t)} = (I_1 \cup I_2 \cdots \cup I_{2^{t-3}}) \cup I_{2^{t-3}+1} \cdots \cup I_{2^{t-2}} = I^{(t-1)} \bigcup (\bigcup_{u=2^{t-3}+1}^{2^{t-2}} I_u) = \bigcup_{s=1}^{2^{t-2}} I_s.$$

Using IA-1 above, we can obtain the following crucial result.

Lemma 8. Let δ_3 and $I^{(t)}$ be given as above. If $x \in \delta_3 + I^{(t)}$, then x is not a coset leader.

Proof. Following the discussions and notations of the *Iterative Algorithm* above, one can derive that

(1):
$$I_{2^j} = \begin{cases} [1, q-2] & \text{if } j=0; \\ [(q-1)(1-q+\cdots+q^{2j-2}), (q-1)^2q^{2j-1}] & \text{if } 1 \leq j \leq t-2. \end{cases}$$
(2): When $t \geq 3$ and $s \in [1, 2^{t-2} - 1]$, we have the 2-adic expansion of s :

$$s = a_0 2^0 + a_1 2^1 + a_2 2^2 + \dots + a_{t-3} 2^{t-3} = (a_0 a_1 a_2 \dots a_{t-3})_2.$$

Set $i = i_s = \min\{j | a_i = 1, 0 \le j \le t - 3\}$. Whence I_s can be given by

$$I_s = I_{2i} + a_{i+1}(q-1)^2 q^{2i+1} + a_{i+2}(q-1)^2 q^{2i+3} + \dots + a_{t-3}(q-1)^2 q^{2t-7}$$

Below, we split into two cases according to a_{i+1} .

Case A: $a_{i+1} = 0$

If $a_{i+1} = 0$, then we have

$$I_{s} = I_{2^{i}} + 0 \times (q-1)^{2} q^{2i+1} + a_{i+2} (q-1)^{2} q^{2i+3} + \dots + a_{t-3} (q-1)^{2} q^{2t-7}$$

$$= I_{2^{i}} + (q-1)^{2} q^{2i+3} \underbrace{(a_{i+2} + a_{i+3} q^{2} + \dots + a_{t-3} q^{2(t-i-5)})}_{\lambda}$$

$$= I_{2^{i}} + (q-1)^{2} q^{2i+3} \lambda, \text{ where } 0 \le \lambda \le 1 + q^{2} + \dots + q^{2(t-i-5)} < q^{2(t-i-4)}$$

Additionally, it is obvious that $I_s = I_{2^{t-2}} = I_{2^i} + (q-1)^2 q^{2i+3} \cdot \lambda$ with i = t-2 and $\lambda = 0$. Hence, for each $s \in [1, 2^{t-2}]$, there exist two integers $0 \le i = i_s \le t-2$ and $0 \le \lambda < q^{2(t-i-3)}$ such that

$$I_s = I_{i,\lambda} = I_{2i} + (q-1)^2 q^{2i+3} \lambda.$$

For a given integer $s \in [1, 2^{t-2}]$, if $l \in I_s = I_{i,\lambda}$, then $x = \delta_3 + l$ can be denoted by

$$x = \delta_3 + l_0 + (q - 1)^2 q^{2i+3} \lambda,$$

where $0 \le i \le t - 2$, $0 \le \lambda < q^{2(t-i-4)}$ and $l_0 \in I_{2^i}$.

When k = 2t - 2i - 1, we have k > 3 and

$$\begin{split} q^k x & \equiv & y_{x,k} \\ & = & q^k x - (\frac{q^k - 1}{2} - q^{k-1} + (q^{k-3} \cdot \dots - q + 1) + (q - 1)^2 q \lambda) n \\ & = & \frac{n}{2} - (q^{2t} - q^{2t-1} + \dots + q^{k+1}) + q^{k-1} - (q^{k-3} - \dots - q + 1) + q^k l_0 - (q - 1)^2 q \lambda. \end{split}$$

Firstly, we study an upper bound of $y_{x,k}$:

$$\begin{array}{ll} y_{x,k} & = & \frac{n}{2} - (q^{2t} - q^{2t-1} + \dots + q^{k+1}) + q^{k-1} - (q^{k-3} - \dots - q + 1) + q^k l_0 - (q - 1)^2 q \lambda \\ & \leq & \frac{n}{2} - (q^{2t} - q^{2t-1} + \dots + q^{k+1}) + q^{k-1} - (q^{k-3} - \dots - q + 1) + q^k l_0 \\ & \leq & \frac{n}{2} - (q^{2t} - q^{2t-1} + \dots + q^{k+1}) + q^{k-1} - (q^{k-3} - \dots - q + 1) + q^k (q - 1)^2 q^{2i-1} \\ & = & \frac{n}{2} - (q^{2t} - q^{2t-1} + \dots + q^{2t-2i}) + q^{2t-2i-2} - (q^{2t-2i-4} - \dots - q + 1) + (q - 1)^2 q^{2t-2} \\ & \leq & \frac{n}{2} - (q^{2t} - q^{2t-1} + \dots + q^4) + q^2 - 1 + (q - 1)^2 q^{2t-2} (\text{choose } i = t - 2) \\ & = & \frac{n}{2} - q^{2t-1} + (q^{2t-3} - \dots + q - 1) - q(q - 1)^2. \end{array}$$

Next, we will investigate a lower bound of $y_{x,k}$:

If i = 0, then $l_0 \in [1, q - 2]$ and $0 \le \lambda < q^{2(t-3)}$, we get that

$$\begin{split} y_{x,k} &= \frac{1}{2}(q^{2t+1}-q^{2t}\cdots+q)+(l_0-1)q^{2t-1}+q^{2t-2}-(q-1)\lambda \\ &> \frac{1}{2}(q^{2t+1}-q^{2t}\cdots+q)+(l_0-1)q^{2t-1}+q^{2t-2}-(q-1)q^{2(t-3)} \\ &\geq \frac{1}{2}(q^{2t+1}-q^{2t}\cdots+q)+(1-1)q^{2t-1}+q^{2t-2}-(q-1)q^{2(t-3)} \\ &= \frac{1}{2}(q^{2t+1}-q^{2t}+q^{2t-1}+q^{2t-2}+(q^{2t-3}-\cdots+q^3-q^2))-(q-1)q^{2(t-3)}. \end{split}$$

If $t \geq 3$ and $1 \leq i \leq t-2$, then $l_0 \in I_{2^i} = [b_i, e_i] = [(q-1)(q^{2i-2} \cdots - q+1), (q-1)^2 q^{2i-1}]$, one can infer that

$$\begin{array}{ll} y_{x,k} & = & \frac{n}{2} - (q^{2t} - q^{2t-1} \cdot \dots + q^{k+1}) + q^{k-1} - (q^{k-3} \cdot \dots - q+1) + q^k l_0 - (q-1)^2 q \lambda \\ \\ & > & \frac{n}{2} - (q^{2t} - q^{2t-1} \cdot \dots + q^{k+1}) + q^{k-1} - (q^{k-3} \cdot \dots - q+1) + q^k l_0 - (q-1)^2 q \times q^{2(t-i-4)} \\ \\ & \geq & \frac{n}{2} - (q^{2t} - q^{2t-1} \cdot \dots + q^{k+1}) + q^{k-1} - (q^{k-3} \cdot \dots - q+1) + q^k b_i - (q-1)^2 q \times q^{2(t-i-4)} \\ \\ & = & \frac{n}{2} - q^{2t} + q^{2t-1} - (q^{2t-3} - \dots + q^k - q^{k-1}) - (q^{k-3} \cdot \dots - q+1) - (q-1)^2 q^{k-6} \\ \\ & \geq & \frac{n}{2} - q^{2t} + q^{2t-1} - (q^{2t-3} - \dots + q^3 - q^{3-1}) - (q^{3-3} \cdot \dots - q+1) - (q-1)^2 q^{3-6} \\ \\ & = & \frac{n}{2} - q^{2t} + q^{2t-1} - (q^{2t-3} - \dots + q-1) + q - 2 - (q-1)^2 q^{-3}. \end{array}$$

Notice that

$$\lceil \frac{(q-1)n/2 - \delta_3}{q} \rceil = \frac{n}{2} - q^{2t} + q^{2t-1} - (q^{2t-3} - \dots + q - 1) \text{ and }$$

$$\frac{(q-1)n/2 + \delta_3}{q} = \frac{n}{2} - q^{2t-1} + (q^{2t-3} - \dots + q - 1).$$

It is easy to observe that $\lceil \frac{(q-1)n/2-\delta_3}{q} \rceil < y_{x,k} < \frac{(q-1)n/2+\delta_3}{q} \text{ for } k=2t-2i-1.$ If $\lceil \frac{(q-1)n/2-\delta_3}{q} \rceil < y_{x,k} < \lfloor \frac{(q-1)n}{2q} \rfloor,$ then $\frac{(q-1)n}{2}-\delta_3 < qy_{x,k} < \frac{(q-1)n}{2},$ we can infer there exists a $j_{x,k} = \frac{(q-1)n}{2} - qy_{x,k} \in C_{y_{x,k}}$ such that $0 < j_{x,k} < \delta_3.$

If $\lceil \frac{(q-1)n}{2a} \rceil \leq y_{x,k} < \frac{(q-1)n/2+\delta_3}{a}$, then $\frac{(q-1)n}{2} \leq qy_{x,k} < \frac{(q-1)n}{2} + \delta_3$, it follows that there exists a $j_{x,k} = qy_{x,k} - \frac{(q-1)n}{2} \in C_{y_{x,k}}$ such that $0 < j_{x,k} < \delta_3$.

For given $s \in [1, 2^{t-2}]$ and $x \in \delta_3 + I_s$, if $a_{i+1} = 1$, we still choose k = 2t - 2i - 1. Similar to the case for $a_{i+1} = 0$, it then can be derived that $\frac{(q+1)n/2 - \delta_3}{q} < y_{x,k} \le \lfloor \frac{(q+1)n/2 + \delta_3}{q} \rfloor$. Also, we shall obtain that there exists either $j_{x,k} = \frac{(q+1)n}{2} - qy_{x,k}$ or $j_{x,k} = qy_{x,k} - \frac{(q+1)n}{2}$ such that $j_{x,k} \in C_{y_{x,k}}$ and $0 < j_{x,k} < \delta_3$.

Combining Cases A and B, for every $x \in [\delta_3 + 1, \delta_3 + (q-1)^2 q^{2t-5}]$, there exists an integer $j_{x,k} \in [0, \delta_3 - 1]$ such that $j_{x,k} \in C_x$, i.e., x is not a coset leader. This completes the proof of this lemma.

Lemma 9. Let δ_1 , δ_2 and δ_3 be given as above, if $x > \delta_3$ and $x \neq \delta_1$ or δ_2 , then x is not a coset leader.

Proof. To attain the desired conclusion, it suffices to verify that there exists $y_{x,k} < x$ or $n-y_{x,k} < x$ for some $k \in [0, m-1]$ and $x > \delta_3$ with $x \neq \delta_1$ or δ_2 . We give the following:

(1): If $x \in [\frac{n}{2} + 1, n - 1]$, then x < n < 2x. Clearly, $n - y_{x,0} = n - x < x$.

(2): If $x \in [\lceil \frac{(q-1)n}{2q} \rceil, \frac{n}{2} - 1]$, then $(q-1)x < \frac{(q-1)n}{2} < qx$, it follows that $y_{x,1} - \frac{(q-1)n}{2} = 1$ $qx - \frac{(q-1)n}{2} < x.$

(3): If $x \in [\delta_2 + 1 = \frac{n}{q+1} \cdot \frac{q-1}{2} + 1, \lfloor \frac{(q-1)n}{2q} \rfloor]$, then $qx < \frac{(q-1)n}{2} < (q+1)x$, whence $n - y_{x,1} = \frac{(q-1)n}{2} - qx < x.$

(4): If $x \in \lceil \frac{(q-1)^2 n}{2q^2} \rceil$, $\delta_2 - 1 = \frac{n}{q+1} \cdot \frac{q-1}{2} - 1 \rceil$, we have $(q^2 - 1)x < \frac{(q-1)^2 n}{2} < q^2 x$, one

can then infer that $y_{x,2} = q^2x - \frac{(q-1)^2n}{2} < x$. (5): If $x \in \lceil \lceil \frac{(q-1)^2n}{2(q^2+1)} \rceil, \lfloor \frac{(q-1)^2n}{2q^2} \rfloor \rceil$, we have $q^2x < \frac{(q-1)^2n}{2} < (q^2+1)x$ and $n-y_{x,2} = 1$ $\frac{(q-1)^2n}{2} - q^2x < x$.

(6): If $x \in \lceil \frac{(q^2-1)(q-1)^2n}{2q^4} \rceil$, $\lfloor \frac{(q-1)^2n}{2(q^2+1)} \rfloor \rceil$, we get that $(q^4-1)x < \frac{(q^2-1)(q-1)^2n}{2} < q^4x$, and then $y_{x,4} = q^4x - \frac{(q^2-1)(q-1)^2n}{2} < x$.

Summarizing the previous six cases, x is not a coset leader for $x \in \lceil \frac{(q^2-1)(q-1)^2n}{2q^4} \rceil, n-1 = 1$ 1] \ $\{\delta_1, \delta_2\}$. It then follows from the lemma above that x is not a coset leader for all x with $x \in \delta_3 + I^{(t)} \bigcup \left[\left\lceil \frac{(q^2 - 1)(q - 1)^2 n}{2q^4} \right\rceil, n - 1 \right] \setminus \{\delta_1, \delta_2\}$.

Notice that for t = 2,

$$\lceil \frac{(q^2 - 1)(q - 1)^2 n}{2q^4} \rceil = \frac{1}{2} (q^5 - 2q^4 + 2q^2 - q) + 1$$

$$\leq \frac{1}{2} (q^5 - 2q^4 + 2q^2 - q) + 1 + \frac{q + 1 - 4}{2}$$

$$= \delta_3 + (q - 2) + 1,$$

and for t > 2, we have

$$\lceil \frac{(q^2 - 1)(q - 1)^2 n}{2q^4} \rceil = \frac{1}{2} (q^{2t+1} - 2q^{2t} + 2q^{2t-2} - q^{2t-3}) + 1$$

$$< \delta_3 + (q - 1)^2 q^{2t-5}.$$

It is easy to know $(\delta_3 + I^{(t)}) \bigcup [\lceil \frac{(q^2 - 1)(q - 1)^2 n}{2q^4} \rceil, n - 1] = [\delta_3 + 1, n - 1]$ and then the desired conclusion can be obtained.

Theorem 10. For given n and q, let $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ and δ_6 be given as above, then they are the first, second, third, fourth, fifth and sixth largest coset leaders, respectively.

Proof. From Lemmas 7-9, it naturally follows that δ_1 , δ_2 and δ_3 are the first, second and third largest coset leaders, respectively. We now proceed to derive the following results.

For
$$1 \le i \le \frac{(q-1)^2}{2} - 1$$
,

$$q^{2t-1}(\delta_3 - i) \equiv q^{2t-1}(\delta_3 - i) - (\frac{q^{2t-1} - 1}{2} - q^{2t-2} + (q^{2t-4} \cdot \dots - q + 1))n$$

$$= \frac{n}{2} - q^{2t} + q^{2t-2} - (q^{2t-4} - \dots - q + 1) - q^{2t-1}i$$

$$\leq \frac{n}{2} - q^{2t} - q^{2t-1} + q^{2t-2} - (q^{2t-4} - \dots - q + 1) < \delta_3 - i.$$

For
$$\frac{(q-1)^2}{2} \le i \le (q-1)^2 - 1 = q^2 - 2q$$
,

$$-q^{2t-1}(\delta_3 - i) \equiv \left(\frac{q^{2t-1} - 1}{2} - q^{2t-2} + (q^{2t-4} \cdot \dots - q + 1)\right)n - q^{2t-1}(\delta_3 - i)$$

$$= q^{2t-1}i - \left(\frac{n}{2} - q^{2t} + q^{2t-2} - (q^{2t-4} - \dots - q + 1)\right)$$

$$\leq q^{2t-1}(q^2 - 2q) - \left(\frac{n}{2} - q^{2t} + q^{2t-2} - (q^{2t-4} - \dots - q + 1)\right) < \delta_3 - i.$$

This implies that there exists an integer $u_x < x$ such that $u_x \in C_x$ for $\delta_4 < x < \delta_3$, i.e., x is not a coset leader for $\delta_4 < x < \delta_3$. In a similar way, one can easily infer that x is not a coset leader for $\delta_5 < x < \delta_4$ or $\delta_6 < x < \delta_5$. Hence, the lemma holds. \square

Lemma 11. If $\delta_i(i = 1, 2, \dots, 6)$ are given as above, then $|C_{\delta_1}| = 1$, $|C_{\delta_2}| = 2$ and $|C_{\delta_i}| = 2m(i = 3, 4, 5, 6)$.

