On mixed radial Moore graphs of diameter 3

J.M. Ceresuela^{a,b}, N. López^a, D. Chemisana^b.

^aDep. de Matemàtica, Universitat de Lleida, Lleida (Spain)

^bApplied Physics Section of the Environm. Sci. Dept. Universitat de Lleida, Lleida (Spain) {jesusmiguel.ceresuela,nacho.lopez,daniel.chemisana}@udl.cat

Abstract

Radial Moore graphs and digraphs are extremal graphs related to the Moore ones where the distance-preserving spanning tree is preserved for some vertices. This leads to classify them according to their proximity to being a Moore graph or digraph. In this paper we deal with mixed radial Moore graphs, where the mixed setting allows edges and arcs as different elements. An exhaustive computer search shows the top ranked graphs for an specific set of parameters. Moreover, we study the problem of their existence by providing two infinite families for different values of the degrees and diameter 3. One of these families turns out to be optimal.

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1 Introduction

Given the values of the maximum out-degree z and the diameter k, there is a natural upper bound $n_{z,k}$ for the largest order of a digraph with these two parameters,

$$n(z,k) = 1 + z + \dots + z^k,$$

referred to as the *Moore bound* for digraphs. Digraphs attaining such a bound are called *Moore digraphs*. In particular, all vertices of a Moore digraph have the same degree (d) and the same eccentricity (k). It is well known that Moore digraphs do only exist in the trivial cases, d = 1 or k = 1, which correspond to the directed cycle of order k + 1 and the complete digraph of order d + 1, respectively (see [16, 2]). This has led to the study of digraphs 'close' to the Moore ones. One way to do it is by allowing the existence of vertices with eccentricity just one more than the value they should have. In this context, regular digraphs of degree z, radius k, diameter at most k + 1 and order equal to n(z, k) are known as radial Moore digraphs. These extremal digraphs were first studied by Knor [12] and they exist for

any value of z and k (see also Gimbert and López [9]).

Lately, the definition of radial Moore digraph was extended to undirected graphs. In this undirected case, given the values of the maximum degree r and the diameter k of a graph, there Moore bound is given by N(r, k) where

$$N(r,k) = 1 + r + r(r-1) + \dots + r(r-1)^{k-1},$$
(1)

Graphs attaining such a bound are referred to as *Moore graphs*. From its definition, all vertices of a Moore graph have the same degree (d) and the same eccentricity (k). So, the definition of *radial Moore graphs* as regular graphs of degree r, radius k, diameter k + 1 and order equal to N(r, k) is just an undirected version of the same definition for digraphs.

Curiously enough, the undirected version of this problem has proved to be much more difficult and it is not obvious whether or not there exists a radial Moore graph for all possible values of degree and diameter. First Capdevila et al. [5] proved the existence of radial Moore graphs for radius 2 and any degree. They also considered some ranking measures of how well a radial Moore graph approximates a Moore graph. Later on, Exoo et al. [8] gave a construction of radial Moore graphs of diameter 3 for all degrees $d \ge 22$ and some other sporadic values of k and d. In 2015, Gomez and Miller [10] presented a construction technique that produces radial Moore graphs for every value of diameter and degree large enough (depending on the diameter). Nevertheless, there exist infinitely many values of r and kfor which the problem of existence of radial Moore graphs remains open.

Terminology and notation

A mixed (or partially directed) graph G with vertex set V may contain a set E of (undirected) edges as well a set A of directed edges (also known as arcs). From this point of view, a graph [resp. directed graph or digraph] has all its edges undirected [resp. directed]. The set of vertices which are adjacent from [to] a given vertex v is denoted by $\Gamma^+(v)$ [$\Gamma^-(v)$]. The undirected degree of a vertex v, denoted by d(v) is the number of edges incident to v. The out-degree [resp. in-degree] of vertex v, denoted by $d^+(v)$ [resp. $d^-(v)$], is the number of arcs emanating from [resp. to] v. If $d^+(v) = d^-(v) = z$ and d(v) = r, for all $v \in V$, then G is said to be totally regular of degrees (r, z)(or simply (r, z)-regular). A walk of length $\ell \geq 0$ from u to v is a sequence of $\ell + 1$ vertices, $u_0 u_1 \dots u_{\ell-1} u_\ell$, such that $u = u_0$, $v = u_\ell$ and each pair $u_{i-1}u_i$, for $i = 1, \ldots, \ell$, is either an edge or an arc of G. A directed walk is a walk containing only arcs. An *undirected walk* is a walk containing only edges. A walk whose vertices are all different is called a *path*. The length of a shortest path from u to v is the *distance* from u to v, and it is denoted by dist(u, v). Note that dist(u, v) may be different from dist(v, u), when

shortest paths between u and v involve arcs. The sum of all distances from a vertex $v, s(v) = \sum_{u \in V} d(v, u)$, is referred to as the *status* of v (see [3]). We define the *status vector* of G, $\mathbf{s}(G)$, as the vector constituted by the status of all its vertices. Usually, when the vector is long enough, we denote it with a short description using superscripts, that is, $\mathbf{s}(G) : s_1^{n_1}, s_2^{n_2}, \ldots, s_k^{n_k}$, where $s_1 > s_2 > \cdots > s_k$, and n_i denotes the number of vertices having s_i as its local status, for all $1 \le i \le k$. The *out-eccentricity* of a vertex u is the maximum distance from u to any vertex in G. A *central vertex* is a vertex having minimum out-eccentricity. The maximum distance between any pair of vertices is the *diameter* of G.

Mixed graphs can be seen as a generalization of both, undirected and directed graphs, see for instance the works by Nguyen and Miller [14] and Buset et al. [4].

