Three classes of BCH codes and their duals

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Abstract

BCH codes are an important class of cyclic codes, and have wide applicantions in communication and storage systems. However, it is difficult to determine the parameters of BCH codes and only a few cases are known. In this paper, we mainly study three classes of BCH codes with $n = q^m - 1$, $\frac{q^{2s} - 1}{q + 1}$, $\frac{q^m - 1}{q - 1}$. On the one hand, we accurately give the parameters of $\mathcal{C}_{(q,n,\delta,1)}$ and its dual codes. On the other hand, we give the sufficient and necessary conditions for $\mathcal{C}_{(q,n,\delta,2)}$ being dually-BCH codes.

Keywords: BCH code, Dual code, Coset leader, Dually-BCH

1. Introduction

Let p be a prime and q > 1 be a p-power. An [n, k, d] linear code \mathcal{C} over \mathbb{F}_q is a k-dimensional subspace of \mathbb{F}_q^n with minimum Hamming distance d. \mathcal{C} is said to be cyclic if $(c_0, c_1, ..., c_{n-1}) \in \mathcal{C}$ implies $(c_{n-1}, c_0, c_1, ..., c_{n-2}) \in \mathcal{C}$. Identify any vector $(c_0, c_1, ..., c_{n-1}) \in \mathbb{F}_q^n$ with

$$c_0 + c_1 x + c_2 x^2 + \dots + c_{n-1} x^{n-1} \in \mathbb{F}_q[x]/\langle x^n - 1 \rangle,$$

i.e., a code \mathcal{C} of length n over \mathbb{F}_q corresponds to a subset of $\mathbb{F}_q[x]/\langle x^n-1\rangle$. Thus \mathcal{C} is a cyclic code if and only if the corresponding subset is an ideal of $\mathbb{F}_q[x]/\langle x^n-1\rangle$. Note that $\mathbb{F}_q[x]/\langle x^n-1\rangle$ is a principal ideal domain, this means that there exists a monic polynomial g(x) of the smallest degree such that $\mathcal{C}=\langle g(x)\rangle$ and $g(x)|(x^n-1)$. g(x) is called the generator polynomial of \mathcal{C} , and $h(x)=(x^n-1)/g(x)$ is called the check polynomial of \mathcal{C} .

For a code C, its dual code, denoted by C^{\perp} , is defined by

$$\mathcal{C}^{\perp} := \{ \mathbf{b} \in \mathbb{F}_q^n : \ \mathbf{b} \cdot \mathbf{c}^T = 0 \text{ for all } c \in \mathcal{C} \},$$

where T denotes the transport and \cdot denotes the standard inner product.

Let $m = \operatorname{ord}_n(q)$, $\mathbb{F}_{q^m}^* = \langle \alpha \rangle$ and $\beta = \alpha^{\frac{q^m-1}{n}}$. Then β is a primitive n-th root of unity. Let $m_i(x)$ denote the minimal polynomial of β^i over \mathbb{F}_q , $0 \le i \le n-1$. For positive integers b and δ , define

$$q_{(a,b,\delta,b)}(x) := \text{lcm}(m_b(x), m_{b+1}(x), ..., m_{b+\delta-2}(x)),$$

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where $2 \leq \delta \leq n$, lcm denotes the least common multiple of $m_i(x)$, $b \leq i \leq b + \delta - 2$. Then $\mathcal{C}_{(q,n,\delta,b)}$ is called a BCH code of length n with designed distance δ . For b = 1, $\mathcal{C}_{(q,n,\delta)} := \mathcal{C}_{(q,n,\delta,1)}$ is narrow-sense BCH code. Suppose $d(\mathcal{C}_{(q,n,\delta,b)})$ denotes the minimum distance of $\mathcal{C}_{(q,n,\delta,b)}$, then $d(\mathcal{C}_{(q,n,\delta,b)}) \geq \delta$.

The importance of the BCH codes in coding theory and communication is apparent, as can be seen in [5] and [13]. However, there are many interesting problems about BCH codes or coset leaders. In [12], Gong et al. proposed the following question:

Question. For a BCH code over \mathbb{F}_q , when is its dual code BCH with respect to the same primitive root of unity?

If the dual of BCH code $\mathcal C$ is still a BCH code with respect to the same primitive root of unity, then $\mathcal C$ is called a dually-BCH code. In [12], the authors gave a necessary and sufficient condition for $\mathcal C_{(q,q^m-1,\delta)}$ being a dually-BCH code. In [19], Wang et al. presented a necessary and sufficient condition for $\mathcal C_{(q,\frac{q^m-1}{q-1},\delta)}$ and $\mathcal C_{(q,\frac{q^{2s}-1}{q+1},\delta)}$ being a dually-BCH codes. But for other lengths or b>1 and $n=q^m-1,\frac{q^{2s}-1}{q+1},\frac{q^m-1}{q-1}$, the condition for $\mathcal C_{(q,n,\delta,b)}$ being a dually-BCH code is still unknown.

It is well known that there is a close relationship between cyclotomic coset leaders and BCH codes. For this reason, many authors determined the cyclotomic cosets for the study of BCH codes, and obtained some pretty results.

- For n = q^m 1, Ding et al.[8] presented the first three largest q-cyclotomic coset leaders modulo
 n = q^m 1. Using those coset leaders, the authors constructed several BCH codes and gave their dimensions and distances (see [6]-[10],[20]).
- For m=2s and $n=\frac{q^{2s}-1}{q+1}$, Wu et al. [18] gave the largest coset leader δ_1 modulo n for q=2,3, and determined the dimension of $\mathcal{C}_{(q,n,\delta)}$ for $2 \nmid s$ and $2 \leq \delta \leq q^s + 1$ or $2 \mid s$ and $2 \leq \delta \leq \lceil \frac{q}{2} \rceil q^{s-1} + 1$. Recently, Wang et al. [19] proved that δ_1 is still the largest q-cyclotomic coset leaders modulo n for $q \geq 4$.
- For n = q^{m-1}/_{q-1}, Ding et al. [17] gave the first two largest coset leaders δ₁ and δ₂ for q = 3, and determined the parameters of C_(q,n,δ) for δ = δ₁ and δ₂. Zhu et al. [21] gave the largest coset leaders δ₁ for q > 3 and m − 1 ≡ 0, 1, q − 2 mod q − 1, and determined the parameters of C_(q,n,δ1).
 Wang et al. [19] presented the largest q-cyclotomic coset leader for any q, and gave the dimension of C_(q,n,δ1).

We shall work on the q-cyclotomic coset leaders modulo $n = q^m - 1, \frac{q^{2s} - 1}{q + 1}, \frac{q^m - 1}{q - 1}$, and the corresponding BCH codes. Our main contributions are:

(A) When $n=q^m-1$. Set $\widetilde{\mathcal{C}}_{(q,q^m-1,\delta,b)}:=\langle (x-1)g_{(q,n,\delta,b)}(x)\rangle$. We prove that the *i*-th largest q-cyclotomic coset leader is $\delta_i=(q-1)q^{m-1}-1-q^{\lfloor\frac{m-1}{2}\rfloor+i-2}$ for $m-(\lfloor\frac{m-1}{2}\rfloor+\lfloor\frac{m-3}{3}\rfloor)\geq i\geq 3$, and give a sufficient and necessary condition for $\widetilde{\mathcal{C}}_{(q,q^m-1,\delta)}^{\perp}$ being a narrow-sense primitive BCH code. Furthermore, when $n\in\{q^m-1,\frac{q^{2s}-1}{q+1},\frac{q^m-1}{q-1}\}$, we give a sufficient and necessary condition for $\mathcal{C}_{(q,n,\delta,2)}$ being dually-BCH codes.

- (B) When $n = \frac{q^{2s}-1}{q+1}$, we obtain the second largest coset leader δ_2 modulo n. Moreover,
 - (B.1) For $\delta_2 \leq \delta \leq \delta_1$, we completely give the dimension and minimum distance of $\mathcal{C}_{(q,n,\delta)}$ and $\mathcal{C}_{(q,n,\delta)}^{\perp}$.
 - (B.2) For $2 \mid s$ and $\lceil \frac{q}{2} \rceil q^{s-1} + 1 \le \delta \le \frac{q^{s+1}+1}{q+1}$, we give the dimension of $\mathcal{C}_{(q,n,\delta)}$.
 - (B.3) For $\delta = a \frac{q^s 1}{q 1}$, $a \frac{q^s + 1}{q + 1}$ if $2 \nmid s$ and $\delta = a \frac{q^s 1}{q^2 1}$ if $2 \mid s$, $1 \le a \le q 1$, we give the dimension and minimum distance of $\mathcal{C}_{(q,n,\delta)}$.
- (C) When $n = \frac{q^m 1}{q 1}$. We obtain the second largest coset leader δ_2 for some special cases, and present the dimension of $\mathcal{C}_{(q,n,\delta)}$ for $\delta_2 \leq \delta \leq \delta_1$.

2. Preliminaries

2.1. Basic Notations

For any positive integer $0 \le s \le q^m - 2$, its q-adic expansion is $s = \sum_{j=0}^{m-1} s_j q^j$, write $s = (s_{m-1}, s_{m-2}, ..., s_1, s_0)$. For integer $0 \le i \le m-1$, we denote $sq^i \pmod{n}$ by $[sq^i]_n$. Then if $n = q^m - 1$, we have

$$[sq^i]_{q^m-1} := (s_{m-1-i}, ..., s_0, s_{m-1}, ..., s_{m-i}).$$

For any $1 \le i \le n-1$, $\delta_{i,n}$ denotes the *i*-th largest *q*-cyclotomic coset leader modulo *n*.

Let $T = \{0 \le i \le n-1 : g_{(q,n,\delta,b)}(\beta^i) = 0\}$ and $T^{-1} = \{n-i : i \in T\}$. Then T and $T^{\perp} = \mathbb{Z}_n \setminus T^{-1}$ are called the defining sets of $\mathcal{C}_{(q,n,\delta,b)}$ and $\mathcal{C}_{(q,n,\delta,b)}^{\perp}$ with respect to β , respectively.

2.2. Cyclotomic Cosets and Coset Leaders

For any t with $0 \le t \le n-1$, the set

$$\{tq^i \pmod{n}: 0 \le i < m\}$$

is called the q-cyclotomic coset modulo n of representative t and is denoted by C_t . The number of elements in C_t is denoted by $|C_t|$. Set $CL(t) := \min\{i : i \in C_t\}$ and $\operatorname{MinRep}_n := \{CL(t) : 0 \le t \le n-1\}$. Then CL(t) is called the coset leader of C_t .

It is well known that the coset leaders are very important to evaluate the dimension and minimum distance of BCH codes. The following four lemmas on coset leaders modulo $q^m - 1$, $\frac{q^{2s} - 1}{q+1}$ and $\frac{q^m - 1}{q+1}$ will be useful for demonstrating our results.

Lemma 1. ([2]) Let n be a positive integer such that gcd(n,q) = 1 and $q^{\lfloor \frac{m}{2} \rfloor} < n \le q^m - 1$. Then s is a coset leader and $|C_s| = m$ for all $1 \le s \le \frac{nq^{\lceil \frac{m}{2} \rceil}}{q^m - 1}$, $s \ne 0 \pmod{q}$.

Lemma 2. ([14, 16]) Let $n = q^m - 1$. Then

(a) The first three largest q-cyclotomic coset leaders modulo n are:

$$\delta_{1,n} = (q-1)q^{m-1} - 1, \ \delta_{2,n} = (q-1)q^{m-1} - 1 - q^{\lfloor \frac{m-1}{2} \rfloor}, \ \delta_{3,n} = (q-1)q^{m-1} - 1 - q^{\lfloor \frac{m+1}{2} \rfloor}.$$

- (b) If the Bose distance of $C_{(n,q,\delta)}$ is $d_i = q^m q^{m-1} q^i 1$, where $\frac{m-2}{2} \le i \le m \lfloor \frac{m}{3} \rfloor 1$. Then the minimum distance of $C_{(n,q,\delta)}$ is d_i .
- (c) Let m = 2s and a be an integer satisfying $q^s + 1 \le a \le q^{s+1}$ and $a \not\equiv 0 \pmod{q}$.
 - (c.1) Set $a = c(q^s + 1)$, $1 \le c \le q 1$, then a is a coset leader and $|C_a| = \frac{m}{2}$.
 - (c.2) Set $a = a_s q^s + a_0$, $1 \le a_0 < a_s \le q 1$, then a is not a coset leader.
 - (c.3) Except for (b.1) and (b.2), the remaining of a are coset leaders and $|C_a| = m$.

