# Terwilliger algebras and some related algebras defined by finite connected simple graphs 

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#### Abstract

For a finite connected simple graph, the Terwilliger algebra is a matrix algebra generated by the adjacency matrix and idempotents corresponding to the distance partition with respect to a fixed vertex. We will consider algebras defined by two other partitions and the centralizer algebra of the stabilizer of the fixed vertex in the automorphism group of the graph. We will give some methods to compute such algebras and examples for various graphs.


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## 1 Introduction

In [10, 11, 12], Paul Terwilliger defined a Terwilliger algebra, originally called a subconstituent algebra, for an association scheme. Especially, it was well studied for $P$ - and $Q$-polynomial association schemes. In [9], Terwilliger algebras for an arbitrary finite connected simple graph were also studied. Recently, Shuang-Dong Li, Yi-Zheng Fan, Tatsuro Ito, Masoud Karimi, and Jing Xu [6] dealt with Terwilliger algebras of trees motivated by the following conjecture.

Conjecture (J. Koolen). For almost all finite connected simple graphs, the Terwilliger algebras coincide with the full matrix algebras.

After that, Jing Xu, Tatsuro Ito, and Shuang-Dong Li 14 considered Terwilliger algebras and centralizer algebras of trees. Then, they investigated when the two algebras coincide.

[^0]For a graph $\Gamma=(X, E)$ with the adjacency matrix $A$ and a fixed vertex $x_{0} \in X$, we will consider the following subalgebras $\mathcal{T}_{\ell}\left(\Gamma, x_{0}\right)(\ell=0,1,2,3,4)$ of $M_{X}(\mathbb{C})$, the full matrix algebra over the complex number field $\mathbb{C}$, rows and columns of whose matrices are indexed by the set $X$. We denote by $E_{x}$ for $x \in X$ the matrix in $M_{X}(\mathbb{C})$ whose $(x, x)$-entry is 1 and the other entries are 0 , and set $E_{Y}=\sum_{y \in Y} E_{y}$ for $Y \subset X$. Let $G$ be the automorphism group of the graph $\Gamma$ and $G_{x_{0}}$ the stabilizer of $x_{0} \in X$ in $G$. Naturally, $G$ acts on $M_{X}(\mathbb{C})$ by permuting rows and columns.

- (the adjacency algebra) Set $\mathcal{T}_{0}\left(\Gamma, x_{0}\right)=\mathbb{C}\langle A\rangle$, the unital $\mathbb{C}$-subalgebra of $M_{X}(\mathbb{C})$ generated by $A$.
- Set $\mathcal{T}_{1}\left(\Gamma, x_{0}\right)=\mathbb{C}\left\langle A, E_{x_{0}}\right\rangle$.
- (the Terwilliger algebra) Consider the distance partition of $X$ with respect to the vertex $x_{0}: X=X_{0} \cup \cdots \cup X_{D}$, where $X_{k}=\left\{x \in X: \partial\left(x_{0}, x\right)=k\right\}$ and $D$ is the diameter of $\Gamma$ with respect to $x_{0}$, the maximal distance from $x_{0}$. Set $\mathcal{T}_{2}\left(\Gamma, x_{0}\right)=\mathbb{C}\left\langle A, E_{X_{0}}, E_{X_{1}}, \ldots, E_{X_{D}}\right\rangle$.
- Let $Y_{1}, \ldots, Y_{r}$ be the $G_{x_{0}}$-orbits on $X$. Set $\mathcal{T}_{3}\left(\Gamma, x_{0}\right)=\mathbb{C}\left\langle A, E_{Y_{1}}, \ldots, E_{Y_{r}}\right\rangle$.
- (the centralizer algebra) Set $\mathcal{T}_{4}\left(\Gamma, x_{0}\right)=\left\{M \in M_{X}(\mathbb{C}): M^{\sigma}=M\right.$ for any $\left.\sigma \in G_{x_{0}}\right\}$.

The vertex $x_{0}$ will be called the base vertex. When there is not fear of the confusion, we omit $\mathcal{T}_{\ell}\left(\Gamma, x_{0}\right)$ with $\mathcal{T}_{\ell}$. Since the distances are preserved by automorphisms, we can see that

$$
\mathcal{T}_{0} \subset \mathcal{T}_{1} \subset \mathcal{T}_{2} \subset \mathcal{T}_{3} \subset \mathcal{T}_{4} \subset M_{X}(\mathbb{C})
$$

The aim of this paper is computing examples of these algebras for various simple graphs and finding examples such that these algebras are different. We remark that all $\mathcal{T}_{\ell}$ are semisimple, because they are closed by transpose and complex conjugate. Since we are considering semisimple algebras over an algebraically closed field, they are isomorphic to direct sums of full matrix algebras. We also remark that the adjacency algebra $\mathcal{T}_{0}$ does not depend on $x_{0}$ and is commutative.

Since the algebras $\mathcal{T}_{\ell}(\ell=0,1,2,3,4)$ are defined as subalgebras of $M_{X}(\mathbb{C})$, they act on $\mathbb{C} X$, the vector space with a formal basis $X$, by right multiplication. The vector space $\mathbb{C} X$ will be called the standard $\mathcal{T}_{\ell}$-module.

The conjecture by Koolen and the paper [6] are about the relationship between $\mathcal{T}_{2}$ and $M_{X}(\mathbb{C})$. And the paper [14] gives a necessary and sufficient condition for $\mathcal{T}_{2}=\mathcal{T}_{4}$ for trees.

This paper is organized as follows. Section 2 describes notations and terminology on finite connected simple graphs and idempotents of semisimple algebras. In Section [3, we list basic facts for $\mathcal{T}_{\ell}(\ell=0,1,2,3,4)$. In Section 4, we deal with the structure of $\mathcal{T}_{1}$. Especially, we determine the structure of $\mathcal{T}_{1}$ for distance-regular graphs. In Section 5, we consider $\mathcal{T}_{2}$ for strongly regular graphs. In Section 6, we consider the structure of $\mathcal{T}_{2}$ and $\mathcal{T}_{3}$ with respect to the base vertex of valency 1. Proposition 6.2 given in this section is used for a construction of an infinite family of examples for $\mathcal{T}_{2} \neq \mathcal{T}_{3}$ (Section
(9). In Section 7, we show the structures of $\mathcal{T}_{\ell}$ for path graphs, star graphs and cycle graphs. Finally, we give infinite families of examples for $\mathcal{T}_{3} \neq \mathcal{T}_{4}$ in Section 8 (Paley graphs) and for $\mathcal{T}_{2} \neq \mathcal{T}_{3}$ in Section 9 .

## 2 Preliminaries

Let $\Gamma=(X, E)$ be a finite connected simple graph. Let $M_{X}(\mathbb{C})$ be the full matrix algebra over the complex number field $\mathbb{C}$, rows and columns of whose matrices are indexed by the set $X$. For $M \in M_{X}(\mathbb{C}), M_{x, y}$ will be the $(x, y)$-entry of $M$. For $x, y \in X$, we write $E_{x, y}$ for the matrix unit. The identity matrix in $M_{X}(\mathbb{C})$ will be denoted by $I$ or $I_{X}$. The all-one matrix in $M_{X}(\mathbb{C})$ will be denoted by $J$ or $J_{X}$. The zero matrix in $M_{X}(\mathbb{C})$ will be denoted by $O$ or $O_{X}$. We write by $A=A(\Gamma) \in M_{X}(\mathbb{C})$ the adjacency matrix of $\Gamma$, namely, $A=\sum_{\{x, y\} \in E}\left(E_{x, y}+E_{y, x}\right)$. For $Y \subset X, E_{Y} \in M_{X}(\mathbb{C})$ will be the diagonal matrix whose diagonal entry $\left(E_{Y}\right)_{x, x}$ is 1 if $x \in Y$ and 0 otherwise. Namely, $E_{Y}=\sum_{y \in Y} E_{y, y}$. When $Y=\{y\}$, we write $E_{y}$ instead of $E_{\{y\}}$. Let $G$ be the automorphism group of the graph $\Gamma$. For $x \in X, G_{x}$ will be the stabilizer of $x$ in $G$. Naturally, $G$ acts on $M_{X}(\mathbb{C})$ by permuting rows and columns.

### 2.1 Finite connected simple graphs

Let $X$ be a finite set and $E$ be a subset of 2-subsets of $X$. In this case, we say that $\Gamma=(X, E)$ a finite simple graph. An element of $X$ is called a vertex and an element of $E$ is called an edge. When $\{x, y\} \in E$, we say that $x$ and $y$ are adjacent in $\Gamma$. A walk of length $\ell$ in $\Gamma$ is a finite sequence $x_{0}, x_{1}, \ldots, x_{\ell}$ in $X$ such that $\left\{x_{i-1}, x_{i}\right\} \in E$ for all $i=1,2, \ldots, \ell$. In this case, the walk is called a walk from $x_{0}$ to $x_{\ell}$. Remark that a walk can contain the same vertices. For $x, y \in X$, the distance of $x$ and $y$ is the minimal length of a walk from $x$ to $y$, and denoted by $\partial(x, y)$. If there is no walk from $x$ to $y$, we set $\partial(x, y)=\infty$. A graph is said to be connected if $\partial(x, y)<\infty$ for all $x, y \in X$. The maximal distance will be called the diameter of $\Gamma$. For $x_{0} \in X$, the diameter of $\Gamma$ with respect to $x_{0}$ is the maximal distance from $x_{0}: \max \left\{\partial\left(x_{0}, y\right): y \in X\right\}$.

Let $\Gamma=(X, E)$ be a finite simple connected graph. An automorphism $\sigma$ of $\Gamma$ is a permutation on $X$ such that $\{x, y\} \in E$ if and only if $\{\sigma(x), \sigma(y)\} \in E$. The automorphism group $\operatorname{Aut}(\Gamma)$ is the group consisting of all automorphisms of $\Gamma$. For $x \in X$, the number of neighbors of $x$ is called the valency of $x$. A graph $\Gamma$ is said to be ( $k$-) regular if all vertices have the same valency $k$. A graph $\Gamma$ is said to be vertextransitive if $\operatorname{Aut}(\Gamma)$ is transitive on $X$. A graph $\Gamma$ is said to be distance-transitive if, for any $x_{1}, x_{2}, y_{1}, y_{2} \in X$ such that $\partial\left(x_{1}, x_{2}\right)=\partial\left(y_{1}, y_{2}\right)$, there exists $\sigma \in \operatorname{Aut}(\Gamma)$ such that $\sigma\left(x_{1}\right)=y_{1}$ and $\sigma\left(x_{2}\right)=y_{2}$. A graph $\Gamma$ is said to be distance-regular if, for $i, j, k=0,1, \ldots, D$, where $D$ is the diameter of $\Gamma$, there exists a non-negative integer $p_{i j}^{k}$ such that $\sharp\{z \in X: \partial(x, z)=i, \partial(z, y)=j\}=p_{i j}^{k}$ for any $x, y \in X$ with $\partial(x, y)=k$. Clearly, a distance-transitive graph is vertex-transitive and distance-regular.