Organization of the paper

This paper is organized as follows: In section 2 the reader will find a generalization of radial Moore graphs to the mixed case. A closeness measure is also presented there. We find all mixed radial Moore graphs for the case of (r, z, k) = (2, 1, 2), and we present the closest ones to being a mixed Moore graph in section 3. Next, in section 4, we study the problem of the existence of mixed radial Moore graphs of diameter 3. To this end, we provide two infinite families H_r (for $r \ge 1$ and z = 1) and G_z (for r = 1 and $z \ge 1$). We use an algebraic method to design H_r . Besides, G_z is constructed performing a convinient swap to a pair of arcs of the family of Kautz digraphs. Next we prove the optimality of this family. Some open problems and concluding remarks are presented in final section 5.

2 Mixed radial Moore graphs

The degree/diameter problem for mixed graphs asks for the largest possible number of vertices n(r, z, k) in a mixed graph with maximum undirected degree r, maximum directed out-degree z, and diameter k. A natural upper bound for n(r, z, k) is derived by counting the number of vertices at every distance from any given vertex v in a mixed graph with given maximum undirected degree r, maximum directed out-degree z, and diameter k. This bound is known as the *Moore bound* for mixed graphs (see [4] and [7]):

$$M(r, z, k) = A \frac{\lambda_1^{k+1} - 1}{\lambda_1 - 1} + B \frac{\lambda_2^{k+1} - 1}{\lambda_2 - 1},$$
(2)

where

$$\tau = (z+r)^2 + 2(z-r) + 1, \tag{3}$$

$$\lambda_1 = \frac{1}{2}(z+r-1-\sqrt{\tau}), \qquad \lambda_2 = \frac{1}{2}(z+r-1+\sqrt{\tau}), \qquad (4)$$

$$A = \frac{\sqrt{\tau} - (z + r + 1)}{2\sqrt{v}}, \qquad B = \frac{\sqrt{\tau} + (z + r + 1)}{2\sqrt{\tau}}.$$
 (5)

It is a matter of routine to check that M(0, z, k) = n(z, k) and M(r, 0, k) = N(r, k). So the mixed Moore bound is a generalization of both undirected Moore bound and directed Moore bound, as expected. *Mixed Moore graphs* are those with order attaining (2), which means that between any pair of vertices there is a unique shortest path of length not greater than the diameter. Mixed Moore graphs must be totally regular of degree (r, z) (see Bosák [1]) and the distance-preserving spanning tree, also known as Moore tree, hanging at any vertex v does not depend on the chosen vertex. (see Fig. 1).



Figure 1: Moore tree of a radial Moore graph of diameter 2.

Next, we define a mixed radial Moore graph as a natural generalization of both radial Moore graph and radial Moore digraph.

Definition 2.1. A totally (r, z)-regular graph of radius k, diameter k + 1 and order M(r, z, k) is a (r, z, k)-mixed radial Moore graph.

Notice that a (0, z, k)-mixed radial Moore graph is simply a radial Moore digraph and a (r, 0, k)- mixed radial Moore graph is a radial Moore graph. The vertex set of any (r, z, k)-mixed radial Moore can be partitioned in two sets where one set contains all central vertices (those vertices with out-eccentricity k, that is, their corresponding distance preserving spannig tree is a Moore tree) and the other set has all non-central vertices (with out-eccentricity k + 1).

Mixed radial Moore graphs can be seen as an approximation of mixed Moore graphs where the diameter constraint has been relaxed a bit. This relaxation may produce a significative number of graphs (as we will see later) that we want to rank in order to see which one is the 'closest' graph to being a Moore graph. The number of central vertices could be a first attempt to measure this closeness, but, as we will see later, this measure is not finer since there are many mixed radial Moore graphs with the same number of central vertices.

Two different ranking measures were introduced in [5] for the undirected case. One ranking measure is related to the girth, since Moore (undirected) graphs attain the minimum order allowed by the girth. But this measure is no longer useful for the mixed case where short directed cycles appear in mixed Moore graphs. The other ranking measure is based on the status of a vertex in a Moore graph. First we give a characterization of a mixed Moore graph in terms of the statuses of their vertices.

Proposition 2.2. Given three positive integers r > 1, z > 1 and k > 1, let G be a (r, z, k)-mixed radial Moore graph. Then, for every vertex v of G we have

$$s(v) \ge A\left(\frac{k\lambda_1^{k+2} - (k+1)\lambda_1^{k+1} + \lambda_1}{(\lambda_1 - 1)^2}\right) + B\left(\frac{k\lambda_2^{k+2} - (k+1)\lambda_2^{k+1} + \lambda_2}{(\lambda_2 - 1)^2}\right)$$

where $\lambda_1, \lambda_2, A, B$ are defined in equations (4) and (5). Moreover, this bound is attained for every vertex if and only if G is a mixed Moore graph.