Lemma 3. ([19]) Let $n = \frac{q^m - 1}{q + 1}$ and m = 2s, the first largest q-cyclotomic coset leader $\delta_{1,n}$ modulo n is:

- If $2 \nmid s$, then $\delta_{1,n} = \frac{(q-1)q^{m-1} q^{\frac{m-2}{2}} 1}{q+1}$ and $|C_{\delta_{1,n}}| = \frac{m}{2}$.
- If $2 \mid s$, then $\delta_{1,n} = \frac{(q-1)q^{m-1} q^{\frac{m}{2}} 1}{q+1}$ and $|C_{\delta_{1,n}}| = m$.

Furthermore, set $(q+1) \mid h$. Then h is a coset leader modulo q^m-1 if and only if $\frac{h}{q+1}$ is a coset leader modulo $\frac{q^m-1}{q+1}$.

Lemma 4. ([19],[21]) Let $n = \frac{q^m - 1}{q - 1}$. Then

(a) Let q > 3. For any integer $1 \le i \le n-1$, take $i = (i_{m-1}, i_{m-2}, \dots, i_0)$. If i is a q-cyclotomic coset leader modulo $\frac{q^m-1}{q-1}$, then $i_{m-1} = 0$.

Furthermore, suppose m-1=a(q-1)+b, where $a\geq 1$ and $0\leq b\leq q-2$. Let $\epsilon=a+1$ when b=q-2 and $\epsilon=a$ when $0\leq b\leq q-3$. If $i_l=q-1$ for all $m-1-\epsilon\leq l\leq m-2$, then $1\leq i_{l-1}\leq i_l$ for all $1\leq l\leq m-2$.

- (b) Let $q \geq 3$ and $m \geq 4$.
 - (b.1) Let $q-1=mt_1+t_2$ and $\sum_{t=1}^{q-1}q^{\left\lceil \frac{mt}{q-1}-1\right\rceil}=\sum_{i=0}^{m-1}a_iq^i$, where $t_1\geq 0$ and $m>t_2\geq 0$.
 - * If $t_2 = 0$, then $a_i = \frac{q-1}{m}$ for all $i \in [0, m-1]$.
 - * If $t_2 \neq 0$, then $a_i = \lceil \frac{q-1}{m} \rceil$ when $i \in \Upsilon$, where $\Upsilon = \{\lceil \frac{m\gamma}{t_2} 1 \rceil, \gamma = 1, 2, \dots, t_2 \}$. Otherwise, $a_i = \lfloor \frac{q-1}{m} \rfloor$.
 - (b.2) The first largest q-cyclotomic coset leader modulo n is $\delta_{1,n} = \frac{q^m 1 \sum_{t=1}^{q-1} q^{\lceil \frac{mt}{q-1} 1 \rceil}}{q-1}$ and $|C_{\delta_{1,n}}| = \frac{m}{\gcd(m,q-1)}$.
- 2.3. Known Results on the Dimension and Minimum Distance of BCH Codes

Given designed distance δ , it is difficult to determine the dimension and minimum distance. But for some special BCH codes, $\dim(\mathcal{C}_{(q,n,\delta)})$ and $d(\mathcal{C}_{(q,n,\delta)})$ can be given. We list them as follows.

Lemma 5. ([17]) Let n be a positive integer such that $q-1 \mid n$ and gcd(n,q) = 1, let δ_b be a divisor of $\frac{n}{q-1}$. Then for $\delta = k\delta_b$, $1 \le k \le q-1$, the minimum distance of $C_{(n,q,\delta)}$ is δ .

Lemma 6. ([18]) Let $n = \frac{q^m - 1}{q + 1}$ and m = 2s. Then

(a) Suppose $2 \le \delta \le q^s + 1$. For $q, s \ge 3$ and s is odd or q = 2 and $s \ge 5$, the dimension k of $\mathcal{C}_{(q,n,\delta,1)}$ is given as follows:

(a.1) If
$$2 \le \delta \le \frac{q^s+1}{q+1}$$
, then

$$k = n - 2s(\delta - 1) + 2s \left| \frac{\delta - 1}{q} \right|.$$

(a.2) If
$$\frac{q^s+1}{q+1} + 1 \le \delta \le (q-1)\frac{q^s+1}{q+1} + 1$$
, then

$$k = n - 2s(\delta - 1) + 2s \left| \frac{\delta - 1}{q} \right| + s \left| \frac{(\delta - 1)(q + 1)}{q^s + 1} \right|.$$

(a.3) If
$$(q-1)\frac{q^s+1}{q+1}+2 \le \delta \le \frac{q^{s+1}-1}{q+1}+2$$
, then

$$k = n - 2s(\delta - 1) + 2s \left| \frac{\delta - 1}{q} \right| + s(q - 1).$$

(a.4) If
$$(q-1)q^{s-1} + \frac{q^s+1}{q+1} + 1 \le \delta \le q^s + 1$$
, then

$$k = n - 2s(\delta - 1) + 2s \left| \frac{\delta - 1}{q} \right| + 3s(q - 1).$$

(b) Suppose $2 \le \delta \le \lceil \frac{q}{2} \rceil q^{s-1} + 1$. For $s \ge 4$ and is even, the dimension k of $\mathcal{C}_{(q,n,\delta)}$ is given as follows:

(b.1) If
$$2 \mid q$$
, then

$$k = n - 2s(\delta - 1) + 2s \left| \frac{\delta - 1}{q} \right|.$$

(b.2) If $2 \nmid q$, then

$$k = \begin{cases} n - 2s(\delta - 1) + 2s \left\lfloor \frac{\delta - 1}{q} \right\rfloor, & \text{if } 2 \le \delta \le \frac{q^s + 1}{2}; \\ n - 2s(\delta - 1) + 2s \left\lfloor \frac{\delta - 1}{q} \right\rfloor + s, & \text{if } \frac{q^s + 1}{2} + 1 \le \delta \le \frac{q + 1}{2}q^{s - 1} + 1. \end{cases}$$

2.4. Known Results on Dually-BCH Codes

For special designed distance of narrow-sense BCH codes, the following two lemmas give the sufficient and necessary conditions on dually-BCH codes.

Lemma 7. ([18]) Let $n = q^m - 1$.

• If q=2 and $m \geq 6$, then $C_{(2,2^m-1,\delta)}$ is a dually-BCH code if and only if

$$\delta = 2, 3, \text{ or } 2^{m-1} - 2^{\lfloor \frac{m-1}{2} \rfloor} \le \delta \le n.$$

• If $q \geq 3$ and $m \geq 2$, then $C_{(q,q^m-1,\delta)}$ is a dually-BCH code if and only if

$$\delta = 2 \ or \ (q-1)q^{m-1} - q^{\lfloor \frac{m-1}{2} \rfloor} \le \delta \le n.$$

Lemma 8. ([19]) Let $n = \frac{q^m - 1}{q + 1}$ and $\delta_{1,n}$ is given in Lemma 3.

(1) If q=2 and $m\geq 4$ is even, then $\mathcal{C}_{(q,n,\delta)}$ is a dually-BCH code if and only if

$$\delta_{1,n} + 1 \le \delta \le n$$
.

(2) If q > 2 and m = 4, then $C_{(q,n,\delta)}$ is a dually-BCH code if and only if

$$\delta = 2, \, \delta_{1,n} \leq \delta \leq n.$$

(3) If q > 2 and $m \neq 4$ is even, then $C_{(q,n,\delta)}$ is a dually-BCH code if and only if

$$\delta_{1,n} + 1 \le \delta \le n$$
.

In addition, let $q \geq 3$, $m \geq 4$, $n = \frac{q^m - 1}{q - 1}$ and $\delta_{1,n}$ is given in Lemma 4. Then $C_{(q,n,\delta)}$ is a dually-BCH code if and only if

$$\delta_{1,n} + 1 \le \delta \le n$$
.

3. The case of $n = q^m - 1$

In this section, we always assume $n = q^m - 1$.

3.1. The computation of $\delta_{i,n}$

Lemma 9. Let q be a prime power and $3 \le i \le m - (\lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{m-3}{3} \rfloor)$,

$$\delta_{i,n} = (q-1)q^{m-1} - 1 - q^{\lfloor \frac{m-1}{2} \rfloor + i - 2},$$

and $\mid C_{\delta_{i,n}} \mid = m$.

Proof. Obviously, the case of i=3 agrees with Lemma 2. By induction, suppose $\delta_{t,n}=(q-1)q^{m-1}-1-q^{\lfloor\frac{m-1}{2}\rfloor+t-2}$, then we need to prove $\delta_{t+1,n}=(q-1)q^{m-1}-1-q^{\lfloor\frac{m-1}{2}\rfloor+t-1}$ for $m-(\lfloor\frac{m-1}{2}\rfloor+\lfloor\frac{m-3}{3}\rfloor)\geq t+1>t\geq 2$. In order to prove this, we divide it into two steps.

Step 1. We claim that $\delta_{i,n}$ is a coset leader for $3 \leq i \leq m - (\lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{m-3}{3} \rfloor)$. Clearly,

$$\delta_{i,n} = (q-2, \underbrace{q-1, \dots, q-1}_{m-\left\lfloor \frac{m-1}{2} \right\rfloor - i}, q-2, \underbrace{q-1, \dots, q-1}_{\left\lfloor \frac{m-1}{2} \right\rfloor + i-2}).$$

Since $i \geq 3$, then $m - \left\lfloor \frac{m-1}{2} \right\rfloor - i < \left\lfloor \frac{m-1}{2} \right\rfloor + i - 2$. Note that

$$\begin{cases}
[\delta_{i}q]_{n} = (\underbrace{q-1, \dots, q-1}_{m-\lfloor \frac{m-1}{2} \rfloor - i}, q-2, \underbrace{q-1, \dots, q-1}_{\lfloor \frac{m-1}{2} \rfloor + i-2}, q-2), \\
[\delta_{i}q^{2}]_{n} = (\underbrace{q-1, \dots, q-1}_{m-\lfloor \frac{m-1}{2} \rfloor - i-1}, q-2, \underbrace{q-1, \dots, q-1}_{\lfloor \frac{m-1}{2} \rfloor + i-2}, q-2, q-1), \\
[\delta_{i}q^{m-\lfloor \frac{m-1}{2} \rfloor - i}]_{n} = (q-2, \underbrace{q-1, \dots, q-1}_{l-1}, q-2, \underbrace{q-1, \dots, q-1}_{m-\lfloor \frac{m-1}{2} \rfloor - i}, \\
[\delta_{i}q^{m-1}]_{n} = (q-1, q-2, \underbrace{q-1, \dots, q-1}_{l-1}, q-2, \underbrace{q-1, \dots, q-1}_{l-1}). \\
[\delta_{i}q^{m-1}]_{n} = (q-1, q-2, \underbrace{q-1, \dots, q-1}_{l-1}, q-2, \underbrace{q-1, \dots, q-1}_{l-1}).
\end{cases}$$

Then $[\delta_{i,n}q^j]_n > \delta_{i,n}$ for any $1 \le j \le m-1$, this implies that $\delta_{i,n}$ is a coset leader and $|C_{\delta_{i,n}}| = m$ for $3 \le i \le m - (\lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{m-3}{3} \rfloor)$.

Step 2: For $m - \lfloor \frac{m-1}{2} \rfloor - t > \lfloor \frac{m-3}{3} \rfloor$, we claim that $J_i := \delta_{t,n} - i$ is not a coset leader for any $1 \le i \le (q-1)q^{\lfloor \frac{m-1}{2} \rfloor + t - 2} - 1$.

Take $h = \lfloor \frac{m-1}{2} \rfloor + t - 2$. Note that $\delta_{t,n} - \delta_{t+1,n} = q^{h+1} - q^h = (q-1)q^h$ and

$$(q-1)q^h - 1 = (q-2)q^h + (q-1)q^{h-1} + (q-1)q^{h-2} + \dots + (q-1)q + q - 1.$$

For any $1 \le i \le (q-1)q^h - 1$, let

$$i := i_h q^h + i_{h-1} q^{h-1} + \dots + i_1 q + i_0,$$

then $0 \le i_j \le q-1$ for any $0 \le j \le h-1$ and $0 \le i_h \le q-2$, and there is at least an $i_j \ne 0$ for $j \in \{0, 1, ..., h\}$. Thus,

$$J_i = (q-2)q^{m-1} + (q-1)q^{m-2} + \dots + (q-1)q^{h+1} + (q-2-i_h)q^h + (q-1-i_{h-1})q^{h-1} + \dots + (q-1-i_1)q + (q-1-i_0).$$

$$(1)$$

For q=2, we have $i_h=0$ and

$$J_i = 2^{m-2} + 2^{m-3} + \dots + 2^{h+1} + (1 - i_{h-1})2^{h-1} + \dots + (1 - i_1)2 + 1 - i_0.$$

If $i_0 = 1$, then $\frac{J_i}{2}$ and J_i are in the same cyclotomic coset and $\frac{J_i}{2} < J_i$. Hence, J_i cannot be a coset leader.