A $k$-regular graph with $n$ vertices is said to be strongly regular, with parameters $(n, k, \lambda, \mu)$ if every adjacent vertices have $\lambda$ common neighbors, and every non-adjacent
vertices have $\mu$ common neighbors. A strongly regular graph is called primitive if both of the graph and its complement are connected. In this article, when we say a strongly regular graph, we mean a primitive strongly regular graph.

The adjacency matrix $A$ of a finite simple connected graph $\Gamma$ is a matrix in $M_{X}(\mathbb{C})$ such that $A_{x, y}=1$ if $\{x, y\} \in E$ and 0 otherwise. For the adjacency matrix $A$ and a non-negative integer $k$, the $(x, y)$-entry of $A^{k}$ is the number of walks of length $k$ from $x$ to $y$ [1, Lemma 2.5]. Let $D$ be the diameter of $\Gamma$. Then $A$ has at least $D+1$ distinct eigenvalues [1, Corollary 2.7]. Remark that all eigenvalues of $A$ are real numbers since $A$ is a real symmetric matrix. An eigenvalue of $A$ is also called an eigenvalue of the graph $\Gamma$.

### 2.2 Idempotents of semisimple algebras

In this subsection, we will summarize basic facts on idempotents of algebras. For details, see [8], for example.

Let $K$ be a field and $\mathcal{A}$ a finite dimensional $K$-algebra. In this article an $\mathcal{A}$-module means a finite dimensional right $\mathcal{A}$-module. A nonzero element $e$ of $\mathcal{A}$ is called an idempotent if $e^{2}=e$. Two idempotents $e$ and $f$ are said to be orthogonal if $e f=f e=0$. An idempotent $e$ is said to be primitive if $e$ is not expressed as a sum of two orthogonal idempotents. We say that $e=e_{1}+\cdots+e_{r}$ is an idempotent decomposition of $e$ if all $e_{i}$ are idempotents and they are orthogonal to each other. An idempotent decomposition $e=e_{1}+\cdots+e_{r}$ is said to be a primitive idempotent decomposition if all $e_{i}$ are primitive. Let $e=e_{1}+\cdots+e_{r}$ be an idempotent decomposition of $e$. Then we have a direct sum decomposition $e \mathcal{A}=e_{1} \mathcal{A} \oplus \cdots \oplus e_{r} \mathcal{A}$ as an $\mathcal{A}$-module. Conversely, a direct sum decomposition of $e \mathcal{A}$ induces an idempotent decomposition. Thus an idempotent $e$ is primitive if and only if $e \mathcal{A}$ is an indecomposable $\mathcal{A}$-module.

By $\operatorname{pi}(\mathcal{A})$, we denote the set of all primitive idempotents of $\mathcal{A}$. For $e, f \in \operatorname{pi}(\mathcal{A})$, we define $e \sim f$ if $e \mathcal{A} \cong f \mathcal{A}$ as $\mathcal{A}$-modules. Clearly, this is an equivalence relation on $\operatorname{pi}(\mathcal{A})$. By $\widetilde{\operatorname{pi}}(\mathcal{A})$, we denote the set of all equivalence classes of $\operatorname{pi}(\mathcal{A})$, and by $[e]$ the equivalence class containing $e \in \operatorname{pi}(\mathcal{A})$. For an $\mathcal{A}$-module $V$ and an idempotent $e$ of $\mathcal{A}$, we have $\operatorname{Hom}_{\mathcal{A}}(V, e \mathcal{A}) \cong V e$ as $K$-spaces. Thus, for $e, f \in \operatorname{pi}(\mathcal{A}), e \sim f$ if and only if $e \mathcal{A} f \neq 0$.

Lemma 2.1. Let $\mathcal{A}$ be a finite dimensional algebra over a field $K$. For an idempotent $e$ of $\mathcal{A}$, e $\mathcal{A} e$ is a $K$-algebra with the identity element $e$. For an idempotent $f$ of e $\mathcal{A} e, f$ is primitive in $e \mathcal{A} e$ if and only if $f$ is primitive in $\mathcal{A}$. For $f, f^{\prime} \in \operatorname{pi}(e \mathcal{A} e), f$ and $f^{\prime}$ are equivalent in e $\mathcal{A} e$ if and only if they are equivalent in $\mathcal{A}$.

Proof. If $f$ is primitive in $\mathcal{A}$, then clearly it is primitive in $e \mathcal{A} e$. Suppose that $f$ is not primitive in $\mathcal{A}, f=g+g^{\prime}$ is an idempotent decomposition of $f \in e \mathcal{A} e$ in $\mathcal{A}$. Since $f \in e \mathcal{A} e$, it follows that $f=f e=e f$. Then $g f=g\left(g+g^{\prime}\right)=g^{2}=g$, and similarly $f g=g$ and $g^{\prime} f=f g^{\prime}=g^{\prime}$. This means that $g=f g f=e g e \in e \mathcal{A} e$ and $g^{\prime} \in e \mathcal{A} e$. Thus $f$ is not primitive in $e \mathcal{A} e$.

Suppose $f, f^{\prime} \in \operatorname{pi}(e \mathcal{A} e)$. If $f$ and $f^{\prime}$ are equivalent in $e \mathcal{A} e$, then clearly they are equivalent in $\mathcal{A}$. Suppose that $f$ and $f^{\prime}$ are equivalent in $\mathcal{A}$. Since $f, f^{\prime} \in e \mathcal{A} e, f=f e$ and $f^{\prime}=e f^{\prime}$, we have $0 \neq f \mathcal{A} f^{\prime}=f(e \mathcal{A} e) f^{\prime}$ and so they are equivalent in $e \mathcal{A} e$.

Every idempotent of a finite dimensional algebra has a primitive idempotent decomposition.

Lemma 2.2. Let $e=e_{1}+\cdots+e_{r}$ be an idempotent decomposition of an idempotent $e$, and $e_{i}=f_{1}^{(i)}+\cdots+f_{r_{i}}^{(i)}$ primitive idempotent decompositions of $e_{i}, i=1, \ldots, r$. Then $e=\sum_{i=1}^{r} \sum_{j=1}^{r_{i}} f_{j}^{(i)}$ is a primitive idempotent decomposition of $e$.
Proof. All $f_{j}^{(i)}$ are primitive idempotents. It is enough to show that the orthogonality of them. Suppose $(i, j) \neq\left(i^{\prime} j^{\prime}\right)$. If $i=i^{\prime}$, then $f_{j}^{(i)} f_{j^{\prime}}^{(i)}=f_{j^{\prime}}^{(i)} f_{j}^{(i)}=0$ since $e_{i}=$ $f_{1}^{(i)}+\cdots+f_{r_{i}}^{(i)}$ is a primitive idempotent decomposition. Suppose that $i \neq i^{\prime}$. We know $f_{j}^{(i)}=e_{i} f_{j}^{(i)} e_{i}$ and $f_{j^{\prime}}^{\left(i^{\prime}\right)}=e_{i^{\prime}} f_{j^{\prime}}^{\left(i^{\prime}\right)} e_{i^{\prime}}$. Thus we have $f_{j}^{(i)} f_{j^{\prime}}^{\left(i^{\prime}\right)}=e_{i} f_{j}^{(i)} e_{i} e_{i^{\prime}} f_{j^{\prime}}^{\left(i^{\prime}\right)} e_{i^{\prime}}=0$ and $f_{j^{\prime}}^{\left(i^{\prime}\right)} f_{j}^{(i)}=e_{i^{\prime}} f_{j^{\prime}}^{\left(i^{\prime}\right)} e_{i^{\prime}} e_{i} f_{j}^{(i)} e_{i}=0$.

Now we suppose that $K=\mathbb{C}$ the complex number field and $\mathcal{A}$ is semisimple. In this case, $\mathcal{A} \cong \bigoplus_{i=1}^{s} M_{n_{i}}(\mathbb{C})$ for some positive integers $n_{i}$ by Wedderburn's Theorem [8, Theorem 1.8.5]. The projections $\pi_{i}: \mathcal{A} \rightarrow M_{n_{i}}(C), i=1 \ldots, s$, are representatives of equivalent classes of irreducible representations of $\mathcal{A}$. Also there is a bijection between $\widetilde{\operatorname{pi}}(\mathcal{A})$ and the set of isomorphism classes of simple $\mathcal{A}$-modules by $[e] \mapsto e \mathcal{A}$. Thus every primitive idempotent belongs to exactly one competent $M_{n_{i}}(\mathbb{C})$. A primitive idempotent of $\mathcal{A}$ is similar to a diagonal matrix unit of some $M_{n_{i}}(\mathbb{C})$. Let $1=e_{1}+\cdots+e_{n}$ be a primitive idempotent decomposition of 1 in $\mathcal{A}$. Then $n_{i}$ is the number of $e_{j}$ 's belonging to $M_{n_{i}}(\mathbb{C})$ and is equal to the dimension of the corresponding simple module.

## 3 Basic facts

In this section, we will state some basic facts. We keep the notations in Introduction. We suppose that $\Gamma=(X, E)$ is a finite simple connected graph and fix $x_{0} \in X$. The automorphism group $G=\operatorname{Aut}(\Gamma)$ also acts on $X \times X$ by $\sigma(x, y)=(\sigma(x), \sigma(y))$.

Theorem 3.1. [1, p.10] Let $\Gamma$ be a finite simple connected graph whose adjacency matrix has $t$ distinct eigenvalues. Then it follows that $\operatorname{dim} \mathcal{T}_{0}\left(\Gamma, x_{0}\right)=t$ and $\mathcal{T}_{0}\left(\Gamma, x_{0}\right) \cong$ $\bigoplus_{i=1}^{t} \mathbb{C}$.

Lemma 3.2. If there is $\sigma \in G$ such that $\sigma\left(x_{0}\right)=y_{0}$, then $\mathcal{T}_{\ell}\left(\Gamma, x_{0}\right) \cong \mathcal{T}_{\ell}\left(\Gamma, y_{0}\right)$. If $\Gamma$ is vertex-transitive, then the structure of $\mathcal{T}_{\ell}\left(\Gamma, x_{0}\right) \quad(\ell=0,1,2,3,4)$ does not depend on the base vertex $x_{0}$.

Lemma 3.3. If $\Gamma$ is distance-transitive, then $\mathcal{T}_{2}\left(\Gamma, x_{0}\right)=\mathcal{T}_{3}\left(\Gamma, x_{0}\right)$.
Proof. In this case, the distance partition with respect to the base vertex $x_{0}$ is just the set of $G_{x_{0}}$-orbits.
 $\left\{A_{Y_{1}}, \ldots, A_{Y_{r}}\right\}$ spans $\mathcal{T}_{4}\left(\Gamma, x_{0}\right)$ and $\operatorname{dim} \mathcal{T}_{4}\left(\Gamma, x_{0}\right)=r$.