Proof. From the proof of Lemma 7, we have derived that $C_{\delta_1} = \{\delta_1\}$ and $C_{\delta_2} = \{\delta_2, \frac{(q+3)}{q}\delta_2\}$. Clearly, $|C_{\delta_1}| = 1$ and $|C_{\delta_2}| = 2$. And when $x \in \{\delta_3, \delta_4, \delta_5\}$, $y_{x,k} > x$ for $1 \le k \le 2m - 1$, it immediately follows that $x(q^k - 1) \not\equiv 0$. Hence, one can know that $|C_x| = 2m$.

Theorem 12. Suppose that $n = q^{2t+1} + 1(t \ge 2)$ and q is an odd prime power. Let δ_i (i = 1, 2, ... 6) be given as above, then the following statements hold:

(1) The narrow-sense BCH codes $C(n, q, \delta, 1)$ have parameters

$$\begin{cases} [n, 6m + 4, d \ge \delta_5]_q & \text{if } \delta_6 + 1 \le \delta \le \delta_5; \\ [n, 4m + 4, d \ge \delta_4]_q & \text{if } \delta_5 + 1 \le \delta \le \delta_4; \\ [n, 2m + 4, d \ge \delta_3]_q & \text{if } \delta_4 + 1 \le \delta \le \delta_3; \\ [n, 4, d \ge \delta_2]_q & \text{if } \delta_3 + 1 \le \delta \le \delta_2; \\ [n, 2, \delta_1 = \frac{n}{2}]_q & \text{if } \delta_2 + 1 \le \delta \le \delta_1; \\ [n, 1, n]_q & \text{if } \delta_1 + 1 \le \delta \le n. \end{cases}$$

(2) The BCH codes $C(n, 2, \delta + 1, 0)$ have parameters

$$\begin{cases} [n,6m+3,d \geq 2\delta_5]_q & \text{if } \delta_6+1 \leq \delta \leq \delta_5; \\ [n,4m+3,d \geq 2\delta_4]_q & \text{if } \delta_5+1 \leq \delta \leq \delta_4; \\ [n,2m+3,d \geq 2\delta_3]_q & \text{if } \delta_4+1 \leq \delta \leq \delta_3; \\ [n,3,d \geq 2\delta_2]_q & \text{if } \delta_3+1 \leq \delta \leq \delta_2; \\ [n,1,2\delta_1=n]_q & \text{if } \delta_2+1 \leq \delta \leq \delta_1. \end{cases}$$

Remark: Consider $n = q^3 + 1$. If δ_1 , δ_2 and δ_3 be given as above. Let $\delta_4 = \delta_3 - 1$, $\delta_5 = \delta_4 - (q-1)$ and $\delta_6 = \delta_5 - 1$. One can show that the lemmas and theorems above hold, too.

BCH codes over \mathbb{F}_q of characteristic two 4.2

In this subsection, let $q = 2^r \ge 4$ and $n = q^{2t+1} + 1$ with $t \ge 2$.

Lemma 13. Suppose that
$$\delta_1 = \frac{n}{q+1} \cdot \frac{q}{2}$$
, $\delta_2 = \delta_1 - \frac{2\delta_1 + (q-1)q}{q^2}$ and $\delta_3 = \delta_2 - q(q-1)$. If $m = 5$, let $\delta_4 = \delta_3 - q$ and $\delta_5 = \delta_1 - \frac{n}{q+1}$. If $m \geq 7$, let $\delta_4 = \delta_3 - q(q^2 - 1)(q - 1)$ and $\delta_5 = \delta_4 - q(q - 1)$. Then $\delta_1, \delta_2, \delta_3, \delta_4$ and δ_5 are all coset leaders.

Proof. See Appendix A.4.

To verify Lemma 14, we first introduce another **Iterative Algorithm** similar to IA-1. Adopting it, one can partition $J^{(t)} = [1, (q-1)q^{2(t-2)}]$ into 2^{t-2} disjoint subintervals.

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Iterative Algorithm 2 (IA-2, for short):

- 1) If t = 2, let $J_1 = J_{2^0} = [1, (q-1)q^{2\times 0}] = [1, q-1] = [a_1, b_1];$
- 2) If t = 3, let $J_2 = J_{2^1} = [a_1 + (q-1)q^{2\times 0}, (q-1)q^{2\times 1}] = [a_2, b_2];$
- 3) If t = 4, consider that

$$J_3 = J_{2^1+1} = J_1 + b_2 = [a_1 + (q-1)q^2, b_1 + (q-1)q^2] = [a_3, b_3],$$

$$J_3 = J_{2^1+1} = J_1 + b_2 = [a_1 + (q-1)q^2, b_1 + (q-1)q^2] = [a_3, b_3],$$

 $J_4 = J_{2^2} = [a_2 + b_2, (q-1)q^{2 \times 2}] = [a_2 + (q-1)q^2, (q-1)q^4] = [a_4, b_4];$

4) Let $t \geq 5$, a partition of $J^{(t-1)}$ is given by

$$J^{(t-1)} = [1, (q-1)q^{2(t-3)}] = J_1 \bigcup J_2 \cdots \bigcup J_{2^{t-3}}, \text{ where } J_j = [a_j, b_j].$$
 For $u = 2^{t-3} + j$ with $1 \le j \le 2^{t-3} - 1$, consider

$$J_u = J_j + b_{2^{t-3}} = [a_j + (q-1)q^{2(t-3)}, b_j + (q-1)q^{2(t-3)}].$$

For
$$u = 2^{t-2}$$
, consider $J_{2^{t-2}} = [a_{2^{t-3}} + b_{2^{t-3}}, (q-1)q^{2(t-2)}] = [a_{2^{t-2}}, b_{2^{t-2}}].$

Then, a partition of $J^{(t)} = [1, (q-1)q^{2(t-2)}]$ can be obtained as follows:

$$J^{(t)} = J^{(t-1)} \bigcup \left(\bigcup_{u=2^{t-3}+1}^{2^{t-2}} J_u \right) = \left(J_1 \cup J_2 \cdots \cup J_{2^{t-3}} \right) \cup J_{2^{t-3}+1} \cdots \cup J_{2^{t-2}}.$$

Lemma 14. Let δ_2 be given as above. If $x \in \delta_2 + J^{(t)} = [\delta_2 + 1, \delta_2 + (q-1)q^{2(t-2)}]$, then x is not a coset leader.

$$\begin{array}{l} \textit{Proof. } \text{For } 0 \leq i \leq t-2 \text{, from } \textit{IA-2} \text{ above, we can infer that} \\ (1): J_{2^j} = \left\{ \begin{array}{ll} [1, q-1] = [1, (q-1)q^{2i}] & \text{if } j=0; \\ [1+(q-1)(1+\cdots+q^{2(j-2)}+q^{2(j-1)}), (q-1)q^{2j}] & \text{if } 1 \leq j \leq t-2. \end{array} \right. \\ (2): \text{ When } t \geq 3 \text{ and } s \in [1, 2^{t-2}-1], \text{ we have the 2-adic expansion of } s: \end{array}$$

$$s = a_0 2^0 + a_1 2^1 + a_2 2^2 + \dots + a_{t-3} 2^{t-3} = (a_0 a_1 a_2 \dots a_{t-3})_2.$$

Let $i = i_s = \min\{j | a_i = 1, 0 \le j \le t - 3\}$, then

$$J_{s} = J_{2^{i}} + a_{i+1}(q-1)q^{2(i+1)} + a_{i+2}(q-1)q^{2(i+2)} + \dots + a_{t-3}(q-1)q^{2(t-3)}$$

$$= J_{2^{i}} + (q-1)q^{2(i+1)}\underbrace{(a_{i+1} + a_{i+2}q^{2} + \dots + a_{t-3}q^{2(t-i-4)})}_{\lambda}$$

$$= J_{2^{i}} + (q-1)q^{2(i+1)}\lambda, \text{ where } 0 \le \lambda \le 1 + q^{2} + \dots + q^{2(t-i-4)} < q^{2(t-i-3)}.$$

Clearly, $J_{2^{t-2}} = J_{2^{t-2}} + (q-1)q^{2(i+1)} \cdot \lambda$ with $\lambda = 0$. Hence, for each $s \in [1, 2^{t-2}]$, there exist two integers $0 \le i \le t-2$ and $0 \le \lambda < q^{2(t-i-3)}$ such that

$$J_s = J_{i\lambda} = J_{2i} + (q-1)q^{2(i+1)}\lambda.$$

Let $x = \delta_2 + l$ with $l \in J^{(t)} = [1, (q-1)q^{2(t-2)}]$, and $J^{(t)} = \bigcup_{s=1}^{2^{t-2}} J_s = J_1 \bigcup J_2 \cdots \bigcup J_{2^{t-2}}$ be a partition of $J^{(t)}$ denoted as above.

For some $s \in [1, 2^{t-2}]$, if $l \in J_s = J_{i,\lambda} = J_{2^i} + (q-1)q^{2(i+1)}\lambda$, then $x = \delta_2 + l$ can be denoted by $x = \delta_2 + (l_0 + (q-1)q^{2(i+1)}\lambda)$, where $0 \le i \le t-2$, $0 \le \lambda < q^{2(t-i-3)}$ and $l_0 \in J_{2^i} = [\frac{q^{2i}+q}{q+1}, (q-1)q^{2i}]$. When k = 2t-2i-1, we have $k \ge 3$ and

$$\begin{split} q^k x &\equiv y_{x,k} \\ &= q^k (\delta_2 + l_0 + (q-1)q^{2(i+1)}\lambda) - (\frac{1}{2}(q^k - q^{k-1} - \frac{q^{k-1} + q}{q+1}) + (q-1)\lambda)n \\ &= \frac{1}{2}(q^{2t+1} - q^{2t} \cdots + q) - q^k + q^{k-1} + q^k l_0 - (q-1)\lambda \\ &= \frac{1}{2}(q^{2t+1} - q^{2t} \cdots + q) + (l_0 - 1)q^{2t-2i-1} + q^{2t-2i-2} - (q-1)\lambda, \end{split}$$

Firstly, we study an upper bound of $y_{\tau k}$:

$$\begin{split} y_{x,k} &= \frac{1}{2}(q^{2t+1}-q^{2t}\cdots+q)+(l_0-1)q^{2t-2i-1}+q^{2t-2i-2}-(q-1)\lambda\\ &\leq \frac{1}{2}(q^{2t+1}-q^{2t}\cdots+q)+(l_0-1)q^{2t-2i-1}+q^{2t-2i-2}\\ &\leq \frac{1}{2}(q^{2t+1}-q^{2t}\cdots+q)+[(q-1)q^{2i}-1]q^{2t-2i-1}+q^{2t-2i-2}\\ &= \frac{1}{2}[q^{2t+1}+q^{2t}-q^{2t-1}-(q^{2t-2}-\cdots+q^2-q)]-(q-1)q^{2t-2i-2}. \end{split}$$

Next, we will investigate a lower bound of $y_{x,k}$:

If i = 0, then $l_0 \in [1, q - 1]$ and $0 \le \lambda < q^{2(t-3)}$, we have

$$\begin{split} y_{x,k} &= \frac{1}{2}(q^{2t+1}-q^{2t}\cdots+q)+(l_0-1)q^{2t-1}+q^{2t-2}-(q-1)\lambda \\ &> \frac{1}{2}(q^{2t+1}-q^{2t}\cdots+q)+(l_0-1)q^{2t-1}+q^{2t-2}-(q-1)q^{2(t-3)} \\ &\geq \frac{1}{2}(q^{2t+1}-q^{2t}\cdots+q)+(1-1)q^{2t-1}+q^{2t-2}-(q-1)q^{2(t-3)} \\ &= \frac{1}{2}[q^{2t+1}-q^{2t}+q^{2t-1}+q^{2t-2}+(q^{2t-3}-\cdots+q^3-q^2)]-(q-1)q^{2(t-3)}. \end{split}$$

If $t \geq 3$ and $1 \leq i \leq t-2$, then $l_0 \in J_{2^i} = \left[\frac{q^{2^i}+q}{q+1}, (q-1)q^{2^i}\right]$, we get that

$$y_{x,k} = \frac{1}{2}(q^{2t+1} - q^{2t} \cdots + q) + (l_0 - 1)q^{2t-1} + q^{2t-2} - (q-1)\lambda$$

$$> \frac{1}{2}(q^{2t+1} - q^{2t} \cdots + q) + (l_0 - 1)q^{2t-1} + q^{2t-2} - (q-1)q^{2(t-3)}$$

$$\geq \frac{1}{2}(q^{2t+1} - q^{2t} \cdots + q) + (\frac{q^{2i} + q}{q+1} - 1)q^{2t-1} + q^{2t-2} - (q-1)q^{2(t-3)}$$

$$= \frac{1}{2}[q^{2t+1} - q^{2t} + q^{2t-1} + (q^{2t-2} - q^{2t-3} \cdots + q^{2t-2i-2})]$$

$$+(q^{2t-2i-3} \cdots - q^2 + q) - (q-1)q^{2(t-i-3)}$$

$$\geq \frac{1}{2}[q^{2t+1} - q^{2t} + q^{2t-1} + (q^{2t-2} - q^{2t-3} \cdots + q^{2t-2(t-2)-2})]$$

$$+(q^{2t-2(t-2)-3} \cdots - q^2 + q) - (q-1)q^{2(t-(t-2)-3)}$$

$$= \frac{1}{2}[q^{2t+1} - q^{2t} + q^{2t-1} + (q^{2t-2} - \cdots - q^3 + q^2)] + q - (q-1)q^{-2}.$$

Note that

$$\frac{qn/2 - \delta_2}{q} = \frac{1}{2}(q^{2t+1} - q^{2t} + q^{2t-1} + (q^{2t-2} - \dots - q^3 + q^2)) - \frac{q}{2} + 1 \text{ and}$$

$$\frac{qn/2 + \delta_2}{q} = \frac{1}{2}(q^{2t+1} + q^{2t} - q^{2t-1} - (q^{2t-2} - \dots + q^2 - q)).$$

It is easy to get that $\frac{qn/2-\delta_2}{q} < y_{x,k} < \frac{qn/2+\delta_2}{q}.$

If $\frac{qn/2-\delta_2}{q} < y_{x,k} \le \frac{n-1}{2}$, then $\frac{qn}{2} - \delta_2 < qy_{x,k} \le \frac{q(n-1)}{2}$, we can infer there exists a $j_{x,k} = \frac{qn}{2} - qy_{x,k} \in C_{y_{x,k}}$ such that $\frac{q}{2} \le j_{x,k} < \delta_2$. If $\frac{n+1}{2} \le y_{x,k} < \frac{qn/2+\delta_2}{q}$, then $\frac{q(n+1)}{2} \le qy_{x,k} < \frac{qn}{2} + \delta_2$, one shall know that there exists a $j_{x,k} = qy_{x,k} - n \equiv qy_{x,k} \in C_{y_{x,k}}$ such that $\frac{q}{2} \le j_{x,k} < \delta_2$.

Concluding the previous discussions, for all $x \in [\delta_2 + 1, \delta_2 + 2^{2t-4}]$, there exists an integer $\frac{q}{2} \leq j_{x,k} < \delta_2$ such that $j_{x,k} \in C_x$. Therefore, x is not a coset leader for $x \in [\delta_2 + 1, \delta_2 + 2^{2t-4}]$.

Lemma 15. Let δ_1 and δ_2 be given as above, if $x > \delta_2$ and $x \neq \delta_1$, then x is not a coset

Proof. To attain the desired conclusion, it suffices to verify there exists $y_{x,k} < x$ or n-1 $y_{x,k} < x$ for some $k \in [0, m-1]$ and $x > \delta_2$ except that $x = \delta_1$. We give our discussions

- (1): If $x \in [\frac{n+1}{2}, n-1]$, then x < n < 2x. Obviously, $n y_{x,0} = n x < x$. (2): If $x \in [\delta_1 + 1 = \frac{n}{q+1} \cdot \frac{q}{2} + 1, \frac{n-1}{2}]$, then $qx < \frac{qn}{2} < (q+1)x$. It follows that $n - y_{x,1} = \frac{qn}{2} - qx < x.$
- (3): If $x \in [\lceil \frac{(q-1)n}{2q} \rceil, \delta_1 1 = \frac{n}{q+1} \cdot \frac{q}{2} 1]$, then $(q^2 1)x < \frac{q(q-1)n}{2} < q^2x$ and $y_{x,2} = q^2 x - \frac{q(q-1)n}{2} < x$
- (4): If $x \in \lceil \lceil \frac{q(q-1)n}{2(q^2+1)} \rceil, \lfloor \frac{(q-1)n}{2q} \rfloor \rceil$, then $q^2x < \frac{q(q-1)n}{2} < (q^2+1)x$, whence $y_{x,2} = \frac{q(q-1)n}{2}$ $\frac{q(q-1)n}{2} - q^2x < x$.
- (5): If $x \in \left[\left\lceil \frac{(q^2-1)(q-1)n}{2q^3} \right\rceil, \left\lfloor \frac{q(q-1)n}{2(q^2+1)} \right\rfloor \right]$, we have $(q^4-1)x < \frac{q(q^2-1)(q-1)n}{2} < q^4x$. Hence, $y_{x,4} = q^4x - \frac{q(q^2-1)(q-1)n}{2} < x$

We then conclude that x is not a coset leader for $x \in \left[\left\lceil \frac{(q^2-1)(q-1)n}{2q^3}\right\rceil, n-1\right] \setminus \{\delta_1\}$. It follows from Lemma 14 that x is not a coset leader for all x with $x \in [\delta_2 + 1, \delta_2 + (q - q)]$ $1)q^{2(t-2)} \bigcup \left[\left\lceil \frac{(q^2-1)(q-1)n}{2q^3} \right\rceil, n-1 \right] \setminus \left\{ \frac{n}{3} \right\}.$

Notice that

$$\lceil \frac{(q^2 - 1)(q - 1)n}{2q^3} \rceil = \frac{1}{2} (q^{2t+1} - q^{2t} - q^{2t-1} + q^{2t-2}) + 1$$

$$< \delta_2 + (q - 1)q^{2(t-2)}.$$

The conclusion can be easily derived from the discussions above.