Proof. The distance-preserving spanning tree of any central vertex v in G is completely determined (see Figure 1 for the case k = 2). Let N_i be the number of vertices at distance i from v, with $N_i = R_i + Z_i$, where R_i is the number of vertices that, in the corresponding tree rooted at v, have an edge with their parents; and Z_i is the number of vertices that have an arc from their parents. Then,

$$N_i = R_i + Z_i = R_{i-1}((r-1) + z) + Z_{i-1}(r+z).$$
(6)

Besides, $Z_i = z(N_{i-1} - Z_{i-1}) + zZ_{i-1} = zN_{i-1}$ and, hence,

$$N_{i} = (r+z)N_{i-1} - R_{i-1} = (r+z)N_{i-1} - (N_{i-1} - Z_{i-1})$$

= $(r+z-1)N_{i-1} + zN_{i-2}, \quad i = 2, 3, \dots$ (7)

with initial values $N_0 = 1$ and $N_1 = r + z$. Hence, the status of vertex v is given by

$$s(v) = \sum_{i=0}^{k} iN_i$$

Taking into account that N_i is defined by the linear recurrence relation (7), we have that $N_i = A\lambda_1^i + B\lambda_2^i$. Hence,

$$\sum_{i=0}^{k} iN_{i} = A \sum_{i=0}^{k} i\lambda_{1}^{i} + B \sum_{i=0}^{k} i\lambda_{2}^{i}$$

Denote $S = \sum_{i=0}^{k} i \lambda_1^i$. Notice that

$$\lambda_1 S - S = k\lambda_1^{k+1} - \sum_{i=1}^k \lambda_1^i = k\lambda_1^{k+1} - \frac{\lambda_1^{k+1} - 1}{\lambda_1 - 1} + 1$$

As a consequence,

$$S = \frac{k\lambda_1^{k+2} - (k+1)\lambda_1^{k+1} + \lambda_1}{(\lambda_1 - 1)^2}$$

The calculation for $\sum_{i=0}^{k} i\lambda_2^i$ is equivalent, obtaining,

$$s(v) = A\left(\frac{k\lambda_1^{k+2} - (k+1)\lambda_1^{k+1} + \lambda_1}{(\lambda_1 - 1)^2}\right) + B\left(\frac{k\lambda_2^{k+2} - (k+1)\lambda_2^{k+1} + \lambda_2}{(\lambda_2 - 1)^2}\right)$$
(8)

A non-central vertex u of G has vertices at distance k+1, so $s(u) \ge s(v)$, completing the proof.

From now on, let $\mathbf{s}_{r,z,k}$ be the vector of dimension n = M(r, z, k) whose components are all equal to s(v), where s(v) denotes de status of any vertex in a mixed Moore graph (see Eq. (8)). Notice that $\mathbf{s}_{r,z,k}$ represent the status vector of a mixed Moore graph.

Let us denote by $\mathcal{RM}(r, z, k)$ the set of all nonisomorphic (r, z, k)-mixed radial Moore graphs. Our purpose is to rank each mixed graph of this 'population' in terms of how close is of being a mixed Moore graph, generalizing one ranking measure given in [5] for the undirected case.

Let $G \in \mathcal{RM}(r, z, k)$. For every positive integer p we define

$$N_p(G) = \|\mathbf{s}(G) - \mathbf{s}_{r,z,k}\|_p.$$

In particular, $N_1(G)$ measures the difference between the total status of G and the one corresponding to a mixed Moore graph.

Given two mixed graphs $G_1, G_2 \in \mathcal{RM}(r, z, k)$, we define G_1 and G_2 to be *status-equivalent*, $G_1 \sim G_2$, if they have the same status vector. In the quotient set of $\mathcal{RM}(r, z, k)$ by \sim , $\mathcal{RM}(r, z, k)/\sim$, we will say that G_1 is closer than G_2 to be a mixed Moore graph if there exists a positive integer l such that

$$N_p(G_1) = N_p(G_2), \ p = 1, \dots, l-1 \text{ and } N_l(G_1) < N_l(G_2),$$

in which case it is denoted by $G_1 < G_2$. Notice that this relation induces a total order in $\mathcal{RM}(r, z, k)/\sim$, since

$$\mathbf{s}(G_1) = \mathbf{s}(G_2) \iff N_p(G_1) = N_p(G_2), \text{ for every } p = 1, \dots, n.$$

3 Ranking of (2, 1, 2)-mixed radial Moore graphs.

To find all mixed radial Moore graphs for r = 2, z = 1 and k = 2, we begin by constructing the mixed Moore tree anchored at a distinguished vertex 0. It has 11 vertices. It remains to consider the arcs and the edges joining vertices at distance 2 from 0 (see figure 2). The directed part of the Moore tree can be completed by a bijection between the set of vertices with zero outdegree $\{4, 5, 6, 7, 8, 9, 10\}$ and the set of vertices with zero indegree $\{0, 1, 3, 4, 6, 8, 10\}$. Of course, there are some bijections that are not valid in order to complete the directed part (for instance, those containing fixed points or the assignation $5 \rightarrow 1$, among others). However, the number of bijections to be taken into account at this step is < 7!. Once a valid bijection b is found, we proceed with the exhaustive search to complete the undirected part. To this end we consider the multiset of vertices $\{4, 5, 5, 6, 7, 7, 8, 9, 9, 10\}$ with undirected degree less than 2, taking into account that those vertices with zero undirected degree appears twice in the set. By considering all possible non ordered pairs of elements of this multiset we get the remaining edges of the mixed graph. Again some of these combinations will be not valid (for instance the ones containing the unordered pair $\{5,5\}$). A combination turns out to be fair depending also on the bijection b, since if some unordered pair matches with an element of b then such combination is dismissed. The total number of pairs of sets of directed arcs and undirected arcs to check is at most $7!\binom{10}{(2,2,2,2,2)} = \frac{10!7!}{2!^5}$, but due to the restrictions, only about 10^6 are valid. Then, these sets are added to the Moore tree and the diameter of the resulting graph is computed. There are about $3 \cdot 10^5$ mixed graphs with diameter 3, but only 9486 are not isomorphic.



Figure 2: Moore tree used as a basis to construct every (2, 1, 2)-mixed radial Moore graph.

It turns out that $\mathcal{RM}(2, 1, 2)$ contains exactly 9486 mixed graphs. According to their number of central vertices c, they are distributed as indicated in Table 1.