Proof. This is shown by direct calculations.
Lemma 3.5. The following statements are equivalent.
(1) $\mathcal{T}_{3}\left(\Gamma, x_{0}\right)=M_{X}(\mathbb{C})$.
(2) $\mathcal{T}_{4}\left(\Gamma, x_{0}\right)=M_{X}(\mathbb{C})$.
(3) $G_{x_{0}}=1$.

Proof. It is clear that (1) implies (2) and (3) implies (1). Also (2) implies (3), by Lemma 3.4, for example.

In [6, Theorem 4], it is shown that $G_{x_{0}}=1$ if and only if $\mathcal{T}_{2}\left(\Gamma, x_{0}\right)=M_{X}(\mathbb{C})$ for trees. This is not true, in general. For example, see Example 5.3.

Lemma 3.6. Let $X=Y_{1} \cup \cdots \cup Y_{r}$ be an arbitrary partition of the vertex set $X$ of a finite simple connected graph $\Gamma=(X, E)$. Suppose that $\left|Y_{i}\right|=1$. We set $\mathcal{T}=$ $\mathbb{C}\left\langle A, E_{Y_{1}}, \ldots, E_{Y_{r}}\right\rangle$. Then the dimension of the simple $\mathcal{T}$-module $E_{Y_{i}} \mathcal{T}$ is at least $r$. Moreover, if $\left|Y_{i}\right|=\left|Y_{j}\right|=1$, then $E_{Y_{i}} \mathcal{T} \cong E_{Y_{j}} \mathcal{T}$.

Proof. By the connectivity of $\Gamma, E_{Y_{i}} \mathcal{T} E_{Y_{k}} \neq 0$ for all $1 \leq k \leq r$. Since $\left|Y_{i}\right|=1, E_{Y_{i}}$ is a primitive idempotent and every $E_{Y_{k}}$ contains an idempotent equivalent to $E_{Y_{i}}$. Thus $E_{Y_{i}}$ has at least $r$ equivalent idempotents in the primitive idempotent decomposition of 1. The last statement is clear.

For the distance partition $X=X_{0} \cup \cdots \cup X_{D}$ with respect to the vertex $x_{0}, X_{0}=\left\{x_{0}\right\}$ and $\left|X_{0}\right|=1$. We can apply Lemma 3.6 and the simple module $E_{X_{0}} \mathcal{T}_{2}$ is called the principal module.

## 4 The structure of $\mathcal{T}_{1}$

In this section, we consider the structure of $\mathcal{T}_{1}=\mathbb{C}\left\langle A, E_{x_{0}}\right\rangle$.
The next lemma is clearly holds.
Lemma 4.1. For $\boldsymbol{v}=\left(v_{x}\right)_{x \in X}, \boldsymbol{v} E_{x_{0}}=\mathbf{0}$ if and only if $v_{x_{0}}=0$.
Proposition 4.2. Let $\Gamma=(X, E)$ be a simple connected graph with $|X|=n$. Fix $a$ base vertex $x_{0} \in X$. Suppose that $\lambda$ is an eigenvalue of the adjacency matrix $A$ of $\Gamma$, and $\boldsymbol{v}$ is a corresponding eigenvector. If the $x_{0}-$ th entry of $\boldsymbol{v}$ is zero, then $\mathbb{C} \boldsymbol{v}$ is a simple $\mathcal{T}_{1}\left(\Gamma, x_{0}\right)$-module not isomorphic to $E_{x_{0}} \mathcal{T}_{1}\left(\Gamma, x_{0}\right)$ as a $\mathcal{T}_{1}\left(\Gamma, x_{0}\right)$-module. If $\boldsymbol{w}$ is an eigenvector corresponding to $\mu$ different from $\lambda$ with the $x_{0}$-th entry zero, then $\mathbb{C} \boldsymbol{w}$ is non-isomorphic to $\mathbb{C} \boldsymbol{v}$ as a $\mathcal{T}_{1}\left(\Gamma, x_{0}\right)$-module.

Proof. Since $\boldsymbol{v}$ is an eigenvector of $A, \boldsymbol{v} A=\lambda \boldsymbol{v} \in \mathbb{C} \boldsymbol{v}$. By assumption, $\boldsymbol{v} E_{x_{0}}=\mathbf{0}$. By $\mathcal{T}_{1}=\mathbb{C}\left\langle A, E_{x_{0}}\right\rangle, \mathbb{C} \boldsymbol{v}$ is a $\mathcal{T}_{1}$-module. The idempotent $E_{x_{0}}$ acts on $E_{x_{0}} \mathcal{T}_{1}$ as nonzero and on $\mathbb{C} \boldsymbol{v}$ as 0 . Thus $\mathbb{C} \boldsymbol{v}$ is not isomorphic to $E_{x_{0}} \mathcal{T}_{1}$. The modules $\mathbb{C} \boldsymbol{v}$ and $\mathbb{C} \boldsymbol{w}$ are non-isomorphic since the actions of $A$ are different.

Corollary 4.3. Suppose that an eigenvalue $\lambda$ of $A$ has the eigenspace $V_{\lambda}$ of dimension $d_{\lambda}$. Set $W_{\lambda}:=\left\{\boldsymbol{v} \in V_{\lambda}: \boldsymbol{v} E_{x_{0}}=\mathbf{0}\right\}$. Then $\operatorname{dim} W_{\lambda}=d_{\lambda}-1$ or $d_{\lambda}$ and $W_{\lambda}$ is a direct sum of isomorphic 1-dimensional simple $\mathcal{T}_{1}\left(\Gamma, x_{0}\right)$-modules.

Proof. Since $\left\{\boldsymbol{v} \in \mathbb{C} X: \boldsymbol{v} E_{x_{0}}=\mathbf{0}\right\}$ has dimension $n-1$, $\operatorname{dim} W_{\lambda}=d_{\lambda}-1$ or $d_{\lambda}$. For an arbitrary basis $\left\{\boldsymbol{v}_{i}\right\}$ of $W_{\lambda}, \mathbb{C} \boldsymbol{v}_{i}$ 's are isomorphic simple $\mathcal{T}_{1}$-modules by Proposition 4.2 and $W_{\lambda}=\bigoplus_{i} \mathbb{C} \boldsymbol{v}_{i}$.

Proposition 4.4. Set $D=\max \left\{\partial\left(x_{0}, y\right): y \in X\right\}$, the diameter with respect to $x_{0}$. Then $\operatorname{dim} E_{x_{0}} \mathcal{T}_{1}\left(\Gamma, x_{0}\right) \geq D+1$.

Proof. For $i=0, \ldots, D$, we consider $E_{x_{0}} A^{i}$. The matrix $E_{x_{0}} A^{i}$ contains $E_{x_{0}, y}$ with $\partial\left(x_{0}, y\right)=i$ and no $E_{x_{0}, y}$ with $\partial\left(x_{0}, y\right)>i$. Thus $\left\{E_{x_{0}} A^{i}: i=0,1, \ldots, D\right\}$ is a linearly independent set, and so $\operatorname{dim} E_{x_{0}} \mathcal{T}_{1} \geq D+1$.

The next result can be applied to $\mathcal{T}_{1}$ for distance-regular graphs.
Theorem 4.5. Let $\Gamma=(X, E)$ be a simple connected graph with diameter $D$ with respect to the base vertex $x_{0}$. Suppose that the graph has exactly $D+1$ eigenvalues $\lambda_{0}, \ldots, \lambda_{D}$ with multiplicities $m_{0}, \ldots, m_{D}$, respectively. Set $t:=\sharp\left\{i: m_{i}>1\right\}$. Then $\mathcal{T}_{1}\left(\Gamma, x_{0}\right) \cong M_{D+1}(\mathbb{C}) \oplus \bigoplus_{i=1}^{t} \mathbb{C}$. Especially, $\operatorname{dim} \mathcal{T}_{1}\left(\Gamma, x_{0}\right)=(D+1)^{2}+t$.

Proof. By Proposition 4.2 and Corollary 4.3, there are $t$ non-isomorphic simple $\mathcal{T}_{1^{-}}$ modules $S_{i}$ for $m_{i}>1$. Set $m_{i}^{\prime}$ the multiplicity of $S_{i}$ in $\mathbb{C} X$. Note that $m_{i}^{\prime}=m_{i}$ or $m_{i}-1$.

We consider $E_{x_{0}} \mathcal{T}_{1}$. By Proposition 4.4, $\operatorname{dim} E_{x_{0}} \mathcal{T}_{1} \geq D+1$. Since the rank of $E_{x_{0}}$ is one, the multiplicity of $E_{x_{0}} \mathcal{T}_{1}$ in $\mathbb{C} X$ is one. Now, considering that $\mathbb{C} X$ contains a submodule isomorphic to $E_{x_{0}} \mathcal{T}_{1} \oplus \bigoplus_{m_{i}>1} m_{i}^{\prime} S_{i}$, we have

$$
n=\sum_{i=0}^{D} m_{i}=(D+1)+\sum_{i=0}^{D}\left(m_{i}-1\right) \leq \operatorname{dim} E_{x_{0}} \mathcal{T}_{1}+\sum_{m_{i}>1} m_{i}^{\prime} \leq n
$$

This shows that $\operatorname{dim} E_{x_{0}} \mathcal{T}_{1}=D+1$ and $\left\{E_{x_{0}} \mathcal{T}_{1}\right\} \cup\left\{S_{i}: m_{i}>1\right\}$ is the set of all representatives of simple $\mathcal{T}_{1}$-modules.

## 5 Terwilliger algebras $\mathcal{T}_{2}$ of strongly regular graphs

The next proposition is essentially proved in [3, Theorem 5.1]. See also [13].

Proposition 5.1. Let $\Gamma=(X, E)$ be an $(n, k, \lambda, \mu)$-strongly regular graph. Fix a vertex $x_{0} \in X$. Let $X=X_{0} \cup X_{1} \cup X_{2}$ be the distance partition of $X$ with respect to $x_{0}$. Then $\mathbb{C}\left\langle E_{X_{1}} A E_{X_{1}}, E_{X_{1}} J_{X} E_{X_{1}}\right\rangle=E_{X_{1}} \mathcal{T}_{2}\left(\Gamma, x_{0}\right) E_{X_{1}}$. Especially, $E_{X_{1}} \mathcal{T}_{2}\left(\Gamma, x_{0}\right) E_{X_{1}}$ is a commutative algebra. (The same things hold also for $X_{2}$.)