Theorem 16. Let $\delta_1, \delta_2, \delta_3, \delta_4$ and δ_5 be given as above, then they are the first, second, third, fourth and fifth largest coset leaders, respectively.

Proof. According to Lemmas 13-15, it is enough to show δ_1 and δ_2 are the first and second largest coset leaders, respectively.

We can easily derive the following.

For $1 \le i \le \frac{q(q-1)}{2} - 1$,

$$q^{2t-1}(\delta_2 - i) \equiv q^{2t-1}(\delta_2 - i) - \frac{1}{2}(q^{2t-1} - q^{2t-2} - \frac{q(q^{2t-3} + 1)}{q+1})n$$

$$= \frac{1}{2}(q^{2t+1} - q^{2t} - q^{2t-1} + q^{2t-2} + (q^{2t-3} - q^{2t-4} + \dots + q)) - q^{2t-1}i$$

$$\leq \frac{1}{2}(q^{2t+1} - q^{2t} - 3q^{2t-1} + q^{2t-2} + (q^{2t-3} - q^{2t-4} + \dots + q)) < \delta_2 - i;$$

For
$$\frac{q(q-1)}{2} \le i \le q(q-1) - 1$$
,

$$-q^{2t-1}(\delta_2 - i) \equiv \frac{1}{2}(q^{2t-1} - q^{2t-2} - \frac{q(q^{2t-3} + 1)}{q+1})n - q^{2t-1}(\delta_2 - i)$$

$$= q^{2t-1}i - \frac{1}{2}(q^{2t+1} - q^{2t} - q^{2t-1} + q^{2t-2} + (q^{2t-3} - q^{2t-4} + \dots + q))$$

$$\leq \frac{1}{2}(q^{2t+1} - q^{2t} - q^{2t-1} - q^{2t-2} - (q^{2t-3} - q^{2t-4} + \dots + q)) < \delta_2 - i.$$

These discussions above imply that there exists an integer $u_x < x$ such that $u_x \in C_x$ for $\delta_3 < x < \delta_2$, i.e., x is not a coset leader for $\delta_3 < x < \delta_2$.

Similarly, one can infer that x is not a coset leader for $\delta_4 < x < \delta_3$ and $\delta_5 < x < \delta_4$. It then immediately follows that $\delta_1, \delta_2, \delta_3, \delta_4$ and δ_5 are the first, second, third, fourth and fifth largest coset leaders, respectively.

Lemma 17. If $x \in \{\delta_2, \delta_3, \delta_4, \delta_5\}$, then $|C_x| = 2m$ except $|C_{\delta_5}| = 2$ for m = 5.

Proof. Combining Lemma 13, one can similarly obtain the desired conclusion. The detailed proof is omitted. \Box

According to the previous results on the coset leaders and their cardinalities, we can give the following result.

Theorem 18. Suppose that $n = q^{2t+1} + 1(t \ge 2)$ and q is a power of 2. Let δ_i (i = 1, 2, 3, 4, 5) be given as above, then the following statements hold:

(1) The narrow-sense BCH codes $C(n, q, \delta, 1)$ have parameters

$$\begin{cases} [n, 6m + 3, d \ge \delta_4]_q & \text{if } \delta_5 + 1 \le \delta \le \delta_4; \\ [n, 4m + 3, d \ge \delta_3]_q & \text{if } \delta_4 + 1 \le \delta \le \delta_3; \\ [n, 2m + 3, d \ge \delta_2]_q & \text{if } \delta_3 + 1 \le \delta \le \delta_2; \\ [n, 3, \delta_1]_q & \text{if } \delta_2 + 1 \le \delta \le \delta_1; \\ [n, 1, n]_q & \text{if } \delta_1 + 1 \le \delta \le n. \end{cases}$$

(2) The BCH codes $C(n, 2, \delta + 1, 0)$ have parameters

$$\left\{ \begin{array}{ll} [n,6m+2,d\geq 2\delta_4]_q & \text{if } \delta_5+1\leq \delta \leq \delta_4; \\ [n,4m+2,d\geq 2\delta_3]_q & \text{if } \delta_4+1\leq \delta \leq \delta_3; \\ [n,2m+2,d\geq 2\delta_2]_q & \text{if } \delta_3+1\leq \delta \leq \delta_2; \\ [n,2,2\delta_1]_q & \text{if } \delta_2+1\leq \delta \leq \delta_1. \end{array} \right.$$

Remark: Consider $n = q^3 + 1$. If δ_1 and δ_2 are given as above. Let $\delta_3 = \delta_2 - (q - 1)^2$, $\delta_4 = \delta_3 - (q - 2)$ and $\delta_5 = \delta_4 - 1$. Notice that $|C_{\delta_3}| = 2$ and $|C_{\delta_4}| = |C_{\delta_5}| = 2m$. The dimensions of $C(n, q, \delta, 1)$ and $C(n, q, \delta, 0)$ can be also similarly given.

5 Some results of BCH codes for $n = q^m + 1$ with m even

This section dedicates to the first largest coset leaders modulo $n = q^m + 1$ with m even. Notice that the structure of the q-cyclotomic cosets modulo n is extremely complex as pointed out in [16]. For investigating the coset leaders, here some new functions and sequences are introduced. However, we only verify the first and second largest coset leaders for odd q, and the first largest coset leader for even q. Some conjectures are proposed, which accurately adapt to all magma examples of q-cyclotomic cosets we have calculated.

5.1 BCH codes over \mathbb{F}_q of odd characteristic

In this subsection, let q be an odd prime power. For giving the main conclusions, we first define a function by

$$\Phi(x,q) = \begin{cases} \frac{q-1}{2} \cdot (q^{2^x} - 1)(q^{2^{x-1}} - 1) \cdots (q^{2^0} - 1) & \text{if } x \ge 0; \\ \frac{q-1}{2} & \text{if } x < 0. \end{cases}$$

When
$$m = 2^r t + 2^{r-1} \ge 6 (r \ge 2, t \ge 1)$$
 is not a power of 2, put $\delta_1 = \frac{n}{2}, \ \delta_2 = \frac{n}{q^{2^{r-1}} + 1} \cdot \Phi(r - 2, q), \ \delta_3 = \delta_2 - 2 \cdot \frac{\delta_2 + \Phi(r - 1, q)}{q^{2^r}}$ and $\delta_4 = \begin{cases} \delta_3 - q(q - 1)^2 & \text{if } t = 1; \\ \delta_3 - 2\Phi(r - 1, q) & \text{if } t > 1. \end{cases}$

When $m = 2^r \ge 4$ is a power of 2, suppose that $\delta_1 = \frac{n}{2}, \ \delta_2 = \frac{n}{q^{2^r} + 1} \cdot \Phi(r - 1, q) = \Phi(r - 1, q), \ \delta_3 = \delta_2 - 2\Phi(r - 3, q)$ and $\delta_4 = \begin{cases} \delta_3 - 1 & \text{if } m = 4; \\ \delta_2 - 2q^{2^{r-2}}\Phi(r - 4, q) & \text{if } m \ge 8. \end{cases}$

Particularly, if m = 2, let $\delta_1 = \frac{n}{2}$, $\delta_2 = \Phi(r - 1, q)$, $\delta_3 = \delta_2 - 1$ and $\delta_4 = \delta_2 - (q - 1)$.

We now first present two critical results of this subsection: Theorem 19 and Conjecture 20. The proof of Theorem 19 will be given after Lemma 21.

Theorem 19. Let q be an odd prime power. Suppose that δ_1 and δ_2 are given as above. Then δ_1 and δ_2 are the first, second largest coset leaders, respectively.

Conjecture 20. Let q be an odd prime power. Suppose that δ_3 and δ_4 are given as above. Then δ_3 and δ_4 are the third, fourth largest coset leaders, respectively.

To verify Theorem 19, a kind of sequences is introduced below. First of all, we introduce some notations.

Consider $a=(a_u,\cdots,a_1,a_0)$ and $b=(b_u,\cdots,b_1,b_0)$, where $a_i,b_i\in\{1,-1\}$. We define a lexicographical comparison over these sequences. Set $c = a - b = (a_u - b_u, \dots, a_1 - b_1, a_0 - a_1)$ b_0). If $c=(0,0,\cdots,0)$, it is said that a=b. If $a\neq b$, define $i=\max\{j|c_i\neq 0\}$. We say that a > b if $a_i - b_i > 0$, and a < b, otherwise. For instance, (1, 1, -1, 1) > (1, -1, -1, 1).

Define such a sequence $S^r = (s^r_{2^{r-1}-1}, \cdots, s^r_1, s^r_0)$, which can be generated by the following recursive operations:

owing recursive operations:
$$S^{1} = (1), S^{2} = (1, -1), S^{3} = (1, -1, -1, 1), \cdots, S^{r-1} = (s_{2^{r-2}-1}^{r-1}, \cdots, s_{1}^{r-1}, s_{0}^{r-1}),$$

$$S^{r} = (S^{r-1}, -S^{r-1}) = (s_{2^{r-2}-1}^{r-1}, \cdots, s_{1}^{r-1}, s_{0}^{r-1}, -s_{2^{r-2}-1}^{r-1}, \cdots, -s_{1}^{r-1}, -s_{0}^{r-1}).$$

Lemma 21. Let $S^r=(s^r_{2^{r-1}-1},\cdots,s^r_1,s^r_0)$ be defined above. For $1\leq k\leq 2^{r-1}-1$, denote

Proof. It can be verified by the mathematical induction. For details, see Appendix A.5. \Box

For clarity, we provide an example for the case r=3.

Example 2. If r=3, note that $S^3=(1,-1,-1,1)$. By definition, we have two k-Left-Shifts as follows.

It is as follows:
$$F_{(1)}^3 = (1,1,-1,1) \text{ and } H_{(1)}^3 = (1,1,-1,-1),$$

$$F_{(2)}^3 = (1,-1,1,-1) \text{ and } H_{(2)}^3 = (1,-1,-1,1),$$

$$F_{(3)}^3 = (1,-1,1,1) \text{ and } H_{(3)}^3 = (1,1,-1,-1).$$
 It is easy to know that $F_{(k)}^3 \geq S^3$ and $H_{(k)}^3 \geq S^3$ for $k = 1,2,3$.

With this result above in hand, Theorem 19 can be well verified below.

Proof of Theorem 19: To attain the desired conclusion, it suffices to verify that there exists $y_{x,k} < x$ or $n - y_{x,k} < x$ for some $k \in [0, m-1]$ and $x > \delta_2$ with $x \neq \delta_1$ or δ_2 . We give the following discussions.

Case 1: m has the form $m = 2^r u + 2^{r-1} (r \ge 2, u \ge 1)$.

Step 1.1: We prove that δ_2 is a coset leader.

Let $S^r = (s^r_{2^{r-1}-1}, \dots, s^r_1, s^r_0)$ and $F^r_{(k)} = (f^{r,k}_{2^{r-1}-1}, \dots, f^{r,k}_1, f^{r,k}_0)$ be defined as above.

$$F^r_{(k)} = \begin{cases} (s^r_{2^{r-1}-1-k}, \cdots, s^r_1, s^r_0, -s^r_{2^{r-1}-1}, -s^r_{2^{r-1}-2}, \cdots, -s^r_{2^{r-1}-k}) & \text{if } s^r_{2^{r-1}-1-k} = 1; \\ (-s^r_{2^{r-1}-1-k}, \cdots, -s^r_1, -s^r_0, s^r_{2^{r-1}-1}, s^r_{2^{r-1}-2}, \cdots, s^r_{2^{r-1}-k}) & \text{if } s^r_{2^{r-1}-1-k} = -1. \end{cases}$$

Utilizing $S^r = (s^r_{2r-1-1}, \cdots, s^r_1, s^r_0)$, we get that

$$\delta_2 = \frac{n}{q^{2^{r-1}} + 1} \cdot \Phi(r - 2, q)$$

$$= \frac{n}{q^{2^{r-1}} + 1} \cdot \frac{q - 1}{2} \cdot (q^{2^{r-2}} - 1)(q^{2^{r-3}} - 1) \cdots (q^{2^0} - 1)$$

$$= \frac{n}{q^{2^{r-1}} + 1} \cdot \frac{q - 1}{2} \cdot \sum_{t=0}^{2^{r-1} - 1} s_t^r q^t.$$

We then show $y_{\delta_2,k}-\delta_2\geq 0$ and $n-y_{\delta_2,k}-\delta_2\geq 0$ by different k for $0\leq k\leq 2^{r-1}-1$: (1): If k=0, it is trivial that $y_{\delta_2,0}=\delta_2$ and $n-2\delta_2>0$.

- (2): If $1 \le k \le 2^{r-1} 1$ and $s_{2^{r-1}-1-k}^r = 1$, one can derive that

$$\begin{split} y_{\delta_2,k} &= q^k \delta_2 - \frac{q-1}{2} \cdot \sum_{t=2^{r-1}-k}^{2^{r-1}-1} s_t^r q^{t+k-2^{r-1}} n, \\ &= \frac{n}{q^{2^{r-1}}+1} \cdot \frac{q-1}{2} \cdot (\sum_{t=0}^{2^{r-1}-1-k} s_t^r q^{t+k} - \sum_{t=2^{r-1}-k}^{2^{r-1}-1} s_t^r q^{t+k-2^{r-1}}) \\ &= \frac{n}{q^{2^{r-1}}+1} \cdot \frac{q-1}{2} \cdot \sum_{t=0}^{2^{r-1}-1} f_t^{r,k} q^t \text{ (see Lemma 21)}, \\ y_{\delta_2,k} - \delta_2 &= \frac{n}{q^{2^{r-1}}+1} \cdot \frac{q-1}{2} \cdot \sum_{t=0}^{2^{r-1}-1} (f_t^{r,k} - s_t^r) q^t \geq 0 \text{ (see Lemma 21)}, \\ n - y_{\delta_2,k} - \delta_2 &= n - \frac{n}{q^{2^{r-1}}+1} \cdot \frac{q-1}{2} \cdot \sum_{t=0}^{2^{r-1}-1} (f_t^{r,k} + s_t^r) q^t \\ &> n - \frac{n}{q^{2^{r-1}}+1} \cdot \frac{q-1}{2} \cdot \sum_{t=0}^{2^{r-1}-1} 2q^t = n - \frac{n(q^{2^{r-1}}-1)}{q^{2^{r-1}}+1} > 0 \end{split}$$

(3): If $1 \le k \le 2^{r-1} - 1$ and $s_{2^{r-1}-1-k}^r = -1$, we infer that

$$\begin{split} y_{\delta_{2,k}} &= q^k \delta_2 - \frac{q-1}{2} \cdot \sum_{t=2^{r-1}-k}^{2^{r-1}-1} s_t^r q^{t+k-2^{r-1}} n + n, \\ &= \frac{n}{q^{2^{r-1}}+1} \cdot \frac{q-1}{2} \cdot (\sum_{t=0}^{2^{r-1}-1-k} s_t^r q^{t+k} - \sum_{t=2^{r-1}-k}^{2^{r-1}-1} s_t^r q^{t+k-2^{r-1}}) + n \\ &= n - \frac{n}{q^{2^{r-1}}+1} \cdot \frac{q-1}{2} \cdot \sum_{t=0}^{2^{r-1}-1} f_t^{r,k} q^t \text{ (see Lemma 21)}, \\ y_{\delta_{2,k}} - \delta_2 &= n - \frac{n}{q^{2^{r-1}}+1} \cdot \frac{q-1}{2} \cdot \sum_{t=0}^{2^{r-1}-1} (f_t^{r,k} + s_t^r) q^t \geq 0 \text{ (similar to (2))}, \\ n - y_{\delta_{2,k}} - \delta_2 &= \frac{n}{q^{2^{r-1}}+1} \cdot \frac{q-1}{2} \cdot \sum_{t=0}^{2^{r-1}-1} (f_t^{r,k} - s_t^r) q^t > 0 \text{ (see Lemma 21)}. \end{split}$$

- **Step 1.2**: We verify that x is not a coset leader for $x > \delta_2$ with $x \neq \delta_1$ through the following discussions:

 - (1): If $x \in [\delta_1 + 1 = \frac{n}{2} + 1, n 1]$, then x < n < 2x. Clearly, $n y_{x,0} = n x < x$. (2): If $x \in [\lceil \frac{\Phi(-1,q)n}{q^{2^0}} \rceil, \delta_1 1 = \frac{n}{2} 1]$, then $(q^{2^0} 1)x < \Phi(-1,q)n < q^{2^0}x$, whence

- $\begin{array}{l} n-y_{x,1}=\frac{(q-1)n}{2}-q^{2^{0}}x < x.\\ (3) \colon \text{If } x \in [\lceil \frac{\Phi(-1,q)n}{q^{2^{0}}+1} \rceil, \lfloor \frac{\Phi(-1,q)n}{q^{2^{0}}} \rfloor], \text{ then } q^{2^{0}}x < \Phi(-1,q)n < (q^{2^{0}}+1)x, \text{ whence } n-y_{x,1}=\frac{(q-1)n}{2}-q^{2^{0}}x < x.\\ \text{For each } i \in [1,r-2] \text{ with } r \geq 3, \text{ the following (4) and (5) hold:}\\ (4) \colon \text{If } x \in [\lceil \frac{\Phi(i-1,q)n}{q^{2^{i}}} \rceil, \lfloor \frac{\Phi(i-2,q)n}{q^{2^{i-1}}+1} \rfloor], \text{ consider that } q^{2^{i}}-1=(q^{2^{i-1}}+1)(q^{2^{i-1}}-1) \text{ and}\\ \Phi(i-1,q)=(q^{2^{i-1}}-1)\Phi(i-2,q). \text{ Then we have } (q^{2^{i}}-1)x < \Phi(i-1,q)n < q^{2^{i}}x, \text{ it } q^{2^{1}}+1 = q^{2^{i}}x = q^{2$
- follows that $y_{x,2^i} \Phi(i-1,q)n = q^{2^i}x \Phi(i-1,q)n < x$. (5): If $x \in [\lceil \frac{\Phi(i-1,q)n}{q^{2^i}+1} \rceil, \lfloor \frac{\Phi(i-1,q)n}{q^{2^i}} \rfloor]$, then $q^{2^i}x < \Phi(i-1,q)n < (q^{2^i}+1)x$, whence $(n - y_{x,2^i}) = \Phi(i-1,q)n - q^{2^i}x < x$
- (6): If $x \in \lceil \frac{\Phi(r-2,q)n}{q^{2^{r-1}}} \rceil, \lfloor \frac{\Phi(r-3,q)n}{q^{2^{r-2}}+1} \rfloor \rceil$, then $(q^{2^{r-1}}-1)x < \Phi(r-2,q)n < q^{2^{r-1}}x$, it follows that $y_{x,2^{r-1}} - \Phi(r-2,q)n = q^{2^{r-1}}x - \Phi(r-2,q)n < x$.
- (7): If $x \in [\delta_2 + 1 = \frac{\Phi(r-2,q)n}{q^{2r-1}+1} + 1, \lfloor \frac{\Phi(r-2,q)n}{q^{2r-1}} \rfloor]$, then $q^{2r-1}x < \Phi(r-2,q)n < (q^{2r-1}+1)x$, whence $n-y_{x,2^{r-1}} = \Phi(r-2,q)n q^{2^{r-1}}x < x$.

 Case 2: $m=2^r$. Case 2 is similar to Case 1 above, then we only present the simplified

- **Step 2.1**: When $m = 2^r$, we have $\delta_2 = \frac{n}{q^{2^r}+1} \cdot \Phi(r-1,q) = \Phi(r-1,q)$. In a very similar way to Step 1.1, one can verify that δ_2 is a coset leader. The detailed proof is
- **Step 2.2**: We show that x is not a coset leader for $x > \delta_2$ and $x \neq \delta_1$ similar to Step 1.2.
 - (1-6) are the same to (1-6) in Step 1.2. Below, we give the remainder of the proof.
- (7): If $x \in [\lceil \frac{\Phi(r-2,q)n}{q^{2^{r-1}}+1} \rceil, \lfloor \frac{\Phi(r-2,q)n}{q^{2^{r-1}}} \rfloor]$, then $q^{2^{r-1}}x < \Phi(r-2,q)n < (q^{2^{r-1}}+1)x$, whence $n-y_{x,2^{r-1}} = \Phi(r-2,q)n q^{2^{r-1}}x < x$.
- (8): If $x \in [\delta_2 = \Phi(r-1,q)n = \frac{\Phi(r-1,q)n}{q^{2^r}+1}, \lfloor \frac{\Phi(r-2,q)n}{q^{2^{r-1}}} \rfloor]$, then $q^{2^{r-1}}x < \Phi(r-2,q)n < 0$ $(q^{2^{r-1}} + 1)x$, it follows that $n - y_{x^{2^{r-1}}} = \Phi(r-2, q)n - q^{2^{r-1}}x < x$.

Combining Cases 1 and 2, Theorem 19 holds.

Lemma 22. Let q be an odd prime power. Suppose that δ_1 and δ_2 be given as above. Then $C_{\delta_1} = \{\delta_1\}$ and $|C_{\delta_2}| = \begin{cases} 2^r & \text{if } m = 2^r u + 2^{r-1} (r \geq 2, u \geq 1); \\ 2m & \text{if } m = 2^r \geq 2. \end{cases}$

Proof. Since $\delta_1 = \frac{n}{2}$ and $\frac{n}{2}(q^2 - 1) = \frac{q^2 - 1}{2}n \equiv 0$, it directly follows that $C_{\delta_1} = \{\delta_1\}$. Note that

$$\delta_2(q^{2^r} - 1) = \frac{\Phi(r - 2, q)n}{q^{2^{r-1}} + 1} \cdot (q^{2^r} - 1) = \Phi(r - 2, q)(q^{2^{r-1}} + 1)n \equiv 0.$$

When $m=2^ru+2^{r-1}$, by the proof of Steps 1.1, we have know that $y_{\delta_2,k}>\delta_2$ for $0\leq k\leq 2^r-1$. It naturally follows that $|C_{\delta_2}|=2^r$. Similarly, we can easily get that $|C_{\delta_2}| = 2^{r+1} = 2m$ for $m = 2^r$.

Now it is sufficient to give the following result.

Theorem 23. Suppose $n = q^m + 1$ and q is an odd prime power. Let δ_i (i = 1, 2) be given as above, then the following statements hold:

(1):
$$m = 2^r u + 2^{r-1} (r \ge 2, u \ge 1)$$

(1.1) The narrow-sense BCH codes $C(n, q, \delta, 1)$ have parameters

$$\begin{cases} [n, 2 + 2^r, d \ge \delta_2]_q & \text{if } \delta = \delta_2; \\ [n, 2, \delta_1]_q & \text{if } \delta_2 + 1 \le \delta \le \delta_1; \\ [n, 1, n]_q & \text{if } \delta_1 + 1 \le \delta \le n. \end{cases}$$

(1.2) The BCH codes $C(n, 2, \delta + 1, 0)$ have parameters

$$\left\{ \begin{array}{ll} [n,1+2^r,d\geq 2\delta_2]_q & \text{if } \delta=\delta_2;\\ [n,1,2\delta_1=n]_q & \text{if } \delta_2+1\leq \delta\leq \delta_1. \end{array} \right.$$

- (2): $m = 2^r$.
- (2.1) The narrow-sense BCH codes $C(n, q, \delta, 1)$ have parameters

$$\begin{cases} [n, 2+2m, d \geq \delta_2]_q & \text{if } \delta = \delta_2; \\ [n, 2, \delta_1]_q & \text{if } \delta_2 + 1 \leq \delta \leq \delta_1; \\ [n, 1, n]_q & \text{if } \delta_1 + 1 \leq \delta \leq n. \end{cases}$$

(2.2) The BCH codes $C(n, 2, \delta + 1, 0)$ have parameters

$$\begin{cases} [n, 1+2m, d \geq 2\delta_2]_q & \text{if } \delta = \delta_2; \\ [n, 1, 2\delta_1 = n]_q & \text{if } \delta_2 + 1 \leq \delta \leq \delta_1; \end{cases}$$

BCH codes over \mathbb{F}_q of characteristic two 5.2

In this section, we only give some conclusions with the detailed proof omitted since the

Define
$$\Phi'(x,q) = \begin{cases} \frac{q}{2} \cdot (q^{2^x} - 1)(q^{2^{x-1}} - 1) \cdots (q^{2^0} - 1) & \text{if } x \ge 0; \\ \frac{q}{2} & \text{if } x < 0. \end{cases}$$

When
$$m$$
 is not the power of 2, let $m = 2^r t + 2^{r-1} (r \ge 2, t \ge 1)$. Suppose that $\delta_1 = \frac{n}{q^{2^{r-1}} + 1} \cdot \Phi'(r - 2, q), \ \delta_2 = \delta_1 - 2 \cdot \frac{\delta_1 + \Phi'(r - 1, q)}{q^{2^r}}$ and $\delta_3 = \begin{cases} \delta_2 - (q - 1)q^2 & \text{if } t = 1; \\ \delta_2 - 2 \cdot \Phi'(r - 1, q) & \text{if } t > 1. \end{cases}$

When $m = 2^r \ge 4$. Suppose that $\delta_1 = \Phi'(r-1,q), \ \delta_2 = \delta_1 - 2\Phi'(r-3,q)$ and $\delta_3 = \begin{cases} \delta_1 - (q-1)(q^2-1) & \text{if } m=4; \\ \delta_1 - 2q^{2^{r-2}}\Phi'(r-4,q) & \text{if } m \ge 8. \end{cases}$ Particularly, if m = 2, then $\delta_1 = \frac{q(q-1)}{2}, \ \delta_2 = \delta_1 - (q-1)$ and $\delta_3 = \delta_2 - 1$.

Particularly if
$$m=2$$
 then $\delta_1 = \frac{q(q-1)}{2}$ $\delta_2 = \delta_1 = (q-1)$ and $\delta_2 = \delta_2 = \frac{q(q-1)}{2}$

Theorem 24. Let q be a power of 2 and δ_1 be given as above. Then δ_1 is the first largest coset leader.

Proof. This result can be obtained similar to verifying that δ_2 is the second largest coset leader in last subsection.

Conjecture 25. Let q be a power of 2. Suppose that δ_2 and δ_3 are given as above. Then δ_2 and δ_3 are the second and third largest coset leaders, respectively.

Remark: Actually, these conclusions in this section can unify some previous results. All theorems and conjectures are applicative for either q=2 or odd m. When q=2, it has been verified in [22] for $m \not\equiv 0 \mod 8$. When m is odd (i.e, m=2t+1), m actually has the form $m=2^rt+2^{r-1}$ and r=1. And we have proved these results in Section 4.

6 Conclusion

By introducing a new kind of sequences and extending some technologies given in [22], this article has generalized many conclusions in [22] from binary field to general finite fields. On one hand, we have determined the coset leaders of C_x with double range of x in [16,17]. On the other hand, we have derived or guessed the first several largest coset leaders modulo $n = q^m + 1$. Then one shall naturally calculate the dimension of some antiprimitive BCH codes as well as their Bose distances.

Rather, a table is given to show our main conclusions.

Table 1. Some results on coset leaders modulo $n = q^m + 1$ over \mathbb{F}_q

	m	q	coset leaders	
Section 3	general	general	$q^{\lceil \frac{m}{2} \rceil} < x \le 2q^{\lceil \frac{m}{2} \rceil} + 2$	Resolved
		odd	$\delta_1, \delta_2, \cdots, \delta_6$	
Section 4	odd			Resolved
		even	$\delta_1,\delta_2,\cdots,\delta_5$	
		odd	δ_1,δ_2	Resolved
			δ_3,δ_4	Conjecture
Section 5	even			
		even	δ_1	Resolved
			δ_2,δ_3	Conjecture

Remark: δ_i is the *i*-th largest coset leader as given in the given section. "resolved" is to say that the corresponding coset leader has been resolved, while "conjecture" denotes that there is just a conjecture for the given coset leader.

For these conjectures, their complete solution would wipe out the difficult problem of determining the largest coset leaders. It is believed that they also hold and can be verified in the similar way to Section 4, though it would be seriously complicated. The interested reader is sincerely welcome to solve the remaining parts. In addition, the new kind of sequences and iterative method adopted in this paper may be useful for other types of codes. For instance, solve the Open Problems 45 and 46 on BCH codes with projective length proposed in [18], calculate the dimension of the LCD negacyclic codes as discussed in [21], and so on. Many more results on cyclic codes are expected in future work.

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A Appendixes

The following lemma is first provided. It is much useful to some derivation processes later and can be naturally obtained by generalizing 2 to any prime power q in Lemma A.1 ([22],

Appendix).

Lemma 26. Let $f(k) = q^{-k}a + q^kb$, where a, b and k are positive real numbers. If $k_2 \ge k_1 \ge \log_a \sqrt{a/b}$, then $f(k_2) \ge f(k_1)$.

A.1The proof of Theorem 1

Proof. To prove the conclusion, it is enough to verify (2)-(5).

- (2) Let $x \in I = [q^{t+1} + q + 1, q^{t+1} + q^t 2]$. To verify (2), we then show $y_{x,k} x \ge 0$ and $n - y_{x,k} - x \ge 0$ in the following cases.
 - (2.1): When $k=0,1,2,\cdots,t-1,$ we have $0 < q^k x < n.$ Then $y_{x,k} = q^k x \ge x$ and

$$\begin{array}{lcl} n-y_{x,k}-x & = & q^{2t+1}+1-(q^k+1)x \\ & \geq & q^{2t+1}+1-(q^k+1)(q^{t+1}+q^t-2) \\ & \geq & q^{2t+1}+1-(q^{t-1}+1)(q^{t+1}+q^t-2) \\ & = & q^{2t+1}-(q^{2t}+q^{2t-1}+q^{t+1}+q^t-2q^{t-1})+3>0. \end{array}$$

(2.2): When k = t, t + 1, we derive $y_{x,k} = q^k x - q^{k-t} n$. It follows that

$$\begin{array}{rcl} y_{x,k}-x & = & (q^k-1)x-q^{k-t}n \\ & \geq & (q^k-1)(q^{t+1}+q+1)-q^{k-t}n \\ & = & q^k(q+1-q^{-t})-(q^{t+1}+q+1) \\ & \geq & q^t(q+1-q^{-t})-(q^{t+1}+q+1) \\ & = & q^t-q-2>0. \end{array}$$

$$\begin{array}{lll} n-y_{x,k}-x & = & (q^{k-t}+1)(q^{2t+1}+1)-(q^k+1)x \\ & \geq & (q^{k-t}+1)(q^{2t+1}+1)-(q^k+1)(q^{t+1}+q^t-2) \\ & = & q^{2t+1}-q^{t+1}-q^t-q^k(q^t-2-q^{-t}) \\ & \geq & q^{2t+1}-q^{t+1}-q^t-q^{t+1}(q^t-2-q^{-t}) \\ & = & q^{t+1}-q^t+q+3>0. \end{array}$$

(2.3): For each $k = t + 2, t + 3, \dots, 2t - 1$. To determine $y_{x,k}$, we divide $I = [q^{t+1} + q^{t+1}]$ $q+1,q^{t+1}+q^t-2$] into q^{k-1-t} disjoint subintervals as follows:

$$I_{1,k} = [q^{t+1} + q + 1, q^{t+1} + q^{2t+1-k} - 1],$$

$$\begin{split} I_{1,k} &= [q^{t+1} + q + 1, q^{t+1} + q^{2t+1-k} - 1], \\ I_{\lambda,k} &= [q^{t+1} + (\lambda - 1)q^{2t+1-k} + 1, q^{t+1} + \lambda q^{2t+1-k} - 1] \text{ for } \lambda \in [2, q^{k-t-1} - 1], \\ I_{\lambda,k} &= [q^{t+1} + (\lambda - 1)q^{2t+1-k} + 1, q^{t+1} + q^t - 2] \text{ for } \lambda = q^{k-t-1}. \end{split}$$

$$I_{\lambda,k} = [q^{t+1} + (\lambda - 1)q^{2t+1-k} + 1, q^{t+1} + q^t - 2] \text{ for } \lambda = q^{k-t-1}.$$

Given $k \in [t+2, 2t-1]$ and $\lambda \in [1, q^{k-t-1}]$, if $x \in I_{\lambda,k} = [l_{\lambda,b}, l_{\lambda,e}]$, it can be inferred that

$$y_{x,k} = q^k x - (q^{k-t} + \lambda - 1)n.$$

The further proof is given below. Firstly, we verify that $y_{x,k} - x > 0$.

