ENUMERATION OF PERMUTATIONS BY THE PARITY OF DESCENT POSITIONS

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Abstract. Noticing that some recent variations of descent polynomials are special cases of Carlitz and Scoville's four-variable polynomials, which enumerate permutations by the parity of descent and ascent positions, we prove a q-analogue of Carlitz-Scoville's generating function by counting the inversion number and a type B analogue by enumerating the signed permutations with respect to the parity of desecnt and ascent positions. As a by-product of our formulas, we obtain a q-analogue of Chebikin's formula for alternating descent polynomials, an alternative proof of Sun's gamma-positivity of her bivariate Eulerian polynomials and a type B analogue, the latter refines Petersen's gamma-positivity of the type B Eulerian polynomials.

Contents

1. Introduction

In the past few years of this century, several variations and refinements of permutation descent, according to the parity of descent positions, have been studied, see [5, 29, 15, 21, 32, 34, 33, 23, 20, 25]. This paper arose from the observation that some of these

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results are related to a work of Carlitz and Scoville [4] dated back to 1973. For example, Chebikin's *alternating descent polynomial* [5] and the *bivariate Eulerian polynomials* in H. Sun [32] and Y. Sun and Zhai [34] are both special cases of Carlitz-Scoville's four-variable polynomials enumerating the permutations according to the parity of both descents and ascents. On the other hand, this connection leads immediately to obtain two equivalent simpler versions of Carlitz-Scoville's generating function. As Carlitz and Scoville's original proof relies on solving a system of differential equations, this prompted us to find a more conceptuel proof, which led up straightforwardly to a q-analogue.

If π is a permutation of $[n] := \{1, \dots, n\}$, an index $i \in [n-1]$ is a *descent position* (resp. *ascent position*) of π if $\pi(i) > \pi(i+1)$ (resp. $\pi(i) < \pi(i+1)$). Let $\operatorname{des}\pi$ (resp. $\operatorname{des}_1\pi$ and $\operatorname{des}_0\pi$) be the number of descents of π (resp. at odd and even positions), i.e.,

$$\operatorname{des}_\nu(\pi) = \#\{i \in [n] | \pi(i) > \pi(i+1) \text{ and } i \equiv \nu \pmod 2\} \quad (\nu \in \{0,1\}).$$

The statistics $\operatorname{asc}\pi$, $\operatorname{asc}_1\pi$ and $\operatorname{asc}_0\pi$ are defined similarly. For $i \in \{2, 3, \dots, n-1\}$, we say $\pi(i)$ is a valley (resp. peak) of π, if $\pi(i-1) > \pi(i) < \pi(i+1)$ (resp. $\pi(i-1) < \pi(i) > \pi(i+1)$) and $\pi(i)$ is a double ascent (resp. double descent) of π, if $\pi(i-1) < \pi(i) < \pi(i+1)$ (resp. $\pi(i-1) > \pi(i) > \pi(i+1)$). Finally we recall that the inversion number of π is $\operatorname{inv}\pi = |\{(i,j) | \pi(i) > \pi(j),\ 1 \le i < j \le n\}|$.

Define the enumerative polynomial of permutations of $\mathfrak{S}_n$ by the parity of ascent and descent positions as

$$P_n(x_0, x_1, y_0, y_1, q) = \sum_{\sigma \in \mathfrak{S}_n} x_0^{\operatorname{asc}_0\sigma} x_1^{\operatorname{asc}_1\sigma} y_0^{\operatorname{des}_0\sigma} y_1^{\operatorname{des}_1\sigma} q^{\operatorname{inv}\sigma}.$$

Recall the following q-exponential series

$$\exp_q(x) = \sum_{n \ge 0} \frac{x^n}{n!_q},$$

where $0!_q = 1$ and $n!_q = \prod_{i=1}^n (1 + q + \cdots + q^{i-1})$ for $n \ge 1$, and the q-trignometric series

$$\cosh_q t = \sum_{n \ge 0} \frac{t^{2n}}{(2n)!_q}, \quad \sinh_q t = \sum_{n \ge 1} \frac{t^{2n-1}}{(2n-1)!_q};$$
$$\cos_q x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!_q}, \quad \sin_q x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!_q}.$$

Theorem 1.1. *Let $\alpha = \sqrt{(y_0 - x_0)(y_1 - x_1)}$. Then*

$$\sum_{n\geq 1} P_n(x_0, x_1, y_0, y_1, q) \frac{t^n}{n!_q}$$
$$= \frac{(x_1 + y_1)\cosh_q(\alpha t) + \alpha \sinh_q(\alpha t) - y_1(\cosh_q^2(\alpha t) - \sinh_q^2(\alpha t)) - x_1}{x_0 x_1 - (x_0 y_1 + x_1 y_0)\cosh_q(\alpha t) + y_0 y_1(\cosh_q^2(\alpha t) - \sinh_q^2(\alpha t))}. \tag{1.1}$$

Remark 1. *When $q = 1$ Eq.* (1.1) *reduces to Carlitz-Scoville's formula* [4, Theorem 3.1] [1]

$$\sum_{n\geq 1} P_n(x_0, x_1, y_0, y_1, 1) \frac{t^n}{n!} = \frac{(x_1 + y_1)\sum_{n\geq 1} \frac{\beta^{n-1} t^{2n}}{(2n)!} + \sum_{n\geq 1} \frac{\beta^{n-1}\, t^{2n-1}}{(2n-1)!}}{1 - (x_0 y_1 + x_1 y_0)\sum_{n\geq 1} \frac{\beta^{n-1} t^{2n}}{(2n)!}}, \tag{1.2}$$

with $\beta = (y_0 - x_0)(y_1 - x_1)$. For the homegeous Eulerian polynomials $P_n(y, y, x, x, 1)$, i.e., $\sum_{\sigma\in\mathfrak{S}_n} x^{des\,\sigma} y^{asc\,\sigma}$, the corresponding formula reads

$$\sum_{n\geq 1} P_n(y, y, x, x, 1) \frac{t^n}{n!} = \frac{e^{xt} - e^{yt}}{x e^{yt} - y e^{xt}}. \tag{1.3}$$

Chen and Fu [6] *recently gave a context-free grammar proof of* (1.3).

Let $\mathrm{UD_n}$ be the set of *up-down* permutations of $12\ldots n$, i.e., permutations $\sigma := \sigma(1)\ldots\sigma(n)$ such that $\sigma(1) < \sigma(2) > \sigma(3) < \cdots$. Obviously

$$P_n(0, 1, 1, 0, q) = \sum_{\sigma\in \mathrm{UD_n}} q^{\mathrm{inv}\,\sigma}$$

and Eq. (1.1) reduces to a q-analogue of André's classical result (see [31, 16, 18]) :

$$1 + \sum_{n\geq 1} P_n(0, 1, 1, 0, q) \frac{x^n}{n!_q} = \frac{1 + \sin_q x}{\cos_q x}. \tag{1.4}$$

For the following two special cases:

$$A_n(x, y, q) := P_n(1, 1, y, x, q) = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}_1\sigma} y^{\mathrm{des}_0\sigma} q^{\mathrm{inv}\,\sigma}, \tag{1.5a}$$

$$\widehat{A}_n(x, y, q) := P_n(y, 1, 1, x, q) = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}_1\sigma} y^{\mathrm{asc}_0\sigma} q^{\mathrm{inv}\,\sigma}, \tag{1.5b}$$

[1] Carlitz and Scoville counted a conventional rise at the beginning as position 0 and a conventional descent at the end as position n (mod 2).

we derive from Theorem 1.1 that

$$\sum_{n\geq 1} A_n(x,y,q)\frac{t^n}{n!_q} = \frac{(1+x)\cosh_q(\alpha t)+\alpha\sinh_q(\alpha t)-x(\cosh_q^2(\alpha t)-\sinh_q^2(\alpha t))-1}{1-(x+y)\cosh_q(\alpha t)+xy(\cosh_q^2(\alpha t)-\sinh_q^2(\alpha t))}, \tag{1.6a}$$

$$\sum_{n\geq 1} \widehat{A}_n(x,y,q)\frac{t^n}{n!_q} = \frac{(1+x)\cos_q(\alpha t)-\alpha\sin_q(\alpha t)-x(\cos_q^2(\alpha t)+\sin_q^2(\alpha t))-1}{y-(xy+1)\cos_q(\alpha t)+x(\cos_q^2(\alpha t)+\sin_q^2(\alpha t))} \tag{1.6b}$$

with $\alpha=\sqrt{(1-x)(1-y)}$.