Proof. It is clear that $\mathbb{C}\left\langle E_{X_{1}} A E_{X_{1}}, E_{X_{1}} J_{X} E_{X_{1}}\right\rangle \subset E_{X_{1}} \mathcal{T}_{2}\left(\Gamma, x_{0}\right) E_{X_{1}}$.
For convenience, we write $A_{(i, j)}$ for $E_{X_{i}} A E_{X_{j}}, J_{(i, j)}$ for $E_{X_{i}} J_{X} E_{X_{j}}$, and $I_{(i, j)}$ for $E_{X_{i}} I_{X} E_{X_{j}}$. Remark that $A_{(1,1)} J_{(1,1)}=J_{(1,1)} A_{(1,1)}=\lambda J_{(1,1)}$ and $A_{(2,2)} J_{(2,2)}=J_{(2,2)} A_{(2,2)}=$ $(k-\mu) J_{(2,2)}$, and so on.

Since $E_{X_{1}} \mathcal{T}_{2} E_{X_{1}}$ is generated by $E_{X_{1}} A E_{X_{s_{1}}} A E_{X_{s_{2}}} \cdots E_{X_{s_{t}}} A E_{X_{1}}$ for $0 \leq s_{u} \leq 2$, it is enough to show that such terms are in $\mathbb{C}\left\langle A_{(1,1)}, J_{(1,1)}\right\rangle$. We may assume that $s_{u} \neq 1$ for all $u$ because we can divide the term into two parts if $s_{u}=1$ for some $u$. Since $A_{(0,0)}=A_{(0,2)}=A_{(2,0)}=O$, this term is $E_{X_{1}} A E_{X_{0}} A E_{X_{1}}=J_{(1,1)}$ or $O$ if $s_{u}=0$ for some $u$. Moreover, if $s_{u}=2$ for some $u$, then this term is $A_{(1,2)} A_{(2,2)}^{m} A_{(2,1)}(m=0,1, \ldots)$ or $O$. Thus, we can see that $E_{X_{1}} \mathcal{T}_{2}\left(\Gamma, x_{0}\right) E_{X_{1}}$ is generated by $A_{(1,1)}, A_{(1,2)} A_{(2,2)}{ }^{m} A_{(2,1)}$ $(m=0,1, \ldots)$ and $J_{(1,1)}$. It is clear that $J_{(1,1)}, A_{(1,1)} \in \mathbb{C}\left\langle A_{(1,1)}, J_{(1,1)}\right\rangle$. We will show that $A_{(1,2)} A_{(2,2)}^{m} A_{(2,1)} \in \mathbb{C}\left\langle A_{(1,1)}, J_{(1,1)}\right\rangle$.

By a known equation $A^{2}=k I+\lambda A+\mu(J-A-I)$ for an $(n, k, \lambda, \mu)$-strongly regular graph, we have

$$
\begin{align*}
A_{(1,1)} A_{(1,2)}+A_{(1,2)} A_{(2,2)} & =(\lambda-\mu) A_{(1,2)}+\mu J_{(1,2)}  \tag{5.1}\\
J_{(1,1)}+A_{(1,1)} A_{(1,1)}+A_{(1,2)} A_{(2,1)} & =(k-\mu) I_{(1,1)}+(\lambda-\mu) A_{(1,1)}+\mu J_{(1,1)} . \tag{5.2}
\end{align*}
$$

Since $\mu J_{(1,2)}=J_{(1,1)} A_{(1,2)}$, the equation (5.1) shows that $A_{(1,2)} A_{(2,2)}=g\left(A_{(1,1)}, J_{(1,1)}\right) A_{(1,2)}$ for some polynomial $g(x, y)$. Thus $A_{(1,2)} A_{(2,2)}^{m}=g\left(A_{(1,1)}, J_{(1,1)}\right)^{m} A_{(1,2)}$. The equation (5.2) shows that $A_{(1,2)} A_{(2,1)}=h\left(A_{(1,1)}, J_{(1,1)}\right)$ for some polynomial $h(x, y)$. Now $A_{(1,2)} A_{(2,2)}{ }^{m} A_{(2,1)}=g\left(A_{(1,1)}, J_{(1,1)}\right)^{m} h\left(A_{(1,1)}, J_{(1,1)}\right) \in \mathbb{C}\left\langle A_{(1,1)}, J_{(1,1)}\right\rangle$. The proof is completed.

Proposition 5.2. Let $\Gamma=(X, E)$ be an $(n, k, \lambda, \mu)$-strongly regular graph. Fix a vertex $x_{0} \in X$. Then $\operatorname{dim} \mathcal{T}_{2}\left(\Gamma, x_{0}\right) \leq 2 n+3$.

Proof. Set $d_{i}=\operatorname{dim} E_{X_{i}} \mathcal{T}_{2} E_{X_{i}}$ for $i=1,2$. Since $E_{X_{i}} \mathcal{T}_{2} E_{X_{i}}$ is a commutative semisimple algebra, $d_{1} \leq k, d_{2} \leq n-k-1$, and $E_{X_{i}}$ is a sum of $d_{i}$ non-equivalent primitive idempotents. Thus $E_{X_{i}} \mathcal{T}_{2}$ is a direct sum of $d_{i}$ non-isomorphic simple $\mathcal{T}_{2}$-modules. Now $E_{X_{1}} \mathcal{T}_{2} E_{X_{2}} \cong \operatorname{Hom}_{\mathcal{T}_{2}}\left(E_{X_{1}} \mathcal{T}_{2}, E_{X_{2}} \mathcal{T}_{2}\right)$ and thus $\operatorname{dim} E_{X_{1}} \mathcal{T}_{2} E_{X_{2}} \leq \min \left\{d_{1}, d_{2}\right\}$. Similarly we have $\operatorname{dim} E_{X_{2}} \mathcal{T}_{2} E_{X_{1}} \leq \min \left\{d_{1}, d_{2}\right\}$. We also remark that $\operatorname{dim} E_{X_{0}} \mathcal{T}_{2} E_{X_{i}}=$ $\operatorname{dim} E_{X_{i}} \mathcal{T}_{2} E_{X_{0}}=1$ for $i=0,1,2$. By $\operatorname{dim} \mathcal{T}_{2}=\sum_{0 \leq i, j \leq 2} \operatorname{dim} E_{X_{1}} \mathcal{T}_{2} E_{X_{2}}$, we have

$$
\operatorname{dim} \mathcal{T}_{2} \leq 5+d_{1}+d_{2}+2 \min \left\{d_{1}, d_{2}\right\} \leq 5+(n-1)+(n-1)=2 n+3
$$

and the assertion holds.
Our algebra $\mathcal{T}_{2}\left(\Gamma, x_{0}\right)$ is just a Terwilliger algebra. A simple $\mathcal{T}_{2}\left(\Gamma, x_{0}\right)$-module $W$ is said to be thin if $\operatorname{dim} W E_{X_{i}} \leq 1$ for all $i=0,1, \ldots, D$ [10, Section 3]. A (distanceregular) graph is said to be thin with respect to $x_{0}$ if every irreducible $\mathcal{T}_{2}\left(\Gamma, x_{0}\right)$-module
is thin. Now $\operatorname{dim} W E_{X_{i}}=\operatorname{dim} \operatorname{Hom}\left(E_{X_{i}} \mathcal{T}_{2}\left(\Gamma, x_{0}\right), W\right)$ and the condition $\operatorname{dim} W E_{X_{i}} \leq 1$ means that the modules $E_{X_{i}} \mathcal{T}_{2}\left(\Gamma, x_{0}\right)$ contain at most one simple submodule isomorphic to $W$. This condition is satisfied for all simple modules $W$ if $E_{X_{i}} \mathcal{T}_{2}\left(\Gamma, x_{0}\right) E_{X_{i}}(i=1,2)$ are commutative. As a consequent of Proposition 5.1, we can say that every strongly regular graph is thin with respect to any vertex. This fact was proved in [13, Lemma 3.3].

Example 5.3. It is known that there are many strongly regular graphs with the trivial automorphism groups. By Lemma 3.5, $\operatorname{dim} \mathcal{T}_{3}=\operatorname{dim} \mathcal{T}_{4}=n^{2}$ for them. By Proposition 5.2, $\operatorname{dim} \mathcal{T}_{2} \leq 2 n+3$. Thus $\mathcal{T}_{2} \subsetneq \mathcal{T}_{3}$ holds for them if $n \geq 4$.

## 6 A base vertex of valency one

In this section, we consider the structures of $\mathcal{T}_{2}$ and $\mathcal{T}_{3}$ in the case that the valency of the base vertex $x_{0}$ is one. There exists the unique neighbor $x_{1}$ of $x_{0}$. We set $X^{\prime}=X \backslash\left\{x_{0}\right\}$, $E^{\prime}=E \backslash\left\{\left\{x_{0}, x_{1}\right\}\right\}$, and consider the graph $\Gamma^{\prime}=\left(X^{\prime}, E^{\prime}\right)$ with the base vertex $x_{1}$. Naturally, we can regard $M_{X^{\prime}}(\mathbb{C})$ as a subset of $M_{X}(\mathbb{C})$.

Lemma 6.1. $E_{X^{\prime}} \mathcal{T}_{\ell}\left(\Gamma, x_{0}\right) E_{X^{\prime}}=\mathcal{T}_{\ell}\left(\Gamma^{\prime}, x_{1}\right)$ for $\ell=2,3$.
Proof. We consider the case $\ell=2$. Let $X=X_{0} \cup X_{1} \cup \cdots \cup X_{D}$ be the distance partition with respect to $x_{0}$ in $\Gamma$. Remark that $X_{0}=\left\{x_{0}\right\}$ and $X_{1}=\left\{x_{1}\right\}$. Then $X^{\prime}=X_{1} \cup \cdots \cup X_{D}$ is a distance partition with respect to $x_{1}$ in $\Gamma^{\prime}$. Thus $\mathcal{T}_{2}\left(\Gamma^{\prime}, x_{1}\right)$ is generated by $E_{X^{\prime}} A E_{X^{\prime}}$ and $E_{X_{1}}, \ldots, E_{X_{D}}$. We have $E_{X^{\prime}} \mathcal{T}_{2}\left(\Gamma, x_{0}\right) E_{X^{\prime}} \supset \mathcal{T}_{2}\left(\Gamma^{\prime}, x_{1}\right)$.

To show $E_{X^{\prime}} \mathcal{T}_{2}\left(\Gamma, x_{0}\right) E_{X^{\prime}} \subset \mathcal{T}_{2}\left(\Gamma^{\prime}, x_{1}\right)$, it is enough to show that

$$
E_{X^{\prime}} A E_{X_{s_{1}}} A E_{X_{s_{2}}} \ldots E_{X_{s_{t}}} A E_{X^{\prime}} \in \mathcal{T}_{2}\left(\Gamma^{\prime}, x_{1}\right)
$$

for $0 \leq s_{u} \leq D$. If $s_{u} \neq 0$ for some $u$, then we can divide the term into two parts. We may assume that $s_{u}=0$ for all $1 \leq u \leq t$. However, $E_{X_{0}} A E_{X_{0}}=O$. Therefore, it is enough to consider $E_{X^{\prime}} A E_{X^{\prime}}$ and $E_{X^{\prime}} A E_{x_{0}} A E_{X^{\prime}}$. Now $E_{X^{\prime}} A E_{X^{\prime}} \in \mathcal{T}_{2}\left(\Gamma^{\prime}, x_{1}\right)$ and $E_{X^{\prime}} A E_{x_{0}} A E_{X^{\prime}}=E_{x_{1}} \in \mathcal{T}_{2}\left(\Gamma^{\prime}, x_{1}\right)$. We have $E_{X^{\prime}} \mathcal{T}_{2}\left(\Gamma, x_{0}\right) E_{X^{\prime}} \subset \mathcal{T}_{2}\left(\Gamma^{\prime}, x_{1}\right)$.