When $\lambda = 1$, let $x \in I_{1,k}$. We then get that $y_{x,k} = q^k x - q^{k-t} n$ and

$$\begin{array}{lll} y_{x,k}-x & = & (q^k-1)x-q^{k-t}n \\ & \geq & (q^k-1)(q^{t+1}+q+1)-q^{k-t}n \\ & = & q^k(q+1-q^{-t})-(q^{t+1}+q+1) \\ & \geq & q^{t+2}(q+1-q^{-t})-(q^{t+1}+q+1) \\ & = & q^{t+3}+q^{t+2}-q^{t+1}-q^2-q-1>0. \end{array}$$

When $\lambda \in [2, q^{k-t-1}]$, we have $x \in I_{\lambda,k} = [l_{\lambda,b}, l_{\lambda,e}] = [q^{t+1} + (\lambda - 1)q^{2t+1-k} + 1, l_{\lambda,e}]$ and $y_{x,k} = q^k x - (q^{k-t} + \lambda - 1)n$, it follows that

$$\begin{array}{lll} y_{x,k}-x & = & (q^k-1)x-(q^{k-t}+\lambda-1)n \\ & \geq & (q^k-1)(q^{t+1}+(\lambda-1)q^{2t+1-k}+1)-(q^{k-t}+\lambda-1)(q^{2t+1}+1) \\ & = & q^k-q^{k-t}+q^{2t+1-k}-q^{t+1}-(q^{2t+1-k}+1)\lambda \\ & \geq & q^k-q^{k-t}+q^{2t+1-k}-q^{t+1}-(q^{2t+1-k}+1)q^{k-t-1} \\ & = & q^{2t+1-k}+q^k(1-q^{-t}-q^{-(t+1)})-q^t-q^{t+1} \\ & \geq & q^{2t+1-(t+2)}+q^{t+2}(1-q^{-t}-q^{-(t+1)})-q^t-q^{t+1} \ (\text{see Lemma 26}) \\ & = & q^{t+2}-q^{t+1}-q^t+q^{t-1}-q^2-q>0. \end{array}$$

Secondly, we show $n - y_{x,k} - x > 0$.

For given $\lambda \in [1, q^{k-t-1}-1]$, since $x \in I_{\lambda,k} = [l_{\lambda,b}, l_{\lambda,e}] = [l_{\lambda,b}, q^{t+1} + \lambda q^{2t+1-k} - 1]$, one can deduce that

$$\begin{array}{lll} n-y_{x,k}-x & = & (q^{k-t}+\lambda)(q^{2t+1}+1)-(q^k+1)x \\ & \geq & (q^{k-t}+\lambda)(q^{2t+1}+1)-(q^k+1)(q^{t+1}+\lambda q^{2t+1-k}-1) \\ & = & q^k+q^{k-t}-q^{t+1}+1-(q^{2t+1-k}-1)\lambda \\ & \geq & q^k+q^{k-t}-q^{t+1}+1-(q^{2t+1-k}-1)(q^{k-t-1}-1) \\ & = & q^{2t+1-k}+q^k(1+q^{-t}+q^{-(t+1)})-q^t-q^{t+1} \\ & \geq & q^{2t+1-(t+2)}+q^{t+2}(1+q^{-t}+q^{-(t+1)})-q^t-q^{t+1} \text{ (see Lemma 26)} \\ & = & q^{t+2}-q^{t+1}-q^t+q^{t-1}+q^2+q>0. \end{array}$$

When $\lambda = q^{k-t-1}$, we have $x \in I_{q^{k-t-1},k}$ and

$$\begin{array}{lcl} n-y_{x,k}-x & = & (q^{k-t}+q^{k-t-1})(q^{2t+1}+1)-(q^k+1)x \\ \\ & \geq & (q^{k-t}+q^{k-t-1})(q^{2t+1}+1)-(q^k+1)(q^{t+1}+q^t-2) \\ \\ & = & q^k(2+q^{-t}+q^{-(t+1)})-q^t-q^{t+1}+2 \\ \\ & \geq & 2q^{t+2}-q^{t+1}-q^t+q^2+q+2>0. \end{array}$$

(2.4): When k=2t, we divide $I=[q^{t+1}+q+1,q^{t+1}+q^t-2]$ into $q^{t-1}-1$ subintervals as follows:

$$\begin{split} &I_{\lambda,k} = [q^{t+1} + \lambda q + 1, q^{t+1} + (\lambda + 1)q - 1] \text{ for } \lambda \in [1, q^{t-1} - 2], \\ &I_{q^{t-1}-1,k} = [q^{t+1} + q^t - q + 1, q^{t+1} + q^t - 2]. \end{split}$$

For some $\lambda \in [1, q^{t-1}-1]$, if $x \in I_{\lambda,k}$, we have $y_{x,k} = q^k x - (q^{k-t} + \lambda)n$. Firstly, we verify that $y_{x,k} - x > 0$.

$$\begin{array}{lll} y_{x,k}-x & = & (q^k-1)x-(q^{k-t}+\lambda)n \\ & \geq & (q^k-1)(q^{t+1}+\lambda q+1)-(q^{k-t}+\lambda)(q^{2t+1}+1) \\ & = & q^{2t}-q^t-q^{t+1}-1-(q+1)\lambda \\ & \geq & q^{2t}-q^t-q^{t+1}-1-(q+1)(q^{t-1}-1) \\ & = & q^{2t}-2q^t-q^{t+1}-q^{t-1}+q>0. \end{array}$$

Secondly, we show $n - y_{x,k} - x > 0$.

For given $\lambda \in [1, q^{t-1} - 2]$, since $x \in I_{\lambda,k}$, one can deduce that

$$\begin{array}{ll} n-y_{x,k}-x & = & (q^t+\lambda+1)(q^{2t+1}+1)-(q^{2t}+1)x \\ & \geq & (q^t+\lambda+1)(q^{2t+1}+1)-(q^{2t}+1)(q^{t+1}+(\lambda+1)q-1) \\ & = & q^{2t}+q^t-q^{t+1}+2-q-(q-1)\lambda \\ & \geq & q^{2t}+q^t-q^{t+1}+2-q-(q-1)(q^{t-1}-2) \\ & = & q^{2t}-q^{t+1}+q^{t-1}+q>0. \end{array}$$

When $\lambda = q^{t-1} - 1$, we have $x \in I_{q^{t-1}-1,k}$ and

$$\begin{array}{lcl} n-y_{x,k}-x & = & (q^t+q^{t-1})(q^{2t+1}+1)-(q^{2t}+1)x \\ & \geq & (q^t+q^{t-1})(q^{2t+1}+1)-(q^{2t}+1)(q^{t+1}+q^t-2) \\ & = & 2q^{t+2}-q^{t+1}+q^{t-1}+2>0. \end{array}$$

Summarizing the four cases above, we can deduce that (2) holds.

- (3) Let $1 \le \alpha \le q-2$, if $q^{t+1}+\alpha q^t+2 \le x \le q^{t+1}+(\alpha+1)q^t-2$, then x is a coset leader.
 - (3.1): When $k=0,1,2,\cdots,t-1,$ we have $y_{x,k}=q^kx\geq x,$ it follows that

$$\begin{array}{lll} n-y_{x,k}-x & = & q^{2t+1}+1-(q^k+1)x \\ & \geq & q^{2t+1}+1-(q^k+1)(q^{t+1}+(\alpha+1)q^t-2) \\ & \geq & q^{2t+1}+1-(q^k+1)(q^{t+1}+((q-2)+1))q^t-2) \\ & \geq & q^{2t+1}+1-(q^{t-1}+1)(q^{t+1}+(q-1)q^t-2) \\ & = & q^{2t+1}-2q^{2t}+q^{2t-1}-2q^{t+1}+2q^{t-1}+q^t+3>0. \end{array}$$

(3.2): When k = t, we have $y_{x,k} = q^k x - n$, hence

$$\begin{array}{rcl} y_{x,k}-x & = & (q^k-1)x-n \\ & \geq & (q^t-1)(q^{t+1}+\alpha q^t+2)-(q^{2t+1}+1) \\ & = & (q^{2t}-q^t)\alpha-q^{t+1}+2q^t-3 \\ & \geq & (q^{2t}-q^t)\cdot 1-q^{t+1}+2q^t-3 \\ & = & q^{2t}-q^{t+1}+q^t-3>0, \\ n-y_{x,k}-x & = & 2n-(q^k+1)x \\ & \geq & 2(q^{2t+1}+1)-(q^t+1)(q^{t+1}+(\alpha+1)q^t-2) \\ & \geq & 2(q^{2t+1}+1)-(q^t+1)(q^{t+1}+((q-2)+1))q^t-2) \\ & = & q^{2t}-2q^{t+1}+3q^t+4>0. \end{array}$$

(3.3): When k=t+1, for given $\alpha\in[1,q-2]$, if $x\in[q^{t+1}+\alpha q^t+2,q^{t+1}+(\alpha+1)q^t-2]$, we have $y_{x,k}=q^kx-(q+\alpha)n$. It follows that

$$\begin{array}{lcl} y_{x,k}-x & = & (q^k-1)x-(q+\alpha)n \\ & \geq & (q^{t+1}-1)(q^{t+1}+\alpha q^t+2)-(q+\alpha)(q^{2t+1}+1) \\ & = & q^{t+1}-q-2-(q^t+1)\alpha \\ & \geq & q^{t+1}-q-2-(q^t+1)(q-2) \\ & = & 2q^t-2q>0, \end{array}$$

$$\begin{array}{lcl} n-y_{x,k}-x & = & (q+\alpha+1)n-(q^k+1)x \\ & \geq & (q+\alpha+1)(q^{2t+1}+1)-(q^{t+1}+1)(q^{t+1}+(\alpha+1)q^t-2) \\ & = & q^{t+1}-q^t+q+3-(q^t-1)\alpha \\ & \geq & q^{t+1}-q^t+q+3-(q^t-1)(q-2) \\ & = & q^t+2q+1>0. \end{array}$$

(3.4): For each $k=t+2,t+3,\cdots,2t$ and $\alpha\in[1,q-2],$ if $x\in I=[q^{t+1}+\alpha q^t+2,q^{t+1}+(\alpha+1)q^t-2],$ we divide I into q^{k-1-t} subintervals as follows:

$$I_{1,k} = [q^{t+1} + \alpha q^t + 2, q^{t+1} + \alpha q^t + q^{2t+1-k} - 1],$$

$$I_{\lambda,k} = [q^{t+1} + \alpha q^t + (\lambda - 1)q^{2t+1-k} + 1, q^{t+1} + \alpha q^t + \lambda q^{2t+1-k} - 1] \text{ for } \lambda \in [2, q^{k-t-1} - 1],$$

$$I_{\lambda,k}^{\gamma,\kappa} = [q^{t+1} + \alpha q^t + (\lambda - 1)q^{2t+1-k} + 1, q^{t+1} + (\alpha + 1)q^t - 2] \text{ for } \lambda = q^{k-t-1}.$$

For some $\lambda \in [1, q^{k-t-1}]$, if $x \in I_{\lambda,k} = [l_{\lambda,b}, l_{\lambda,e}]$, we have

$$y_{x,k} = q^k x - (q^{k-t} + \alpha q^{k-t-1} + \lambda - 1)n.$$

First, we will show $y_{x,k} - x > 0$.

If $\lambda = 1$ and $x \in I_{\lambda,k}$, we get that

$$\begin{array}{lll} y_{x,k}-x & = & (q^k-1)x-(q^{k-t}+\alpha q^{k-t-1})n \\ & \geq & (q^k-1)(q^{t+1}+\alpha q^t+2)-(q^{k-t}+\alpha q^{k-t-1})(q^{2t+1}+1) \\ & = & 2q^k-q^{t+1}-q^{k-t}-2-(q^{k-t-1}+q^t)\alpha \\ & \geq & 2q^k-q^{t+1}-q^{k-t}-2-(q^{k-t-1}+q^t)(q-2) \\ & = & 2q^k(2-q^{-t}+q^{-t-1})-2q^{t+1}+2q^t-2 \\ & \geq & 2q^{t+2}(1-q^{-t}+q^{-t-1})-2q^{t+1}+2q^t-2 \\ & = & 2(q^t-1)(q^2-q+1)>0. \end{array}$$

If $\lambda \in [2, q^{k-t-1}]$ and $x \in I_{\lambda,k} = [l_{\lambda,b}, l_{\lambda,e}] = [q^{t+1} + \alpha q^t + (\lambda - 1)q^{2t+1-k} + 1, l_{\lambda,e}]$, then

$$\begin{array}{l} y_{x,k}-x\\ &= (q^k-1)x-(q^{k-t}+\alpha q^{k-t-1}+\lambda-1)n\\ &\geq (q^k-1)(q^{t+1}+\alpha q^t+(\lambda-1)q^{2t+1-k}+1)-(q^{k-t}+\alpha q^{k-t-1}+\lambda-1)n\\ &= q^k-q^{t+1}-q^{k-t}+q^{2t+1-k}-(q^{k-t-1}+q^t)\alpha-(q^{2t+1-k}+1)\lambda\\ &\geq q^k-q^{t+1}-q^{k-t}+q^{2t+1-k}-(q^{k-t-1}+q^t)\alpha-(q^{2t+1-k}+1)q^{k-t-1}\\ &\geq q^k-q^{t+1}-q^{k-t}+q^{2t+1-k}-(q^{k-t-1}+q^t)(q-2)-(q^{2t+1-k}+1)q^{k-t-1}\\ &\geq q^k-q^{t+1}-q^{k-t}+q^{2t+1-k}-(q^{k-t-1}+q^t)(q-2)-(q^{2t+1-k}+1)q^{k-t-1}\\ &= q^{2t+1-k}+q^k(1-2q^{-t}+q^{-t-1})-2q^{t+1}+q^t\\ &\geq q^{2t+1-(t+2)}+q^{t+2}(1-2q^{-t}+q^{-t-1})-2q^{t+1}+q^t \text{ (see Lemma 26)}\\ &= q^{t+2}-2q^{t+1}+q^t+q^{t-1}-2q^2+q>0, \end{array}$$

Next, we will show $n - y_{x.k} - x > 0$.

If $\lambda \in [1, q^{k-t-1} - 1]$ and $x \in I_{\lambda,k} = [l_{\lambda,b}, l_{\lambda,e}] = [l_{\lambda,b}, q^{t+1} + \alpha q^t + \lambda q^{2t+1-k} - 1]$, then

$$\begin{array}{ll} n-y_{x,k}-x \\ &= (q^{k-t}+\alpha q^{k-t-1}+\lambda)n-(q^k+1)x \\ &\geq (q^{k-t}+\alpha q^{k-t-1}+\lambda)(q^{2t+1}+1)-(q^k+1)(q^{t+1}+\alpha q^t+\lambda q^{2t+1-k}-1) \\ &= q^k+q^{k-t}-q^{t+1}+1-(q^t-q^{k-t-1})\alpha-(q^{2t+1-k}-1)\lambda \\ &\geq q^k+q^{k-t}-q^{t+1}+1-(q^t-q^{k-t-1})\alpha-(q^{2t+1-k}-1)(q^{k-t-1}-1) \end{array}$$

$$\geq q^k + q^{k-t} - q^{t+1} + 1 - (q^t - q^{k-t-1})(q-2) - (q^{2t+1-k} - 1)(q^{k-t-1} - 1)$$

$$= q^{2t+1-k} + q^k(1 + 2q^{-t} - q^{-t-1}) - 2q^{t+1} + q^t$$

$$\geq q^{2t+1-(t+2)} + q^{t+2}(1 + 2q^{-t} - q^{-t-1}) - 2q^{t+1} + q^t \text{ (see Lemma 26)}$$

$$= q^{t+2} - 2q^{t+1} + q^t + q^{t-1} + 2q^2 - q > 0.$$

If $\lambda = q^{k-t-1}$ and $x \in I_{\lambda_k}$, then

$$\begin{array}{l} n-y_{x,k}-x\\ &= (q^{k-t}+\alpha q^{k-t-1}+q^{k-t-1})n-(q^k+1)x\\ \geq (q^{k-t}+\alpha q^{k-t-1}+q^{k-t-1})(q^{2t+1}+1)-(q^k+1)(q^{t+1}+\alpha q^t+q^t-2)\\ &= 2q^k+q^{k-t}+q^{k-t-1}-q^{t+1}-q^t+2-(q^t-q^{k-t-1})\alpha\\ \geq 2q^k+q^{k-t}+q^{k-t-1}-q^{t+1}-q^t+2-(q^t-q^{k-t-1})(q-2)\\ &= q^k(2+2q^{-t}-q^{-t-1})-2q^{t+1}+q^t+2\\ \geq q^{2t}(2+2q^{-t}-q^{-t-1})-2q^{t+1}+q^t+2\\ &= 2q^{2t}-2q^{t+1}+3q^t-q^{t-1}+2>0. \end{array}$$

Concluding the previous five cases, we can attain the result of (3).