Notice that the number of central vertices only takes four distinct values being 4 their maximum. It would be interesting to find an upper bound for the cardinality of the center of a mixed radial Moore graph as a fraction

c = 1	c = 2	c = 3	c = 4	Total
8529 (89.91%)	906~(9.55%)	13~(0.14%)	38~(0.40%)	9486~(100%)

Table 1: Distribution of mixed radial Moore graphs according to their number of central vertices.

of its order. On the other hand, we remark that around 90% of the mixed graphs have just one central vertex.

The first two mixed graphs G_1 and G_2 in the status ranking are shown in Figure 3. Both graphs have the same status sequence: $\mathbf{s}(G_1): 17^4, 18^6, 19^1$, so $G_1 \sim G_2$. They attain the minimum value for the status norm $N_1(G_1) =$ 8. The undirected part of both graphs is the union of two cycles of lengths 5 and 6, but their directed part is different: G_1 has three directed cycles of lengths 4, 4 and 3, meanwhile G_2 has two directed cycles of lengths 7 and 4. Curiously enough, G_1 has a non trivial automorphism (as it can be seen in figure 3, where G_1 shows an axial symmetry) but G_2 has none. The spectrum of both G_1 and G_2 is $\{3^1, 1^2, (-1)^2, \lambda_1^3, \lambda_2^3\}$, where $\lambda_1 = \frac{-1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{-1-\sqrt{5}}{2}$. The set of eigenvalues $\{3, \lambda_1, \lambda_2\}$ is precisely the one that a mixed Moore graph should have in this case. G_2 is cospectral with G_1 , and it can be obtained by applying a recent method to obtain cospectral digraphs with a locally line digraph. For more details, see Dalfó and Fiol [6].



Figure 3: The first two graphs G_1 and G_2 according to the status ranking.

Proposition 3.1. The mixed graphs G_1 and G_2 depicted in Fig. 3 have status vector $\mathbf{s}(G_1)$: $17^4, 18^6, 19^1$ and hence status norm $N_1(G_1) = 18$. They are the first two ranked (2, 1, 2)-mixed radial Moore graphs according to the status norm.

4 Existence of (r, z, 2)-mixed radial Moore graphs

4.1 A family of (r, 1, 2)-mixed radial Moore graphs.

We begin by constructing the mixed Moore tree anchored at a distinguished vertex 0. The out-neighbors of 0 is the set of integers $\{1, 2, \ldots, r+1\}$ being r+1 adjacent with an arc from 0. The vertices at distance 2 from the root are labelled with as an ordered pair of integers (i, j) where *i* indicates that the vertex is pending from vertex *i* and *j* is used to establish an order among the vertices pending from *i*. Note that the subtree that pends from vertex r+1 has an extra vertex compared with the other subtrees pending from 1 to *r* (see figure 4).



Figure 4: Moore tree for r > 2, z = 1 and k = 2, rooted at vertex 0.

The proposed graph family H_r is constructed through the addition of the following edges to the Moore tree for r > 2, z = 1 and k = 2:

1. Edges $(i, j) \sim (u, v)$ such that

$$j - i \equiv v - u \pmod{r+1},$$

where i, u = 1, 2, ..., r + 1 and j, v = 1, 2, ..., r. Notice that vertex (r + 1, r + 1) is not included in this list.

2. $(i,r) \sim (r+1,r+1)$ for every $i = 1, \ldots, r$.

The following arcs are also included in H_r :

- 3. The arc $(r+1, r+1) \to 0$.
- 4. Arcs $(r+1, i) \to r+1-i$, for every i = 1, ..., r.
- 5. Arcs from (i, r) to (r + 1, r + 1 i), for every i = 1, ..., r.

- 6. $(i, j) \rightarrow (r + i j, r j)$, for every i = 1, ..., r and j = 1, ..., r 1such that $j - i \not\equiv r \pmod{r+1}$, where indices in the last vertex must be taken modulus r + 1.
- 7. $(i, j) \rightarrow (i 1, j)$, for every $i = 1, \dots, r$ and $j = 1, \dots, r 1$ such that $j i \equiv r \pmod{r+1}$.



Figure 5: Graph H_3 . In order to keep the figure clean some adjacencies are indicated through colors (distinct from black). There is an edge joining vertices of the same color.

Proposition 4.1. Let r > 2 be a positive integer. Then H_r is a totally (r, 1)-regular mixed graph.

Proof. Let us start proving the undirected regularity. In the Moore tree it is clear that vertices $i = 0, 1, \ldots, r+1$ have indeed undirected degree r. All vertices (i, j) such that $i = 1, \ldots, r+1$ and $j = 1, \ldots, r-1$ and (r+1, r) have undirected degree 1. Finally (i, r) vertices such that $i = 1, \ldots, r$ and (r+1, r+1) have undirected degree 0. The edges added in step 2 increase the undirected degree of (i, r) vertices such that $i = 1, \ldots, r$ to 1, and the undirected degree of (r+1, r+1) to r. By now all vertices have degree r but (i, j) s.t. $i = 1, 2, \ldots, r+1$ and $j = 1, 2, \ldots, r$ which have undirected degree 1. This set has cardinal r(r+1). Precisely, the relationship stated in step 1 is an equivalence relation in this set and thus, it establishes a partition of r+1 classes, each one made up of r vertices. For every element, an edge is drawn to the other members of the class, adding r-1 to the undirected degree, resulting a total of r.