If $i_0 = 0$, suppose l is the largest integer such that $i_l = 1, 1 \le l \le t - 1$. Then

$$J_i = (0, \underbrace{1, 1, \dots, 1}_{m-2-h}, 0, \underbrace{1, 1, \dots, 1}_{h-l-1}, 0, \underbrace{1 - i_{l-1}, \dots, 1 - i_1, 1}_{l}).$$

Note that $m-2-h=m-\left\lfloor\frac{m-1}{2}\right\rfloor-t$ and $h-l-1=\left\lfloor\frac{m-1}{2}\right\rfloor+t-3-l$. Since $m-\left(\left\lfloor\frac{m-1}{2}\right\rfloor+\left\lfloor\frac{m-3}{3}\right\rfloor\right)\geq t+1$, then $m-\left\lfloor\frac{m-1}{2}\right\rfloor-t\geq \left\lfloor\frac{m-3}{3}\right\rfloor+1>\left\lfloor\frac{m-3}{3}\right\rfloor$.

Note that

$$h - l + 1 + l = m - 3 - (m - \left| \frac{m-1}{2} \right| - t) < m - 3 - \left| \frac{m-3}{3} \right| \le 2 \left| \frac{m-3}{3} \right|.$$
 (2)

Then $h-l+1 \leq \left\lfloor \frac{m-3}{3} \right\rfloor$ if $h-l+1 \leq l$ or $l \leq \left\lfloor \frac{m-3}{3} \right\rfloor$ if $h-l+1 \geq l$, then m-2-h>h-l-1 or m-2-h>l.

If m - 2 - h > h - l - 1, then

$$[J_i q^{m-1-h}]_{q^m-1} = (0, \underbrace{1, 1, \dots, 1}_{h-l-1}, 0, \underbrace{1 - i_{l-1}, \dots, 1 - i_1, 1}_{l}, 0, \underbrace{1, 1, \dots, 1}_{m-2-h}) < J_i.$$
(3)

If m-2-h>l, then

$$[J_i q^{m-1-l}]_{q^m-1} = (0, \underbrace{1 - i_{l-1}, \dots, 1 - i_1, 1}_{l}, 0, \underbrace{1, 1, \dots, 1}_{m-2-h}, 0, \underbrace{1, 1, \dots, 1}_{h-l-1}) < J_i.$$

$$(4)$$

By Eqs. (3) and (4), we know that J_i cannot be a coset leader.

For q > 2, by Eq. (1),

- If $i_h \ge 1$, we have $q 2 > q 2 i_h$, then $J_i q^{m-1-h} \mod q^m 1 < J_i$.
- If there exists an integer j such that $i_j \geq 2$, $0 \leq j \leq t-1$, we have $q-2 > q-1-i_j$, then $J_i q^{m-1-j} \mod q^m -1 < J_i$.

Thus, J_i cannot be a coset leader. Next we consider $i_h = 0$ and $i_j \in \{0, 1\}$ for any $0 \le j \le t - 1$. Note that $i \ge 1$, suppose l is the largest index such that $i_l = 1, 1 \le l \le t - 1$. Then

$$J_i = (q-2, \underbrace{q-1, \dots, q-1}_{m-2-h}, q-2, \underbrace{q-1, \dots, q-1}_{h-l-1}, q-2, \underbrace{q-1-i_{l-1}, \dots, q-1-i_1, q-1-i_0}_{l}).$$

By Eqs. (2), (3) and (4), J_i cannot be a coset leader. Thus we complete the proof.

In particular, for i = 4, 5, we have

Corollary 10. (1) Let $m \ge 10$. Then $\delta_{4,n} = (q-1)q^{m-1} - 1 - q^{\lfloor \frac{m-1}{2} \rfloor + 2}$ and $|C_{\delta_{4,n}}| = m$.

(2) Let
$$m \ge 14$$
. Then $\delta_{5,n} = (q-1)q^{m-1} - 1 - q^{\lfloor \frac{m-1}{2} \rfloor + 3}$ and $|C_{\delta_{5,n}}| = m$.

Remark 1. The proof of Lemma 9 has been given in [14]. For completeness, we provide a proof which is different from [14], and we give $|\delta_{i,n}|$.

3.2. BCH Codes and Dually-BCH Codes

The following theorem provides the information on the parameters of the BCH code $\mathcal{C}_{(q,n,\delta_{i,n})}$.

Theorem 11. Let q be a prime power and $3 \le i \le m - (\lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{m-3}{3} \rfloor)$, then $C_{(q,n,\delta_{i,n})}$ has parameters $[q^m - 1, k, \delta_{i,n}]$, where

$$k = \begin{cases} im, & \text{if } 2 \nmid m; \\ (i - \frac{1}{2})m, & \text{if } 2 \mid m. \end{cases}$$

Proof. Form Lemmas 2 and 9, we can obtain the results directly.

For b = 1, the conditions of $C_{(q,q^m-1,\delta)}$ being dually-BCH codes have been given by Lemma 7. For b = 2, we have

Theorem 12. Let $q \geq 3$ and $m \geq 2$. Then $C_{(q,n,\delta,2)}$ is a dually-BCH code if and only if

$$(q-1)q^{m-1} - q^{\lfloor \frac{m-1}{2} \rfloor} - 1 \le \delta \le q^m - 2.$$

Proof. By the definition, the defining set of $C_{(q,q^m-1,\delta,2)}$ with respect to α is $T = C_2 \cup C_3 \cup \cdots \cup C_{\delta}$, $2 \leq \delta \leq n-1$. Note that $0 \notin T$, i.e., $0 \in T^{\perp}$, this means $C_0 \subset T^{\perp}$. Therefore, if $C_{(q,n,\delta,2)}$ is a dually-BCH code, then there must exist an integer $r \geq 1$ such that $T^{\perp} = C_0 \cup C_1 \cup \cdots \cup C_{r-1}$.

If $q \leq \delta \leq n-1$, then $C_1 = C_q \subset T$, i.e., $T = C_1 \cup C_2 \cup \cdots \cup C_\delta$. Hence, by Lemma 7, we know that $C_{(q,q^m-1,\delta,2)}$ is a dually-BCH code if and only if $(q-1)q^{m-1} - q^{\lfloor \frac{m-1}{2} \rfloor} - 1 \leq \delta \leq n-1$.

If $2 \leq \delta \leq q-1$. Since $[1,q-1] \subset [1,q^{\lceil \frac{m}{2} \rceil}]$, then $i \in \text{MinRep}_n$ and $|C_i| = m$ for any $1 \leq i \leq q-1$ by Lemma 1. We thus have $C_2 \subset T$ and $C_1 \not\subset T$, i.e., $m \leq \dim(\mathcal{C}_{(q,n,\delta,2)}) \leq n-m < n$ and $C_{CL(n-1)} \subset T^{\perp}$. Clearly, $CL(n-1) = (q-1)q^{m-1} - 1 = \delta_{1,n}$. Therefore, if $C_{(q,n,\delta,2)}$ is a dually-BCH

code, then $T^{\perp} = C_0 \cup C_1 \cup \cdots \cup C_{\delta_{1,n}}$. However, $\dim(\mathcal{C}_{(q,n,\delta,2)}) \leq n - m$, which contradicts the fact that $\dim(\mathcal{C}_{(q,n,\delta,2)}) + \dim(\mathcal{C}_{(q,n,\delta,2)}^{\perp}) = n$. Thus we complete the proof.

Theorem 13. Let q=2 and $m\geq 6$. Then $\mathcal{C}_{(q,n,\delta,2)}$ is a dually-BCH code if and only if

$$\delta = 2 \ or \ 2^{m-1} - 2^{\lfloor \frac{m-1}{2} \rfloor} - 1 < \delta < n-1.$$

Proof. The proof is very similar to that of Theorem 12, hence we omit it.

For b=1, we then study the condition of $\widetilde{\mathcal{C}}_{(2,n,\delta)}^{\perp}$ being a narrow-sense BCH code.

Theorem 14. Let q=2 and $m\geq 6$. Then $\widetilde{\mathcal{C}}_{(2,n,\delta)}^{\perp}$ is a narrow-sense BCH code if and only if

$$\delta = 2, 3, \ or \ 2^{m-1} - 2^{\lfloor \frac{m-1}{2} \rfloor} \le \delta \le 2^{m-1} - 1.$$

Proof. By definition, the defining set of $\widetilde{C}_{(2,n,\delta)}$ with respect to α is $T = C_0 \cup C_1 \cup C_2 \cup \cdots \cup C_{\delta-1} = T' \cup C_0$, where $T' = C_1 \cup C_2 \cup \cdots \cup C_{\delta-1}$. Hence,

$$T^{\perp} = \mathbb{Z}_{n} \setminus T^{-1} = (\mathbb{Z}_{n} \setminus T)^{-1} = ((\mathbb{Z}_{n} \setminus (T'))^{-1}) \setminus C_{0}.$$

For $\delta=2,3$, we know that $T=C_0\cup C_1$ and $T^{-1}=C_0\cup C_{2^{m-1}-1}$. Note that $2^{m-1}-1$ is the largest coset leader modulo 2^m-1 , then

$$T^{\perp} = \mathbb{Z}_n \setminus T^{-1} = C_1 \cup C_2 \cup \dots \cup C_{2^{m-1}-2}.$$

Thus, $\widetilde{\mathcal{C}}_{(2,n,\delta)}^{\perp}$ is a BCH code with b=1 and designed distance $\delta=2^{m-1}-1$. For $2^{m-1}-2^{\lfloor\frac{m-1}{2}\rfloor}\leq\delta\leq n$, we divide it into two cases:

- If $2^{m-1} \leq \delta \leq n$, then $T = \mathbb{Z}_n$, which leads to $\widetilde{C}_{(2,n,\delta)}^{\perp} = \{\mathbf{0}\}.$
- If $2^{m-1} 2^{\lfloor \frac{m-1}{2} \rfloor} \le \delta \le 2^{m-1} 1$, note that $\delta_{2,n} + 1 = 2^{m-1} 2^{\lfloor \frac{m-1}{2} \rfloor}$. Then

$$T^{\perp} = (\mathbb{Z}_n \setminus T)^{-1} = (\mathbb{Z}_n \setminus (C_0 \cup C_1 \cup C_2 \cup \cdots \cup C_{\delta_{2,n}}))^{-1} = (C_{\delta_{1,n}})^{-1} = C_{CL(n-\delta_{1,n})} = C_1.$$

Obviously, $\widetilde{\mathcal{C}}_{(2,n,\delta)}^{\perp}$ is a narrow-sense BCH code.

For $3 < \delta < 2^{m-1} - 2^{\lfloor \frac{m-1}{2} \rfloor}$, we claim that $\widetilde{\mathcal{C}}_{(2,n,\delta)}^{\perp}$ is not a narrow-sense BCH code.

By Lemma 2, we know that $\delta_{1,n}=2^{m-1}-1>2^{m-1}-2^{\lfloor\frac{m-1}{2}\rfloor}>\delta$, then $\delta_{1,n}\notin T$. Note that $CL(n-\delta_{1,n})=1$, i.e., $C_1\in T^{\perp}$. If $\widetilde{C}_{(2,n,\delta)}^{\perp}$ is a narrow-sense BCH code, then there must exist an integer $r\geq 1$ such that $T^{\perp}=C_1\cup C_2\cup\cdots\cup C_{r-1}$.

By Lemma 7, we know that $C_{(2,n,\delta)}$ is not a dually-BCH code for all any $3 < \delta < 2^{m-1} - 2^{\lfloor \frac{m-1}{2} \rfloor}$, which means there is no $r \geq 1$ such that the defining set of $C_{(2,n,\delta)}^{\perp}$ is equal to $C_0 \cup C_1 \cup \cdots \cup C_{r-1}$. Note that $T^{\perp} = ((\mathbb{Z}_n \setminus (T'))^{-1}) \setminus C_0$, then there is no integer $r \geq 1$ such that $T^{\perp} = C_1 \cup C_2 \cup \cdots \cup C_{r-1}$. i.e., $\widetilde{C}_{(2,n,\delta)}^{\perp}$ is not a narrow-sense BCH code. Thus we complete the proof.