We consider the case $\ell=3$. Set $G=\operatorname{Aut}(\Gamma)$. Since $G_{x_{0}}$ fixes $x_{1}$, we can see that the stabilizer of $x_{1}$ in $\operatorname{Aut}\left(\Gamma^{\prime}\right)$ coincides with $G_{x_{0}}$. Suppose that $X=Y_{0} \cup Y_{1} \cup \cdots \cup Y_{r}$ is a partition of $X$ into $G_{x_{0}}$-orbits where $Y_{0}=\left\{x_{0}\right\}$ and $Y_{1}=\left\{x_{1}\right\}$. Then $X^{\prime}=Y_{1} \cup \cdots \cup Y_{r}$ is a partition of $X^{\prime}$ into $G_{x_{0} \text {-orbits. Thus the same arguments as above can be applied }}$ and the statement for $\mathcal{T}_{3}$ holds.

Proposition 6.2. Suppose, for $\ell=2,3$, that $\mathcal{T}_{\ell}\left(\Gamma^{\prime}, x_{1}\right) \cong M_{n_{0}}(\mathbb{C}) \oplus \bigoplus_{i=1}^{r} M_{n_{i}}(\mathbb{C})$, where the primitive idempotent $E_{x_{1}}$ belongs to $M_{n_{0}}(\mathbb{C})$. Then $\mathcal{T}_{\ell}\left(\Gamma, x_{0}\right) \cong M_{n_{0}+1}(\mathbb{C}) \oplus$ $\bigoplus_{i=1}^{r} M_{n_{i}}(\mathbb{C})$ and the primitive idempotent $E_{x_{0}}$ belongs to $M_{n_{0}+1}(\mathbb{C})$.

Proof. Suppose $\ell$ is 2 or 3. Let $I_{X^{\prime}}=e_{1}+\cdots+e_{s}$ be a primitive idempotent decomposition of $I_{X^{\prime}}$ in $\mathcal{T}_{\ell}\left(\Gamma^{\prime}, x_{1}\right)$. By Lemma 2.1 and Lemma 6.1, $I_{X}=E_{x_{0}}+e_{1}+\cdots+e_{s}$ is a primitive idempotent decomposition of $I_{X}$ in $\mathcal{T}_{\ell}\left(\Gamma, x_{0}\right)$. Since $E_{x_{0}} A E_{x_{1}} \neq 0$, we can say that $E_{x_{0}}$ and $E_{x_{1}}$ are equivalent. The assertion holds.

## 7 Some specific graphs

In this section, we investigate the structures of $\mathcal{T}_{1}, \mathcal{T}_{2}, \mathcal{T}_{3}$, and $\mathcal{T}_{4}$ for path graphs, star graphs, and cycle graphs.

### 7.1 Path graphs

Let $P_{n}$ be the path graph with $n$ vertices for $n \geq 2$ [2, Section 1.4.4]. We set the vertex set $\{1,2, \ldots, n\}$ and the edge set $E=\{\{i, j\}:|i-j|=1\}$. Let $m$ be the base vertex. By symmetry, we may assume that $2 m-1 \leq n$.

Let $\zeta$ be a primitive $(2 n+2)$ th root of unity in $\mathbb{C}$. Then the eigenvalues of the adjacency matrix $A$ of the path graph $P_{n}$ are

$$
\lambda_{i}:=\zeta^{i}+\zeta^{-i} \quad(i=1, \ldots, n)
$$

and the corresponding eigenvectors are

$$
\boldsymbol{v}_{i}:=\left(\zeta^{i}-\zeta^{-i}, \zeta^{2 i}-\zeta^{-2 i}, \ldots, \zeta^{n i}-\zeta^{-n i}\right)
$$

Moreover all eigenvalues are of multiplicity 1 . Thus, we know $\mathcal{T}_{0}\left(P_{n}\right) \cong \bigoplus_{i=1}^{n} \mathbb{C}$ from Theorem 3.1.

First, we determine the structure of $\mathcal{T}_{1}$.
Theorem 7.1. $\mathcal{T}_{1}\left(P_{n}, m\right) \cong M_{n-t}(\mathbb{C}) \oplus \bigoplus_{i=1}^{t} \mathbb{C}$, where $t:=\sharp\{s: 1 \leq s \leq n,(n+$ 1) $\mid s m\}=\left\lfloor\frac{n \operatorname{gcd}(m, n+1)}{n+1}\right\rfloor$. Especially, $\operatorname{dim} \mathcal{T}_{1}\left(P_{n}, m\right)=(n-t)^{2}+t$.

Proof. Since $\mathbb{C} X=\bigoplus_{i=1}^{n} \mathbb{C} \boldsymbol{v}_{i}$, we can define projections $\pi_{i}: \mathbb{C} X \rightarrow \mathbb{C} X$ such that $\pi_{i}\left(\boldsymbol{v}_{j}\right)=\delta_{i, j} \boldsymbol{v}_{i}$. Let $\varepsilon_{i}$ be the matrix corresponding to $\pi_{i}$, namely $\pi_{i}(\boldsymbol{w})=\boldsymbol{w} \varepsilon_{i}$ for all $\boldsymbol{w} \in \mathbb{C} X$. Then we know $\varepsilon_{i} \in \mathbb{C}\langle A\rangle=\mathcal{T}_{0}$ by theory of linear algebra. Since $\lambda_{i}$ is a simple root, $\varepsilon_{i}$ has rank one and is a primitive idempotent in $\mathcal{T}_{1}$.

We will apply Proposition 4.2. The $m$-th entry of $\boldsymbol{v}_{i}$ is $\zeta^{m i}-\zeta^{-m i}$, and this is zero if and only if $2 n+2 \mid 2 m i$. Thus we have $t$ non-isomorphic 1 -dimensional simple $\mathcal{T}_{1}$-modules.

We consider the case of $\zeta^{m i}-\zeta^{-m i} \neq 0$. We remark that $\mathbb{C} \boldsymbol{v}_{i}=\mathbb{C} X \varepsilon_{i}$. Then $\boldsymbol{v}_{i} E_{x_{0}} \neq \mathbf{0}$ and hence $\varepsilon_{i} E_{x_{0}} \neq O$. This means that $\varepsilon_{i}$ and $E_{x_{0}}$ are equivalent primitive idempotents. Thus the primitive idempotent decomposition $I_{X}=\sum_{i=1}^{n} \varepsilon_{i}$ has $(n-t)$ idempotents equivalent to $E_{x_{0}}$. This means that the dimension of the simple $\mathcal{T}_{1}$-module $E_{x_{0}} \mathcal{T}_{1}$ is $n-t$. The assertion holds.

Next, we consider the structures of $\mathcal{T}_{\ell}\left(P_{n}, m\right)(\ell=2,3,4)$ of the path graph $P_{n}$. Let $X=X_{0} \cup \cdots \cup X_{D}$ is the distance partition of $P_{n}$ with respect to the base vertex $m$.

Lemma 7.2. The following statements hold.
(1) $E_{m, m} \in \mathcal{T}_{1}\left(P_{n}, m\right)$.
(2) $E_{m, m-k}+E_{m, m+k} \in \mathcal{T}_{1}\left(P_{n}, m\right)$ for $k=1, \ldots, m-1$.
(3) $E_{m-k, m}+E_{m+k, m} \in \mathcal{T}_{1}\left(P_{n}, m\right)$ for $k=1, \ldots, m-1$.
(4) $E_{m-k, m-k^{\prime}}+E_{m-k, m+k^{\prime}}+E_{m+k, m-k^{\prime}}+E_{m+k, m+k^{\prime}} \in \mathcal{T}_{1}\left(P_{n}, m\right)$ for $k, k^{\prime}=1, \ldots, m-1$.
(5) $E_{m-k, m-k^{\prime}}+E_{m+k, m+k^{\prime}} \in \mathcal{T}_{2}\left(P_{n}, m\right)$ for $k, k^{\prime}=1, \ldots, m-1$.

Moreover, elements in the above statements are linearly independent.
Proof. The statement (1) is clear by definition.
We prove (2) by the induction on $k$. First, we have $E_{m, m-1}+E_{m, m+1}=E_{X_{0}} A \in$ $\mathcal{T}_{1}$. Suppose (2) holds for all $k^{\prime}<k$. We have $\mathcal{T}_{1} \ni\left(E_{m, m-k+1}+E_{m, m+k-1}\right) A=$ $\left(E_{m, m-k+2}+E_{m, m+k-2}\right)+\left(E_{m, m-k}+E_{m, m+k}\right)$ and $E_{m, m-k+2}+E_{m, m+k-2} \in \mathcal{T}_{1}$ by the inductive hypothesis. Thus $E_{m, m-k}+E_{m, m+k} \in \mathcal{T}_{1}$ holds

Similarly, (3) holds.
By (2) and (3), we have $E_{m-k, m-k^{\prime}}+E_{m-k, m+k^{\prime}}+E_{m+k, m-k^{\prime}}+E_{m+k, m+k^{\prime}}=\left(E_{m+k, m}+\right.$ $\left.E_{m-k, m}\right)\left(E_{m, m-k^{\prime}}+E_{m, m+k^{\prime}}\right) \in \mathcal{T}_{1}$ and (4) holds.

We have $\partial\left(m-k, m-k^{\prime}\right)=\partial\left(m+k, m+k^{\prime}\right)=\left|k-k^{\prime}\right|$ and $\partial\left(m-k, m+k^{\prime}\right)=$ $\partial\left(m+k, m-k^{\prime}\right)=\left|k+k^{\prime}\right|>\left|k-k^{\prime}\right|$. Thus $\mathcal{T}_{2} \ni E_{X_{k}} A^{\left|k-k^{\prime}\right|} E_{X_{k^{\prime}}}=E_{m-k, m-k^{\prime}}+E_{m+k, m+k^{\prime}}$. (5) holds.

It is easy to see these elements are linearly independent.
Theorem 7.3. Suppose $n>2 m-1$. Then $\mathcal{T}_{2}\left(P_{n}, m\right)=\mathcal{T}_{3}\left(P_{n}, m\right)=\mathcal{T}_{4}\left(P_{n}, m\right)=$ $M_{X}(\mathbb{C})$.