Remark 2. *Formulae* (1.2), (1.6a) *and* (1.6b) *are actually equivalent. Indeed, for any* $\sigma\in\mathfrak{S}_n$ *it is clear that*

$$des_0\,\sigma + asc_0\,\sigma = \lfloor (n-1)/2\rfloor, \tag{1.7a}$$

$$des_1\,\sigma + asc_1\,\sigma = \lfloor n/2\rfloor. \tag{1.7b}$$

Hence the distribution of the quadruple statistics $(asc_0\,, asc_1\,, des_0\,, des_1\,)$ *is equivalent to any pair of the statistics in* $\{des_1\,, asc_1\,\}\times\{des_0\,, asc_0\,\}$. *In particular, we have*

$$\widehat{A}_n(x,y,q) = y^{\lfloor (n-1)/2\rfloor}A_n(x,1/y,q), \tag{1.8}$$

and

$$P_n(x_0,x_1,y_0,y_1,q) = x_0^{\lfloor (n-1)/2\rfloor}x_1^{\lfloor n/2\rfloor}A_n\left(\frac{y_1}{x_1},\frac{y_0}{x_0},q\right). \tag{1.9}$$

The polynomial $A_n(x,x,q):=\sum_{\sigma\in\mathfrak{S}_n}x^{des\,\sigma}q^{inv\,\sigma}$ *is a classical* q*-analogue of Eulerian polynomials and Eq.* (1.6a) *yields Stanley's formula* [30, 28],

$$1+\sum_{n\geq 1} xA_n(x,x,q)\frac{t^n}{n!_q} = \frac{1-x}{1-x\exp_q((1-x)t)}, \tag{1.10}$$

of which another refinement was given in [26].

As a variation of descent, Chebikin [5] *introduced the* alternating descent set *of permutation* $\pi\in\mathfrak{S}_n$ *by*

$$\widehat{D}(\pi)=\{i\in[n-1]|\pi(i)>\pi(i+1) \text{ and } i \text{ is odd or } \pi(i)<\pi(i+1) \text{ and } i \text{ is even}\}.$$

Hence, the number of alternating descents $\widehat{des}\,\pi = |\widehat{D}(\pi)|$ *equals* $des_1\sigma + asc_0\sigma$ *and formula* (1.6b) *with* $x=y$ *and* $q=1$ *reduces to*

$$1+\sum_{n\geq 1} x\widehat{A}_n(x,x,1)\frac{t^n}{n!} = \frac{1-x}{1-x(\sec(1-x)t+\tan(1-x)t)}, \tag{1.11}$$

which is equivalent to [5, Theorem 4.2], *see also* [15, Eq. (22)]. *As Chebikin, being unaware of the work of Carlitz and Scoville, Sun* [32] *and Sun and Zhai* [34] *reconsidered* *the polynomials* $A_n(x, y, 1)$*, and a cumbersome formula for* (1.6a) *is given in* [34, Theorem 2.2]. *Other proofs of formula* (1.11) *and generalizations appeared in* [29, 15, 20, 25].

As the original proof of (1.2) with $q = 1$ in [4] is not easy (see also the solution of Exercise 4.3.14 in [16]), we shall give a more conceptual proof of (1.6a), which is equivalent to Theorem 1.2, by exploring a *sieve method*, see [30, 14, 5, 31].

Our second goal is to give a type B analogue of Carlitz and Scoville's formula, i.e., Theorem 1.1 with $q = 1$. Denote by $\mathcal{B}_n$ the collection of type B permutations σ of the set $[\pm n] := \{\pm 1, \dots, \pm n\}$ such that $\sigma(-i) = -\sigma(i)$ for all $i \in [n]$, obviously, $|\sigma| := |\sigma(1)| \dots |\sigma(n)| \in \mathfrak{S}_n$. As usual (see [3, 28]), we always assume that type B permutations are prepended by 0. That is, we identify an element $\sigma = \sigma(1) \dots \sigma(n)$ in $\mathcal{B}_n$ with the word $\sigma(0)\sigma(1) \dots \sigma(n)$, where $\sigma(0) = 0$. We say that $\sigma \in \mathcal{B}_n$ has a descent (resp. ascent) at position i, if $\sigma(i) > \sigma(i + 1)$ (resp. $\sigma(i) < \sigma(i + 1)$) for $i \in \{0\} \cup [n - 1]$. By abuse of notation, in this section, we use $\operatorname{des}\sigma$ (resp. $\operatorname{des}_1\sigma$ and $\operatorname{des}_0\sigma$) to denote the number of descents of σ (resp. at odd and even positions). The statistics $\operatorname{asc}\sigma$, $\operatorname{asc}_1\sigma$ and $\operatorname{asc}_0\sigma$ are defined similarly for the ascents.

Define the enumerative polynomials

$$B_n(x, y) := \sum_{\sigma \in \mathcal{B}_n} x^{\operatorname{des}_1\sigma} y^{\operatorname{des}_0\sigma}. \tag{1.12}$$

Theorem 1.2. *Let* $\alpha = \sqrt{(1 - x)(1 - y)}$. *Then*

$$\sum_{n \ge 1} B_{2n}(x, y) \frac{t^{2n}}{(2n)!} = \frac{(x + y)\cosh(2\alpha t) + (1 - x)(1 - y)\cosh(\alpha t) - (1 + xy)}{(1 + xy) - (x + y)\cosh(2\alpha t)}, \tag{1.13a}$$

$$\sum_{n \ge 1} B_{2n-1}(x, y) \frac{t^{2n-1}}{(2n - 1)!} = \frac{\alpha(1 + y)\sinh(\alpha t)}{(1 + xy) - (x + y)\cosh(\alpha t)}. \tag{1.13b}$$

Remark 3. *When* $x = y$*, the polynomial* $B_n(x, x) := \sum_{\sigma \in \mathcal{B}_n} x^{des\,\sigma}$ *is the usual* Eulerian polynomial of type B *and Theorem 1.2 is equivalent to the known generating function, see* [7, Corollary 3.9] *or* [28, Theorem 13.3],

$$\sum_{n \ge 0} B_n(x, x) \frac{t^n}{n!} = \frac{(x - 1)e^{t(x-1)}}{x - e^{2t(x-1)}}. \tag{1.14}$$

Now, consider the following variant of $B_n(x, y)$

$$\widehat{B}_n(x, y) := \sum_{\sigma \in \mathcal{B}_n} x^{\operatorname{des}_1\sigma} y^{\operatorname{asc}_0\sigma} = y^{\lfloor (n+1)/2 \rfloor} B_n(x, 1/y). \tag{1.15}$$

From Theorem 1.2 we derive plainly the generating function of the latter polynomials.

Theorem 1.3. *Let $\alpha = \sqrt{(1-x)(1-y)}$. Then*

$$\sum_{n\geq 1} \widehat{B}_{2n}(x,y)\frac{t^{2n}}{(2n)!} = \frac{(1+xy)\cos(2\alpha t) - (1-x)(1-y)\cos(\alpha t) - (x+y)}{(x+y) - (1+xy)\cos(2\alpha t)}, \tag{1.16a}$$

$$\sum_{n\geq 1} \widehat{B}_{2n-1}(x,y)\frac{t^{2n-1}}{(2n-1)!} = \frac{-\alpha(1+y)\,\sinh(\alpha t)}{(x+y) - (1+xy)\cos(2\alpha t)}. \tag{1.16b}$$

Remark 4. *Similar to Chebikin's alternating descent set of type A (see* [5]*), we can define the* alternating descent set *of any $\sigma \in \mathcal{B}_n$ by*

$$\widehat{D}_B(\pi) = \{i \in \{0\} \cup [n-1] | \pi(i) > \pi(i+1) \text{ if } i \text{ is odd or } \pi(i) < \pi(i+1) \text{ if } i \text{ is even}\}.$$

Let $\widehat{des}_B(\sigma) = |\widehat{D}_B(\sigma)|$. Clearly $\widehat{B}_n(x,x) = \sum_{\sigma\in\mathcal{B}_n} x^{\widehat{des}_B(\sigma)}$, which is the n-th alternating Eulerian polynomial of type B *in* [21]*, and Theorem 1.3 reduces to the generating function in* [23, 9, 25],

$$\sum_{n\geq 0} \hat{B}_n(x,x)\frac{u^n}{n!} = \frac{x-1}{(x-1)\cos(u(1-x)) + (x+1)\sin(u(1-x))}. \tag{1.17}$$

Define the general enumerative polynomials of permutations by the parity of the ascent and descent positions:

$$P_n^B(x_0,x_1,y_0,y_1) = \sum_{\sigma\in\mathcal{B}_n} x_0^{\mathrm{asc}_0\,\sigma} x_1^{\mathrm{asc}_1\,\sigma} y_0^{\mathrm{des}_0\,\sigma} y_1^{\mathrm{des}_1\,\sigma}. \tag{1.18}$$

For any $\sigma \in \mathcal{B}_n$ we have

$$\begin{aligned} \mathrm{des}_0\,\sigma + \mathrm{asc}_0\,\sigma &= \lfloor (n+1)/2 \rfloor, \\ \mathrm{des}_1\,\sigma + \mathrm{asc}_1\,\sigma &= \lfloor n/2 \rfloor. \end{aligned} \tag{1.19}$$

Hence the distribution of the quadruple statistics $(\mathrm{asc}_0, \mathrm{asc}_1, \mathrm{des}_0, \mathrm{des}_1)$ is equivalent to any of the four pairs in $\{\mathrm{des}_1, \mathrm{asc}_1\} \times \{\mathrm{des}_0, \mathrm{asc}_0\}$. It follows that

$$P_n^B(x_0,x_1,y_0,y_1) = x_0^{\lfloor (n+1)/2\rfloor} x_1^{\lfloor n/2\rfloor} B_n\left(\frac{y_1}{x_1}, \frac{y_0}{x_0}\right). \tag{1.20}$$

We derive plainly the following generating function from Theorem 1.2.