The only vertex with indegree and outdegree equal to 1 in the Moore tree is r+1. Step 3 completes the directed degrees of vertices 0 and (r+1, r+1). Vertices *i* for i = 1, ..., r already have outdegree 1, and the indegree turns 1 after step 4. This step also increases by 1 the outdegree of all vertices (r+1, j). Step 5 completes the directed degrees of these vertices, since every (r+1, j) receives an arc, and also completes vertices (i, r) for $i = 1, \ldots r$, since they already had indegree 1 and after step 5 their outdegree is also 1. It is easy to see that steps 6 and 7 make the outdegree of (i, j) vertices s.t. $i = 1, \ldots, r$ and $j = 1, \ldots, r-1$ to be 1, but it is less obvious that it also increases the indegree to 1. To prove so, it is only necessary to show that there are not different arcs in steps 6 and 7 that are incident to the same vertex. Suppose that (a, b) and (c, d) with the conditions of step 6 are different vertices incident to the same vertex through the assignation of step 6. Then,

$$r+a-b \equiv r+c-d \pmod{r+1}$$
 and $r-b \equiv r-d \pmod{r+1}$.

The second equivalence implies b = d since $1 \le b, d \le r-1$ are integers that cannot differ by r+1. Introducing b = d in the first equivalence entails $a \equiv c \pmod{r+1}$. Again this implies a = c, since $1 \le a, c \le r$ are integers that cannot differ by r+1. Thus (a, b) and (c, d) represent the same vertex, a contradiction that proves the statement. To end the proof, it is enough to see that the set of vertices that receive the arcs in step 6 are different from the ones that receive the arcs of step 7. In step 7 arcs are incident to vertices $(\alpha, \beta) = (i - 1, j)$ s.t. $j - i \equiv r \pmod{r+1}$ and thus, they belong to the class of vertices such that $\beta - \alpha \equiv j - i + 1 \equiv 0 \pmod{r+1}$ while in step 6 arcs are incident to vertices $(\gamma, \delta) = (r + i - j, r - j)$ s.t. $j - i \not\equiv r$ $(\mod r+1)$ and thus $\delta - \gamma \equiv -i \not\equiv 0 \pmod{r+1}$ since $1 \le i \le r$.

Proposition 4.2. For every r > 2, H_r is a (r, 1, 2)-mixed radial Moore graph with status vector:

$$\mathbf{s}(H_r): \quad (2r^2+3r+3)^2, (2r^2+3r+4)^{r-1}, (2r^2+4r+2)^2, (3r^2+5)^{r^2-2r-1} \\ (3r^2+6)^5, (3r^2+7)^{2r-3}, (3r^2+r+5)^{r-3}, (3r^2+r+6)^2.$$

Proof. The proof will proceed as follows. For every vertex $u \in V(H_r)$ we will find the number of vertices $v \in V(H_r)$ such that $d(u, v) \leq 2$ trough a sketch of the first and second neighbors. Then, every vertex which is not present in the sketch $w \in V(H_r)$ such that d(u, w) > 2 is listed and we proof that, indeed, d(u, w) = 3 identifying the explicit u - w path.

We will proceed in the mentioned manner, lumping vertices in subsets if the calculation can be made analogously. The number associated to a given edge in figures 6, 7 and 8 indicates the step in the construction of H_r where this connection was added.

- Vertex 0 is clearly a central vertex, so it status is $\mathbf{s}_{r,1,2} = 2r^2 + 3r + 3$.
- For vertices i = 1, ..., r the sketch is shown in figure 6. Vertex 1 is a central vertex, but it is not the case for i = 2, ..., r, where only vertex (i 1, r) remains at distance three through the path:

$$i \xrightarrow{0} 0 \xrightarrow{0} i - 1 \xrightarrow{0} (i - 1, r).$$



Figure 6: Neighbors at distance two of vertex $i = 2, \ldots, r$.

Hence, this contributes to the status vector in $2r^2 + 3r + 3$, $(2r^2 + 3r + 4)^{r-1}$.

• For vertices (i, r) s.t. i = 2, ..., r - 1 the sketch is shown in figure 7.



Figure 7: Neighbors at distance one and two of vertex (i, r) with $i = 2, \ldots, r-1$.

Remaining vertices:

- Vertex (i, i):

$$(i,r) \xrightarrow{5} (r+1,r+1-i) \xrightarrow{4} i \xrightarrow{0} (i,i)$$

- Vertices (r+1, j) s.t. j = 1, ..., r and $j \neq r-i, r+1-i$, a total of r-2 vertices:

$$(i,r) \xrightarrow{5} (r+1,r+1-i) \xrightarrow{0} r+1 \xrightarrow{0} (r+1,j)$$

- Vertices (i-1,j) s.t. $j = 1, \ldots, r-2$, a total of r-2 vertices:

$$(i,r) \xrightarrow{1} (i-1,r-1) \xrightarrow{0} i-1 \xrightarrow{0} (i-1,j)$$

- Vertices (i+1, j) s.t. $j = 2, \ldots, r$, a total of r-2 vertices:

$$(i,r) \xrightarrow{1} (r+1,r-i) \xrightarrow{0} i+1 \xrightarrow{0} (i+1,j)$$

- Vertices (α, β) s.t. $\alpha = 1, \ldots, r-1$; $\beta = 1, \ldots, r-1$ and $\beta - \alpha \not\equiv r - i, r - i + 1 \pmod{r+1}$ and $\alpha \neq i - 1, i, i + 1$, a total of (r-3)(r-3) vertices:

$$(i,r) \xrightarrow{1} (\alpha, \alpha + r - i) \xrightarrow{0} \alpha \xrightarrow{0} (\alpha, \beta)$$

There is a total of $r^2 - 3r + 4$ vertices at distance 3. So, they contribute in $(3r^2 + 7)^{r-2}$ to the status vector.