Proof. It is enough to show that $\mathcal{T}_{2}=M_{X}(\mathbb{C})$. Recall that $X=X_{0} \cup \cdots \cup X_{n-m}$ is the distance partition with respect to the vertex $m$. We have $\mathcal{T}_{2} \ni E_{X_{0}}=E_{m, m}$, $\mathcal{T}_{2} \ni E_{X_{k}}=E_{m-k, m-k}+E_{m+k, m+k}$ for $1 \leq k \leq m-1$, and $\mathcal{T}_{2} \ni E_{X_{k}}=E_{m+k, m+k}$ for $m \leq k \leq n-m$. For $1 \leq k \leq m-1$, we have

$$
E_{m+k, m+k}=E_{m+k, 2 m} E_{2 m, m+k}=\left(E_{X_{k}} A^{m-k} E_{2 m, 2 m}\right)\left(E_{2 m, 2 m} A^{m-k} E_{X_{k}}\right) \in \mathcal{T}_{2}
$$

Consequently, $E_{k, k} \in \mathcal{T}_{2}$ for all $1 \leq k \leq n$.
Now, for every $1 \leq i, j \leq n$, we have $E_{i, i} A^{|i-j|} E_{j, j} \in \mathcal{T}_{2}$ is a nonzero multiple of $E_{i, j}$ and thus $E_{i, j} \in \mathcal{T}_{2}$. This leads to $\mathcal{T}_{2}=M_{X}(\mathbb{C})$.

Since the path graph $P_{n}$ is a tree and the stabilizer $G_{m}=1$ in the case of $n>2 m-1$, Theorem 7.3 holds also by [6, Theorem 4].

Theorem 7.4. Suppose $n=2 m-1$. Then $\mathcal{T}_{2}\left(P_{n}, m\right)=\mathcal{T}_{3}\left(P_{n}, m\right)=\mathcal{T}_{4}\left(P_{n}, m\right) \cong$ $M_{m}(\mathbb{C}) \oplus M_{m-1}(\mathbb{C}) . \quad$ Especially, $\operatorname{dim} \mathcal{T}_{2}\left(P_{n}, m\right)=\operatorname{dim} \mathcal{T}_{3}\left(P_{n}, m\right)=\operatorname{dim} \mathcal{T}_{4}\left(P_{n}, m\right)=$ $2 m^{2}-2 m+1=\left(n^{2}+1\right) / 4$.


Proof. The set of matrices in Lemma 7.2 are in $\mathcal{T}_{2}$ and linearly independent. Thus $\operatorname{dim} \mathcal{T}_{2} \geq 2 m^{2}-2 m+1$.

Set $g=(1, n)(2, n-1) \ldots(m-1, m+1)$. The stabilizer $G_{m}$ of $m$ in $G=\operatorname{Aut}\left(P_{n+1}\right)$ is $G_{m}=G=\langle g\rangle$ of order 2. For the permutation character $\rho$, we have $\rho(1)=2 m-1$ and $\rho(g)=1$. Thus $\rho=m 1_{G_{m}}+(m-1) \chi_{G_{m}}$, where $1_{G_{m}}$ is the trivial character and $\chi_{G_{m}}$ is the unique non-trivial irreducible character of $G_{m}$. Thus $\mathcal{T}_{4} \cong M_{m}(\mathbb{C}) \oplus M_{m-1}(\mathbb{C})$. We have $\operatorname{dim} \mathcal{T}_{2} \leq \operatorname{dim} \mathcal{T}_{3} \leq \operatorname{dim} \mathcal{T}_{4}=2 m^{2}-2 m+1$.

### 7.2 Star graphs

The star graph with $n$ vertices is a complete bipartite graph $K_{1, n-1}$ [1, p. 49], [2, Section 1.4.2]. We set the vertex set $\{1,2, \ldots, n\}$ and the edge set $\{\{1, i\}: i=2,3, \ldots, n\}$. By symmetry, it is enough to consider the case that the base vertex is 1 or 2 . Since the star graph is equal to the path graph $P_{n}$ if $n \leq 3$, we may assume that $n \geq 4$. The eigenvalues of $K_{1, n-1}$ are $-\sqrt{n-1}, 0, \sqrt{n-1}$ with multiplicities $1, n-2,1$, respectively. Therefore we have $\mathcal{T}_{0}\left(K_{1, n-1}\right) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ from Theorem 3.1.

Theorem 7.5. For $n \geq 4, \mathcal{T}_{\ell}\left(K_{1, n-1}, 1\right) \cong M_{2}(\mathbb{C}) \oplus \mathbb{C}$ for $\ell=1,2,3,4$.
Proof. It is clear that $E_{1,1} \in \mathcal{T}_{1}, \sum_{i=2}^{n} E_{1, i}=E_{1,1} A \in \mathcal{T}_{1}, \sum_{i=2}^{n} E_{i, 1}=A E_{1,1} \in \mathcal{T}_{1}$ and $\sum_{i=2}^{n} \sum_{j=2}^{n} E_{i, j}=\left(A E_{1,1}\right)\left(E_{1,1} A\right) \in \mathcal{T}_{1}$. Also we have $\sum_{i=2}^{n} E_{i, i}=I_{X}-E_{1,1} \in \mathcal{T}_{1}$. Thus, we know that $\operatorname{dim} \mathcal{T}_{1} \geq 5$ since these matrices are linearly independent.

The stabilizer $G_{1}$ of 1 in $G=\operatorname{Aut}\left(K_{1, n-1}\right)$ is the symmetric group on $\{2,3, \ldots, n\}$. Therefore the orbits of $G_{1}$ on $X \times X$ are $\{(1,1)\},\{(1, i): i=2, \ldots, n\},\{(i, 1): i=$ $2, \ldots, n\},\{(i, i): i=2, \ldots, n\}$, and $\{(i, j): i, j=2, \ldots, n, i \neq j\}$. Thus $\operatorname{dim} \mathcal{T}_{4}=5$. This means that $\mathcal{T}_{1}=\mathcal{T}_{2}=\mathcal{T}_{3}=\mathcal{T}_{4}$. Since $\mathcal{T}_{\ell}(\ell=1,2,3,4)$ is a non-commutative 5-dimensional semisimple $\mathbb{C}$-algebra, it follows that $\mathcal{T}_{\ell} \cong M_{2}(\mathbb{C}) \oplus \mathbb{C}$.

Theorem 7.6. For $n \geq 4, \mathcal{T}_{\ell}\left(K_{1, n-1}, 2\right) \cong M_{3}(\mathbb{C}) \oplus \mathbb{C}$ for $\ell=1,2,3,4$.
Proof. In this case, we can apply Theorem 4.5. So we obtain that $\mathcal{T}_{1} \cong M_{3}(\mathbb{C}) \oplus \mathbb{C}$ and $\operatorname{dim} \mathcal{T}_{1}=10$.

The stabilizer $G_{2}$ of 2 in $G=\operatorname{Aut}\left(K_{1, n-1}\right)$ is the symmetric group on $\{3,4, \ldots, n\}$, and the orbits of $G_{2}$ on $X \times X$ are $\{(1,1)\},\{(1,2)\},\{(2,1)\},\{(2,2)\},\{(1, i): i=$ $3, \ldots, n\},\{(i, 1): i=3, \ldots, n\},\{(2, i): i=3, \ldots, n\},\{(i, 2): i=3, \ldots, n\},\{(i, i): i=$ $3, \ldots, n\}$, and $\{(i, j): i, j=3, \ldots, n, i \neq j\}$. Therefore we know that $\operatorname{dim} \mathcal{T}_{4}=10$ and $\mathcal{T}_{1}=\mathcal{T}_{2}=\mathcal{T}_{3}=\mathcal{T}_{4}$.

### 7.3 Cycle graphs

Let $C_{n}$ be the cycle graph with $n$ vertices $(n \geq 3)$ [2, Section 1.4.3]. The graph $C_{n}$ defines a $P$ - and $Q$-polynomial association scheme and the structure of the Terwilliger algebra $\mathcal{T}_{2}$ is determined in [12]. We set the vertex set $X=\{1,2, \ldots, n\}$ and the edge set $\{\{1,2\},\{2,3\}, \ldots,\{n-1, n\},\{n, 1\}\}$. By symmetry, it is enough to consider only the case that the base vertex is 1 . Let $\xi$ be a primitive $n$-th root of unity in $\mathbb{C}$. The diameter of $C_{n}$ is $D=\left\lfloor\frac{n}{2}\right\rfloor$. The eigenvalues of the adjacency matrix $A$ are $\lambda_{i}:=\xi^{i}+\xi^{-i} \quad(i=0,1, \ldots, D)$. The multiplicity of $\lambda_{0}$ is 1 and the multiplicity of $\lambda_{i}$ is 2 for $i=1, \ldots, D-1$. The multiplicity of $\lambda_{D}$ is 1 if $n$ is even and 2 if $n$ is odd.

We remark that $C_{n}$ is a distance-regular graph and so can be applied Theorem 4.5.
Theorem 7.7. $\mathcal{T}_{0}\left(\Gamma, x_{0}\right) \cong \bigoplus_{i=1}^{D+1} \mathbb{C}$ and

$$
\mathcal{T}_{1}\left(\Gamma, x_{0}\right) \cong \begin{cases}M_{D+1}(\mathbb{C}) \oplus \bigoplus_{i=1}^{D-1} \mathbb{C} & \text { if } n \text { is even } \\ M_{D+1}(\mathbb{C}) \oplus \bigoplus_{i=1}^{D} \mathbb{C} & \text { if } n \text { is odd }\end{cases}
$$

where $D=\left\lfloor\frac{n}{2}\right\rfloor$ is the diameter of $C_{n}$.
Proof. It is clear from Theorem 3.1 and Theorem 4.5.
Next, we consider the structures of $\mathcal{T}_{2}, \mathcal{T}_{3}$ and $\mathcal{T}_{4}$.
Theorem 7.8. We have

$$
\mathcal{T}_{2}\left(\Gamma, x_{0}\right)=\mathcal{T}_{3}\left(\Gamma, x_{0}\right)=\mathcal{T}_{4}\left(\Gamma, x_{0}\right) \cong \begin{cases}M_{D+1}(\mathbb{C}) \oplus M_{D-1}(\mathbb{C}) & \text { if } n \text { is even } \\ M_{D+1}(\mathbb{C}) \oplus M_{D}(\mathbb{C}) & \text { if } n \text { is odd }\end{cases}
$$

where $D=\left\lfloor\frac{n}{2}\right\rfloor$ is the diameter of $C_{n}$.
Proof. By [12, Example 6.1 (23), (24)], we have that $\mathcal{T}_{2} \cong M_{D+1}(\mathbb{C}) \oplus M_{D-1}(\mathbb{C})$ if $n$ is even and $\mathcal{T}_{2} \cong M_{D+1}(\mathbb{C}) \oplus M_{D}(\mathbb{C})$ if $n$ is odd.