Theorem 15. Let $q \geq 3$ and $m \geq 2$. Then $\widetilde{\mathcal{C}}_{(2,n,\delta)}^{\perp}$ is a narrow-sense BCH code if and only if

$$\delta = 2 \ or \ (q-1)q^{m-1} - q^{\lfloor \frac{m-1}{2} \rfloor} \le \delta \le (q-1)q^{m-1} - 1.$$

Proof. The proof is very similar to that of Theorem 14, hence we omit it.

4. The case of $n = \frac{q^{2s}-1}{q+1}$

In this section, we always assume $m=2s, n=\frac{q^m-1}{q+1}$ and $\beta_1=\alpha^{q+1}$.

4.1. The computation of $\delta_{2,n}$

For s=2 and $2 \nmid q$, $\delta_{2,n}$ has been determined in [19]. Next we give $\delta_{2,n}$ for $s \geq 3$.

Lemma 16. Let q be a prime power, then

$$\delta_{2,n} = \begin{cases} \frac{(q-1)q^{2s-1} - q^{s+1} - 1}{q+1}, & \text{if } 2 \nmid s \text{ and } s > 4; \\ \frac{(q-1)q^{2s-1} - q^{s+2} - 1}{q+1}, & \text{if } 2 \mid s \text{ and } s > 6, \end{cases}$$

and $\mid C_{\delta_{2,n}} \mid = 2s$.

Proof. If $2 \nmid s$, we have $(q+1) \mid (q^{s+1}-1)$ and

$$(q+1) \mid (q^{2s}-1) - (q^{2s-1}+1) - (q^{s+1}-1) = (q-1)q^{2s-1} - q^{s+1} - 1.$$

Note that by Corollary 10, $\delta_{4,q^m-1} = (q-1)q^{2s-1} - q^{s+1} - 1$, this implies that $\frac{(q-1)q^{2s-1} - q^{s+1} - 1}{q+1} \in \text{MinRep}_n$ by Lemma 3.

We claim that $\delta_{2,n} = \frac{(q-1)q^{2s-1}-q^{s+1}-1}{q+1}$. Suppose there exists an integer a such that $a \in \text{MinRep}_n$ and $\delta_{2,n} < a < \delta_{1,n}$. So $a(q+1) \in \text{MinRep}_{q^m-1}$, $\delta_{4,q^m-1} = \delta_{2,n}(q+1) < a(q+1) < \delta_{2,q^m-1} = \delta_{1,n}(q+1)$. This means

$$a(q+1) = \delta_{3,q^m-1} = (q-1)q^{2s-1} - 1 - q^s.$$
(5)

Note that $\gcd(q+1,(q-1)q^{2s-1}-1-q^s)=\gcd(q+1,q-1)< q+1$, then $(q+1)\nmid (q-1)q^{2s-1}-1-q^s$. By Eq. (5), this is impossible. Thus, $\delta_{2,n}=\frac{(q-1)q^{2s-1}-q^{s+1}-1}{q+1}$.

Let $\mid C_{\delta_{2,n}} \mid = l$, then

$$\frac{q^{2s}-1}{q+1} \mid \left(\frac{(q-1)q^{2s-1}-q^{s+1}-1}{q+1}\right)(q^l-1) \Longleftrightarrow (q^{2s}-1) \mid \left((q-1)q^{2s-1}-q^{s+1}-1\right)(q^l-1).$$

Thus l = 2s. The case $2 \mid s$ is similar.

Thus we complete the proof.

Lemma 17. Let $2 \nmid q$.

(1) If
$$s = 3$$
, then $\delta_{2,n} = \frac{(q-1)q^5 - q^4 - 1}{q+1}$ and $|C_{\delta_{2,n}}| = 6$.

(2) If
$$s = 4$$
, then $\delta_{2,n} = \frac{(q-1)q^7 - q^6 - 1}{q+1}$ and $|C_{\delta_{2,n}}| = 8$.

(3) If
$$s = 6$$
, then $\delta_{2,n} = \frac{(q-1)q^{11}-q^7-1}{q+1}$ and $|C_{\delta_{2,n}}| = 12$.

Proof. We just give the proof for Case (2), since the proofs for the other cases are similar.

Note that $[((q-1)q^7-q^6-1)q^j]_{q^8-1} > (q-1)q^7-q^6-1$ for any $1 \le j \le 7$, this implies that $(q-1)q^7-q^6-1 \in \text{MinRep}_{q^8-1}$ and $|C_{(q-1)q^7-q^6-1}|=8$. Since $(q+1) \mid (q-1)q^7-q^6-1$, this implies that $\frac{(q-1)q^7-q^6-1}{q+1} \in \text{MinRep}_n$ and $|C_{\frac{(q-1)q^7-q^6-1}{q+1}}|=8$ by Lemma 3.

We claim that $\delta_{2,n} = \frac{(q-1)q^7 - q^6 - 1}{q+1}$. Suppose there exists an integer a such that $a \in \text{MinRep}_n$ and $\delta_{2,n} < a < \delta_{1,n}$, then $a(q+1) \in \text{MinRep}_{q^8 - 1}$.

Note that all coset leaders modulo $q^8 - 1$ between $\delta_{1,n}(q+1) = (q-1)q^7 - q^4 - 1$ and $\delta_{2,n}(q+1) = (q-1)q^7 - q^6 - 1$ are $(q-1)q^7 - q^5 - q^2 - 1$, $(q-1)q^7 - q^5 - q^3 - 1$ and $(q-1)q^7 - q^5 - q^3 - q - 1$, this means

$$(q+1)a = (q-1)q^7 - q^5 - q^2 - 1, (q-1)q^7 - q^5 - q^3 - 1 \text{ or } (q-1)q^7 - q^5 - q^3 - q - 1.$$
 (6)

Note that $(q+1) \nmid (q-1)q^7 - q^5 - q^2 - 1$, $(q+1) \nmid (q-1)q^7 - q^5 - q^3 - 1$ and $(q+1) \nmid (q-1)q^7 - q^5 - q^3 - q - 1$, this is impossible by Eq. (6). Therefore, $\delta_{2,n} = \frac{(q-1)q^7 - q^6 - 1}{q+1}$. Thus we complete the proof.

4.2. BCH Codes and Dually-BCH Codes

Theorem 18. Let q be a prime power and s > 4. For $2 \nmid s$,

- If $\delta_{2,n} + 1 \leq \delta \leq \delta_{1,n}$, then $C_{(q,n,\delta)}$ has parameters $\left[\frac{q^{2s}-1}{q+1}, s+1, d(C_{(q,n,\delta)}) \geq \delta\right]$ and its dual $C_{(q,n,\delta)}^{\perp}$ has parameters $\left[\frac{q^{2s}-1}{q+1}, \frac{q^{2s}-1}{q+1} s 1, 2\right]$.
- If $\delta = \delta_{2,n}$, then $C_{(q,n,\delta_{2,n})}$ has parameters $\left[\frac{q^{2s}-1}{q+1}, 3s+1, d(C_{(q,n,\delta_{2,n})}) \ge \delta_{2,n}\right]$ and its dual $C_{(q,n,\delta_{2,n})}^{\perp}$ has parameters $\left[\frac{q^{2s}-1}{q+1}, \frac{q^{2s}-1}{q+1} 3s 1, 3 \le d(C_{(q,n,\delta_{2,n})}^{\perp}) \le 4\right]$.

For $2 \mid s \text{ and } s \neq 6$,

- If $\delta_{2,n}+1 \leq \delta \leq \delta_{1,n}$, then $\mathcal{C}_{(q,n,\delta)}$ has parameters $\left[\frac{q^{2s}-1}{q+1},2s+1,d(\mathcal{C}_{(q,n,\delta)})\geq \delta\right]$ and its dual $\mathcal{C}_{(q,n,\delta)}^{\perp}$ has parameters $\left[\frac{q^{2s}-1}{q+1},\frac{q^{2s}-1}{q+1}-2s-1,3\leq d(\mathcal{C}_{(q,n,\delta)}^{\perp})\leq 4\right]$.
- If $\delta = \delta_{2,n}$, then $C_{(q,n,\delta_{2,n})}$ has parameters $\left[\frac{q^{2s}-1}{q+1}, 4s+1, d(C_{(q,n,\delta_{2,n})}) \ge \delta_{2,n}\right]$ and its dual $C_{(q,n,\delta_{2,n})}^{\perp}$ has parameters $\left[\frac{q^{2s}-1}{q+1}, \frac{q^{2s}-1}{q+1} 4s 1, 3 \le d(C_{(q,n,\delta_{2,n})}^{\perp}) \le 4\right]$.

Proof. Form Lemmas 3 and 16, the parameters of $C_{(q,n,\delta)}$ can be obtained directly. Next we consider the parameters of $C_{(q,n,\delta)}^{\perp}$.

Case 1. For $\delta_{2,n} + 1 \leq \delta \leq \delta_{1,n}$, by definition, we know the defining set of $\mathcal{C}_{(q,n,\delta)}^{\perp}$ with respect to β_1 is $T^{\perp} = (\mathbb{Z}_n \setminus T)^{-1} = C_0 \cup C_{CL(n-\delta_{1,n})}$. Then $d(\mathcal{C}_{(q,n,\delta)}^{\perp}) \geq 2$.

1.1) If $2 \nmid s$, then

$$n - \delta_{1,n} = \frac{q^{2s} - 1 - ((q-1)q^{2s-1} - q^{s-1} - 1)}{q+1} = \frac{q^{2s-1} + q^{s-1}}{q+1}.$$

Thus $CL(n-\delta_{1,n})=\frac{q^s+1}{q+1}$ and $\mid C_{\frac{q^s+1}{q+1}}\mid =s.$ This means

$$\dim(\mathcal{C}_{(q,n,\delta)}^{\perp}) = n - |T^{\perp}| = n - s - 1.$$

Form the sphere packing bound,

$$q^{n-(s+1)} \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} (q-1)^i \binom{n}{i} \le q^n \Longrightarrow \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} (q-1)^i \binom{n}{i} \le q^{s+1}. \tag{7}$$

Suppose d = 3, then by Eq. (7), we have

$$1 + (q-1)\frac{q^{2s} - 1}{q+1} \le q^{s+1},$$

which is impossible due to s > 4. Thus d = 2.

1.2) If $2 \mid s$ and $s \neq 6$, then

$$n - \delta_{1,n} = \frac{q^{2s} - 1 - ((q-1)q^{2s-1} - q^s - 1)}{q+1} = \frac{q^{2s-1} + q^s}{q+1}.$$

Thus $CL(n-\delta_{1,n})=\frac{q^{s-1}+1}{q+1}$ and $\mid C_{\frac{q^{s-1}+1}{q+1}}\mid =2s.$ This means

$$\dim(\mathcal{C}_{(q,n,\delta)}^{\perp}) = n - |T^{\perp}| = n - 2s - 1.$$

Suppose $d(\mathcal{C}_{(q,n,\delta)}^{\perp}) = 5$, by Eq. (7), we know that

$$1 + (q-1)\frac{q^{2s} - 1}{q+1} + (q-1)^2 \frac{q^{2s} - 1}{q+1} \frac{q^{2s} - q - 2}{q+1} > q^{2s+1}.$$
 (8)

Thus $2 \leq d(\mathcal{C}_{(q,n,\delta)}^{\perp}) \leq 4$. Set $s_1 = \frac{q^{s-1}+1}{q+1}$, then the parity-check matrix of $\mathcal{C}_{(q,n,\delta)}^{\perp}$ is

$$H = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \beta_1^{s_1} & \beta_1^{2s_1} & \cdots & \beta_1^{(n-1)s_1} \end{pmatrix}.$$

We claim that any two columns are linear independent over \mathbb{F}_q . Suppose there exists $(c_1, c_2) \in \mathbb{F}_q \times \mathbb{F}_q \setminus \{(0,0)\}$ such that $c_1 + c_2 = 0$ and $c_1 \beta_1^{is_1} + c_2 \beta_1^{js_1} = 0$ for any $0 \le i < j \le n-1$, which equals to

$$\beta_1^{s_1 i} = \beta_1^{s_1 j} \iff \alpha^{(j-i)(q^{s-1}+1)} = 1 \iff (j-i)(q^{s-1}+1) \equiv 0 \pmod{q^{2s}-1}. \tag{9}$$

Since $\gcd(2s,s-1) = \gcd(s,s-1) = 1$, then $\gcd(q^{s-1}+1,q^{2s}-1) = q^{\gcd(s-1,2s)} + 1 = q+1$. Eq. (9) holds if and only if $i-j \equiv 0 \pmod{\frac{q^{2s}-1}{q+1}}$, this means i=j. Thus $d(\mathcal{C}_{(q,n,\delta)}^{\perp}) \geq 3$.