Figure 8: Neighbors at distance one and two of vertex (i, i - 1) with $i = 3, \ldots, r - 1$. The stars point out the fact that there are repeated vertices in the tagged subsets.

- For vertices (i, i − 1) s.t. i = 3,...,r − 1 the sketch is shown in Fig. 8. Remaining vertices:
 - Vertex (1, j) s.t. $j = 2, \ldots, r$, a total of r 1 vertices:

$$(i, i-1) \xrightarrow{1} (r+1, r) \xrightarrow{4} 1 \xrightarrow{0} (1, j)$$

- Vertices (r+1, j) s.t. $j = 1, \ldots, r-1, r+1$, a total of r vertices:

$$(i, i-1) \xrightarrow{1} (r+1, r) \xrightarrow{0} r+1 \xrightarrow{0} (r+1, j)$$

- Vertices (α, β) s.t. $\alpha = 2, ..., r-1; \beta = 1, ..., r$ and $\beta - \alpha \neq 0, r$ (mod r+1) and $(\alpha, \beta) \neq (r, r+1-i)$, a total of (r-2)(r-2)-1vertices:

$$(i, i-1) \xrightarrow{1} (\alpha, \alpha-1) \xrightarrow{0} \alpha \xrightarrow{0} (\alpha, \beta)$$

There is a total of $r^2 - 2r + 2$ vertices at distance 3.

Analogously, one should proceed for every subset V_i presented in table 2. Column 3 contains the cardinal of each subset, and column 4 contains the number of vertex at distance 3 from every vertex in V_i , denoted as $n_3(V_i)$. The collection of all subsets V_i constitutes a partition of the vertex set. The status vector of H_r can easily computed from this table, obtaining

$$\mathbf{s}(H_r): \quad (2r^2+3r+3)^2, (2r^2+3r+4)^{r-1}, (2r^2+4r+2)^2, (3r^2+5)^{r^2-2r-1} \\ (3r^2+6)^5, (3r^2+7)^{2r-3}, (3r^2+r+5)^{r-3}, (3r^2+r+6)^2.$$

 $N_1(H_r)$ can be also obtained from table 2 through $N_1(H_r) = \sum_{0 \le i \le 9} |V_i| \cdot n_3(V_i)$ obtaining $N_1(H_r) = r^4 - 2r^3 + 8r - 2$.

i	V_i	$ V_i $	$n_3(V_i)$
0	$\{0,1\}$	2	0
1	$\{i \mid i=2,\ldots,r\}$	r-1	1
2	$\{r+1, (r+1, r+1)\}$	2	r-1
3	$\{(i,i) \mid i = 2, \dots, r-1\}$	r-2	$r^2 - 3r + 2$
4	$\{(i,j) \neq (1,r-1) \mid j-i \not\equiv 0, r \pmod{r+1}\}$	$r^2 - 3r + 1$	$r^2 - 3r + 2$
5	$\{(r+1,r), (1,r), (r,r), (1,1), (1,r-1)\}$	5	$r^2 - 3r + 3$
6	$\{(r+1,j) \mid j = 1, \dots, r-1\}$	r-1	$r^2 - 3r + 4$
7	$\{(i,r)\mid i=2,\ldots,r-1\}$	r-2	$r^2 - 3r + 4$
8	$\{(i, i-1) \mid i \neq 2, r\}$	r-3	$r^2 - 2r + 2$
9	$\{(2,1), (r,r-1)\}$	2	$r^2 - 2r + 3$

Table 2: Partition of the vertices set of H_r showing the number of vertices at distance 3 for any given vertex of V_i .

4.2 An optimal family of (1, z, 2)-mixed radial Moore graphs.

Since mixed radial Moore graphs are presented as a kind of approximation of mixed Moore graphs, it is natural to think that their existence is guaranteed in the cases when a mixed Moore graph indeed exists. Moreover, the 'closest' mixed radial Moore graph to being a mixed Moore graph could be obtained through a slight modification of the latter.



Figure 9: The unique mixed Moore graph G on 6 vertices and a mixed radial Moore graph G^* obtained by an arc swap to G. Moreover, G^* is the closest graph to G according to the status norm.

For instance, the mixed graph depicted in Fig. 9 is precisely the unique mixed Moore graph for parameters (r, z, k) = (1, 1, 2), which corresponds to the Kautz digraph on six vertices. It is not difficult to check that the mixed graph on its right is a mixed radial Moore graph and moreover, it is the closest mixed graph according to the status norm. This mixed graph has been obtained by a simple swap of two arcs of the Moore graph. In this new graph only two vertices have a slight modification of their status, incrementing just in one unit the value that they had in the original graph. The remaining vertices preserve the status that they had.

This particular result can be generalized. To see this, let (u, u', v, v') be four (ordered) vertices of a mixed graph D = (V, E, A) so that $uv, u'v' \in A$ and $uv', u'v \notin A \cup E$. The arc swap operation removes the two arcs uv, u'v'and adds uv' and u'v to A (Fig. 10). Notice that this operation preserves the degrees of the vertices in the resulting mixed graph.



Figure 10: Arc swap operation.

It turns out that, under certain conditions, every mixed graph constructed by an arc swapping to a proper mixed Moore graph of diameter 2 is a (r, z, 2)-mixed radial Moore graph, and moreover, this mixed graph is the closest graph to being a Moore graph. **Theorem 4.3.** Let G^* be the mixed graph obtained by an arc swapping of some vertices (u, u', v, v') in a proper mixed Moore graph G of diameter 2, such that $\Gamma^-(u) = \Gamma^-(u')$ and $\Gamma^+(v) = \Gamma^+(v')$. Then G^* is a (r, z, 2)-mixed radial Moore graph with $N_1(G^*) = 2$.