We determine the structure of $\mathcal{T}_{4}$. First, we consider the case of $n=2 D$, namely $n$ is even. Let $g=(2,2 D)(3,2 D-1) \cdots(D, D+2)$. Then the stabilizer $G_{1}$ of 1 in the group $G=\operatorname{Aut}\left(C_{n}\right)$ is $\langle g\rangle$. For the permutation character $\rho$, we know that $\rho(1)=2 D$ and $\rho(g)=2$. So $\rho=(D+1) 1_{G_{1}}+(D-1) \chi_{G_{1}}$, where $1_{G_{1}}$ is the trivial character and $\chi_{G_{1}}$ is the non-trivial irreducible character of $G_{1}$. Therefore, $\mathcal{T}_{4} \cong M_{D+1}(\mathbb{C}) \oplus M_{D-1}(\mathbb{C})$.

Next, we consider the case of $n=2 D+1$. Let $h=(2,2 D+1)(3,2 D) \cdots(D+1, D+2)$. Then the stabilizer $H_{1}$ of 1 in $H=\operatorname{Aut}\left(C_{n}\right)$ is $\langle h\rangle$. Let $\rho$ be the permutation character. Since it follows that $\rho(1)=2 D+1$ and $\rho(h)=1$, we have $\rho=(D+1) 1_{H_{1}}+D \varphi_{H_{1}}$, where $1_{H_{1}}$ is the trivial character and $\varphi_{H_{1}}$ is the non-trivial irreducible character of $H_{1}$. Thus $\mathcal{T}_{4} \cong M_{D+1}(\mathbb{C}) \oplus M_{D}(\mathbb{C})$.

## 8 Paley graphs: Examples for $\mathcal{T}_{3} \neq \mathcal{T}_{4}$

In this section, we consider Paley graphs [1, page 129], 5]. We show in Corollary 8.6 that Paley graphs with $p$ vertices ( $p$ is a prime such that $p \geq 7$ ) give examples for $\mathcal{T}_{3} \neq \mathcal{T}_{4}$.

Let $\operatorname{Paley}\left(p^{a}\right)=(X, E)$ be the Paley graph with $|X|=p^{a} \equiv 1(\bmod 4)$, where $p$ is a prime number. It is a strongly regular graph, and thus we can apply Theorem 4.5 for $\mathcal{T}_{1}$.

We describe the automorphism group $G=\operatorname{Aut}\left(\operatorname{Paley}\left(p^{a}\right)\right)$. The vertex set $X$ is identified with the finite field $\operatorname{GF}\left(p^{a}\right)$. We fix a primitive element $\xi$ of $\operatorname{GF}\left(p^{a}\right)$. The group $G$ is generated by $\mu_{\alpha}: x \mapsto x+\alpha\left(\alpha \in \operatorname{GF}\left(p^{a}\right)\right), \sigma: x \mapsto x \xi^{2}$, and $\tau: x \mapsto x^{p}$. The orders of $\sigma$ and $\tau$ are $\left(p^{a}-1\right) / 2$ and $a$, respectively. The stabilizer of 0 in $G$ is $G_{0}=$ $\langle\sigma, \tau\rangle$. The Paley graph is distance-transitive and so the structure of $\mathcal{T}_{\ell}\left(\operatorname{Paley}\left(p^{a}\right), x_{0}\right)$ $(\ell=0,1,2,3,4)$ does not depend on a choice of the base vertex $x_{0}$. Thus we fix the base vertex 0 and write $\mathcal{T}_{\ell}\left(\operatorname{Paley}\left(p^{a}\right)\right)$ for $\mathcal{T}_{\ell}\left(\operatorname{Paley}\left(p^{a}\right), 0\right)$. We keep these notations in this section.

Theorem 8.1. $\mathcal{T}_{1}\left(\operatorname{Paley}\left(p^{a}\right)\right) \cong M_{3}(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$ and $\operatorname{dim} \mathcal{T}_{1}\left(\operatorname{Paley}\left(p^{a}\right)\right)=11$.
Proof. The graph Paley $\left(p^{a}\right)$ has eigenvalues $\left(p^{a}-1\right) / 2,\left(-1+\sqrt{p^{a}}\right) / 2,\left(-1-\sqrt{p^{a}}\right) / 2$ with multiplicities $1,\left(p^{a}-1\right) / 2,\left(p^{a}-1\right) / 2$, respectively. By Theorem 4.5, we have the result.

Theorem 8.2. $\operatorname{dim} \mathcal{T}_{4}\left(\operatorname{Paley}\left(p^{a}\right)\right) \leq 2 p^{a}+3$. For the case $|X|=p$, we have the equality $\operatorname{dim} \mathcal{T}_{4}(\operatorname{Paley}(p))=2 p+3$ and $\mathcal{T}_{4}(\operatorname{Paley}(p)) \cong M_{3}(\mathbb{C}) \oplus \bigoplus_{i=1}^{(p-3) / 2} M_{2}(\mathbb{C})$.

Proof. Set $k=\left(p^{a}-1\right) / 2$. We consider the centralizer algebra $\mathcal{U}$ of $H=\langle\sigma\rangle \subset G_{0}$. Then $\mathcal{T}_{4}$ is a subalgebra of $\mathcal{U}$. We fix an ordering of $X=\operatorname{GF}\left(p^{a}\right)$ by

$$
\left\{0, \quad 1, \xi^{2}, \xi^{4}, \ldots, \xi^{p^{a}-3}, \quad \xi, \xi^{3}, \ldots, \xi^{p^{a}-2}\right\} .
$$

Then the permutation matrix given by $\sigma$ with respect to this ordering is


Thus the elements in $\mathcal{U}$ are of the form

$$
\left(\begin{array}{c|cccc|cccc}
a & b & \ldots & \ldots & b & c & \ldots & \ldots & c  \tag{8.1}\\
\hline b^{\prime} & d_{0} & d_{1} & \ldots & d_{k-1} & e_{0} & e_{1} & \ldots & e_{k-1} \\
\vdots & d_{k-1} & \ddots & \ddots & \vdots & e_{k-1} & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & d_{1} & \vdots & \ddots & \ddots & e_{1} \\
b^{\prime} & d_{1} & \ldots & d_{k-1} & d_{0} & e_{1} & \ldots & e_{k-1} & e_{0} \\
\hline c^{\prime} & e_{0}^{\prime} & e_{1}^{\prime} & \ldots & e_{k-1}^{\prime} & d_{0}^{\prime} & d_{1}^{\prime} & \ldots & d_{k-1}^{\prime} \\
\vdots & e_{k-1}^{\prime} & \ddots & \ddots & \vdots & d_{k-1}^{\prime} & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & e_{1}^{\prime} & \vdots & \ddots & \ddots & d_{1}^{\prime} \\
c^{\prime} & e_{1}^{\prime} & \ldots & e_{k-1}^{\prime} & e_{0}^{\prime} & d_{1}^{\prime} & \ldots & d_{k-1}^{\prime} & d_{0}^{\prime}
\end{array}\right) .
$$

The number of parameters is $5+4 k=2 p^{a}+3$ and this is the dimension of $\mathcal{U}$. Thus $\operatorname{dim} \mathcal{T}_{4} \leq 2 p^{a}+3$.

Suppose $|X|=p$. In this case, $G_{0}=H$ and $\mathcal{U}=\mathcal{T}_{4}$, and thus the equality holds. Remark that all $E_{X_{i}} \mathcal{T}_{4} E_{X_{j}}(i, j=1,2)$ are isomorphic to the group algebra of the cyclic group of order $k$, though $E_{X_{i}} \mathcal{T}_{4} E_{X_{j}}$ are not algebras if $i \neq j$ but we identify them. Let $\varepsilon_{0}, \ldots, \varepsilon_{k-1}$ be the primitive idempotent of the group algebra, where $\varepsilon_{0}$ corresponds to the trivial representation, and set

$$
\mathcal{A}_{0}=\left\{\left(\begin{array}{c|lll|lll}
a & b & \ldots & b & c & \ldots & c \\
\hline b^{\prime} & & & & & \\
\vdots & d \varepsilon_{0} & & e \varepsilon_{0} & \\
b^{\prime} & & & & \\
\hline c^{\prime} & & & \\
\vdots & e^{\prime} \varepsilon_{0} & & d^{\prime} \varepsilon_{0}
\end{array}\right)\right\}, \mathcal{A}_{s}=\left\{\left(\begin{array}{c|ccc|cc}
0 & 0 & \ldots & 0 & 0 & \ldots \\
\hline 0 & & & 0 \\
c^{\prime} & & & & \\
0 & & & & & e \varepsilon_{s} \\
0 & & & \\
\hline 0 & & & \\
\vdots & & e^{\prime} \varepsilon_{s} & & d^{\prime} \varepsilon_{s} \\
0 & & &
\end{array}\right)\right\}
$$

for $1 \leq s \leq k-1$. Then $\mathcal{U}=\bigoplus_{s=0}^{k-1} \mathcal{A}_{s}$ as algebras, and $\mathcal{A}_{0} \cong M_{3}(\mathbb{C}), \mathcal{A}_{s} \cong M_{2}(\mathbb{C})$ for $1 \leq s \leq k-1$. We have $\mathcal{T}_{4} \cong M_{3}(\mathbb{C}) \oplus \bigoplus_{i=1}^{(p-3) / 2} M_{2}(\mathbb{C})$.