Case 2. For $\delta = \delta_{2,n}$, the defining set of $\mathcal{C}_{(q,n,\delta_{2,n})}^{\perp}$ is $T^{\perp} = (\mathbb{Z}_n \setminus T)^{-1} = C_0 \cup C_{CL(n-\delta_{1,n})} \cup C_{CL(n-\delta_{2,n})}$.

2.1) If $2 \nmid s$, then

$$n - \delta_{2,n} = \frac{q^{2s} - 1 - ((q-1)q^{2s-1} - q^{s+1} - 1)}{q+1} = \frac{q^{2s-1} + q^{s+1}}{q+1}.$$

Clearly, $CL(n-\delta_{2,n})=\frac{q^{s-2}+1}{q+1}$ and $\mid C_{\frac{q^{s-2}+1}{q+1}}\mid =2s.$ We have

$$\dim(\mathcal{C}_{(q,n,\delta_2)}^{\perp}) = n - |T^{\perp}| = n - 3s - 1.$$

By Eq. (8), we know $2 \le d(\mathcal{C}_{(q,n,\delta_{2,n})}^{\perp}) \le 4$. Take $s_1 = \frac{q^s+1}{q+1}$ and $s_2 = \frac{q^{s-2}+1}{q+1}$, then the parity-check matrix of $\mathcal{C}_{(q,n,\delta_2)}^{\perp}$ is

$$H = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \beta_1^{s_1} & \beta_1^{2s_1} & \cdots & \beta_1^{(n-1)s_1} \\ 1 & \beta_1^{s_2} & \beta_1^{2s_2} & \cdots & \beta_1^{(n-1)s_2} \end{pmatrix}.$$

Note that $d(\mathcal{C}_{(q,n,\delta_{2,n})}^{\perp})=2$ if and only if there exist i,j such that $0\leq i\neq j\leq n-1$ and

$$\begin{cases} \beta_1^{is_1} = \beta_1^{js_1} \\ \beta_1^{is_2} = \beta_1^{js_2} \end{cases} \Longrightarrow \begin{cases} \beta_1^{(j-i)s_1} = 1 \\ \beta_1^{(j-i)s_2} = 1 \end{cases}$$
 (10)

Eq. (10) equals to

$$(i-j)(q^s+1) \equiv 0 \pmod{q^{2s}-1}$$
 and (11)

$$(i-j)(q^{s-2}+1) \equiv 0 \pmod{q^{2s}-1}.$$
 (12)

By (11), we know that $i - j \equiv 0 \pmod{q^s - 1}$ where $0 \le i < j \le n - 1$. Thus Eq. (12) holds if and only if there exists an integer k such that $0 \le k \le \frac{q^s + 1}{q + 1} - 1$ and

$$k(q^{s-2}+1) \equiv 0 \pmod{q^s+1}.$$
 (13)

Note that $2 \nmid (s-2)$, then $\gcd(q^s+1,q^{s-2}+1)=q+1$, i.e., Eq. (13) holds if and only if $k \equiv 0 \pmod{\frac{q^s+1}{q+1}}$, this is impossible. Thus $3 \leq d(\mathcal{C}_{(q,n,\delta_{2,n})}^{\perp}) \leq 4$.

2.2) The case $2 \mid s$ and $s \neq 6$ is similar to 2.1).

Thus we complete the proof.

Theorem 19. Let q be a prime power and $2 \nmid q$, then we have the following.

- If s = 3, then $C_{(q,n,\delta)}$ with $\delta_{2,n} + 1 \le \delta \le \delta_{1,n}$ has parameters $\left[\frac{q^6-1}{q+1}, 4, d \ge \delta\right]$ and $C_{(q,n,\delta_{2,n})}$ has parameters $\left[\frac{q^6-1}{q+1}, 10, d \ge \delta_{2,n}\right]$.
- If s=4, then $C_{(q,n,\delta)}$ with $\delta_{2,n}+1\leq \delta\leq \delta_{1,n}$ has parameters $\left[\frac{q^8-1}{q+1},9,d\geq \delta\right]$ and $C_{(q,n,\delta_{2,n})}$ has parameters $\left[\frac{q^8-1}{q+1},17,d\geq \delta_{2,n}\right]$.
- If s = 6, then $C_{(q,n,\delta)}$ with $\delta_{2,n} + 1 \le \delta \le \delta_{1,n}$ has parameters $\left[\frac{q^{12}-1}{q+1}, 13, d \ge \delta\right]$ and $C_{(q,n,\delta_{2,n})}$ has parameters $\left[\frac{q^{12}-1}{q+1}, 25, d \ge \delta_{2,n}\right]$.

Proof. Form Lemmas 3 and 17, we can obtain the results directly.

Theorem 20. Let q be a prime power and s > 4, then we have the following.

• If $2 \nmid s$, then the code $\widetilde{C}_{(q,n,\delta)}^{\perp}$ with $\delta_{2,n} + 1 \leq \delta \leq \delta_{1,n}$ has parameters

$$\left[\frac{q^{2s}-1}{q+1}, \frac{q^{2s}-1}{q+1}-s, 2\right].$$

• If $2 \mid s$ and s > 6, then the code $\widetilde{C}_{(q,n,\delta)}^{\perp}$ with $\delta_{2,n} + 1 \leq \delta \leq \delta_{1,n}$ has parameters

$$\left[\frac{q^{2s}-1}{q+1}, \frac{q^{2s}-1}{q+1}-2s, 3\right].$$

Proof. By definition, we know that the defining set of $\widetilde{\mathcal{C}}_{(q,n,\delta)}^{\perp}$ with respect to β_1 is $T^{\perp} = (\mathbb{Z}_n \setminus T)^{-1} = C_{CL(n-\delta_{1,n})}$, then $d(\widetilde{\mathcal{C}}_{(q,n,\delta)}) \geq 2$. Note that $\widetilde{\mathcal{C}}_{(q,n,\delta)} \subset \mathcal{C}_{(q,n,\delta)}$, which means $\widetilde{\mathcal{C}}_{(q,n,\delta)}^{\perp} \supset \mathcal{C}_{(q,n,\delta)}^{\perp}$, then $d(\widetilde{\mathcal{C}}_{(q,n,\delta)}^{\perp}) \leq d(\mathcal{C}_{(q,n,\delta)}^{\perp})$.

If $2 \nmid s$, we have $2 \leq d(\widetilde{\mathcal{C}}_{(q,n,\delta)}^{\perp}) \leq d(\mathcal{C}_{(q,n,\delta)}^{\perp}) = 2 \Longrightarrow d(\widetilde{\mathcal{C}}_{(q,n,\delta)}^{\perp}) = 2$. Since $CL(n - \delta_{1,n}) = \frac{q^s + 1}{q + 1}$ and $|C_{\frac{q^s + 1}{q + 1}}| = s$. This means

$$\dim(\widetilde{\mathcal{C}}_{(a,n,\delta)}^{\perp}) = n - |C_{CL(n-\delta_1)}| = n - s.$$

Then the parameters of code $\widetilde{\mathcal{C}}_{(q,n,\delta)}^{\perp}$ are determined.

If $2 \mid s \text{ and } s > 6$, note that $CL(n - \delta_{1,n}) = \frac{q^{s-1} + 1}{q+1}$ and $|C_{\frac{q^{s-1} + 1}{q+1}}| = 2s$. This means

$$\dim(\widetilde{\mathcal{C}}_{(q,n,\delta)}^{\perp}) = n - |C_{CL(n-\delta_1)}| = n - 2s.$$

Set $s_1 = \frac{q^{s-1}+1}{q+1}$, then the parity-check matrix is

$$H = \begin{pmatrix} 1 & \beta_1^{s_1} & \beta_1^{2s_1} & \cdots & \beta_1^{(n-1)s_1} \end{pmatrix}.$$

Since $\gcd(q^{s-1}+1,q^{2s}-1)=q^{\gcd(s-1,2s)}+1=q+1$, then $\gcd(s_1,n)=1$. Therefore, $\beta_1^{s_1}$ is a primitive n-th root of unity, then $\widetilde{\mathcal{C}}_{(q,n,\delta)}^{\perp}$ is the Hamming code and $d(\widetilde{\mathcal{C}}_{(q,n,\delta)}^{\perp})=3$. Then the parameters of code $\widetilde{\mathcal{C}}_{(q,n,\delta)}^{\perp}$ are determined. Thus we complete the proof.

Example 21. Let (q, s) = (2, 5), we have n = 341 then $\delta_{1,n} = 165$ and $\delta_{2,n} = 149$. The BCH code $C_{(2,341,\delta)}$ with $149 \le \delta \le 165$ has parameters $[341, 6, d \ge \delta]$, the BCH code $C_{(2,341,149)}$ has parameters $[341, 16, d \ge 149]$, the code $C_{(2,341,149)}^{\perp}$ has parameters $[341, 325, 4 \ge d \ge 3]$.

Example 22. Let (q, s) = (3, 3), we have n = 182. Then $\delta_{2,n} = 101$. The BCH code $C_{(3,182,101)}$ has parameters $[182, 10, \geq 101]$.

For $2 \mid s$ and $s \geq 4$, we will investigate the dimension of the code $C_{(q,n,\delta)}$, where $\lceil \frac{q}{2} \rceil q^{s-1} \leq \delta \leq \frac{q^{s+1}+1}{q+1}$.

Lemma 23. Let $\lceil \frac{q}{2} \rceil q^{s-1} \le i < \frac{q^{s+1}+1}{q+1}$ and $[i]_q \ne 0$, then we have the following.

- (1) If $2 \mid q$ and let $i = \frac{a_s q^s + a_0}{q+1}$, where $a_s = q t$ and $a_0 = t + 1$ for all $1 \le t \le \frac{q-2}{2}$, then i is not a coset leader. Otherwise, i is a coset leader and $|C_i| = 2s$.
- (2) If $2 \nmid q$ and let $i = \frac{a_s q^s + a_0}{q+1}$, where $a_s = q t$ and $a_0 = t + 1$ for all $1 \leq t \leq \frac{q-3}{2}$, then i is not a coset leader. Otherwise, i is a coset leader and $|C_i| = 2s$.

Proof. We just give the proof for Case (1), since the proof for Case (2) is similar.

Note that $i \in \text{MinRep}_n$ if and only if $i(q+1) \in \text{MinRep}_{q^m-1}$ by Lemma 3, then we need to find a such that $a \in \text{MinRep}_{q^m-1}$, $\frac{q^s(q+1)}{2} \le a \le q^{s+1}$ and $(q+1) \mid a$.

Note that $\left[\frac{q^s(q+1)}{2}, q^{s+1}\right] \subset [q^s+1, q^{s+1}]$, then we divide $\frac{q^s(q+1)}{2} \leq a \leq q^{s+1}$ into the following three cases by Lemma 2.

- 1) $a = c(q^s + 1)$ for $1 \le c \le q 1$. Since $2 \mid q$ and $2 \mid s$, then $gcd(q^s + 1, q^2 1) = 1$, which implies that $gcd(q^s + 1, q + 1) = 1$. Hence, $q + 1 \nmid c(q^s + 1)$ for all $1 \le c \le q 1$, then we do not consider these values.
- 2) $a = a_s q^s + a_0$ for $1 \le a_0 < a_s \le q 1$. Since $a = a_s (q^s 1) + a_s + a_0$, then $q + 1 \mid a$ if $\begin{cases} a_s = q t \\ a_0 = t + 1 \end{cases}$ for all $1 \le t \le \frac{q}{2} 1$. Note that

$$a_s q^s + a_0 \ge (\frac{q}{2} + 1)q^s + \frac{q}{2} > \frac{q^s(q+1)}{2},$$

then $q+1 \mid a$ and $a \in \left[\frac{q^s(q+1)}{2}, q^{s+1}\right]$ if $\begin{cases} a_s = q-t \\ a_0 = t+1 \end{cases}$ for all $1 \le t \le \frac{q}{2}-1$, but $a \notin \text{MinRep}_{q^m-1}$.

3) Otherwise, for any $a \in [\frac{q^s(q+1)}{2}, q^{s+1}]$ and $q+1 \mid a$, we know that $a \in \text{MinRep}_{q^m-1}$ and $|C_a| = 2s$.

For $\frac{q^s(q+1)}{2} \le a < q^{s+1}+1$, we know that $a \notin \text{MinRep}_{q^m-1}$ and $(q+1) \mid a$ if and only if $a = a_s q^s + a_0$ and $\begin{cases} a_s = q - t \\ a_0 = t+1 \end{cases}$ for all $1 \le t \le \frac{q}{2} - 1$. Combining all the cases above, the desired conclusion then follows.

Theorem 24. Let $2 \mid s$ and $s \geq 4$. Then the dimension k of $C_{(q,n,\delta)}$ is given as follows, where $\lceil \frac{q}{2} \rceil q^{s-1} + 1 \leq \delta \leq \frac{q^{s+1}+1}{q+1}$.