Proof. Assume that (u, u', v, v') are four different vertices such that $\Gamma^{-}(u) = \Gamma^{-}(u')$ and $\Gamma^{+}(v) = \Gamma^{+}(v')$. Let us see what happens with the shortest paths in G^* . We divide the proof according to the length of the shortest paths in G:

- Let w_0, w_1, w_2 be a shortest path (of length 2) in G. Once the arc swap (u, u', v, v') is done, we will see that either the same path w_0, w_1, w_2 is a shortest path in G^* or there is an alternative path with length ≤ 2 . For the case $w_0 = u$ [$w_0 = u'$] and $w_2 = v'$ [$w_2 = v$], then the arc swap produces a shortest path. For the remaining cases, we will see that the length of the path is preserved. Indeed, if $w_i \neq u, u'$ for i = 0, 1 then the same path is a shortest path in G^* . If $w_1 = u$ and $w_2 \neq v$ the same path is also a shortest path in G^* , and the shortest path w_0, u, v in G becomes w_0, u', v in G^* since $w_0 \in \Gamma^-(u)$ and by assumption $\Gamma^-(u) = \Gamma^-(u')$. The same argument applies when $w_1 = u'$ by exchanging u by u'. It remains to consider the case when $w_0 = u$ or $w_0 = u'$ (again, by symmetry, we can consider just one case, say $w_0 = u$): when $w_1 \neq v, v'$ the same shortest path is valid and for $w_1 = v$ or v' we obtain a shortest path in G_z just exchanging v by v' (due to the arc swapping) since $\Gamma^+(v) = \Gamma^+(v')$.
- For shortest paths of length one, we only have to consider paths u, v and u', v' (of length 1) in G, which no longer exist in G*. We will prove that there is a path of length three from u to v and there is no one of length 2 (the proof for the case u', v' follows the same ideas). Since Γ⁺(v) = Γ⁺(v') and r ≥ 1, then it exists w ∈ Γ⁺(v) such that vw is an edge and v'w is an arc. So u, v', w, v is the desired path. Assume that there is a path of length 2 in G* from u to v, then it exists w ∈ Γ⁺(u) ∩ Γ⁻(v). Notice that this is only possible if w = v' or w = u', since otherwise the shortest path from u to v in G would have length 2. But if the path u, v', v exists in G* then we would have the arc v'v in G contradicting the fact that Γ⁺(v) = Γ⁺(v'). The same applies to the case u, u', v where we would have a contraction with Γ⁻(u) = Γ⁻(u').

Notice that under the assumption $\Gamma^{-}(u) = \Gamma^{-}(u')$ and $\Gamma^{+}(v) = \Gamma^{+}(v')$ there are only two paths of length 3 in G^* , those starting at u and u' and ending at v and v', respectively. As a consequence, every vertex in G^* is a central vertex, except u and u', having just one vertex at distance 3 each, that is, G^* has radius 2, diameter 3 and $N_1(G^*) = 2$.

Theorem 4.3 can be applied at least at the unique infinite family of proper mixed Moore graphs known until now. The family of *Kautz digraphs*. The vertices of the *Kautz digraph* $Ka(d, k), d \ge 1, k \ge 1$, are words of length k on the alphabet $\Sigma = \{0, 1, \ldots, d\}$ without two consecutive identical numbers. There is an arc from vertex $(v_0, v_1, \ldots, v_{k-1})$ to vertices $(v_1, \ldots, v_{k-1}, x)$, where $x \in S \setminus \{v_{k-1}\}$. It is known that Ka(d, k) has order $(d+1)d^{k-1}$ and diameter k. Kautz digraphs Ka(d, 2) are the unique mixed Moore graph of diameter 2 for every $d \ge 2$ (taking into account that every oriented digon is replaced by an edge, that is, r = 1 and z = d - 1) as noted in [15].

We may construct a new family of (1, z, 2)-mixed radial Moore graphs by performing a convinient *arc swap* to Ka(1+z, 2) such that the conditions of theorem 4.3 are fullfilled. Moreover, this is the best possible, as we will see later. First we provide a helpful lemma.

Lemma 4.4. Let G be a (r, z, k)-mixed radial Moore graph and $w \in V(G)$ be a central vertex of G. Suppose w is adjacent to a non-central vertex $u \in V(G)$ through an edge. Then there exists another vertex in the outneighborhood of w different from u which is non-central.

Proof. Let $\Gamma^+(w) = \{a_1, \ldots, a_z, e_1, \ldots, e_{r-1}, u\}$ be the out-neighborhood of w, such that vertices a_i are adjacent from w through an arc and vertices e_j are adjacent from w through an edge. By contradiction suppose all vertices in $\Gamma^+(w) \setminus \{u\}$ are central vertices.

Since u is a non-central vertex and G is a (r, z, k)-mixed radial Moore graph there must exist a vertex $v \in V(G)$ such that d(u, v) = k + 1 and thus d(w, v) = k. The distance preserving spanning tree rooted at w must be a Moore tree, and v must be located at distance k, belonging to any of the subtrees rooted at $\Gamma^+(w) \setminus \{u\}$. Without loss of generality, we can suppose that it hangs from the subtree rooted at a_1 . Since all vertices in $\Gamma^+(w) \setminus \{u\}$ are central there must be a path of length not greater than k from every a_i and e_i to v. For a_1 it already exists a (k-1)-path in the Moore Tree but it is forced that for every vertex a_i [resp. e_i] in $\Gamma^+(w) \setminus \{u, a_1\}$ there exists a vertex a'_i [resp. e'_i] such that $d(a_i, a'_i) = k - 1$ [resp. $d(e_i, e'_i) = k - 1$] and there is an adjacency (arc or edge) from a_i [resp. e_i] to v (see figure 11).