Theorem 1.4. *We have*

$$\sum_{n\geq 1} P_{2n}^B(x_0,x_1,y_0,y_1)\frac{t^{2n}}{(2n)!} = \frac{(x_0y_1 + x_1y_0)\sum_{n\geq 0}\frac{\alpha^n(2t)^{2n}}{(2n)!} + \sum_{n\geq 0}\frac{\alpha^{n+1}t^{2n}}{(2n)!} - (x_1x_0 + y_0y_1)}{(x_0x_1 + y_0y_1) - (y_1x_0 + x_1y_0)\sum_{n\geq 0}\frac{\alpha^n(2t)^{2n}}{(2n)!}}, \tag{1.21}$$

and

$$\sum_{n\geq 1} P^B_{2n-1}(x_0,x_1,y_0,y_1)\frac{t^{2n-1}}{(2n-1)!} = \frac{(y_0^2-x_0^2)(y_1-x_1)\sum_{n\geq 0}\frac{\alpha^n t^{2n+1}}{(2n+1)!}}{(x_0x_1+y_0y_1)-(x_0y_1+x_1y_0)\sum_{n\geq 0}\frac{\alpha^n(2t)^{2n}}{(2n)!}}, \quad (1.22)$$

where $\alpha=(y_0-x_0)(y_1-x_1)$.

In view of (1.15) and (1.20), Theorem 1.2, Theorem 1.3 and Theorem 1.4 are equivalent. We shall give a proof of Theorem 1.2 in the same vein as the proof of (1.6a) with $q=1$.

An important feature of Eulerian polynomials is the gamma-nonnegativity [28]. More recently, Sun [33] proved that the bivariate Eulerian polynomials $(1+y)A_{2n}(x,y,1)$ and $A_{2n+1}(x,y,1)$ are γ-positive (see Theorem 2.3). Our third goal is to derive some symmetric expansion formulae for bivariate polynomials allied to the above four families of bi-Eulerian polynomials. This will be done by applying their generating functions and combinatorics of André permutations [12, 13, 17].

The rest of this paper is organised as follows. We will first study the symmetric and gamma expansions of the two sequences of bi-polynomials as well as their type analogues in Section 2 and postpone the proof of (1.6a) and Theorem 1.2 to Section 3 and Section 4, respectively. We conclude with some open problems in Section 5.

As suggested by a referee, for reader's convenience, we list the main permutation statistics of this paper in the following table.

$\operatorname{des}_0\pi$	the number of descents of π at even positions
$\operatorname{des}_1\pi$	the number of descents of π at odd positions
$\operatorname{asc}_0\pi$	the number of ascents of π at even positions
$\operatorname{asc}_1\pi$	the number of ascents of π at odd positions
$\operatorname{inv}\pi$	the number of inversions of π
$\operatorname{lpk}(\pi)$	the number of left peaks of π, see (2.14)

Table 1. Main statistics of $\pi\in\mathfrak{S}_n$

2. Symmetric and positive expansions of bi-Eulerian polynomials

Define two families of bi-Eulerian polynomials $(\widetilde{A}_n(x,y))_{n\geq 1}$ and $(\overline{A}_n(x,y))_{n\geq 1}$ by

$$\widetilde{A}_{2n}(x,y)=(1+y)A_{2n}(x,y,1),\quad \widetilde{A}_{2n-1}(x,y)=A_{2n-1}(x,y,1), \quad (2.1a)$$

$$\overline{A}_{2n}(x,y)=(1+y)\widehat{A}_{2n}(x,y,1),\quad \overline{A}_{2n-1}(x,y)=\widehat{A}_{2n-1}(x,y,1); \quad (2.1b)$$

and their type B analogues $(\widetilde{B}_n(x,y))_{n\geq 1}$ and $(\overline{B}_n(x,y))_{n\geq 1}$ by

$$\widetilde{B}_{2n}(x,y) = B_{2n}(x,y), \quad \widetilde{B}_{2n-1}(x,y) = (1+y)^{-1}B_{2n-1}(x,y), \tag{2.2a}$$

$$\overline{B}_{2n}(x,y) = \widehat{B}_{2n}(x,y), \quad \overline{B}_{2n-1}(x,y) = (1+y)^{-1}\widehat{B}_{2n-1}(x,y). \tag{2.2b}$$

By (1.6a) and (1.6b) (resp. Theorem 1.2 and Theorem 1.3) both polynomials $\widetilde{A}_n(x,y)$ and $\overline{A}_n(x,y)$ (resp. $\widetilde{B}_n(x,y)$ and $\overline{B}_n(x,y)$) are symmetric in x and y.

Recall that a polynomial with real coefficients $P(x) = \sum_{i=0}^n a_i x^i$ is *gamma-positive* (resp. *semi-gamma-positive*) if there are nonnegative numbers γ_i such that $P(x) = \sum_i \gamma_i x^i (1+x)^{n-2i}$ (resp. $P(x) = (1+x)^\nu \sum_i \gamma_i x^i (1+x^2)^{\lfloor n/2\rfloor - i}$ with $\nu = 0$ or 1.), see [28] and [22] respectively. It is known that the gamma-positivity is stronger than the semi-gamma-positivity [22].

In this section, we shall first derive the semi-gamma-positive formulae for the bi-Eulerian polynomials $\widetilde{A}_n(x,y), \overline{A}_n(x,y), \widetilde{B}_n(x,y)$ and $\overline{B}_n(x,y)$ from their generating functions and then apply Hetyei-Reiner's min-max tree model [17] for permutations to derive the corresponding γ-positive formulae for $\widetilde{A}_n(x,y)$ and $\overline{A}_n(x,y)$ as well as their type B analogues by refining Petersen's proof for the γ-positivity of type B Eulerian polynomials [28].

2.1. **Semi-gamma-positivity of bi-Eulerian polynomials.** The following generalizes the semi-gamma-positivity of Eulerian polynomials to bi-Eulerian polynomials.

Theorem 2.1. *Let $a(n,j)$ (resp. $\bar{a}(n,j)$) be the number of permutations in $\mathfrak{S}_n$ with j odd descents and without even descents (resp. ascents) for $n \geq 1$ and $0 \leq 2j \leq n$. Then*

$$\widetilde{A}_n(x,y) = \sum_{j=0}^{\lfloor \frac{n}{2}\rfloor} a(n,j)\,(x+y)^j(1+xy)^{\lfloor \frac{n}{2}\rfloor - j}; \tag{2.3a}$$

$$\overline{A}_n(x,y) = \sum_{j=0}^{\lfloor \frac{n}{2}\rfloor} \bar{a}(n,j)\,(x+y)^j(1+xy)^{\lfloor \frac{n}{2}\rfloor - j}, \tag{2.3b}$$

and

$$\bar{a}(n,j) = a(n, \lfloor n/2\rfloor - j) \quad \textit{for} \quad 0 \leq j \leq \lfloor n/2\rfloor. \tag{2.3c}$$

Proof. Let $\alpha(x,y) = (1-x)(1-y)$. Then

$$\alpha(x,y) = (1+xy)\cdot \alpha\left(\frac{x+y}{1+xy}, 0\right).$$

It follows from (1.6) that

$$\widetilde{A}_n(x,y) = (1+xy)^{\lfloor \frac{n}{2} \rfloor} A_n\left(\frac{x+y}{1+xy}, 0, 1\right), \tag{2.4a}$$

$$\overline{A}_n(x,y) = (1+xy)^{\lfloor \frac{n}{2} \rfloor} \widehat{A}_n\left(\frac{x+y}{1+xy}, 0, 1\right), \tag{2.4b}$$

which are obviously equivalent to (2.3a) and (2.3b), respectively.