(4) Let $x \in I = [q^{t+1} + (q-1)q^t + 2, 2q^{t+1} - 2q - 1]$. To verify (4), one needs to show $y_{x,k} - x \ge 0$ and $n - y_{x,k} - x \ge 0$ in the following cases:

(4.1): When $k=0,1,2,\cdots,t-1$, we have $y_{x,k}=2^kx\geq x$, it follows that

$$\begin{array}{lcl} n-y_{x,k}-x & = & n-(q^k+1)x \\ & \geq & q^{2t+1}+1-(q^k+1)(2q^{t+1}-2q-1) \\ & \geq & q^{2t+1}+1-(q^{t-1}+1)(2q^{t+1}-2q-1) \\ & = & q^{2t+1}-(2q^{2t}+2q^{t+1}-2q^t-q^{t-1}-2q)+2>0. \end{array}$$

(4.2): When k = t, we have $y_{x,k} = q^k x - n$. Hence,

$$\begin{array}{rcl} y_{x,k}-x & = & (q^k-1)x-n \\ & \geq & (q^t-1)(q^{t+1}+(q-1)q^t+2)-(q^{2t+1}+1) \\ & = & q^{2t+1}-q^{2t}-2q^{t+1}+3q^t-3>0, \\ n-y_{x,k}-x & = & 2n-(q^k+1)x \\ & \geq & 2(q^{2t+1}+1)-(q^t+1)(2q^{t+1}-2q-1) \\ & = & q^{2t}+2q+3>0. \end{array}$$

(4.3): When k = t+1 and $x \in [2q^{t+1} - q^t + 2, 2q^{t+1} - 2q - 1]$, we have $y_{x,k} = q^k x - (2q-1)n$. It then can be inferred that

$$\begin{array}{rcl} y_{x,k}-x & = & (q^k-1)x-(2q-1)n \\ & \geq & (q^{t+1}-1)(2q^{t+1}-q^t+2)-(2q-1)(q^{2t+1}+1) \\ & = & q^t-2q-1>0. \\ n-y_{x,k}-x & = & 2qn-(q^k+1)x \\ & \geq & 2q(q^{2t+1}+1)-(q^{t+1}+1)(2q^{t+1}-2q-1) \\ & = & 2q^{t+2}-q^{t+1}+4q+1>0. \end{array}$$

(4.4): For each $k = t + 2, t + 3, \dots, 2t - 1$, we divide $I = [2q^{t+1} - q^t + 2, 2q^{t+1} - 2q - 1]$ into q^{k-1-t} subintervals as follows:

$$\begin{split} I_{1,k}^{-} &= [2q^{t+1} - q^t + 2, 2q^{t+1} - q^t + q^{2t+1-k} - 1], \\ I_{\lambda,k}^{-} &= [2q^{t+1} - q^t + (\lambda - 1)q^{2t+1-k} + 1, 2q^{t+1} - q^t + \lambda q^{2t+1-k} - 1] \text{ for } \lambda \in [2, q^{k-t-1} - 1], \\ I_{\lambda,k}^{-} &= [2q^{t+1} - q^t + (\lambda - 1)q^{2t+1-k} + 1, 2q^{t+1} - 2q - 1] \text{ for } \lambda = q^{k-t-1}. \end{split}$$

For some $\lambda \in [1, q^{k-t-1}]$, if $x \in I_{\lambda,k} = [l_{\lambda,b}, l_{\lambda,e}]$, we have

$$y_{x,k} = q^k x - (2q^{k-t} - q^{k-t-1} + \lambda - 1)n.$$

First, we will show $y_{x,k} - x > 0$.

If $\lambda = 1$ and $x \in I_{\lambda,k} = I_{1,k}$, then

$$\begin{array}{lll} y_{x,k}-x & = & (q^k-1)x-(2q^{k-t}-q^{k-t-1})n \\ & \geq & (q^k-1)(2q^{t+1}-q^t+2)-(2q^{k-t}-q^{k-t-1})(q^{2t+1}+1) \\ & = & q^k(2-2q^{-t}+q^{-t-1})-2q^{t+1}+q^t-2 \\ & \geq & q^{t+2}(2-2q^{-t}+q^{-t-1})-2q^{t+1}+q^t-2 \\ & = & 2q^{t+2}-2q^{t+1}+q^t-2q^2+q-2>0. \end{array}$$

If $\lambda \in [2, q^{k-t-1}]$ and $x \in I_{\lambda,k} = [l_{\lambda,b}, l_{\lambda,e}] = [2q^{t+1} - q^t + (\lambda - 1)q^{2t+1-k} + 1, l_{\lambda,e}]$, we have

$$\begin{array}{ll} y_{x,k}-x & = & (q^k-1)x-(2q^{k-t}-q^{k-t-1}+\lambda-1)n \\ & \geq & (q^k-1)(2q^{t+1}-q^t+(\lambda-1)q^{2t+1-k}+1)-(2q^{k-t}-q^{k-t-1}+\lambda-1)n \\ & = & q^k-2q^{t+1}-2q^{k-t}+q^{2t+1-k}+q^{k-t-1}+q^t-(q^{2t+1-k}+1)\lambda \\ & \geq & q^k-2q^{t+1}-2q^{k-t}+q^{2t+1-k}+q^{k-t-1}+q^t-(q^{2t+1-k}+1)q^{k-t-1} \\ & = & q^{2t+1-k}+q^k(1-2q^{-t})-2q^{t+1} \\ & \geq & q^{2t+1-(t+2)}+q^{t+2}(1-2q^{-t})-2q^{t+1} \text{ (see Lemma 26)} \\ & = & q^{t+2}-2q^{t+1}+q^{t-1}-2q^2>0. \end{array}$$

Next, we will show $n - y_{x,k} - x > 0$.

If $\lambda \in [1, q^{k-t-1} - 1]$ and $x \in I_{\lambda,k} = [l_{\lambda,b}, l_{\lambda,e}] = [l_{\lambda,b}, 2q^{t+1} - q^t + \lambda q^{2t+1-k} - 1]$, then

$$\begin{array}{ll} n-y_{x,k}-x & = & (2q^{k-t}-q^{k-t-1}+\lambda)n-(q^k+1)x \\ & \geq & (2q^{k-t}-q^{k-t-1}+\lambda)(q^{2t+1}+1)-(q^k+1)(2q^{t+1}-q^t+\lambda q^{2t+1-k}-1) \\ & = & q^k+2q^{k-t}-2q^{t+1}+1+q^t-q^{k-t-1}-(q^{2t+1-k}-1)\lambda \\ & \geq & q^k+2q^{k-t}-2q^{t+1}+1+q^t-q^{k-t-1}-(q^{2t+1-k}-1)(q^{k-t-1}-1) \\ & = & q^{2t+1-k}+q^k(1+2q^{-t})-2q^{t+1} \\ & \geq & q^{2t+1-(t+2)}+q^{t+2}(1+2q^{-t})-2q^{t+1} \text{ (see Lemma 26)} \\ & = & q^{t+2}-2q^{t+1}+q^{t-1}+2q^2>0. \end{array}$$

If $\lambda = q^{k-t-1}$ and $x \in I_{\lambda,k}$, we obtain that

$$\begin{array}{lcl} n-y_{x,k}-x & = & (2q^{k-t}-q^{k-t-1}+\lambda)n-(q^k+1)x \\ & \geq & 2q^{k-t}(q^{2t+1}+1)-(q^k+1)(2q^{t+1}-2q-1) \\ & = & q^k(2q+1+2q^{-t})-2q^{t+1}+2q+1 \\ & \geq & q^{t+2}(2q+1+2q^{-t})-2q^{t+1}+2q+1 \\ & = & 2q^{t+3}+q^{t+2}-2q^{t+1}+2q^2+2q+1>0. \end{array}$$

(4.4): When k = 2t, we partition $I = [2q^{t+1} - q^t + 2, 2q^{t+1} - 2q - 1]$ into $q^{t-1} - 2$ subintervals as follows:

$$\begin{split} I_{1,k} &= [2q^{t+1} - q^t + 2, 2q^{t+1} - q^t + q - 1], \\ I_{\lambda,k} &= [2q^{t+1} - q^t + (\lambda - 1)q + 1, 2q^{t+1} - q^t + \lambda q - 1] \text{ for } \lambda \in [2, q^{t-1} - 2]. \\ \text{For some } \lambda \in [1, q^{t-1} - 2], \text{ if } x \in I_{\lambda,k} = [l_{\lambda,b}, l_{\lambda,e}], \text{ we derive that} \end{split}$$

$$y_{x,k} = q^k x - (2q^t - q^{t-1} + \lambda - 1)n.$$

Firstly, we will show $y_{x,k} - x > 0$. If $\lambda = 1$ and $x \in I_{\lambda,k} = I_{1,k}$, then

$$\begin{array}{lcl} y_{x,k}-x & = & (q^k-1)x-(2q^t-q^{t-1})n \\ & \geq & (q^{2t}-1)(2q^{t+1}-q^t+2)-(2q^t-q^{t-1})(q^{2t+1}+1) \\ & = & 2q^{2t}-2q^{t+1}-q^t+q^{t-1}-2>0. \end{array}$$

If $\lambda \in [2, q^{t-1} - 2]$ and $x \in I_{\lambda,k} = [2q^{t+1} - q^t + (\lambda - 1)q + 1, l_{\lambda,e}]$, we have

$$\begin{array}{lll} y_{x,k}-x & = & (q^k-1)x-(2q^t-q^{t-1}+\lambda-1)n \\ & \geq & (q^{2t}-1)(2q^{t+1}-q^t+(\lambda-1)q+1)-(2q^t-q^{t-1}+\lambda-1)(q^{2t+1}+1) \\ & = & q^{2t}-2q^{t+1}-q^t+q+q^{t-1}-(q+1)\lambda \\ & \geq & q^{2t}-2q^{t+1}-q^t+q+q^{t-1}-(q+1)(q^{t-1}-2) \\ & = & q^{2t}-2q^{t+1}-2q^t+3q+2>0. \end{array}$$

Next, we show $n-y_{x,k}-x>0$. For any $\lambda\in[1,q^{t-1}-2],$ if $x\in I_{\lambda,k}=[l_{\lambda,b},l_{\lambda,e}]=[l_{\lambda,b},2q^{t+1}-q^t+\lambda q-1],$ then

$$\begin{array}{lll} n-y_{x,k}-x & = & (2q^t-q^{t-1}+\lambda)n-(q^{2t}+1)x \\ & \geq & (2q^t-q^{t-1}+\lambda)(q^{2t+1}+1)-(q^{2t}+1)(2q^{t+1}-q^t+\lambda q-1) \\ & = & q^{2t}-2q^{t+1}+3q^t-q^{t-1}+1-(q-1)\lambda \\ & \geq & q^{2t}-2q^{t+1}+3q^t-q^{t-1}+1-(q-1)(q^{t-1}-2) \\ & = & q^{2t}-2q^{t+1}+2q^t+2q-1>0. \end{array}$$

According to the four cases above, one can deduce that (4) holds.

(5) It is easy to check the following statements:

$$\begin{array}{rcl} -(q^{t+1}+\alpha q^t-1)q^{t+1} & \equiv & (q+\alpha)n-(q^{t+1}+\alpha q^t-1)q^{t+1} \\ & = & q^{t+1}+q+\alpha < q^{t+1}+\alpha q^t-1, \\ (q^{t+1}+\alpha q^t+1)q^{t+1} & \equiv & (q^{t+1}+\alpha q^t+1)q^{t+1}-(q+\alpha)n \\ & = & q^{t+1}-q-\alpha < q^{t+1}+\alpha q^t+1. \end{array}$$

When $\beta = 1$ or 2 and $1 \le \gamma \le \beta q - 1$, from $t \ge 2$, we can easily get $(\beta q - \gamma)q^t \ge q^t > \beta + \gamma$. It follows that

$$\begin{aligned} -(\beta q^{t+1} - \gamma)q^t &\equiv \beta n - (\beta q^{t+1} - \gamma)q^t = \gamma q^t + \beta < \beta q^{t+1} - \gamma, \\ (\beta q^{t+1} + \gamma)q^t &\equiv (\beta q^{t+1} + \gamma)q^t - \beta n = \gamma q^t - \beta < \beta q^{t+1} + \gamma. \end{aligned}$$

From definition, the congruence expressions above imply that there exists some integer $y \in [1, x - 1]$ satisfying $y \in C_x$ for each x in (5). Hence, x is not a coset leader and (5) follows.

A.2 The proof of Theorem 4

Proof. Since (1) has been derived by [16, 17], it suffices to prove (2) and (3).

- (2): To verify (2), one only needs to show $y_{x,k} x \ge 0$ and $n y_{x,k} x \ge 0$ for all $x \in I = [q^t + 2, 2q^t 2]$ and $k \in [0, 2t 1]$. Next, we will give the proof through the following subcases by different k.
- (2.1): When $k=0,1,2,\cdots,t-1$, it is easy to get $x \leq q^k x < n$. Hence, for each x we have $y_{x,k}=q^k x \geq x$ and

$$\begin{array}{lcl} n-y_{x,k}-x & = & q^{2t}+1-(q^k+1)x \\ & \geq & q^{2t}+1-(q^{t-1}+1)x \\ & \geq & q^{2t}+1-(q^{t-1}+1)(2q^t-2) \\ & = & (q^t-2q^{t-1}-2)q^t+2q^{t-1}+3>0. \end{array}$$

(2.2): When k = t, we derive $y_{x,k} = q^k x - n$, it then follows that

$$\begin{aligned} y_{x,k} - x &=& (q^k - 1)x - n \geq (q^t - 1)(q^t + 2) - (q^{2t} + 1) = q^t - 3 > 0, \\ n - y_{x,k} - x &=& 2n - (q^k + 1)x \geq 2(q^{2t} + 1) - (q^t + 1)(2q^t - 2) = 4 > 0. \end{aligned}$$

(2.3): When $k=t+1,t+2,\cdots,2t-1$, observing the intractability to determine $y_{x,k}$, we first partition $I=[q^t+2,2q^t-2]$ into q^{k-t} disjoint subintervals below.

$$\begin{split} I_{\lambda,k} &= [q^t + 2, q^t + \lambda q^{2t-k} - 1] \text{ for } \lambda = 1, \\ I_{\lambda,k} &= [q^t + (\lambda - 1)q^{2t-k} + 1, q^t + \lambda q^{2t-k} - 1] \text{ for } \lambda \in [2, q^{k-t} - 1], \\ I_{\lambda,k} &= [q^t + (\lambda - 1)q^{2t-k} + 1, q^t + \lambda q^{2t-k} - 1] \text{ for } \lambda \in [2, q^{k-t} - 1], \\ I_{\lambda,k} &= [q^t + (\lambda - 1)q^{2t-k} + 1, 2q^t - 2] \text{ for } \lambda = q^{k-t}. \end{split}$$

Given $k \in [t+1, 2t-1]$, if $x \in I_{\lambda,k} = [l_{\lambda,b}, l_{\lambda,e}]$ for $\lambda \in [1, q^{k-t}]$, it is not difficult to obtain that

$$y_{x,k} = q^k x - (q^{k-t} + \lambda - 1)n.$$

Then we shall further verify the remainder of the proof. Firstly, we will show that $y_{x,k}-x>0.$

If $\lambda=1$, we have $x\in I_{\lambda,k}=I_{1,k}=[q^t+2,q^t+q^{2t-k}-1],$ it follows that

$$\begin{array}{rcl} y_{x,k}-x & = & (q^k-1)x-q^{k-t}n \\ & \geq & (q^k-1)(q^t+2)-q^{k-t}(q^{2t}+1) \\ & = & q^k(2-q^{-t})-q^t-2 \\ & \geq & q^{t+1}(2-q^{-t})-q^t-2 \\ & = & 2q^{t+1}-q^t-q-2>0. \end{array}$$

If $\lambda \in [2, q^{k-t}]$, then $x \in I_{\lambda,k} = [l_{\lambda,b}, l_{\lambda,e}] = [q^t + (\lambda - 1)q^{2t-k} + 1, l_{\lambda,e}]$, we get that

$$\begin{array}{lll} y_{x,k}-x & = & (q^k-1)x-(q^{k-t}+\lambda-1)n \\ & \geq & (q^k-1)(q^t+(\lambda-1)q^{2t-k}+1)-(q^{k-t}+\lambda-1)(q^{2t}+1) \\ & = & q^{2t-k}+q^k-q^t-q^{k-t}-(q^{2t-k}+1)\lambda \\ & \geq & q^{2t-k}+q^k-q^t-q^{k-t}-(q^{2t-k}+1)q^{k-t} \\ & = & q^{2t-k}+q^k(1-2q^{-t})-2q^t \\ & \geq & q^{2t-(t+1)}+q^{t+1}(1-2q^{-t})-2q^t \text{ (see Lemma 26)} \\ & = & q^{t+1}-2q^t+q^{t-1}-2q>0. \end{array}$$

Next, we will show $n - y_{x,k} - x > 0$.

If $\lambda \in [1, q^{k-t} - 1]$, then $x \in I_{\lambda k} = [l_{\lambda k}, l_{\lambda k}] = [l_{\lambda k}, q^t + \lambda q^{2t-k} - 1]$, we derive that

$$\begin{array}{ll} n-y_{x,k}-x & = & (q^{k-t}+\lambda)n-(q^k+1)x \\ & \geq & (q^{k-t}+\lambda)(q^{2t}+1)-(q^k+1)(q^t+\lambda q^{2t-k}-1) \\ & = & q^k+q^{k-t}+1-q^t-(q^{2t-k}-1)\lambda \\ & \geq & q^k+q^{k-t}+1-q^t-(q^{2t-k}-1)(q^{k-t}-1) \\ & = & q^{2t-k}+q^k(1+2q^{-t})-2q^t \\ & \geq & q^{2t-(t+1)}+q^{t+1}(1+2q^{-t})-2q^t \text{ (see Lemma 26)} \\ & = & q^{t+1}-2q^t+q^{t-1}+2q>0. \end{array}$$

If $\lambda=q^{k-t}$, we have $x\in I_{\lambda,k}=I_{q^{k-t}}=[q^t+(\lambda-1)q^{2t-k}+1,2q^t-2].$ Then

$$\begin{array}{lcl} n-y_{x,k}-x & = & 2q^{k-t}n-(q^k+1)x \\ & \geq & 2q^{k-t}(q^{2t}+1)-(q^k+1)(2q^t-2) \\ & = & 2(q^k+q^{k-t}-q^t+1) \\ & \geq & 2(q^{t+1}+q^{(t+1)-t}-q^t+1) \\ & = & 2(q^{t+1}-q^t+q+1) > 0. \end{array}$$

This completes the proof of (2.3).