These forced adjacencies increase in r + z - 2 the in-degree of v, and counting the arc or edge that keeps v hanging from the tree it results that the total in-degree of v is r + z - 1. Since G is a totally (r, z)-regular graph, there must be another vertex v' adjacent to v but:

- Vertex v' must hold d(w, v') = k, otherwise w would not be a central vertex.
- It cannot pend from the subtree rooted at u. This would mean d(u, v) = k, which would be a contradiction.



Figure 11: Distance preserving spanning tree rooted at w. Dashed lines are used to represent adjacencies that might be either arcs or edges.

- It cannot pend from a vertex at distance k 1 from a_1 , since a_1 is a central vertex and there would be two different (a_1, v) -paths of length less than k.
- It cannot pend from vertex a_i [resp. e_i] in $\Gamma^+(w) \setminus \{u, a_1\}$, since a_i [resp. e_i] is a central vertex and there would be two different (a_i, v) -paths [resp. (e_i, v) -paths] of length k.

Corollary 4.5. Let G_z be a mixed graph obtained by an arc swapping of vertices 01, 03, 12, 32 in Ka(z + 1, 2), $z \ge 2$. Then G_z is a (1, z, 2)-mixed radial Moore graph, which is the closest graph to a mixed Moore graph (according to the status norm).

Proof. Clearly $\Gamma^{-}(01) = \Gamma^{-}(03) = \{x0 \mid x \in \Sigma, x \neq 0\}$ and $\Gamma^{+}(12) = \Gamma^{+}(32) = \{2y \mid y \in \Sigma, y \neq 2\}$. By theorem 4.3 we have that G_z is a (1, z, 2)-mixed radial Moore graph with $N_1(G_z) = 2$. According to lemma 4.4, every mixed radial Moore graph G has at least 2 non-central vertices, so $N_1(G) \geq 2$, and the result follows.

For the undirected case, every infinite family of radial Moore graphs of diameter 2 constructed until know has a norm status that polynomially increases with the degree. The family of mixed radial Moore graphs G_z has a constant norm status equal to 2, independently of z. This family of mixed graphs is cospectral with the Kautz digraphs and preserve some symmetries, as the one depicted in Fig. 12.



Figure 12: The digraph G_z for z = 2, obtained from the Kautz digraph Ka(3,2) (the dashed arcs are replaced by the green ones in the optimal swap).

5 Final remarks and open problems

Mixed radial Moore graphs are extremal graphs that are interesting by its own definition, but they are also a nice approximation to the well known Moore graphs. In this paper we provide two infinite families H_r and G_z of mixed radial Moore graphs of diameter 2, but it would be very interesting to know if they exist for any combination of the parameters (r, z, k), even for diameter greater than two.

Question 1. Are there (r, z, k)-mixed radial Moore graphs for any $r, z \ge 1$ and $k \ge 2$?

The arc swap method used in subsection 4.2 provides an optimal family G_z of (1, z, 2)-mixed radial Moore graphs, but in could be useful for other set of parameters (r, z, k) where mixed Moore graphs exist. For instance, we have applied this method to the unique mixed Moore graph on 18 vertices, the one for the case (3, 1, 2) (described in [1]). This mixed Moore graph B_{18} has an status vector of $\mathbf{s}(B_{18}) : 30^{18}$. It turns out that an optimal arc swap produces a mixed graph B'_{18} with status vector $\mathbf{s}(B'_{18}) : 30^{10}, 31^6, 32^2$, that is, $N_1(B'_{18}) = 10$. It is known that B_{18} contains many copies of the Kautz digraph on 6 vertices (see [13]). It turns out that B'_{18} is constructed from B_{18} by the arc swap depicted in Fig. 9 to any of this inside copies, that is, the optimal arc swap to the family of kautz digraphs described in subsection 4.2 produces the closest mixed radial Moore graph to B_{18} so far (among all possible arc swaps that in can be done). Nevertheless, we do not know if B'_{18} is the top ranked graph in $\mathcal{RM}(3, 1, 2)$, since we only have checked those graphs obtained by a single arc swap to B_{18} . This method has been

applied also to the two mixed Moore graphs of order 108 known until know (see [11]) with status vector $\mathbf{s}_{3,7,2}$: 204¹⁰⁸, obtaining a mixed radial Moore graph with status vector 204⁹⁸, 205⁸, 209², that is, norm status equal to 18.

Problem 5.1. Find those mixed radial Moore graphs best ranked in $\mathcal{RM}(r, z, k)$ for specific values $r, z \ge 1$ and $k \ge 2$, in those cases where mixed radial Moore graphs exist.

We have answered to this question for the cases (2, 1, 2) and (1, z, 2) in this paper, but there are infinitely many open cases. So, table 3 is presented below with the best results found so far.

$z \backslash r$	1	2	3	4	5	6	7	
1	2	8	10	158	413	910	1769	
2	2	?	?	?	?	?	?	
3	2	?	?	?	?	?	18	
4	2	?	?	?	?	?	?	
÷	:	÷	÷	÷	÷	÷	÷	

Table 3: Values of $N_1(G)$ of the closest mixed radial Moore graphs G found so far. Red numbers correspond to optimal values. Blue numbers show the status norm of the closest mixed radial Moore graphs obtained by arc swapping to a Moore graph. Green values are the status norm values of the H_r family.

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