Define the completion σ^c of $\sigma \in \mathfrak{S}_n$ by $\sigma^c(i) = n+1-\sigma(i)$ for $1 \leq i \leq n$. It is clear that the mapping $\varphi : \sigma \mapsto \sigma^c$ is an involution on $\mathfrak{S}_n$ and satisfies $\mathrm{des}_i\, \sigma = \mathrm{asc}_i\, \sigma^c$ for $i \in \{0,1\}$. Thus

$$(\mathrm{des}_1 \sigma^c,\ \mathrm{asc}_0 \sigma^c) = (\mathrm{asc}_1 \sigma,\ \mathrm{des}_0 \sigma) = (\lfloor n/2 \rfloor - \mathrm{des}_1 \sigma,\ \mathrm{des}_0 \sigma)\,.$$

Eq. (2.3c) follows by restricting φ on the set of permutations in $\mathfrak{S}_n$ with j odd descents and without even descent. $\square$

Remark 5. *The combinatorial interpretation of $a_{n,j}$ actually follows from the existence of formula* (2.3a)*, which was first conjectured by Sun* [32] *and then proved by Sun and Zhai* [34]*.*

Similarly, we have the following B-analogue of Theorem 2.1.

Theorem 2.2. *Let $b(n,j)$ (resp. $\bar{b}(n,j)$) be the number of permutations in $\mathcal{B}_n$ with j odd descents and without even descents (resp. even ascents). Then*

$$\widetilde{B}_n(x,y) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} b(n,j)\,(x+y)^j (1+xy)^{\lfloor \frac{n}{2} \rfloor - j}, \tag{2.5a}$$

$$\overline{B}_n(x,y) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \bar{b}(n,j)\,(x+y)^j (1+xy)^{\lfloor \frac{n}{2} \rfloor - j}, \tag{2.5b}$$

and

$$\bar{b}(n,j) = b(n, \lfloor n/2 \rfloor - j) \quad \text{for} \quad 0 \leq j \leq \lfloor n/2 \rfloor\,. \tag{2.5c}$$

Proof. Let $\alpha(x,y) = (1-x)(1-y)$. Then

$$\alpha(x,y) = (1+xy) \cdot \alpha((x+y)/(1+xy), 0).$$

We derive from Theorem 1.2 and Theorem 1.3 immediately

$$\widetilde{B}_n(x,y) = (1+xy)^{\lfloor \frac{n}{2} \rfloor} B_n\left(\frac{x+y}{1+xy}, 0\right), \tag{2.6a}$$

$$\overline{B}_n(x,y) = (1+xy)^{\lfloor \frac{n}{2} \rfloor} \widehat{B}_n\left(\frac{x+y}{1+xy}, 0\right), \tag{2.6b}$$

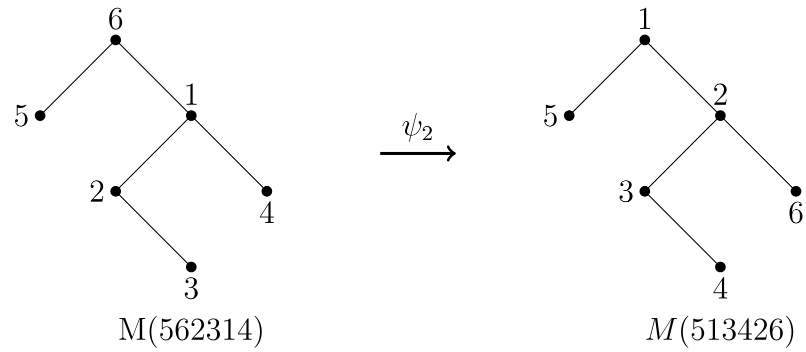

FIGURE 1. The action of operator ψ_2 at tree $M(562314)$.

which are what (2.5a) and (2.5b) mean.

Consider the negation $\bar{\sigma}$ of $\sigma \in \mathcal{B}_n$ by $\bar{\sigma}(i) = -\sigma(i)$ for $1 \le i \le n$. It is clear that the mapping $\phi : \sigma \mapsto \bar{\sigma}$ is an involution on $\mathfrak{S}_n$ and satisfies $\operatorname{des}_i \sigma = \operatorname{asc}_i \bar{\sigma}$ for $i \in \{0, 1\}$. Thus

$$(\operatorname{des}_1 \bar{\sigma},\ \operatorname{asc}_0 \bar{\sigma}) = (\operatorname{asc}_1 \sigma,\ \operatorname{des}_0 \sigma) = (\lfloor n/2 \rfloor - \operatorname{des}_1 \sigma,\ \operatorname{des}_0 \sigma)\,.$$

Eq. (2.5c) follows by restricting ϕ on the set of permutations in $\mathcal{B}_n$ with j odd descents and without even descent. $\square$

We note that Eq. (2.3a) does not directly reduce to the known γ-positivity formula of Eulerian polynomials $A_n(x, x)$ when $x = y$. To derive the latter expansion we shall appeal to the min-max tree representations of permutations due to Hetyei and Reiner [17]. Similarly, to derive gamma-positivity formula of type B Eulerian polynomials $B_n(x, x)$ from Theorem 2.2 we shall appeal to an action on permutations due to Petersen [28].

2.2. **Gamma-positivity of bi-Eulerian polynomials of type A.** We define the *min-max tree* $M(w)$ associated to a sequence of distincte integers $w = w_1 \dots w_n$ as follows.

(1) First, $M(w)$ is a binary tree with vertices labelled $w_1, \dots, w_n$. Let i be the least integer for which either $w_i = min\{w_1, w_2, \dots, w_n\}$ or $w_i = \max\{w_1, w_2, \dots, w_n\}$. Define w_i to be the root of $M(w)$.
(2) Then recursively define $M(w_1, \dots, w_{i-1})$ and $M(w_{i+1}, \dots, w_n)$ to be the left and right subtree of w_i, respectively.

Conversely, the left-first order reading of the tree $M(w)$ yields the sequence w, see [17, 10] and [31, pp. 57-61].

An interior vertex in $M(w)$ is called a *min (resp. max)* vertex if it is the minimum (resp. maximum) label among all its descendants. Let $M(w_i)$ $(resp.\ M_l(w_i), M_r(w_i))$ denote the subtree (resp. the left subtree, the right subtree) of $M(w)$ with root w_i.

For $1 \le i \le n$, we define the *operator* ψ_i permuting the labels of $M(w)$ as in the following.

(1) If w_i is a min vertex, then replace w_i by the largest element of $M_r(w_i)$, permute the remaining elements of $M_r(w_i)$ such that they keep their same relative orders and all other vertices in $M(w)$ are fixed.
(2) If w_i is a max vertex, then replace w_i by the smallest element of $M_r(w_i)$ such that they keep their same relative order, and all other vertices in $M(w)$ are fixed.

An illustration of operator ψ_2 is given in Figure 1.

Given a permutation $\pi = \pi(1)\pi(2)\dots\pi(n)$ of $Y = \{y_1, y_2, \dots, y_n\}_<$, which is a set of positive integers. The $\pi(i)$-factorization of π is the sequence $(w_1, w_2, \pi(i), w_4, w_5)$, $1 \le i \le n$, where

(1) the concatenation product $w_1 w_2 \pi(i) w_4 w_5$ is equal to π;
(2) w_2 is the longest right factor of $\pi(1)\pi(2)\dots\pi(i-1)$, all letters of which are greater than $\pi(i)$;
(3) w_4 is the longest left factor of $\pi(i+1)\pi(i+2)\dots\pi(n)$, all letters of which are greater than $\pi(i)$.

Note that above any of w_1, w_2, w_4 or w_5 may be empty.

Definition 2.1 (see [13, 10])**.** *A permutation $\pi \in \mathfrak{S}_n$ is an* André permutation *(of kind I) if π has no double descents and ends with ascent, i.e., $\pi(n-1) < \pi(n)$, and if $i \in \{2, \dots, n\}$ is a valley of π and $(w_1, w_2, \pi(i), w_4, w_5)$ is the $\pi(i)$-factorization of π, then the maximum letter of $w_2 w_4$ is in w_4.*

For example, the André permutations of length 4 are 1234, 1324, 2314, 2134 and 3124.

Fact 2.2. *The operators ψ_i are commuting involutions acting on $M(w)$ and generate an abelien group G_w isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{l(w)}$, where $l(w)$ is the number of internal certices of $M(w)$. Those ψ_i for which w_i is an internal vertex are a minimal set S_w of generators for G_w. For any subset $S \subseteq G_w$ we define the HR action ψ_S by $\psi_S(M(w)) = \prod_{i \in S} \psi_i(M(w))$.*

For $\pi \in \mathfrak{S}_n$, let $\mathrm{Orb}(\pi)$ be the set of permutations w such that $M(w)$ is in the orbit of $M(\pi)$ under the HR-action. Thus, for any $\pi \in \mathfrak{S}_n$, there is a unique permutation π^A in $\mathrm{Orb}(\pi)$ such that all its interior vertices in $M(\pi^A)$ are min vertices.