Remark. The equality in Theorem 8.2 does not hold for $p^{a}, a>1$. For example,

| $p^{a}$ | 9 | 25 | 49 | 81 | 121 | 125 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\operatorname{dim} \mathcal{T}_{4}$ | 15 | 33 | 59 | 51 | 135 | 93 |

Theorem 8.3. $\mathcal{T}_{2}\left(\operatorname{Paley}\left(p^{a}\right)\right)=\mathcal{T}_{3}\left(\operatorname{Paley}\left(p^{a}\right)\right)$.
Proof. Since Paley $\left(p^{a}\right)$ is distance-transitive, the statement holds by Lemma 3.5.
Theorem 8.4. $\operatorname{dim} \mathcal{T}_{2}\left(\operatorname{Paley}\left(p^{a}\right)\right) \leq p^{a}+8$.
Proof. By Proposition 5.1, $E_{X_{1}} \mathcal{T}_{2} E_{X_{1}}$ is commutative and generated by symmetric matrices. Thus all matrices in $E_{X_{1}} \mathcal{T}_{2} E_{X_{1}}$ are symmetric. By the form (8.1) of matrices,
$E_{X_{1}} \mathcal{T}_{2} E_{X_{1}}$ is contained in the adjacency algebra of the cycle $C_{\left(p^{a}-1\right) / 2}$. Therefore $d:=$ $\operatorname{dim} E_{X_{1}} \mathcal{T}_{2} E_{X_{1}} \leq\left(p^{a}+3\right) / 4$. The algebra $E_{X_{1}} \mathcal{T}_{2} E_{X_{1}}$ has $d$ non-isomorphic 1-dimensional simple modules. The idempotent $E_{X_{1}}$ is a sum of $d$ non-equivalent primitive idempotents in $E_{X_{1}} \mathcal{T}_{2} E_{X_{1}}$. By Lemma 2.1, $E_{X_{1}}$ is a sum of $d$ non-equivalent primitive idempotents in $\mathcal{T}_{2}$, and the same thing holds for $E_{X_{2}}$. Now $E_{X_{1}} \mathcal{T}_{2} E_{X_{2}} \cong \operatorname{Hom}_{\mathcal{T}_{2}}\left(E_{X_{1}} \mathcal{T}_{2}, E_{X_{2}} \mathcal{T}_{2}\right)$ and $E_{X_{1}} \mathcal{T}_{2}$ and $E_{X_{2}} \mathcal{T}_{2}$ are sums of $d$ non-isomorphic simple $\mathcal{T}_{2}$-modules. We have $\operatorname{dim} E_{X_{1}} \mathcal{T}_{2} E_{X_{2}} \leq d$ and similarly $\operatorname{dim} E_{X_{2}} \mathcal{T}_{2} E_{X_{1}} \leq d$. Now we can conclude that

$$
\operatorname{dim} \mathcal{T}_{2}=\sum_{i=0}^{2} \sum_{j=0}^{2} \operatorname{dim} E_{X_{i}} \mathcal{T}_{2} E_{X_{j}} \leq 5+4 d \leq p^{a}+8
$$

and the result holds.
Example 8.5. (1) $\operatorname{dim} \mathcal{T}_{2}=p+8$ holds for $|X|=p=5,13,17,29,41,53,89,109$, $113,137,149$.
(2) $\operatorname{dim} \mathcal{T}_{2}=p+4$ holds for $|X|=p=37,61,73,97,101$.
(3) $\operatorname{dim} \mathcal{T}_{2}=p^{a}+6$ for $p^{a}=9, \operatorname{dim} \mathcal{T}_{2}=p^{a}$ for $p^{a}=25, \operatorname{dim} \mathcal{T}_{2}=p^{a}-14$ for $p^{a}=49$, $\operatorname{dim} \mathcal{T}_{2}=p^{a}-48$ for $p^{a}=81$, $\operatorname{dim} \mathcal{T}_{2}=p^{a}-54$ for $p^{a}=121$, $\operatorname{dim} \mathcal{T}_{2}=p^{a}-72$ for $p^{a}=125$,
Corollary 8.6. For $\operatorname{Paley}(p)$ with a prime number $p \geq 7, \mathcal{T}_{1}(\operatorname{Paley}(p)) \subsetneq \mathcal{T}_{2}(\operatorname{Paley}(p))=$ $\mathcal{T}_{3}(\operatorname{Paley}(p)) \subsetneq \mathcal{T}_{4}(\operatorname{Paley}(p))$.
Proof. This is clear by Theorems 8.1, 8.2, 8.3, 8.4.

## 9 Examples for $\mathcal{T}_{2} \neq \mathcal{T}_{3}$

We consider the following graph $\Delta_{n}$ with $n$ vertices for $n \geq 5$. The graph $\Delta_{5}$ with the base vertex 5 is the example of the minimum vertices for $\mathcal{T}_{2} \neq \mathcal{T}_{3}$. (We calculated by using McKay's database [7] for connected simple graphs and GAP4 [4]).


We would like to show that $\mathcal{T}_{2}\left(\Delta_{n}, n\right) \subsetneq \mathcal{T}_{3}\left(\Delta_{n}, n\right)$. By Proposition 6.2, it is enough to show it for the case $n=5$.

We set $\mathcal{T}_{\ell}:=\mathcal{T}_{\ell}\left(\Delta_{5}, 5\right)$ for $\ell=0,1,2,3,4$, and $X_{0}:=\{5\}, X_{1}:=\{1,2,3,4\}$. Obviously $\mathcal{T}_{1}=\mathcal{T}_{2}$ holds. The adjacency matrix of $\Delta_{5}$ is

$$
\left(\begin{array}{llll|l}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
\hline 1 & 1 & 1 & 1 & 0
\end{array}\right)
$$

We set

$$
B_{1}=\left(\begin{array}{c|c}
O_{4} & { }^{t} \mathbf{0}_{4} \\
\hline \mathbf{0}_{4} & 1
\end{array}\right), B_{i}=\left(\begin{array}{c|c}
O_{4} & { }^{t} \boldsymbol{v}_{i} \\
\hline \mathbf{0}_{4} & 0
\end{array}\right) \quad(i=2,3), B_{i}=B_{i-2}^{T}(i=4,5),
$$

and

$$
B_{i}=\left(\begin{array}{c|c}
B_{i}^{\prime} & { }^{t} \mathbf{0}_{4} \\
\hline \mathbf{0}_{4} & 0
\end{array}\right) \quad(i=6, \ldots, 13)
$$

where $\mathbf{0}_{4}=(0,0,0,0), O_{4}$ is the zero matrix of degree $4, \boldsymbol{v}_{2}=(1,1,1,1), \boldsymbol{v}_{3}=(0,1,1,0)$, and

$$
\begin{gathered}
B_{6}^{\prime}=\left(\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right), B_{7}^{\prime}=\left(\begin{array}{llll}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array}\right), B_{8}^{\prime}=\left(\begin{array}{llll}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{array}\right), \\
B_{9}^{\prime}=\left(\begin{array}{llll}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array}\right), B_{10}^{\prime}=\left(\begin{array}{llll}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{array}\right), B_{11}^{\prime}=\left(\begin{array}{llll}
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0
\end{array}\right), \\
B_{12}^{\prime}=\left(\begin{array}{llll}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{array}\right), B_{13}^{\prime}=\left(\begin{array}{llll}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right) .
\end{gathered}
$$

Direct calculation shows the following.
Lemma 9.1. The set $\left\{B_{1}, \ldots, B_{11}\right\}$ is a basis of $\mathcal{T}_{2}\left(\Delta_{5}, 5\right)$, and $\left\{B_{1}, \ldots, B_{13}\right\}$ is a basis of $\mathcal{T}_{3}\left(\Delta_{5}, 5\right)$. Especially, $\operatorname{dim} \mathcal{T}_{2}\left(\Delta_{5}, 5\right)=11<13=\operatorname{dim} \mathcal{T}_{3}\left(\Delta_{5}, 5\right)$.

Lemma 9.2. $\mathcal{T}_{1}\left(\Delta_{5}, 5\right)=\mathcal{T}_{2}\left(\Delta_{5}, 5\right) \cong M_{3}(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$ and $\mathcal{T}_{3}\left(\Delta_{5}, 5\right) \cong M_{3}(\mathbb{C}) \oplus M_{2}(\mathbb{C})$.
Proof. The algebra $E_{X_{1}} \mathcal{T}_{2} E_{X_{1}}$ contains the adjacency algebra of the path graph of length 4. Thus $E_{X_{1}}$ is decomposed into four primitive idempotents $E_{X_{1}}=e_{1}+e_{2}+e_{3}+e_{4}$ in $E_{X_{1}} \mathcal{T}_{2} E_{X_{1}}$. and thus the identity matrix $I$ is decomposed into five primitive idempotents $I=E_{X_{0}}+e_{1}+e_{2}+e_{3}+e_{4}$ in $\mathcal{T}_{2}$. Since $\operatorname{Hom}_{\mathcal{T}_{2}}\left(E_{X_{0}} \mathcal{T}_{2}, E_{X_{1}} \mathcal{T}_{2}\right) \cong E_{X_{0}} \mathcal{T}_{2} E_{X_{1}}=\mathbb{C} B_{2} \oplus \mathbb{C} B_{3}$ has dimension 2 , exactly two of $\left\{e_{1}, e_{2}, e_{3}, e_{4}\right\}$, say $e_{1}, e_{2}$, are equivalent to $E_{X_{0}}$. Thus $\mathcal{T}_{2} \cong M_{3}(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$ if $e_{3}$ and $e_{4}$ are non-equivalent, or $\mathcal{T}_{2} \cong M_{3}(\mathbb{C}) \oplus M_{2}(\mathbb{C})$ if $e_{3}$ and $e_{4}$ are equivalent. However, we know that $\operatorname{dim} \mathcal{T}_{2}=11$. We have the result for $\mathcal{T}_{2}$.

By the same argument shows $\mathcal{T}_{3} \cong M_{3}(\mathbb{C}) \oplus M_{2}(\mathbb{C})$.
Now we can determine $\mathcal{T}_{\ell}\left(\Delta_{n}, n\right), \ell=1,2,3,4$.
Theorem 9.3. For $n \geq 5, \mathcal{T}_{1}\left(\Delta_{n}, n\right)=\mathcal{T}_{2}\left(\Delta_{n}, n\right) \cong M_{n-2}(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$ and $\mathcal{T}_{3}\left(\Delta_{n}, n\right)=$ $\mathcal{T}_{4}\left(\Delta_{n}, n\right) \cong M_{n-2}(\mathbb{C}) \oplus M_{2}(\mathbb{C})$. Especially, $\operatorname{dim} \mathcal{T}_{1}\left(\Delta_{n}, n\right)=\operatorname{dim} \mathcal{T}_{2}\left(\Delta_{n}, n\right)=n^{2}-4 n+$ $6<n^{2}-4 n+8=\operatorname{dim} \mathcal{T}_{3}\left(\Delta_{n}, n\right)=\operatorname{dim} \mathcal{T}_{4}\left(\Delta_{n}, n\right)$ holds.

Proof. By Proposition 6.2 and Lemma 9.2, we have the results for $\mathcal{T}_{2}\left(\Delta_{n}, n\right)$ and $\mathcal{T}_{3}\left(\Delta_{n}, n\right)$. By induction, we can prove that $E_{k} \in \mathcal{T}_{1}\left(\Delta_{n}, n\right)$ for $5 \leq k \leq n$. Thus $\mathcal{T}_{1}\left(\Delta_{n}, n\right)=$ $\mathcal{T}_{2}\left(\Delta_{n}, n\right)$. The automorphism group of $\Delta_{n}$ is a cyclic group of order 2. By the similar argument to the proof of Theorem 7.4 for the permutation character, we have the result for $\mathcal{T}_{4}\left(\Delta_{n}, n\right)$.

Remark. We can find no examples for $\mathcal{T}_{3} \neq \mathcal{T}_{4}$ among all connected simple graphs with vertices less than or equal to 9 (by using the McKay's database [7] and GAP4 [4]). Thus we find no examples for $\mathcal{T}_{2} \neq \mathcal{T}_{3} \neq \mathcal{T}_{4}$.

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