(1) If $2 \mid q$, then we have

$$k = n - 2s\left(\delta + \frac{q-2}{2} - \left\lfloor \frac{(\delta-1)(q+1)}{q^s} \right\rfloor\right) + 2s \left\lfloor \frac{\delta-1}{q} \right\rfloor.$$

(2) If $2 \nmid q$, then we have

$$k = n - 2s\left(\delta + \frac{q-1}{2} - \left| \frac{(\delta-1)(q+1)}{q^s} \right| \right) + 2s \left| \frac{\delta-1}{q} \right| + s.$$

Proof. Form Lemmas 6 and 23, we can obtain the results directly.

Example 25. Let (q, s) = (2, 4), we have n = 85. The BCH code $C_{(2,85,9)}$ has parameters $[85, 53, \ge 9]$. The code is almost optimal in the sense that the minimum distance of the optimal binary linear code with length 85 and dimension 53 is 10 according to the tables of best codes known in ([11]) when the equality holds.

We will give the dimension and the minimum distance for special designed distance in the following theorem.

Theorem 26. Let a be an integer and $1 \le a \le q - 1$. Then the following holds.

(1) For $2 \nmid s$ and s > 3 (s > 5 if <math>q = 2),

(1.1) if
$$\delta = a \frac{q^s - 1}{q - 1}$$
, then $C_{(q, n, \delta)}$ has parameters $\left[\frac{q^{2s} - 1}{q + 1}, k, \delta\right]$, where

$$k = \begin{cases} n - 2s(\delta - 1) + 2s \left\lfloor \frac{\delta - 1}{q} \right\rfloor + s \left\lfloor \frac{(\delta - 1)(q + 1)}{q^s + 1} \right\rfloor, & \text{if } 1 \le a \le q - 3; \\ n - 2s(\delta - 1) + 2s \left\lfloor \frac{\delta - 1}{q} \right\rfloor + s(q + 1), & \text{if } a = q - 2; \\ n - s \left((q - 1)(2q^{s - 1} - 3) - 2 \right), & \text{if } a = q - 1. \end{cases}$$

(1.2) if $\delta = a \frac{q^s + 1}{q+1}$, then $C_{(q,n,\delta)}$ has parameters $\left[\frac{q^{2s} - 1}{q+1}, k, \delta\right]$, where

$$k = \begin{cases} n - 2s(\delta - 1) + 2s \left\lfloor \frac{\delta - 1}{q} \right\rfloor, & \text{if } a = 1; \\ n - s(2\delta - a - 1) + 2s \left\lfloor \frac{\delta - 1}{q} \right\rfloor, & \text{if } 2 \le a \le q - 1. \end{cases}$$

(2) For $2 \mid s$ and $s \geq 4$, if $\delta = a \frac{q^s - 1}{q^2 - 1}$, then $\mathcal{C}_{(q,n,\delta)}$ has parameters $\left[\frac{q^{2s} - 1}{q + 1}, k, \delta\right]$, where

$$k=n-2s(\delta-1)+2s\left\lfloor \frac{\delta-1}{q}\right\rfloor$$
.

Proof. We only prove Case (1), since the proof for Case (2) is similar.

Note that $q+1 \mid q^s+1$ and $q-1 \mid q^s-1$ if $2 \nmid s$, then $\frac{q^s-1}{q-1}$ and $\frac{q^s+1}{q+1}$ are integers. Since $q^s-1 \mid \frac{q^s+1}{q+1}(q^s-1)$ and $q^s+1 \mid \frac{q^s-1}{q-1}(q^s+1)$, then $\frac{q^s-1}{q-1} \mid \frac{n}{q-1}$ and $\frac{q^s+1}{q+1} \mid \frac{n}{q-1}$. Obviously, $q^2-1 \mid q^{2s}-1$, then $\gcd(n,q-1)=q-1$ and $\gcd(n,q)=1$. Hence, if $\delta=a\frac{q^s-1}{q-1}$ and $a\frac{q^s+1}{q+1}$, then we have $d(\mathcal{C}_{(q,n,\delta)})=\delta$ by Lemma 5.

For $\delta = a \frac{q^s - 1}{q - 1}$, we divide into the following two cases by Lemma 6.

- 1) For q = 2 and $s \ge 5$, then a = 1 and $\delta = \frac{2^s 1}{2 1}$. Note that, $(2 1)2^{s 1} + \frac{2^s + 1}{2 + 1} + 1 \le \delta \le 2^s + 1$, and $\dim(\mathcal{C}_{(2,n,\delta)}) = n s(2^s 5)$.
- 2) For q > 3 and s > 3,
 - if 1 < a < q 3, note that

$$\frac{q^s+1}{q+1}+1 \leq \frac{q^s-1}{q-1} \leq \delta \leq (q-3)\frac{q^s-1}{q-1} \leq (q-1)\frac{q^s+1}{q+1}+1,$$

and dim
$$(\mathcal{C}_{(q,n,\delta)}) = n - 2s(\delta - 1) + 2s \left| \frac{\delta - 1}{q} \right| + s \left| \frac{(\delta - 1)(q+1)}{q^s + 1} \right|$$
.

• if a = q - 2, note that

$$(q-1)\frac{q^s+1}{q+1}+2 \le (q-2)\frac{q^s-1}{q-1} \le \frac{q^{s+1}-1}{q+1}+2,$$

and dim
$$(\mathcal{C}_{(q,n,\delta)}) = n - 2s(\delta - 1) + 2s \left| \frac{\delta - 1}{q} \right| + s(q+1).$$

• if a = q - 1, note that

$$(q-1)q^{s-1} + \frac{q^s+1}{q+1} + 1 \le (q-1)\frac{q^s-1}{q-1} \le q^s+1,$$

and dim
$$(\mathcal{C}_{(q,n,q^s-1)}) = n - s((q-1)(2q^{s-1}-3)-2).$$

For $\delta = a \frac{q^s + 1}{q + 1}$, we still obtain the dimension by determining the range of δ , i.e., the result corresponds to (1.2). Thus we complete the proof.

Example 27. We have the following examples for the code of Theorem 26.

- Let (q, s) = (2, 4), we have n = 85. The BCH code $C_{(2,85,69)}$ has parameters [85, 69, 5]. The code is almost optimal according to the tables of best codes known in ([11]).
- Let (q, s) = (2, 5), we have n = 341. The BCH code $C_{(2,341,206)}$ has parameters [341,206,31]. The BCH code $C_{(2,341,291)}$ has parameters [341,291,11].

For b=1, the conditions of $\mathcal{C}_{(q,n,\delta)}$ being dually-BCH codes have been given by Lemma 8. For b=2, we have

Theorem 28. Let $q \geq 3$ and $s \geq 2$. Then the following statements hold.

(1) If s = 2, then $C_{(q,n,\delta,2)}$ is a dually-BCH code if and only if

$$\delta_{1,n} - 1 \le \delta \le n - 1.$$

(2) If $s \neq 2$, then $C_{(q,n,\delta,2)}$ is a dually-BCH code if and only if

$$\delta_{1,n} \leq \delta \leq n-1$$
.

Proof. We just prove Case (2), as the conclusion for Case (1) can be similarly proved.

By the definition, the defining set of $C_{(q,n,\delta,2)}$ with respect to β_1 is $T=C_2\cup C_3\cup\cdots\cup C_{\delta}$, $2\leq \delta\leq n-1$. Note that $0\notin T$, i.e., $0\in T^{\perp}$, this means $C_0\subset T^{\perp}$. Therefore, if $C_{(q,n,\delta,2)}$ is a dually-BCH code, then there exists $r\geq 1$ such that $T^{\perp}=C_0\cup C_1\cup\cdots\cup C_{r-1}$.

If $q \leq \delta \leq n-1$, then $C_1 = C_q \subset T$, i.e., $T = C_1 \cup C_2 \cup \cdots \cup C_\delta$. By Lemma 8, we know that $C_{(q, \frac{q^{2s}-1}{q+1}, \delta, 2)}$ is a dually-BCH code if and only if $\delta_{1,n} \leq \delta \leq n-1$.

If $2 \leq \delta \leq q-1$, note that $\frac{nq^{\lceil \frac{m}{2} \rceil}}{q^m-1} = \frac{q^s}{q+1}$. Since $s \geq 2$, then $[1,q-1] \subset \left[1,\frac{nq^{\lceil \frac{m}{2} \rceil}}{q^m-1}\right]$. Hence, $i \in \text{MinRep}_n$ and $|C_i| = 2s$ for all $1 \leq i \leq q-1$ by Lemma 1. We have $C_2 \subset T$ and $C_1 \not\subset T$, i.e., $2s \leq \dim(\mathcal{C}_{(q,n,\delta,2)}) \leq n-2s < n$. We consider two cases.

- 1) If $2 \nmid s$, note that $\frac{q^s+1}{q+1}$ is a coset leader modulo n and $\frac{q^s+1}{q+1} > q-1 \ge \delta$, i.e., $\frac{q^s+1}{q+1} \notin T$. Since $CL(n-\frac{q^s+1}{q+1}) = CL(\frac{q^{2s}-q^s-2}{q+1}) = \frac{(q-1)q^{2s-1}-q^{s-1}-1}{q+1} = \delta_{1,n}$, then $C_{\delta_{1,n}} \subset T^{\perp}$. Therefore, if $C_{(q,n,\delta,2)}$ is a dually-BCH code, then $T^{\perp} = C_0 \cup C_1 \cup \cdots \cup C_{\delta_{1,n}}$. However, $\dim(C_{(q,n,\delta,2)}) \le n-2s$, which contradicts the equation of $\dim(C_{(q,n,\delta,2)}) + \dim(C_{(q,n,\delta,2)}) = n$.
- 2) If $2 \mid s$, note that $\frac{q^{s-1}+1}{q+1}$ is a coset leader modulo n and $\frac{q^{s-1}+1}{q+1} > q-1 \ge \delta$, i.e., $\frac{q^{s-1}+1}{q+1} \notin T$. Since $CL(n-\frac{q^{s-1}+1}{q+1}) = \frac{(q-1)q^{2s-1}-q^s-1}{q+1} = \delta_{1,n}$, then $C_{\delta_{1,n}} \subset T^{\perp}$. Hence, if $C_{(q,n,\delta,2)}$ is a dually-BCH code, then $T^{\perp} = C_0 \cup C_1 \cup \cdots \cup C_{\delta_{1,n}}$. However, $\dim(C_{(q,n,\delta,2)}) \le n-2s$, which contradicts the equation of $\dim(C_{(q,n,\delta,2)}) + \dim(C_{(q,n,\delta,2)}) = n$.

Thus we complete the proof.

Theorem 29. Let q=2 and $s\geq 2$ be even. Then $C_{(q,n,\delta,2)}$ is a dually-CBH code if and only if

$$\delta_{1,n} \leq \delta \leq n-1$$
.

Proof. The proof is very similar to that of Theorem 28, hence we omit it.

5. The case of $n = \frac{q^m - 1}{q - 1}$

In this section, we always suppose $n = \frac{q^m - 1}{q - 1}$ and $\beta_2 = \alpha^{q - 1}$, where $q \ge 3$ and $m \ge 4$.

5.1. The computation of $\delta_{2,n}$

Lemma 30. Let $0 \le t < n$ and t be of the form

$$t = (0, \underbrace{q-1, \dots, q-1}_{n_{q-1}}, \underbrace{q-2, \dots, q-2}_{n_{q-2}}, \dots, \underbrace{i, \dots, i}_{n_i}, \dots, \underbrace{1, \dots, 1}_{n_1}).$$

- If $\sum_{i=l}^{q-1} n_i < j < \sum_{i=l+1}^{q-1} n_i$, where $l \in \{1, 2, \dots, q-1\}$, then $[tq^j]_n > t$.
- If $j = \sum_{i=l}^{q-1} n_i$, $q 1 \ge l \ge 1$, then

$$[tq^j]_n = (0, \underbrace{q-1, \dots, q-1}_{n_{l-1}}, \dots, \underbrace{q-l+1, \dots, q-l+1}_{n_1+1}, \underbrace{q-l, \dots, q-l}_{n_{q-1}}, \dots, \underbrace{1, \dots, 1}_{n_l-1}).$$

Proof. Note that $\frac{q^m-1}{q-1}|tq^i-[tq^i]_{q^m-1},$ i.e., $[tq^i]_{q^m-1}\equiv [tq^j]_n.$

1) If $\sum_{i=l}^{q-1} n_i > j > \sum_{i=l+1}^{q-1} n_i$, where $l \in \{2, 3, \dots, q-1\}$, then $[tq^j]_n$ is congruent to

$$[tq^j]_{q^m-1} = (\underbrace{l, \dots, l}_{n_l-u}, \underbrace{l-1, \dots, l-1}_{n_{l-1}}, \dots, 0, \underbrace{q-1, \dots, q-1}_{n_{q-1}}, \dots, \underbrace{l, \dots, l}_{u}),$$

where $u = j - \sum_{i=l+1}^{q-1} n_i - 1, 0 \le u < n_l - 1$. Note that $ln > [tq^j]_{q^m-1} > n$, then $[tq^j]_n = [tq^j]_{q^m-1} - (l-1)n > q^{m-1} > t$.