Fact 2.3. *A permutation $\pi \in \mathfrak{S}_n$ is an André permutation if and only if all interior vertices of min-max tree $M(\pi)$ are min vertices.*

It follows that $\cup_{\pi \in \mathrm{And}_n} \mathrm{Orb}(\pi) = \mathfrak{S}_n$, where And_n is the set of André permutations in $\mathfrak{S}_n$. Let $\mathfrak{S}_n^*$ (resp. $\mathrm{Orb}^*(\pi)$) be the subset of permutations in $\mathfrak{S}_n$ (resp. $\mathrm{Orb}(\pi)$) which have no even descents. By restriction on the permutations which have only odd-descents we have

$$\mathfrak{S}_n^* = \cup_{\pi\in \mathrm{And}_n} \mathrm{Orb}^*(\pi). \tag{2.7}$$

For any subset $S \subseteq [n]$ and André permutation π, since all interior vertices of $M(\pi)$ are min vertices, we have $\operatorname{des}\psi_S(M(\pi)) \geq \operatorname{des}\pi$.

Let $\pi \in \mathrm{And}_n$. So all the interior vertices of $M(\pi)$ are min vertices, if $\pi(k)$ is a valley of π with the $\pi(k)$-factorization $(w_1, w_2, \pi(k), w_4, w_5)$, then the position of the last letter of w_2 is a descent position, and the HR action ψ_k on $M(\pi)$ will shift the descent position $k-1$ to k, since the vertex in $M(\pi)$ corresponding to $\pi(k)$ will be relabelled by the largest letter of its subtree, and all other vertices keep their same relative order. Thus, the HR action ψ_S with S being the set of indices of odd-valley-positions in π will evacuate all the even descent positions, and the total number of descents will remain the same, let $\psi_S(\pi) = \pi'$, clearly $\pi' \in \mathrm{Orb}^*(\pi)$.

Fact 2.4. *For $\pi \in \mathrm{And}_n$ we have*

$$\mathrm{Orb}^*(\pi) = \mathrm{Orb}^*(\pi') = \prod_{i\in S}(1+\psi_i)\{\pi'\},$$

where S is the set of odd ascent positions of π'. Moreover, as $des_0\psi_i(\pi') = des_0(\pi')+1$, the folllowing identity holds

$$\sum_{\sigma\in \mathrm{Orb}^*(\pi)} p^{des\,\sigma} = (1+p)^{\lfloor n/2\rfloor - des\,\pi} p^{des\,\pi}. \tag{2.8}$$

Recall that $a(n,j)$ is the number of permutations in $\mathfrak{S}_n$ with j odd descents and without even descents.

Lemma 2.5. *Let $d(n,j)$ be the number of André permutations in $\mathfrak{S}_n$ with j descents for $0 \leq 2j \leq n$. Then*

$$a(n,j) = \sum_{i=0}^{j} \binom{\lfloor n/2\rfloor - i}{j-i} d(n,i). \tag{2.9}$$

Proof. Applying the above facts

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_n^*} p^{\operatorname{des}(\sigma)} &= \sum_{\pi\in\mathrm{And}_n}\sum_{\sigma\in\mathrm{Orb}^*(\pi)} p^{\operatorname{des}(\sigma)}\\ &= \sum_{\pi\in\mathrm{And}_n}(1+p)^{\lfloor n/2\rfloor-\operatorname{des}\pi}p^{\operatorname{des}\pi}.\end{aligned}$$

We derive (2.9) by extracting the coefficient of p^j. $\square$

Lemma 2.6. *If $\bar{d}(n,i)$ is the number of min-max trees on $[n]$ having i max interior vertices with two children, then*

$$\bar{d}(n,i) = \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} \binom{j}{i} d(n,j). \tag{2.10}$$

Proof. If $\pi \in \mathfrak{S}_n$ is an André permutation, then the number of interior vertices with two children of $M(\pi)$ equals $\operatorname{des}(\pi)$. Any permutation $\pi \in \mathfrak{S}_n$ such that $M(\pi)$ has i max interior vertices with two children can be obtained from the André permutation π^A in $\operatorname{Orb}(\pi)$ by choosing i interior vertices with two children among the interior vertices with two children of $M(\pi^A)$ and then applying HR operator on these i vertices (to transform them into max vertices). Hence, in each orbite of an André min-max tree (i.e., the tree $M(w)$ associated to an André permutation w) with j interior vertices having two children, there are $\binom{j}{i}$ min-max trees on $[n]$ having i max interior vertices with two children. The result follows by summing over all the orbits. □

Recall that a permutation w of $[n]$ is an *André permutation of kind II* if, for $1 \le k \le n$,

(1) the subsequence of the smallest k elements in w has no double descent ;
(2) the subsequence of the smallest k elements in w ends with an ascent.

The permutation w is called *Simsun* if it satisfies condition (1), [8, 10, 27]. For example, the five Simsun 3-permutations are: $231, 132, 312, 123, 213$ and the five André 4-permutations of the second kind are: 1234, 1423, 3124, 3412, 4123.

Actually, the number of n-André permutations and that of $(n-1)$-simsun permutations are both equal to the *Euler number* E_n, which can be defined by

$$\sum_{n\ge 0} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

Let $D_n(x)$ (resp. $rs_n(x)$) be the descent polynomial of André permutations (resp. Simsun permutations) of length n. By means of generating function argument, Chow and Shiu [8] proved that the descent number is equidistributed over $(n-1)$-simsun permutations and n-André permutations, i.e.,

$$D_n(x) = rs_{n-1}(x) = \sum_{i=0}^{n-1} d(n,i)x^i \quad (n \ge 2) \tag{2.11}$$

with $D_1(x) = 1$.

Combining Theorem 2.1 and Lemma 2.5 we obtain an alternative proof of the following result of H. Sun [33].

Theorem 2.3. *Let $d(n,j)$ be the number of André permutations in $\mathfrak{S}_n$ with j descents for $0 \le 2j \le n$ and $\bar{d}(n,i)$ be the number of min-max trees on n vertices having i max interior vertices with two children, then*

$$\widetilde{A}_n(x,y) = \sum_{j=0}^{\lfloor n/2 \rfloor} d(n,j)(x+y)^j(1+x+y+xy)^{\lfloor \frac{n}{2} \rfloor - j}, \tag{2.12a}$$

$$\overline{A}_n(x,y) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \bar{d}(n,i)(x+y)^i(1+x+y+xy)^{\lfloor n/2 \rfloor - i}, \tag{2.12b}$$

and $rs_{n-1}(1+x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \bar{d}(n,i)x^i$, where $rs_{n-1}(x)$ is the descent polynomial of Simsun permutations.

Proof. Plugging (2.9) in (2.3a) we obtain

$$\begin{aligned}
\widetilde{A}_n(x,y) &= \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{i=0}^{j} \binom{\lfloor n/2 \rfloor - i}{j-i} d(n,i)(x+y)^j(1+xy)^{\lfloor \frac{n}{2} \rfloor - j} \\
&= \sum_{i=0}^{\lfloor n/2 \rfloor} d(n,i)(x+y)^i \sum_{j \ge 0} \binom{\lfloor n/2 \rfloor - i}{j} (x+y)^j (1+xy)^{\lfloor n2 \rfloor - i - j} \\
&= \sum_{i=0}^{\lfloor n/2 \rfloor} d(n,i)(x+y)^i(1+x+y+xy)^{\lfloor n/2 \rfloor - i},
\end{aligned}$$

which is the right-hand side of (2.12a) upon replacing i by j.

By (1.8) and (2.1) we have $\overline{A}_n(x,y) = y^{\lfloor n/2 \rfloor} \widetilde{A}_n(x, 1/y)$. Hence

$$\overline{A}_n(x,y) = \sum_{j=0}^{\lfloor n/2 \rfloor} d(n,j)(1+xy)^j(1+x+y+xy)^{\lfloor n/2 \rfloor - j}.$$

Now, rewriting $(1+xy)^j$ in the last sum as

$$(1+xy)^j = \sum_{i=0}^{j} (-1)^i \binom{j}{i} (1+x+y+xy)^{j-i}(x+y)^i,$$

we obtain the right-hand side of (2.12b). □

Remark 6. *Comparing* (2.13) *with* [27, Theorem 2] *we notice that $d(n,j)$ is also the number of André permutations of kind II of $[n]$ with j descents. This result is implicit in* [12, 13, 10]. *If $x = y$, Theorem 2.3 plainly reduces to the classical γ-formula of Eulerian* *polynomials, see* [27, Theorem 1],

$$A_n(x,x) = \sum_{j=0}^{\lfloor n/2 \rfloor} d(n,j)\, 2^j\, x^j (1+x)^{n-1-2j}. \tag{2.13}$$

Also, Lin et al. [20, Theorem 1.1] *proved the $x = y$ case of* (2.12b).