(3) It is easily derived that

$$(q^{t}+1)q^{t} = q^{2t} + q^{t} \equiv q^{t} - 1,$$

$$(2q^{t}-1)q^{3t} = 2q^{4t} - q^{3t} \equiv q^{t} + 2,$$

$$(2q^{t}+1)q^{t} = 2q^{2t} + q^{t} \equiv q^{t} - 2,$$

$$2(q^{t}+1)q^{t} = 2q^{2t} + 2q^{t} \equiv 2(q^{t} - 1).$$

Thus, if $x = q^t + 1, 2q^t - 1, 2q^t + 1$ or $2q^{t+1} + 2$, it is obvious that there exists an integer less than x in C_x . That is to say, x is not a coset leader and then (3) holds.

The proof of Lemma 7 A.3

Proof. It is easy to derive that $C_{\delta_1} = \{\delta_1\}$ and $C_{\delta_2} = \{\delta_2, \frac{q+3}{q}\delta_2\}$, which implies that both δ_1 and δ_2 are coset leaders. The remainder of the proof is to verify that δ_3 , δ_4 , δ_5 and δ_6 are also coset leaders by the following steps.

Step 1: Consider that

$$\delta_3 = \delta_2 - \frac{2\delta_2 + (q-1)^2}{q^2} = \frac{n}{2} - q^{2t} + (q^{2t-2} - q^{2t-3} + \dots + q^2 - q).$$

We then show $y_{\delta_3,k}-\delta_3\geq 0$ and $n-y_{\delta_3,k}-\delta_3\geq 0$ by four cases below. (2.1): If k=0, it is obvious that $y_{\delta_3,k}=\delta_3$ and $n-y_{\delta_3,k}=n-\delta_3>\delta_3$.

(2.2): If k = 1, 2, we get

$$y_{\delta_3,k} = q^k \delta_3 - (\frac{q^k - 1}{2} - q^{k-1})n$$

= $\frac{n}{2} + (q^{2t-2} - q^{2t-3} + \dots + q^2 - q)q^k + q^{k-1},$

$$\begin{split} y_{\delta_3,k} - \delta_3 &= q^{2t} + (q^{2t-2} - q^{2t-3} + \dots + q^2 - q)(q^k - 1) + q^{k-1} > 0, \\ n - y_{\delta_3,k} - \delta_3 &= q^{2t} - (q^{2t-2} - q^{2t-3} + \dots + q^2 - q)(q^k + 1) - q^{k-1} \\ &\geq q^{2t} - (q^{2t-2} - q^{2t-3} + \dots + q^2 - q)(q^2 + 1) - q^{2-1} \\ &= (q^{2t-3} + \dots - q^2 + q)(q^2 + 1) - q^{2t-2} - q > 0. \end{split}$$

(2.3): If $k = 3, 5, \dots, 2t - 1$, then one can deduce that

$$\begin{array}{rcl} y_{\delta_3,k} & = & q^k \delta_3 - (\frac{q^k-1}{2} - q^{k-1} + (q^{k-3} \cdot \cdot \cdot - q + 1))n \\ & = & \frac{n}{2} - (q^{2t} - q^{2t-1} + \cdot \cdot \cdot + q^{k+1}) + q^{k-1} - (q^{k-3} - \cdot \cdot \cdot - q + 1), \\ y_{\delta_3,k} - \delta_3 & = & q^{2t-1} - 2(q^{2t-2} - q^{2t-3} \cdot \cdot \cdot - q) - 1 - q^k + 2q^{k-1} - q^{k-2} \\ & \geq & q^{2t-1} - 2(q^{2t-2} - q^{2t-3} \cdot \cdot \cdot - q) - 1 - q^{2t-1} + 2q^{(2t-1)-1} - q^{(2t-1)-2} \\ & = & q^{2t-3} - 2(q^{2t-4} \cdot \cdot \cdot + q^2 - q) - 1 > 0, \\ n - y_{\delta_3,k} - \delta_3 & = & 2q^{2t} - q^{2t-1} + q^k - 2q^{k-1} + q^{k-2} + 1 \\ & \geq & 2q^{2t} - q^{2t-1} + q^{2t-1} - 2q^{(2t-1)-1} + q^{(2t-1)-2} + 1 \\ & = & 2(q^{2t} - q^{2t-2}) + q^{2t-3} + 1 > 0. \end{array}$$

(2.4): If $t \geq 3$ and $k = 4, 6, \dots, 2t$, we can similarly infer that

$$\begin{aligned} y_{\delta_3,k} &= q^k \delta_3 - (\frac{q^k - 1}{2} - q^{k-1} + (q^{k-3} \cdots + q - 1))n \\ &= \frac{n}{2} + (q^{2t} - q^{2t-1} + \cdots - q^{k+1}) + q^{k-1} - (q^{k-3} - \cdots + q - 1), \\ y_{\delta_3,k} - \delta_3 &= 2q^{2t} - q^{2t-1} - q^k + 2q^{k-1} - q^{k-2} + 1 \\ &\geq 2q^{2t} - q^{2t-1} - q^{2t} + 2q^{2t-1} - q^{2t-2} + 1 \\ &= q^{2t} + q^{2t-1} - q^{2t-2} + 1 > 0, \end{aligned}$$

$$n - y_{\delta_3,k} - \delta_3 = q^{2t-1} - 2(q^{2t-2} \cdots + q^2 - q) + q^k - 2q^{k-1} + q^{k-2} - 1$$

$$\geq q^{2t-1} - 2(q^{2t-2} \cdots + q^2 - q) + q^4 - 2q^{4-1} + q^{4-2} - 1$$

$$= q^{2t-1} - 2(q^{2t-2} \cdots + q^2 - q) + q^4 - 2q^3 + q^2 - 1 > 0.$$

From the four cases above, one can conclude that δ_3 is a coset leader.

Step 2: Note that $\delta_4 = \delta_3 - (q-1)^2 = \frac{n}{2} - q^{2t} + (q^{2t-2} - q^{2t-3} + \dots + q^2 - q) - (q-1)^2$. We can further show $y_{\delta_4,k} - \delta_4 \ge 0$ and $n - y_{\delta_4,k} - \delta_4 \ge 0$ in six cases.

(2.1): If k=0, it is easy to know that $y_{\delta_4,k}=\delta_4$ and $n-y_{\delta_4,k}=n-\delta_4>\delta_4$.

(2.2): If k = 1, 2, we get that

$$\begin{array}{rcl} y_{\delta_4,k} & = & q^k \delta_4 - (\frac{q^k-1}{2} - q^{k-1})n \\ & = & \frac{n}{2} + (q^{2t-2} - q^{2t-3} + \dots + q^2)q^k - q^{k-1}(q^3 - q^2 + q - 1), \\ y_{\delta_4,k} - \delta_4 & = & q^{2t} + (q^{2t-2} - q^{2t-3} + \dots + q^2)(q^k - 1) - q^{k-1}(q^3 - q^2 + q - 1) + q^2 - q + 1 \\ & \geq & q^{2t} + (q^{2t-2} - q^{2t-3} + \dots + q^2)(q^1 - 1) - q^{1-1}(q^3 - q^2 + q - 1) + q^2 - q + 1 \\ & = & q^{2t} + (q^{2t-2} - q^{2t-3} + \dots + q^3)(q - 1) + (q - 1)^2 + 1 > 0, \end{array}$$

$$n - y_{\delta_4,k} - \delta_4 = q^{2t} - (q^{2t-2} - q^{2t-3} + \dots - q^3)(q^k + 1) - (q - 1)(q^k + 1) - q^{k-1}$$

$$\geq q^{2t} - (q^{2t-2} - q^{2t-3} + \dots - q^3)(q^2 + 1) - (q - 1)(q^2 + 1) - q^{2-1}$$

$$= q^{2t} - (q^{2t-2} - q^{2t-3} + \dots - q)(q^2 + 1) + (q - 1)^2(q^2 + 1) - q$$

$$= q^{2t} - (q^{2t-2} - q^{2t-3} + \dots - q + 1)(q^2 + 1) + ((q - 1)^2 + 1)(q^2 + 1) - q$$

$$= q^{2t} - \frac{q^{2t-1} + 1}{q+1}(q^2 + 1) + ((q - 1)^2 + 1)(q^2 + 1) - q$$

$$= \frac{q^{2t} - q^{2t-1} - q^2 - 1}{q+1}(q^2 + 1) + ((q - 1)^2 + 1)(q^2 + 1) - q > 0.$$

(2.3): If $t \geq 3$ and $k = 3, 5, \dots, 2t - 3$, it shall be deduced

$$\begin{array}{lll} y_{\delta_4,k} & = & q^k \delta_4 - (\frac{q^k-1}{2} - q^{k-1} + (q^{k-3} \cdot \cdot \cdot - q + 1))n \\ & = & \frac{n}{2} - (q^{2t} - \cdot \cdot \cdot - q + 1) - q^{k-2}(q^4 - 2q^3 + 2q^2 - 2q + 1), \\ y_{\delta_4,k} - \delta_4 & = & q^{2t-1} - 2(q^{2t-2} - q^{2t-3} \cdot \cdot \cdot - q^3) - q^2 - q^{k-2}(q^4 - 2q^3 + 2q^2 - 2q + 1) \\ & \geq & q^{2t-1} - 2(q^{2t-2} - q^{2t-3} \cdot \cdot \cdot - q^3) - q^2 - q^{(2t-3)-2}(q^4 - 2q^3 + 2q^2 - 2q + 1) \\ & = & 2(q^{2t-5} \cdot \cdot \cdot - q^2 + q) - q^{2t-5} + q^2 - 2q > 0, \\ n - y_{\delta_4,k} - \delta_4 & = & 2q^{2t} - q^{2t-1} + (q-1)^2 + 1 + q^{k-2}(q^4 - 2q^3 + 2q^2 - 2q + 1) > 0. \end{array}$$

(2.4): If $t \geq 3$ and $k = 4, 6, \dots, 2t - 2$, we then can similarly obtain that

$$y_{\delta_4,k} = q^k \delta_4 - (\frac{q^k - 1}{2} - q^{k-1} + (q^{k-3} \cdots + q - 1))n$$

= $\frac{n}{2} + (q^{2t} - \cdots - q + 1) - q^{k-2}(q^4 - 2q^3 + 2q^2 - 2q + 1),$

$$\begin{array}{rcl} y_{\delta_4,k} - \delta_4 & = & 2q^{2t} - q^{2t-1} + (q-1)^2 + 1 - q^{k-2}(q^4 - 2q^3 + 2q^2 - 2q + 1) \\ & \geq & 2q^{2t} - q^{2t-1} + (q-1)^2 + 1 - q^{(2t-2)-2}(q^4 - 2q^3 + 2q^2 - 2q + 1) \\ & = & q^{2t} + q^{2t-1} - 2q^{2t-2} + 2q^{2t-3} - q^{2t-4} + (q-1)^2 + 1 > 0, \\ n - y_{\delta_4,k} - \delta_4 & = & q^{2t-1} - 2(q^{2t-2} - q^{2t-3} \cdots - q^3) - q^2 + q^{k-2}(q^4 - 2q^3 + 2q^2 - 2q + 1) \\ & \geq & q^{2t-1} - 2(q^{2t-2} - q^{2t-3} \cdots - q^3) - q^2 + q^{4-2}(q^4 - 2q^3 + 2q^2 - 2q + 1) \\ & = & q^{2t-1} - 2(q^{2t-2} - q^{2t-3} \cdots - q^3) + q^2(q^4 - 2q^3 + 2q^2 - 2q) > 0. \end{array}$$

(2.5): If k = 2t - 1, it then follows that

$$\begin{split} y_{\delta_4,k} &= q^k \delta_4 - (\frac{q^k-1}{2} - q^{k-1} + (q^{k-3} \cdots - q + 1) - 1)n \\ &= \frac{n}{2} + q^{2t} - q^{2t-1} + q^{2t-2} - (q^{2t-4} - \cdots - q + 1) + 1, \\ y_{\delta_4,k} - \delta_4 &= 2q^{2t} - q^{2t-1} - q^{2t-3} + 2(q^{2t-3} \cdots - q^2 + q) + (q-1)^2 > 0, \\ n - y_{\delta_4,k} - \delta_4 &= q^{2t-1} - 2q^{2t-2} + q^{2t-3} + (q-1)^2 > 0. \end{split}$$

(2.6): If k = 2t, then one can get that

$$y_{\delta_4,k} = q^k \delta_4 - (\frac{q^k - 1}{2} - q^{k-1} + (q^{k-3} \cdots - q^2 + q) - q + 1)n$$

= $\frac{n}{2} - q^{2t} + q^{2t-1} - (q^{2t-3} - \cdots + q - 1) + q - 2,$

$$\begin{aligned} y_{\delta_4,k} - \delta_4 &= q^{2t-1} - q^{2t-2} + q^2 - q > 0, \\ n - y_{\delta_4,k} - \delta_4 &= 2q^{2t} - q^{2t-1} - q^{2t-2} + 2(q^{2t-3} \cdot \cdot \cdot \cdot - q^2 + q) + q^2 - 3q + 2 > 0. \end{aligned}$$

Concluding the discussions above, one can infer that δ_4 is a coset leader.

In the similar way to the proofs of Steps 1 and 2, we can also derive that both δ_5 and δ_6 are coset leaders. The detailed proofs are omitted here.

The proof of Lemma 13

Proof. It is easy to derive that $C_{\delta_1} = \{\delta_1, \frac{q+2}{q}\delta_1\}$, which implies that $|C_{\delta_1}| = 2$ and δ_1 is a coset leader. Next, we will verify that δ_2 , δ_3 , δ_4 and δ_5 are all also coset leaders by the following discussions.

Step 1: Notice that

$$\delta_2 = \delta_1 - \frac{2\delta_1 + (q-1)q}{q^2} = \frac{1}{2}((q^{2t+1} - q^{2t}) - (q^{2t-1} - q^{2t-2} + \dots - q^2 + q)).$$

We then split into five cases to show $y_{\delta_2,k} - \delta_2 \ge 0$ and $n - y_{\delta_2,k} - \delta_2 \ge 0$ by different k: (1.1): If k = 0, it is clear that $y_{\delta_2,k} = \delta_2$ and $n - y_{\delta_2,k} = n - \delta_2 > \delta_2$.

(1.2): If k=1, we easily get that

$$\begin{aligned} y_{\delta_{2,k}} &= q\delta_{2} - \frac{q-2}{2}n = \frac{1}{2}((q^{2t+1} - q^{2t}) + (q^{2t-1} - q^{2t-2} + \dots - q^{2}) - q + 2), \\ y_{\delta_{2,k}} - \delta_{2} &= q^{2t-1} - q^{2t-2} + \dots - q^{2} + 1 > 0, \\ n - y_{\delta_{2,k}} - \delta_{2} &= q^{2t} + q > 0. \end{aligned}$$

(1.3): If k=2, it is not difficult to obtain that

$$\begin{split} y_{\delta_2,k} &= q^2 \delta_2 - \frac{q^2 - q - 2}{2} n \\ &= \frac{1}{2} q^{2t+1} + \frac{1}{2} ((q^{2t} - q^{2t-1} + q^{2t-2} \cdot \cdot \cdot - q^3) - q^2 + q + 2), \\ y_{\delta_2,k} - \delta_2 &= q^{2t} - q^2 + q + 1 > 0, \\ n - y_{\delta_2,k} - \delta_2 &= q^{2t-1} - q^{2t-2} + \cdot \cdot \cdot + q^3 > 0. \end{split}$$

(1.4): If $k = 3, 5, \dots, 2t - 1$, we shall infer that

$$\begin{split} y_{\delta_2,k} &= q^k \delta_2 - \frac{1}{2} (q^k - q^{k-1} - \frac{q(q^{k-2} + 1)}{q+1}) n \\ &= \frac{1}{2} (q^{2t+1} - q^{2t} + q^{2t-1} \cdots - q^2 + q) - q^k + q^{k-1}, \\ y_{\delta_2,k} - \delta_2 &= (q^{2t-1} - q^{2t-2} + q^{2t-3} \cdots - q^2 + q) - (q^k - q^{k-1}) \\ &\geq (q^{2t-1} - q^{2t-2} + q^{2t-3} \cdots - q^2 + q) - (q^{2t-1} - q^{2t-2}) \\ &= q^{2t-3} - q^{2t-4} + q^{2t-5} \cdots - q^2 + q > 0, \\ n - y_{\delta_2,k} - \delta_2 &= q^{2t} + q^k - q^{k-1} + 1 \geq q^{2t} + (q^3 - q^2) + 1 > 0. \end{split}$$

(1.5): If $k = 4, 6, \dots, 2t$, it can be similarly derived that

$$y_{\delta_{2,k}} = 2^{k} \delta_{2} - \frac{1}{2} (q^{k} - q^{k-1} - q^{k-2} + \frac{q(q^{k-3} + 1)}{q+1}) n + n$$

$$= \frac{1}{2} q^{2t+1} + \frac{1}{2} (q^{2t} - q^{2t-1} + \dots + q^{2} - q) - q^{k} + q^{k-1} + 1,$$

$$\begin{split} y_{\delta_2,k} - \delta_2 &= q^{2t} - (q^k - q^{k-1}) + 1 \\ &\geq q^{2t} - (q^{2t} - q^{2t-1}) + 1 = q^{2t-1} + 1 > 0, \\ n - y_{\delta_2,k} - \delta_2 &= (q^{2t-1} - q^{2t-2} \cdots - q^2 + q) + q^k - q^{k-1} \\ &\geq (q^{2t-1} - q^{2t-2} \cdots - q^2 + q) + (q^4 - q^3) > 0. \end{split}$$

Collecting these discussions above, one can conclude that δ_2 is a coset leader. Step 2: Observe that

$$\delta_3 = \delta_2 - q(q-1) = \frac{1}{2}((q^{2t+1} - q^{2t}) - (q^{2t-1} - q^{2t-2} + \dots + q^3) - (q^2 - q)).$$

We then give the proof by the following three cases:

(2.1): If k=0, clearly, $y_{\delta_3,k}=\delta_3$ and $n-y_{\delta_3,k}=n-\delta_3>\delta_3$.