If $\sum_{i=1}^{q-1} n_i > j > \sum_{i=2}^{q-1} n_i$, then $[tq^j]_n$ is congruent to

$$[tq^j]_{q^m-1} = (\underbrace{1,\ldots,1}_{n_l-u},0,\underbrace{q-1,\ldots,q-1}_{n_{q-1}},\ldots,\underbrace{2,\ldots,2}_{n_2},\underbrace{l,\ldots,l}_{u}),$$

where $u = j - \sum_{i=2}^{q-1} n_i - 1$, $0 \le u < n_1 - 1$. Note that $[tq^j]_{q^m - 1} < n$, then $[tq^j]_n = [tq^j]_{q^m - 1} > q^{m-1} > t$.

2) If $j = \sum_{i=l}^{q-1} n_i$, $q - 1 \ge l \ge 1$, then

$$[tq^{j}]_{n} = [tq^{j}]_{q^{m}-1} - (l-1)n$$

$$= (0, \underbrace{q-1, \dots, q-1}_{n_{l-1}}, \dots, \underbrace{q-l+1, \dots, q-l+1}_{n_{1}+1}, \underbrace{q-l, \dots, q-l}_{n_{q-1}}, \dots, \underbrace{1, \dots, 1}_{n_{l}-1}).$$
(14)

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Thus we complete the proof.

Lemma 31. Let q > 3 and $m \ge q$. Suppose m - 1 = a(q - 1) + b, where $a \ge 1$ and $0 \le b \le q - 2$.

(1) If $a \ge 3$ and b = 0, i.e., m = a(q - 1) + 1, then

$$\delta_{2,n} = \frac{q^m - 1 - q^{m-1} - q^{m-a} - \sum_{l=1}^{q-3} q^{al-1}}{q - 1},$$

and $\mid C_{\delta_{2,n}} \mid = m$.

(2) If
$$b = 1$$
, i.e., $m = a(q - 1) + 2$, let $A = \lfloor \frac{q-1}{2} \rfloor$, then
$$\delta_{2,n} = \frac{q^m - 1 - q^{m-1} - \sum_{l=1}^{A-1} q^{al} - \sum_{l=A}^{q-2} q^{al+1}}{q-1},$$

and $|C_{\delta_2}| = m$.

(3) If b = 2, i.e., m = a(q-1) + 3, let $A = \lfloor \frac{q-1}{2} \rfloor$, then

$$\delta_{2,n} = \begin{cases} \frac{q^m - 1 - \sum_{t=1}^{q-1} q^{\lceil \frac{mt}{q-1} - 1 \rceil}}{q-1} - q^{(2A+1)a+1} + q^{(A+1)a}, & \text{if } q \equiv 0 \ (mod \ 3); \\ \frac{q^m - 1 - \sum_{t=1}^{q-1} q^{\lceil \frac{mt}{q-1} - 1 \rceil}}{q-1} - q^{2Aa+1}, & \text{if } q \equiv 1 \ (mod \ 3); \\ \frac{q^m - 1 - \sum_{t=1}^{q-1} q^{\lceil \frac{mt}{q-1} - 1 \rceil}}{q-1} - q^{Aa}, & \text{if } q \equiv 2 \ (mod \ 3), \end{cases}$$

and $\mid C_{\delta_{2,n}} \mid = m$.

Proof. We just provide the proof for Case (1), since the proofs for the other cases are similar. Note that

$$\frac{q^{m-1-q^{m-1}-q^{m-a}-\sum_{l=1}^{q-3}q^{al-1}}}{q-1} = (0, \underbrace{q-1, \ldots, q-1}_{a-1}, \underbrace{q-2, \ldots, q-2}_{a+2}, \ldots, \underbrace{i, \ldots, i}_{a}, \ldots, \underbrace{1, \ldots, 1}_{a-1}).$$

It then follows that $\delta = \frac{q^m - 1 - q^{m-1} - q^{m-a} - \sum_{l=1}^{q-3} q^{al-1}}{q-1} \in \text{MinRep}_n \text{ and } | C_{\delta} | = m \text{ by Lemma 30.}$ We claim that $\delta_{2,n} = \frac{q^m - 1 - q^{m-1} - q^{m-a} - \sum_{l=1}^{q-3} q^{al-1}}{q-1}$. Suppose there exists an integer s such that $s \in \text{MinRep}_n$ and $\delta_{2,n} < s < \delta_{1,n}$. Let the q-adic expansion of s be $\sum_{i=1}^{m-1} s_i q^i$. Note that

$$\delta_{1,n} = (0, \underbrace{q-1, \dots, q-1}_{c}, \underbrace{q-2, \dots, q-2}_{c}, \dots, \underbrace{i, \dots, i}_{c}, \dots, \underbrace{1, \dots, 1}_{c}).$$

Then we have $s_{m-1} = 0$, $s_i = q - 1$ for all $m - a \le i \le m - 2$ and $s_{m-1-a} = q - 1$ or $s_{m-1-a} = q - 2$.

1) If $s_{m-1-a}=q-1$, then $i_l=q-1$ for all $m-1-a\leq l\leq m-2$. By Lemma 4, s must be of the form

$$(0, \underbrace{q-1,\ldots,q-1}_{a}, \underbrace{q-2,\ldots,q-2}_{n_{q-2}},\ldots,\underbrace{i,\ldots,i}_{n_{i}},\ldots,\underbrace{1,\ldots,1}_{n_{1}}).$$

Since $[sq^j]_n \ge s$ for $j = \sum_{i=l}^{q-1} n_i$, then $n_l \ge a$ for all $2 \le l \le q-2$ and $n_1 \ge a-1$ by Eq. (14). In addition, m = a(q-1) + 1, then $n_1 = a$ or $n_1 = a - 1$. Hence, if $n_1 = a$ then $s = \delta_{1,n}$; if $n_1 = a - 1$ then $s > \delta_{1,n}$, which contradicts the fact that $s < \delta_{1,n}$.

2) If $s_{m-1-a} = q - 2$, we divide into two steps to prove.

Step 1. We claim that s is of the form

$$(0, \underbrace{q-1, \dots, q-1}_{a-1}, \underbrace{q-2, \dots, q-2}_{n_{q-2}}, \dots, \underbrace{i, \dots, i}_{n_i}, \dots, \underbrace{1, \dots, 1}_{n_1}). \tag{15}$$

If there exists an integer i such that $s_i \neq 0$ and $s_i < s_{i-1}$, then we have $[sq^{m-1-i}]_n =$ $[sq^{m-1-i}]_{q^m-1} - s_i n < s$, which contradicts the fact that $s \in \text{MinRep}_n$. Hence we know that $s_i \geq s_{i-1}$ if $s_i \neq 0$. Then $s = (I_e, I_{e-1}, \dots, I_0)$, where

$$I_t = (0, \underbrace{q-1, \dots, q-1}_{n_{t,q-1}}, \underbrace{q-2, \dots, q-2}_{n_{t,q-2}}, \dots, \underbrace{1, \dots, 1}_{n_{t,1}})$$

for all $0 \le t \le e$ and $n_{t,j} \ge 0$ for all $1 \le j \le q - 1$. Clearly, $n_{e,q-1} = a - 1$.

Let $k = \sum_{j=1}^{q-1} n_{0,j}$. Since $[sq^{m-1-k}]_n \ge s$, then $n_{0,q-1} \ge n_{e,q-1} = a-1$. Similarly, we get $n_{t,q-1} \ge a-1$ for all $e \ge t \ge 0$. Denote the q-adic expansion of $[sq^{n_{e,q-1}}]_n$ by $\sum_{i=1}^{m-1} s_i' q^i$, then $(s_{m-1}', s_{m-2}', \ldots, s_1')$ must be of the form $(I_e', I_{e-1}', \ldots, I_0')$ by Eq. (14), where

$$I'_{e} = (0, \underbrace{q-1, \dots, q-1}_{n_{e,q-2}}, \dots, \underbrace{3, \dots, 3}_{n_{e,2}}, \underbrace{2, \dots, 2}_{n_{e,1}+1}, \underbrace{1, \dots, 1}_{n_{e-1,q-1}-1}),$$

$$I_0' = (0, \underbrace{q-1, \dots, q-1}_{n_{0,q-2}}, \dots, \underbrace{3, \dots, 3}_{n_{0,2}}, \underbrace{2, \dots, 2}_{n_{0,1}+1}, \underbrace{1, \dots, 1}_{n_{e,q-1}-1}).$$

Since $[sq^{n_{e,q-1}}]_n \ge s$, then $n_{e,q-2} \ge a-1$.

Similarly, we have $n_{t,j} \ge a-1$ for all $e \ge t \ge 0$ and $2 \le j \le q-2$ and $n_{t,1} \ge a-2$ for all $e \ge t \ge 0$. Therefore,

$$m = \sum_{t=0}^{e} \left(\sum_{j=1}^{q-1} n_{t,j} + 1 \right) \ge (e+1)(q-1)(a-1).$$

If $e \ge 1$, we have $m = a(q-1) + 1 \ge 2(q-1)(a-1)$, which contradicts to $a \ge 3$. Hence, s is of the form Eq. (15), where $n_i \ge a-1$ for all $q-2 \ge i \ge 2$ and $n_1 \ge a-2$.

Step 2. Since $s > \delta_{2,n}$, then $n_{q-2} \ge a+3$, or $n_{q-2} = a+2$ and there exists an integer i such that $n_i \ge a+1$ and $n_j = a$ for all $q-3 \ge j > i$.

(2.1) If $n_{q-2} \ge a+3$, since m=a(q-1)+1, then there exists $n_j=a-1$ with $q-3 \ge j \ge 2$, or $n_t=a$ for all $q-3 \ge t \ge 2$ and $n_1=a-2$.

If there exists $n_j = a - 1$ with $q - 3 \ge j \ge 2$. Let $|\{n_i \ge a + 1 : 1 \le i < q - 2\}| = k$, then $|\{n_i = a - 1 : 1 \le i < q - 2\}| \ge k + 2$. Hence, there are two integers u and v such that $n_u, n_v = a - 1$ and $n_i \le a$ for all $u \ge i \ge v$. Note that

$$\left[sq^{\sum_{i=u+1}^{q-1}n_i}\right]_n = (0, \underbrace{q-1, \dots, q-1}_{a-1}, \underbrace{q-2, \dots, q-2}_{n_{u-1}}, \dots, \underbrace{q+v-u-1, \dots, q+v-u-1}_{a-1}, \dots, \underbrace{1, \dots, 1}_{n_{u+1}-1}).$$

Clearly, $\left[sq^{\sum_{i=u+1}^{q-1}n_i}\right]_n < s$, which contradicts the fact that $s \in \text{MinRep}_n$.

If $n_t = a$ for all $q - 3 \ge t \ge 2$ and $n_1 = a - 2$, we have $\left[sq^{\sum_{i=2}^{q-1} n_i} \right]_n < s$, which contradicts the fact that $s \in \text{MinRep}_n$.

(2.2) If $n_{q-2} = a+2$ and there exists an integer i such that $n_i \geq a+1$, $n_j = a$ for all $q-3 \geq j > i$, the case is similar to (2.1).

Thus we complete the proof.

Lemma 32. Let q > 3 and $m \ge q$. Suppose m - 1 = a(q - 1) + b, where $a \ge 1$ and $0 \le b \le q - 2$.

(1) If
$$b = q - 4$$
, i.e., $m = a(q - 1) + q - 3$, let $A = \lfloor \frac{q}{2} \rfloor$, then

$$\delta_{2,n} = \frac{q^m - 1 - \sum_{t=1}^{q-1} q^{\lceil \frac{mt}{q-1} - 1 \rceil}}{q-1} - q^{A(a+1)-2},$$

and $\mid C_{\delta_{2,n}} \mid = m$.

(2) If b = q - 3, i.e., m = a(q - 1) + q - 2, then

$$\delta_{2,n} = \frac{q^m - 1 - \sum_{t=1}^{q-1} q^{\lceil \frac{mt}{q-1} - 1 \rceil}}{q-1} - q^a,$$

and $\mid C_{\delta_{2,n}} \mid = m$.