2.3. **Gamma-positivity of bi-Eulerian polynomials of type B.** We refine Petersen's proof of gamma-nonnegativity of type B Eulerian polynomials in [28].

Given a permutation $u \in \mathfrak{S}_n$ we denote by $\mathcal{B}(u)$ the set of all permutations $\omega \in \mathcal{B}(u)$ such that $\omega(i) = \sigma_i u(i)$ with $\sigma_i \in \{-,+\}$ for $1 \leq i \leq n$. Then we have the following observations:

- if $u(i-1) < u(i)$, then $\omega(i-1) > \omega(i)$ if and only if $\sigma_i = -$,
- if $u(i-1) > u(i)$, then $\omega(i-1) > \omega(i)$ if and only if $\sigma_{i-1} = +$.

To put it another way, the sign σ_j controls the descent in position $j-1$ if and only if $j-1$ is *not* a descent position of u, and it controls the descent in position j if and only if j *is* a descent position of u.

Consider the example of $u = 31472865$. Then there is a descent in position 0 if and only if $\sigma_1 = -$ while there is a descent in position 1 if and only if $\sigma_1 = +$. Since $u(2) = 1$ is smaller than the elements on either side of it, the sign σ_2 has no effect whatever on the descent set. With $u(3) = 4$, we find that $\omega(2) > \omega(3)$ if and only if $\sigma_3 = -$, but that σ_3 does not control whether $\omega(3)$ is greater than $\omega(4)$ (σ_4 does that). By considering the sign of each letter in turn.

We summarize the above consideration more precisely in the following

Observation 2.7. *Let $u \in \mathfrak{S}_n$. If $\omega \in \mathcal{B}_n(u)$ with $\omega(j) = \sigma_j u(j)$, then*

- *If $u(j-1) < u(j) > u(j+1)$, then σ_j controls both the descent in position $j-1$ and position j. That is, if $\sigma_j = +$, then in ω, $j-1$ is not a descent position, but j is a descent position. If $\sigma_j = -$, then in ω, $j-1$ is a descent position but j is not. This means, σ_j does not change the number of descents, but it controls the parity of descent position.*
- *If $u(j-1) < u(j) < u(j+1)$, then σ_j controls the descent on position $j-1$, but no effect on position j. That is, if $\sigma_j = +$, then $j-1$ is not a descent position, if $\sigma_j = -$, then $j-1$ is a descent position.*
- *If $u(j-1) > u(j) > u(j+1)$, then σ_j controls the descent on position j, but no effect on position $j-1$. That is, if $\sigma_j = +$, then j is a descent position, if $\sigma_j = -$, then j is not a descent position.*
- *If $u(j-1) > u(j) < u(j+1)$, then σ_j has no effect on the descent set.*

The *number of left peaks* of permutation $u \in \mathfrak{S}_n$ is defined by

$$\mathrm{lpk}(u) = |\{1 \le i < n : u(i-1) < u(i) > u(i+1)\}|, \tag{2.14}$$

where $u(0) = 0$, $u(n+1) = n+1$.

Lemma 2.8. *If $\omega \in \mathcal{B}_n$ is a permutation with j odd descents and without even descents, then $|\omega|$ is a permutation in $\mathfrak{S}_n$ with $\mathrm{lpk}(|\omega|) \le j$.*

Proof. Since ω does not have descents on even positions, we have $\omega(1) > 0$ and ω does not have double descents. Suppose $\omega(i)$ is the first valley with $\sigma_i = -$ and $\omega(k)$ is the peak closest to $\omega(i)$ on the right. Then $\omega(i)\omega(i+1)\dots\omega(k)$ is an increasing subsequence, and there has no peak in $|\omega(i)||\omega(i+1)|\dots|\omega(k)|$. Let $\omega_0 = \omega(1)\omega(2)\dots\omega(i-1)|\omega(i)||\omega(i+1)|\dots|\omega(k)|\omega(k+1)\dots\omega(n)$ then, the difference of peak sets of ω_0 and ω happens on $\omega(i-1)$, $|\omega(i)|$ and $|\omega(k)|$, $\omega(k+1)$. As it is not possible that both $\omega(i-1)$ and $|\omega(i)|$ are peaks in ω_0 (but $\omega(i-1)$ is a peak in ω). Since $|\omega(k)| \ge \omega(k) > \omega(k+1)$, so $|\omega(k)|$ is the only possible peak candidate of $|\omega(k)|$ and $\omega(k+1)$ in ω_0 ($\omega(k)$ is a peak in ω). In summary, we have $\mathrm{lpk}(\omega_0) \le \mathrm{lpk}(\omega)$. We repeat this process on ω_0, finally, we obtain $\mathrm{lpk}(|\omega|) \le \mathrm{lpk}(\omega) = j$. $\square$

Lemma 2.9. *Let $g(n,i) = |\{u \in \mathfrak{S}_n : \mathrm{lpk}\,(u) = i\}|$. Then*

$$b(n,j) = \sum_{i=0}^{j} \binom{\lfloor n/2 \rfloor - i}{j-i} g(n,i) 2^i.$$

Proof. Let u be a permutation in $\mathfrak{S}_n$ with $\mathrm{lpk}(u) = i \le j$. We can use the following process to transform it to a permutation of $\mathcal{B}_n$ with j odd descent and without even descents.

Process A

(1) Firstly, we sign the i valleys of u with either $-$ or $+$, which gives ω_1.
(2) Secondly, in ω_1, we sign the peaks at even positions with $-$, then we obtain ω_2 with all the peaks at odd positions (by Remark 2.7).
(3) Thirdly, choose a $j-i$ elements subset D of $C := \{1, 3, \dots, 2\lfloor \frac{n}{2} \rfloor - 1\} \setminus \mathrm{LPK}(\omega_2)$, where $\mathrm{LPK}(\omega_2)$ is the position set of peaks of ω_2. For $l \in D$, if $\omega_2(l)$ is a descent then we do nothing with $\omega_2(l)$, if $\omega_2(l)$ is an ascent then we sign $\omega_2(l+1)$ (*it must be a double ascent in u*) with $-$. For $l \notin D$ but $l \in C$, if $\omega_2(l)$ is a descent then we sign $\omega_2(l)$ (*it must be a double descent in u*)with $-$, if $\omega_2(l)$ is an ascent, then we do nothing with $\omega_2(l)$, which gives ω_3.
(4) Lastly, in ω_3 we sign all the double descents at even positions with $-$, which gives ω_4.

By Observation 2.7, we see that ω_4 is a permutation in $\mathcal{B}_n$ with j odd descents and without even descents.

In this process, no letter in u is repeatedly signed. And we can see that for a fixed $u \in \mathfrak{S}_n$ with i peaks, by Process A, it can produce $\binom{\lfloor n/2\rfloor - i}{j-i} \cdot 2^i$ different permutations in $\mathcal{B}_n$ with j odd descents and without even descents. By Lemma 2.8, for $\omega \in \mathcal{B}_n$ with j odd descents and without even descents, we have $|\mathrm{lpk}(|\omega|)| \le j$ and by Remark 2.7, the descent positions in ω are totally controlled by the signs of peaks, double descents and double ascents of $|\omega|$, that is ω can be constructed by $|\omega|$ through Process A. This completes the proof. $\square$

Theorem 2.4. *Let* $g(n,j) = |\{u \in \mathfrak{S}_n : \mathrm{lpk}(u) = j\}|$. *Then*

$$\widetilde{B}_n(x,y) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} g(n,j) 2^j (x+y)^j (1+x+y+xy)^{\lfloor \frac{n}{2} \rfloor - j}, \tag{2.15}$$

$$\overline{B}_n(x,y) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \bar{g}(n,j) 2^j (x+y)^j (1+x+y+xy)^{\lfloor \frac{n}{2} \rfloor - j} \tag{2.16}$$

with

$$\bar{g}(n,j) = \sum_{i=0}^{\lfloor n/2 \rfloor - j} \binom{i+j}{j} g(n, i+j) 2^i. \tag{2.17}$$

Proof. By Theorem 2.2 and Lemma 2.9, we obtain (2.15). To prove (2.16), by (1.15), (2.2a) and (2.2b), we first note

$$\overline{B}_n(x,y) = y^{\lfloor \frac{n}{2} \rfloor} \widetilde{B}_n(x, 1/y).$$