(2.2): If k = 1, we have then

$$\begin{array}{rcl} y_{\delta_3,k} & = & q\delta_3 - \frac{q-2}{2}n \\ & = & \frac{1}{2}((q^{2t+1}-q^{2t})\!\!+\!(q^{2t-1}-q^{2t-2}+\cdots-q^2)\!\!-\!\!q+2)\!\!-\!\!q^2(q-1), \\ y_{\delta_3,k} - \delta_3 & = & ((q^{2t-1}\!\!-\!\!q^{2t-2}\!\!+\cdots+q^3)-q^3)+q^2-q+1>0, \\ n-y_{\delta_3,k} - \delta_3 & = & q^{2t}+q^3>0. \end{array}$$

(2.3): If k=2, we get that

$$\begin{array}{rcl} y_{\delta_3,k} & = & q^2\delta_3 - \frac{q^2-q-2}{2}n \\ & = & \frac{1}{2}q^{2t+1} + \frac{1}{2}((q^{2t}-q^{2t-1}+\cdots-q^3)-q^2+q+2)-q^3(q-1), \\ y_{\delta_3,k} - \delta_3 & = & q^{2t}-q^4+q^3+1>0, \\ n-y_{\delta_2,k} - \delta_3 & = & (q^{2t-1}-q^{2t-2}+\cdots+q^3)+q(q^2+1)(q-1)>0. \end{array}$$

(2.4): If $t \geq 3$ and $k = 3, 5, \dots, 2t - 3$, it then follows that

$$\begin{array}{lll} y_{\delta_3,k} & = & q^k \delta_3 - \frac{1}{2} (q^k - q^{k-1} - \frac{q(q^{k-2}+1)}{q+1}) n \\ & = & \frac{1}{2} (q^{2t+1} - q^{2t} \cdot \cdot \cdot - q^2 + q) - (q^{k+2} - q^{k+1} + q^k - q^{k-1}), \\ y_{\delta_3,k} - \delta_3 & = & (q^{2t-1} - q^{2t-2} + q^{2t-3} \cdot \cdot \cdot + q) - (q^{k+2} - q^{k+1} + q^k - q^{k-1}) + q(q-1) \\ & \geq & (q^{2t-1} - q^{2t-2} \cdot \cdot \cdot + q) - (q^{(2t-3)+2} - q^{(2t-3)+1} + q^{2t-3} - q^{(2t-3)-1}) + q(q-1) \\ & = & (q^{2t-5} - q^{2t-6} \cdot \cdot \cdot + q) + q(q-1) > 0, \\ n - y_{\delta_3,k} - \delta_3 & = & q^{2t} + (q^{k+2} - q^{k+1} + q^k - q^{k-1}) + 1 + q(q-1) \\ & \geq & q^{2t} + (q^{3+2} - q^{3+1} + q^3 - q^{3-1}) + 1 + q(q-1) \\ & = & q^{2t} + q^5 - q^4 + q^3 - q + 1 > 0. \end{array}$$

(2.5): If $t \geq 3$ and $k = 4, 6, \dots, 2t - 2$, then one can similarly infer that

$$\begin{split} y_{\delta_{3,k}} &= 2^k \delta_3 - \frac{1}{2} (q^k - q^{k-1} - q^{k-2} + \frac{q(q^{k-3} + 1)}{q+1}) n + n, \\ &= \frac{1}{2} q^{2t+1} + \frac{1}{2} (q^{2t} - q^{2t-1} + \dots + q^2 - q) - (q^{k+2} - q^{k+1} + q^k - q^{k-1}) + 1, \end{split}$$

$$\begin{array}{rcl} y_{\delta_3,k}-\delta_3 &=& q^{2t}-(q^{k+2}-q^{k+1}+q^k-q^{k-1})+1+q(q-1)\\ &\geq& q^{2t}-(q^{(2t-2)+2}-q^{(2t-2)+1}+q^{2t-2}-q^{(2t-2)-1})+1+q(q-1)\\ &=& (q^{2t-3}+1)(q^2-q+1)>0,\\ n-y_{\delta_3,k}-\delta_3 &=& (q^{2t-1}-q^{2t-2}\cdot\cdot\cdot-q^2+q)+(q^{k+2}-q^{k+1}+q^k-q^{k-1})+q(q-1)\\ &\geq& (q^{2t-1}-q^{2t-2}\cdot\cdot\cdot-q^2+q)+(q^{4+2}-q^{4+1}+q^4-q^{4-1})+q(q-1)\\ &=& (q^{2t-1}-q^{2t-2}\cdot\cdot\cdot\cdot+q^5)+q^6-q^5>0. \end{array}$$

(2.6): If k = 2t - 1, we have then

$$\begin{array}{rcl} y_{\delta_3,k} & = & q^k \delta_3 - \frac{1}{2} (q^k - q^{k-1} - \frac{q(q^{k-2}+1)}{q+1}) n + n \\ & = & \frac{1}{2} (q^{2t+1} + q^{2t} - q^{2t-1} + q^{2t-2} + (q^{2t-3} \cdot \cdot \cdot - q^2 + q)) + 1, \\ y_{\delta_3,k} - \delta_3 & = & q^{2t} + (q^{2t-3} - q^{2t-2} \cdot \cdot \cdot + q) + q^2 - q + 1 > 0, \\ n - y_{\delta_2,k} - \delta_3 & = & q^{2t-1} - q^{2t-2} + q^2 - q > 0. \end{array}$$

(2.7): If k = 2t, it is not difficult to obtain that

$$\begin{split} y_{\delta_3,k} &= q^k \delta_3 - \frac{1}{2} (q^k - q^{k-1} - q^{k-2} + \frac{q(q^{k-3} + 1)}{q+1}) n + q n \\ &= \frac{1}{2} (q^{2t+1} - q^{2t} + q^{2t-1} + (q^{2t-2} - q^{2t-3} + \dots + q^2) + q), \\ y_{\delta_3,k} - \delta_3 &= q^{2t-1} + q^2 > 0, \\ n - y_{\delta_3,k} - \delta_3 &= q^{2t} - (q^{2t-2} - q^{2t-3} \dots + q^2) + q(q-1) + 1 > 0. \end{split}$$

To sum up, one shall know that δ_3 is a coset leader.

In the similar way to the proofs of Steps 1 and 2, we can also derive that both δ_4 and δ_5 are coset leaders. Therefore, the detailed proofs are omitted here.

A.5 The proof of Lemma 21

 ${\it Proof.}$ It shall be verified by the mathematical induction here.

If r = 2, then $S^2 = (1, -1)$, it is trivial that $F_{(1)}^2 = (1, 1)$ and $H_{(1)}^2 = (1, -1)$. This obviously implies that $F_{(1)}^2 \ge S_2$ and $H_{(1)}^2 \ge S_2$.

Now assume that $F_{(k)}^{(r)} \geq S^r$ and $H_{(k)}^r \geq S^r$ for $r \geq 3$ and $1 \leq k \leq 2^{r-1} - 1$. The remainder of the proof is to verify $F_{(k)}^{r+1} \geq S^{r+1}$ and $H_{(k)}^{r+1} \geq S^{r+1}$ for $1 \leq k \leq 2^r - 1$. By definition, if $S^r = (s_{2^{r-1}-1}^r, \dots, s_1^r, s_0^r)$, then

$$S^{r+1} = (S^r, -S^r) = (s_{2^{r-1}-1}^r, \cdots, s_1^r, s_0^r, -s_{2^{r-1}-1}^r, \cdots, -s_1^r, -s_0^r).$$

We split into the following cases.

Case 1: $1 \le k \le 2^{r-1} - 1$.

Subcase 1.1: If $s_{2^{r-1-k}}^{r+1} = s_{2^{r-1}-1-k}^{r} = 1$, we obtain that

$$F_{(k)}^{r+1} = (s_{2^r-1-k}^r, \cdots, s_1^r, s_0^r, -s_{2^{r-1}-1}^r, \cdots, -s_1^r, -s_0^r, -s_{2^{r-1}-1}^r, -s_{2^{r-1}-2}^r, \cdots, -s_{2^{r-1}-k}^r)$$
 and
$$H_{(k)}^{r+1} = (s_{2^r-1-k}^r, \cdots, s_1^r, s_0^r, -s_{2^{r-1}-1}^r, \cdots, -s_1^r, -s_0^r, s_{2^{r-1}-1}^r, s_{2^{r-1}-2}^r, \cdots, s_{2^{r-1}-k}^r).$$
 Notice that
$$S^{r+1} = (S^r, -S^r) = (s_{2^{r-1}-1}^r, \cdots, s_1^r, s_0^r, -s_{2^{r-1}-1}^r, \cdots, -s_1^r, -s_0^r).$$

According to the assumption, one has known that

$$\begin{split} F^r_{(k)} &= (s^r_{2^{r-1}-1-k}, \cdots, s^r_1, s^r_0, -s^r_{2^{r-1}-1}, -s^r_{2^{r-1}-2}, \cdots, -s^r_{2^{r-1}-k}) \geq S^r. \\ \text{Clearly, it follows that } F^{r+1}_{(k)} &\geq S^{r+1} \text{ and } H^{r+1}_{(k)} \geq S^{r+1}. \end{split}$$

Subcase 1.2: If $s_{2^{r-1}-k}^{r+1} = s_{2^{r-1}-1-k}^{r} = -1$, we have

$$F_{(k)}^{r+1} = \left(-s_{2^{r}-1-k}^{r}, \cdots, -s_{1}^{r}, -s_{0}^{r}, s_{2^{r}-1-1}^{r}, \cdots, s_{1}^{r}, s_{0}^{r}, s_{2^{r}-1-1}^{r}, s_{2^{r}-1-2}^{r}, \cdots, s_{2^{r}-1-k}^{r}\right) \text{ and }$$

$$H_{(k)}^{r+1} = (-s_{2^{r-1-k}}^r, \cdots, -s_1^r, -s_0^r, s_{2^{r-1-1}}^r, \cdots, s_1^r, s_0^r, -s_{2^{r-1-1}}^r, -s_{2^{r-1-2}}^r, \cdots, -s_1^r, -s_0^r).$$
 Notice that $S^{r+1} = (S^r, -S^r) = (s_{2^{r-1-1}}^r, \cdots, s_1^r, s_0^r, -s_{2^{r-1-1}}^r, \cdots, -s_1^r, -s_0^r).$

By assumption, we have known that

$$F_{(k)}^r = (-s_{2r-1-1-k}^r, \cdots, -s_1^r, -s_0^r, s_{2r-1-1}^r, s_{2r-1-2}^r, \cdots, s_{2r-1-k}^r) \ge S^r.$$

$$\begin{split} F^r_{(k)} &= (-s^r_{2^{r-1}-1-k}, \cdots, -s^r_1, -s^r_0, s^r_{2^{r-1}-1}, s^r_{2^{r-1}-2}, \cdots, s^r_{2^{r-1}-k}) \geq S^r. \\ \text{Consequently, it is easy to know } F^{r+1}_{(k)} \geq S^{r+1} \text{ and } H^{r+1}_{(k)} \geq S^{r+1}. \end{split}$$

Case 2: $k = 2^{r-1}$.

Notice that $s_{2^{r-1}-k}^{r+1} = -s_{2^{r-1}-1}^{r} = -1$. Then we have

$$F_{r+1}^{(k)}=(s_{2^{r-1}-1}^r,\cdots,s_1^r,s_0^r,s_{2^{r-1}-1}^r,\cdots,s_1^r,s_0^r)=(S^r,S^r)$$
 and

$$H_{r+1}^{(k)} = (s_{2^{r-1}-1}^r, \cdots, s_1^r, s_0^r, -s_{2^{r-1}-1}^r, \cdots, -s_1^r, -s_0^r) = (S^r, -S^r) = S_{r+1}$$
It obviously follows that $F_{(k)}^{r+1} \ge S^{r+1}$ and $H_{(k)}^{r+1} \ge S^{r+1}$.

Case 3: $2^{r-1} + 1 \le k \le 2^r - 1$. Put $u = k - 2^{r-1}$. Then $1 \le u \le 2^{r-1} - 1$. Subcase 3.1 If $s_{2^{r-1}-k}^{r+1} = -s_{2^{r-1}-1-u}^{r} = 1$, we get that

$$\begin{split} F_{(k)}^{r+1} &= (-s_{2^r-1-u}^r, \cdots, -s_1^r, -s_0^r, -s_{2^{r-1}-1}^r, \cdots, -s_1^r, -s_0^r, s_{2^{r-1}-1}^r, s_{2^{r-1}-1}^r, s_{2^{r-1}-2}^r, \cdots, s_{2^{r-1}-u}^r) \\ &\text{and } H_{(k)}^{r+1} = (-s_{2^r-1-u}^r, \cdots, -s_1^r, -s_0^r, s_{2^{r-1}-1}^r, \cdots, s_1^r, s_0^r, -s_{2^{r-1}-1}^r, -s_{2^{r-1}-2}^r, \cdots, -s_{2^{r-1}-u}^r) \\ &\text{According to the assumption, one has known} \end{split}$$

According to the assumption, one has known
$$H^r_{(k)} = (-s^r_{2^{r-1}-1-u}, \cdots, -s^r_1, -s^r_0, -s^r_{2^{r-1}-1}, -s^r_{2^{r-1}-2}, \cdots, -s^r_{2^{r-1}-u}) \geq S^r \text{ and } F^r_{(k)} = (-s^r_{2^{r-1}-1-u}, \cdots, s^r_1, -s^r_0, s^r_{2^{r-1}-1}, s^r_{2^{r-1}-2}, \cdots, s^r_{2^{r-1}-u}) \geq S^r.$$
 Combining $S^{r+1} = (S^r, -S^r) = (s^r_{2^{r-1}-1}, \cdots, s^r_1, s^r_0, -s^r_{2^{r-1}-1}, \cdots, -s^r_1, -s^r_0)$, one can easily derive that $F^{r+1}_{(k)} \geq S^{r+1}$ and $H^{r+1}_{(k)} \geq S^{r+1}$. Subcase 3.2 If $s^{r+1}_{2^{r-1}-k} = -s^r_{2^{r-1}-1-u} = -1$, we have

$$F_{(k)}^{r+1} = (s_{2r-1-u}^r, \cdots, s_1^r, s_0^r, s_{2r-1-1}^r, \cdots, s_1^r, s_0^r, -s_{2r-1-1}^r, -s_{2r-1-1}^r, -s_{2r-1-2}^r, \cdots, -s_{2r-1-u}^r) \text{ and } H_{(k)}^{r+1} = (s_{2r-1-u}^r, \cdots, s_1^r, s_0^r, -s_{2r-1-1}^r, \cdots, -s_1^r, -s_0^r, s_{2r-1-1}^r, s_{2r-1-2}^r, \cdots, s_{2r-1-u}^r)$$
 Similarly, we shall also get that $F_{(k)}^{r+1} \geq S^{r+1}$ and $H_{(k)}^{r+1} \geq S^{r+1}$. To sum up, one can conclude that $F_{(k)}^{r+1} \geq S^{r+1}$ and $H_{(k)}^{r+1} \geq S^{r+1}$ for any $1 \leq k \leq 2^{r-1}$.

This completes the proof.

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