(3) If b = q - 2, i.e., m = (a + 1)(q - 1), then

$$\delta_{2,n} = \frac{q^m - 1 - q^{m-1} - q^{m-1-a} - \sum_{l=1}^{q-3} q^{(a+1)l-1}}{q-1},$$

and $\mid C_{\delta_{2,n}} \mid = m$.

Proof. We just give the proof for Case (1), since the proofs for the other cases are similar. If $2 \nmid q$, then $\delta = \frac{q^m - 1 - \sum_{t=1}^{q-1} q^{\lceil \frac{mt}{q-1} - 1 \rceil}}{q-1} - q^{A(a+1)-2}$ is of the form

$$(0,\underbrace{q-1,\ldots,q-1}_{a},\ldots,\underbrace{t,\ldots,t}_{a+1},\ldots,\underbrace{\frac{q+1}{2},\ldots,\frac{q+1}{2}}_{,},\underbrace{\frac{q-1}{2},\ldots,\frac{q-1}{2}}_{,},\ldots,\underbrace{i,\ldots,i}_{a+1},\ldots,\underbrace{1,\ldots,1}_{a}).$$

It then follows that $\delta \in \text{MinRep}_n$ and $|C_{\delta}| = m$ by Lemma 30. We claim that $\delta_{2,n} = \frac{q^m - 1 - \sum_{t=1}^{q-1} q^{\lceil \frac{mt}{q-1} - 1 \rceil}}{q-1} - q^{A(a+1)-2}$. Suppose there exists an integer s such that $s \in \text{MinRep}_n$ and $\delta_{2,n} < s < \delta_{1,n}$. Let the q-adic expansion of s be $\sum_{i=1}^{m-1} s_i q^i$. Note that

$$\delta_{1,n} = (0, \underbrace{q-1, \dots, q-1}_{a}, \dots, \underbrace{t, \dots, t}_{a+1}, \dots, \underbrace{\frac{q+1}{2}, \dots, \frac{q+1}{2}}_{2}, \underbrace{\frac{q-1}{2}, \dots, \frac{q-1}{2}}_{2}, \dots, \underbrace{i, \dots, i}_{a+1}, \dots, \underbrace{1, \dots, 1}_{a}).$$

Since $\delta_{2,n} < s < \delta_{1,n}$, we have $s_i = q-1$ for all $m-1-a \le i \le m-2$. By Lemma 4, s must be of the form

$$(0,\underbrace{q-1,\ldots,q-1}_{a},\ldots,\underbrace{t,\ldots,t}_{a+1},\ldots,\underbrace{\frac{q+1}{2},\ldots,\frac{q+1}{2}}_{n_{\frac{q+1}{2}}},\underbrace{\frac{q-1}{2},\ldots,\frac{q-1}{2}}_{n_{\frac{q-1}{2}}},\ldots,\underbrace{i,\ldots,i}_{n_{i}},\ldots,\underbrace{1,\ldots,1}_{n_{1}}).$$

Note that

$$\left[sq^{\sum_{i=2}^{q-1}n_i}\right]_n = (0, \underbrace{q-1, \dots, q-1}_{n_1+1}, \underbrace{q-2, \dots, q-2}_{a}, \dots, \underbrace{t, \dots, t}_{a+1}, \underbrace{\frac{q-1}{2}, \dots, \frac{q-1}{2}}_{n_{\frac{q+1}{2}}}, \dots, \underbrace{1, \dots, 1}_{n_2-1}),$$

then $n_1 \ge a$ since $\left[sq^{\sum_{i=2}^{q-1} n_i} \right]_n \ge s$. Similarly, we have $n_i \ge a$ for all $\frac{q+1}{2} \ge i \ge 1$. Since $\delta_{2,n} < s < \delta_{1,n}$, then $n_{\frac{q+1}{2}} = a+1$ or $n_{\frac{q+1}{2}} = a$.

1) When $n_{\frac{q+1}{2}} = a+1$. Since $\delta_{1,n} > s$, then there exists an integer l such that $n_l = a$ and $n_j = a+1$ for all $\frac{q-1}{2} > j > l > 1$. Note that

$$\left[sq^{\sum_{i=\frac{q+1}{2}}^{q-1}n_i}\right]_n = (0, \underbrace{q-1, \ldots, q-1}_{a}, \ldots, \underbrace{t, \ldots, t}_{a+1}, \ldots, \underbrace{l+\frac{q-1}{2}, \ldots, l+\frac{q-1}{2}}_{q}, \ldots, \underbrace{1, \ldots, 1}_{a}),$$

Since $q-1-\frac{q-1}{2}>q-1-\frac{q-1}{2}-l$, we have $\left[sq^{\sum_{i=\frac{q+1}{2}}^{q-1}n_i}\right]< s$, which contradicts the fact that s is a coset leader.

2) When $n_{\frac{q+1}{2}} = a$.

If $n_{\frac{q-1}{2}} \ge a+2$, there are two integers u and v such that $n_u, n_v = a$ and $n_i \le a+1$ for all $\frac{q-3}{2} \ge u \ge i \ge v \ge 1$. Note that

$$\left[sq^{\sum_{i=u+1}^{q-1}n_i}\right]_n = (0, \underbrace{q-1, \dots, q-1}_{a}, \underbrace{q-2, \dots, q-2}_{n_{u-1}}, \dots, \underbrace{q+v-u-1, \dots, q+v-u-1}_{a}, \dots, \underbrace{1, \dots, 1}_{n_{u+1}-1}).$$

Since $q-1-\frac{q+1}{2}>u-v$, then we have $\left[sq^{\sum_{i=u+1}^{q-1}n_i}\right]_n < s$, which contradicts the fact that s is a coset leader.

If $n_{\frac{q-1}{2}} = a+1$, there must exist an integer l such that $n_l \geq a+2$ and $n_i = a+1$ for all $\frac{q-3}{2} \geq i > l > 1$. We have $|\{n_i = a : l > i \geq 1\}| \geq 2$. Hence, there are two integers u and v such that $n_u, n_v = a$ and $n_i \leq a+1$ for all $l > u \geq i \geq v \geq 1$. We can get that $\left[sq^{\sum_{i=u+1}^{q-1} n_i}\right]_n < s$, which contradicts the fact that s is coset leader.

In the same way, we can prove $\delta_{2,n}$ is the second largest coset leader when $2 \mid q$. This completes the proof.

5.2. BCH Codes and Dually-BCH Codes

Then we have the following conclusion when the length of the BCH codes $C_{(q,n,\delta)}$ satisfy the cases of Lemmas 31 and 32.

Theorem 33. Let q > 3 and m be of the form given by Lemmas 31 or 32. Then the BCH code $C_{(q,n,\delta)}$ with $\delta_{2,n} + 1 \le \delta \le \delta_{1,n}$ has parameters

$$\left[\frac{q^m-1}{q-1}, \frac{m}{\gcd(m,q-1)} + 1, d \ge \delta\right]$$

and the BCH code $C_{(q,n,\delta_{\delta_2,n})}$ has parameters

$$\left[\frac{q^m-1}{q-1}, \frac{m}{qcd(m,q-1)} + m + 1, d \ge \delta_{2,n}\right],$$

Proof. Form Lemmas 4, 31 and 32, we can obtain the results directly.

Example 34. Let (q, m) = (4, 5), we have n = 341. The BCH code $C_{(4,341,\delta)}$ with $230 \le \delta \le 233$ has parameters $[341, 6, \ge \delta]$, and the code $C_{(4,341,229)}$ has parameters $[341, 11, \ge 229]$.

Lemma 35. Let $q \geq 3$ and $m \geq 4$. If $2 \leq \delta \leq q-1$, then we have $\delta_{1,n} \in T^{\perp}$.

Proof. Let $m=a(q-1)+b,\ a\geq 0$ and $0\leq b\leq q-2$. Since $T^{\perp}=\mathbb{Z}_n\backslash T^{-1}$, then $C_{\delta_{1,n}}\subset T^{\perp}$ if and only if $C_{(n-\delta_{1,n})}\not\subseteq T$. By Lemma 4, we have $(q-1)(n-\delta_{1,n})=\sum_{t=1}^{q-1}q^{\lceil\frac{mt}{q-1}-1\rceil}=\sum_{i=0}^{m-1}a_iq^i$, where $a_i=\left\lceil\frac{q-1}{m}\right\rceil$ or $a_i=\left\lfloor\frac{q-1}{m}\right\rfloor$. Let $\delta'\in \mathrm{MinRep}_{q^m-1}$ and $a\in C_{(q-1)(n-\delta_{1,n})}$.

If a=0, then q-1>m. Note that $a_i=\left\lceil\frac{q-1}{m}\right\rceil\geq 2$ or $a_i=\left\lfloor\frac{q-1}{m}\right\rfloor\geq 1$ for all $0\leq i\leq m-1$, then $\delta'\geq \sum_{i=0}^{m-1}q^i$ by Lemma 4. Then

$$CL(n - \delta_{1,n}) = \frac{\delta'}{q-1} \ge \frac{\sum_{i=0}^{m-1} q^i}{q-1} > q+1 > \delta.$$

If $a \geq 1$, similarly, we have $CL(n-\delta_{1,n})$ has the form $(\ldots,1,\underbrace{0,\ldots,0}_a)$, i.e., $CL(n-\delta_{1,n}) > q^{m-a-1}-1$. It is easy to check that $m-a-1\geq a+1$ when $m\geq 4$. We have

$$CL(n-\delta_{1,n}) = \frac{\delta'}{q-1} > \frac{q^{m-a-1}}{q-1} \ge \frac{q^{a+1}-1}{q-1} > q+1 > \delta.$$

Therefore, $C_{(n-\delta_{1,n})} \nsubseteq T$, i.e., $\delta_{1,n} \in T^{\perp}$. This completes the proof.

For b=1, the conditions of $\mathcal{C}_{(q,n,\delta)}$ being dually-BCH codes have been given by Lemma 8. For b=2, we have

Theorem 36. Let $q \geq 3$ and $m \geq 4$, then $C_{(q,n,\delta,2)}$ is a dually-CBH code if and only if

$$\frac{q^m - 1 - \sum_{t=1}^{q-1} q^{\left\lceil \frac{mt}{q-1} - 1 \right\rceil}}{q-1} \le \delta \le n-1.$$

Proof. The proof is very similar to that of Theorem 28, we omit it.

6. Conclusions

The main contributions of this paper are as follows:

- For the codes of length $n = q^m 1$, we found the *i*-th largest *q*-cyclotomic coset leader is $\delta_i = (q-1)q^{m-1} 1 q^{\lfloor \frac{m-1}{2} \rfloor + i 2}$. The parameters of $\mathcal{C}_{(q,n,\delta_{i,n})}$ was investigated (see Theorem 11).
- For the codes of length $n = \frac{q^{2s}-1}{q+1}$, we find the second largest coset leader $\delta_{2,n}$. The parameters of $\mathcal{C}_{(q,n,\delta)}$ with $\delta_{2,n} \leq \delta \leq \delta_{1,n}$ and its dual code was settled (see Theorems 18-20). The dimension of $\mathcal{C}_{(q,n,\delta)}$ were determined, where $\lceil \frac{q}{2} \rceil q^{s-1} \leq \delta \leq \frac{q^{s+1}+1}{q+1}$ and $2 \mid s$ (see Theorem 24). Finally, we gave the dimension and the minimum distance of three subclasses of $\mathcal{C}_{(q,n,\delta)}$ for $\delta = a\frac{q^s-1}{q-1}$, $a\frac{q^s+1}{q+1}$ if $2 \nmid s$ and $\delta = a\frac{q^s-1}{q^2-1}$ if $2 \mid s$, $1 \leq a \leq q-1$ (see Theorem 26).
- For the codes of length $n = \frac{q^m 1}{q 1}$, we found the second largest coset leader $\delta_{2,n}$ for some special cases. The parameters of $\mathcal{C}_{(q,n,\delta)}$ with $\delta_{2,n} \leq \delta \leq \delta_{1,n}$ were investigated (see Theorem 33).
- Sufficient and necessary conditions for $C_{(q,n,\delta,2)}$ being dually-BCH codes were given, where $n=q^m-1,\frac{q^m-1}{q-1}$ and $\frac{q^{2s}-1}{q+1}$ (see Theorems 12, 13, 28, 29, 36). Moreover, we found the sufficient and necessary conditions for the dual code of $\widetilde{C}_{(q,q^m-1,\delta)}$ to be a narrow-sence primitive BCH code (see Theorems 14 and 15).

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