It follows from (2.15) that

$$\overline{B}_n(x,y) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} g(n,j) 2^j \, (1+xy)^j (1+x+y+xy)^{\lfloor \frac{n}{2} \rfloor - j}. \tag{2.18}$$

The rest of the proof is the same as that of Eq. (2.12b), so it is omitted. $\square$

Remark 7. *When $x = y$ identity* (2.18) *reduces to Proposition 10 in* [23]. *Identity* (2.17) *is equivalent to the polynomial identity:*

$$\begin{aligned} \sum_{j=0}^{\lfloor n/2 \rfloor} \bar{g}(n,j) x^j &= \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - j} \binom{i+j}{j} g(n, i+j) 2^i x^j \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} g(n,k) (2+x)^k. \end{aligned} \tag{2.19}$$

If $x = y$ identity (2.15) *reduces to Petersen's formula for type B Eulerian polynomial* $B_n(x,x)$*, see* [28, Theorem 13.5],

$$B_n(x,x) = \sum_{j=0}^{\lfloor n/2 \rfloor} g(n,j)\,(4x)^j(1+x)^{n-2j}, \tag{2.20}$$

and Eq. (2.16) *reduces to Ma et al.'s formula for type B alternating descent polynomials, see* [23, Theorem 12]

$$\widehat{B}_n(x,x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \bar{g}(n,j)(-4x)^j(1+x)^{n-2j}. \tag{2.21}$$

3. Counting permutations of type A by the parity of descent positions

If $\sigma = \sigma_1 \cdots \sigma_n$ is a permutation in $\mathfrak{S}_n$, the descent set $\mathrm{Des}(\sigma)$ of σ is $\mathrm{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}\} \subseteq [n-1]$. We denote by $\mathrm{Des}_0(\sigma)$ (resp. $\mathrm{Des}_1(\sigma)$) the set of even (resp. odd) descents of σ. For brevity we denote their cardinalities by $\mathrm{des}_0(\sigma) = |\mathrm{Des}_0(\sigma)|$ and $\mathrm{des}_1(\sigma) = |\mathrm{Des}_1(\sigma)|$.

Any subset $S = \{s_1, \dots, s_k\}_< \subseteq [n-1]$ can be encoded by the composition $\mathrm{co}(S) := (s_1, s_2 - s_1, \cdots, s_k - s_{k-1}, n - s_k)$ of n. Clearly this correspondence is a bijection. For any composition $\lambda = (\lambda_1, \dots, \lambda_l)$ of n, let S_λ be the subset $\{\lambda_1, \lambda_1 + \lambda_2, \dots, \lambda_1 + \cdots \lambda_{l-1}\}$ of $[n-1]$ and define the q-multinomial coefficient

$$\binom{n}{\lambda}_q := \binom{n}{\mathrm{co}(S_\lambda)}_q = \frac{n!_q}{\lambda_1!_q \cdots \lambda_l!_q}.$$

For any subset $S \subseteq [n-1]$, let $\Delta_n(S) := \{\sigma \in \mathfrak{S}_n \mid \mathrm{Des}(\sigma) \subseteq S\}$ and $R_n(S)$ be the set of rearrangements of word $1^{\lambda_1} \dots l^{\lambda_l}$, where $\lambda_i = s_i - s_{i-1}$ for $i \in [l]$ with $l = k+1$, $s_0 = 0$ and $s_l = n$. There is a bijection $\psi : \sigma \mapsto w$ from $\Delta_n(S)$ to $R_n(S)$ defined by $w(j) = i$ if $\sigma(j) \in \{\sigma(s_{i-1}+1), \dots, \sigma(s_i)\}_<$ for $j \in [n]$ and $i \in [l]$. Clearly the number of inversions of w, i.e., $|\{i < j \mid w(i) > w(j), i, j \in [n]\}|$, is equal to $\mathrm{inv}\,\sigma$. By a theorem of MacMahon (see [2, p. 41]) we obtain the following known result (see [31, p. 227]).

Lemma 3.1. *Let $S = \{s_1, s_2, \dots, s_k\}_< \subseteq [n-1]$ and $\alpha_n(S,q) = \sum_{\sigma \in \Delta_n(S)} q^{\mathrm{inv}\sigma}$. Then*

$$\alpha_n(S,q) = \binom{n}{\mathrm{co}(S)}_q.$$

To prove (1.6a) we need three more lemmas. For convenience, for any subset $S \subseteq \mathbb{N}$ let $S_{\mathrm{e}} = S \cap 2\mathbb{N}$ and $S_{\mathrm{o}} = S \cap (2\mathbb{N}+1)$ be the subsets of even and odd integers of S, respectively. For $n \in \mathbb{N}$, let $\mathrm{O}[n]$ (resp. $\mathrm{E}[n]$) be the collection of odd (resp. even) elements of $[n]$. Consider the polynomial

$$P_n(x,y,q) := \sum_{S\subseteq[n-1]} \alpha_n(S,q)x^{|S_o|}y^{|S_e|}. \tag{3.1}$$

Lemma 3.2. *For $n \geq 1$ we have*

$$A_n(x,y,q) = (1-x)^{\lfloor \frac{n}{2}\rfloor}(1-y)^{\lfloor \frac{n-1}{2}\rfloor} P_n\left(\frac{x}{1-x}, \frac{y}{1-y}, q\right). \tag{3.2}$$

Proof. By Lemma 2.1 we have

$$\begin{aligned} P_n(x,y,q) &= \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}_1(\sigma)}y^{\mathrm{des}_0(\sigma)}q^{\mathrm{inv}\,(\sigma)} \sum_{S\subseteq[n-1]\setminus D(\sigma)} x^{|S_o|}y^{|S_e|} \\ &= \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}_1(\sigma)}y^{\mathrm{des}_0(\sigma)}q^{\mathrm{inv}\,(\sigma)}(1+x)^{\lfloor \frac{n}{2}\rfloor-\mathrm{des}_1(\sigma)}(1+y)^{\lfloor \frac{n-1}{2}\rfloor-\mathrm{des}_0(\sigma)} \end{aligned}$$

as there are $\lfloor \frac{n}{2}\rfloor - \mathrm{des}_1(\sigma)$ odd (resp. $\lfloor \frac{n-1}{2}\rfloor - \mathrm{des}_0(\sigma)$ even) integers in $[n-1]\setminus D(\sigma)$. In other words, we can write $P_n(x,y,q)$ as

$$P_n(x,y,q) = (1+x)^{\lfloor \frac{n}{2}\rfloor}(1+y)^{\lfloor \frac{n-1}{2}\rfloor} A_n\left(\frac{x}{1+x}, \frac{y}{1+y}, q\right),$$

which is equivalent to (3.2). □

Remark 8. *Let $P_n(x) = \sum_{S\subseteq[n-1]} \alpha_n(S,1)x^{|S|}$. It is not diffucult to see that*

$$P_n(x) = \sum_{k=0}^{n-1}(k+1)!S(n,k+1)x^k,$$

where $S(n,k)$ denotes the Stirling number of the second kind, i.e., the number of ways to partition a set of n objects into k non-empty subsets (see [31]*). So, when $x=y$, formula* (3.2) *reduces to the* Frobenius formula*, see* [11]*,*

$$A_n(x) = \sum_{k=1}^{n} k!S(n,k)x^{k-1}(1-x)^{n-k}. \tag{3.3}$$

Lemma 3.3. *We have*

$$B(t,x) := \sum_{n\geq 1} P_{2n}(x,0,q)\frac{t^{2n}}{(2n)!_q} = \frac{(\cosh_q t - 1)(1 - x(\cosh_q t - 1)) + x\sinh_q^2 t}{1 - x(\cosh_q t - 1)}, \tag{3.4}$$

$$C(t,x) := \sum_{n\geq 1} P_{2n-1}(x,0,q)\frac{t^{2n-1}}{(2n-1)!_q} = \frac{\sinh_q t}{1 - x(\cosh_q t - 1)}. \tag{3.5}$$

Proof. There is a bijection between the set of compositions $\gamma = (\gamma_1, \cdots, \gamma_l)$ of $2n$ such that γ_1, $\gamma_1+\gamma_2$, $\dots$, $\gamma_1+\gamma_2+\cdots+\gamma_{l-1}$ are odd numbers and the set of subsets S_γ of $\mathrm{O}[2n]$. Hence

$$\begin{aligned}
\sum_{n\geq 1} P_{2n}(x,0,q)\frac{t^{2n}}{(2n)!_q} &= \sum_{n\geq 1}\left(\sum_{S\subseteq \mathrm{O}[2n]} \alpha_{2n}(S,q)x^{|S|}\right)\frac{t^{2n}}{(2n)!_q}\\
&= \sum_{l\geq 1}\left(\sum_{\gamma}\frac{t^{\gamma_1}}{\gamma_1!_q}\cdots\frac{t^{\gamma_l}}{\gamma_l!_q}\right)x^{l-1}\\
&= \sum_{i\geq 1}\frac{t^{2i}}{2i!_q} + x\sum_{l\geq 2}\left(\sum_{i\geq 1}\frac{t^{2i-1}}{(2i-1)!_q}\right)^2\left(x\sum_{i\geq 1}\frac{t^{2i}}{2i!_q}\right)^{l-2}\\
&= \cosh_q t - 1 + \frac{x\sinh_q^2 t}{1-x(\cosh_q t-1)},
\end{aligned}$$

which gives (3.4).

In the same vein, wa have

$$\begin{aligned}
\sum_{n\geq 1} P_{2n-1}(x,0,q)\frac{t^{2n-1}}{(2n-1)!_q} &= \sum_{n\geq 1}\sum_{S\subseteq \mathrm{O}[2n-1]} \alpha_{2n-1}(S,q)x^{|S|}\frac{t^{2n-1}}{(2n-1)!_q}\\
&= \sum_{l\geq 1}\left(\sum_{\gamma}\frac{t^{\gamma_1}}{\gamma_1!_q}\cdots\frac{t^{\gamma_l}}{\gamma_l!_q}x^{l-1}\right)\\
&= \sum_{l\geq 1}\left(\sum_{i\geq 1}\frac{t^{2i-1}}{(2i-1)!_q}\right)\left(x\sum_{i\geq 1}\frac{t^{2i}}{(2i)!_q}\right)^{l-1},
\end{aligned}$$

which is clearly equal to (3.5). $\square$

Next we generalize (3.4) and (3.5) to the general y.

Lemma 3.4. *We have*

$$\sum_{n\geq 1} P_{2n}(x,y,q)\frac{t^{2n}}{(2n)!_q} = \frac{B(t,x)}{1-yB(t,x)}, \tag{3.6}$$

$$\sum_{n\geq 1} P_{2n-1}(x,y,q)\frac{t^{2n-1}}{(2n-1)!_q} = \frac{C(t,x)}{1-yB(t,x)}. \tag{3.7}$$

Proof. Consider

$$P_n(x,y,q) = \sum_{(\sigma,S)} x^{|S_o|}y^{|S_e|}q^{\mathrm{inv}\,\sigma} \qquad (\sigma\in\mathfrak{S}_n \text{ and } D(\sigma)\subseteq S\subseteq[n-1]).$$

There is a bijection between the set of subsets S of $[n-1]$ with fixed even integers $S_{\mathrm{e}} = \{m_1 < \cdots < m_{l-1}\} \subset \mathrm{E}[n-1]$ and the set of sequences of compositions of $m_i - m_{i-1}$ with odd parts for $i \in [l]$ with $m_0 = 0$ and $m_l = n$. Let $\mathrm{co}(S_{\mathrm{e}}) = (n_1, \ldots, n_l)$ be the corresponding composition of n. Then

$$\sum_{n\geq 1} P_{2n}(x,y,q)\frac{t^{2n}}{(2n)!_q} = \sum_{l\geq 1}\prod_{i=1}^{l-1}\left[\sum_{S_i\subseteq \mathrm{O}[2n_i]} \alpha_{2n_i}(S_i,q)x^{|S_i|}\frac{t^{2n_i}}{(2n_i)!_q}y\right] \times \left[\sum_{S_l\subseteq \mathrm{O}[2n_l]} \alpha_{2n_l}(S_l,q)x^{|S_l|}\frac{t^{2n_l}}{(2n_l)!_q}\right],$$

which is equal to $\sum_{l\geq 1} y^{l-1}\cdot B(t,x)^l = \frac{B(t,x)}{1-yB(t,x)}$.

Similary, we have

$$\sum_{n\geq 1} P_{2n-1}(x,y,q)\frac{t^{2n-1}}{(2n-1)!_q} = \sum_{l\geq 1}\prod_{i=1}^{l-1}\left(\sum_{S_i\subseteq \mathrm{O}[2n_i]} \alpha_{2n_i}(S_i,q)x^{|S_i|}\frac{t^{2n_i}}{(2n_i)!}y\right) \times \left(\sum_{S_l\subseteq \mathrm{O}[2n_l-1]} \alpha_{2n_l-1}(S_l,q)x^{|S_l|}\frac{t^{2n_l-1}}{(2n_l-1)!_q}\right),$$

which can be written as $\sum_{l\geq 1} y^{l-1}\cdot B(t,x)^{l-1}\cdot C(t,x) = \frac{C(t,x)}{1-yB(t,x)}$. □

We obtain (1.6a) by combining Lemma 3.2, Lemma 3.3 and Lemma 3.4.

4. Counting permutations of type B by the parity of descent positions

Let $\mathcal{B}_n^+$ (resp. $\mathcal{B}_n^-$) be the subset of permutations in $\mathcal{B}_n$ whose first entry is positive (resp. negative). Clearly the doubleton $\{\mathcal{B}_n^-, \mathcal{B}_n^+\}$ is a partition of $\mathcal{B}_n$. Introduce the corresponding enumerative polynomials:

$$B_n^-(x,y) = \sum_{\sigma\in\mathcal{B}_n^-} x^{\mathrm{des}_1\sigma}y^{\mathrm{des}_0\sigma}, \quad B_n^+(x,y) = \sum_{\sigma\in\mathcal{B}_n^+} x^{\mathrm{des}_1\sigma}y^{\mathrm{des}_0\sigma}.$$

Then $B_n(x,y) = B_n^-(x,y) + B_n^+(x,y)$.

For $\tau \in \mathcal{B}_n$, let τ^- be the permutation in $\mathcal{B}_n$ such that $\tau^-(i) = -\tau(i)$ for $i \in [n]$. It is clear that the mapping $\rho : \tau \longmapsto \tau^-$ is an involution on $\mathcal{B}_n$ such that

$$\begin{aligned} \mathrm{des}_1\tau + \mathrm{des}_1\tau^- &= \lfloor n/2\rfloor, \\ \mathrm{des}_0\tau + \mathrm{des}_0\tau^- &= \lfloor (n+1)/2\rfloor. \end{aligned} \tag{4.1}$$

Besides, the restriction of ρ on $\mathcal{B}_n^+$ sets up a bijection $\rho : \mathcal{B}_n^+ \to \mathcal{B}_n^-$, therefore

$$B_n^-(x,y) = x^{\lfloor n/2\rfloor}y^{\lfloor (n+1)/2\rfloor}B_n^+(1/x,1/y). \tag{4.2}$$

So, we need only to compute the exponential generating functions of $B_n^+(x,y)$.

For $\sigma \in B_n$, we denote by $D(\sigma)$ the set of descents of σ. If S is a subset of $[n-1]$ let $\alpha_n^+(S)$ be the number of permutations $\sigma \in \mathcal{B}_n^+$ such that $D(\sigma) \subseteq S$. A set composition of set Ω is an ℓ-tuple $(\Omega_1,\dots,\Omega_\ell)$ of subsets of Ω such that $\{\Omega_1,\dots,\Omega_\ell\}$ is a set partition of Ω.

Lemma 4.1. *Let $S=\{s_1<\cdots<s_k\} \subseteq [n-1]$, and $s_0=0$ and $s_{k+1}=n$. Then*

$$\alpha_n^+(S) = \binom{n}{\mathrm{co}(S)} 2^{n-s_1}. \tag{4.3}$$

Proof. We can construct the permutations $\sigma \in \mathcal{B}_n^+$ with $D(\sigma) \subseteq S$ as in the following:

- partition $[n]$ to obtain a set-composition $(\Omega_1,\dots,\Omega_{k+1})$ of $[n]$ with $|\Omega_i| = s_i$ for $1 \le i \le k$ and $|\Omega_{k+1}| = n - s_k$,
- sign the elements in Ω_i by $\epsilon \in \{-1,1\}$ for $i = 2,\dots k+1$.
- arrange the elements in each block Ω_i increasingly.

It is clear that the number of such permutations is

$$\binom{n}{s_1-s_0, s_2-s_1, \dots, s_{k+1}-s_k} 2^{n-s_1}.$$

This is the desired formula. $\square$

Similar to permutations of type A (see (3.1)), consider the polynomial

$$Q_n^+(x,y) = \sum_{S\subseteq[n]} \alpha_n^+(S) x^{|S_o|} y^{|S_e|}. \tag{4.4}$$

Lemma 4.2. *We have*

$$B_{2n}^+(x,y) = (1-x)^n (1-y)^{n-1} Q_{2n}^+\left(\frac{x}{1-x}, \frac{y}{1-y}\right), \tag{4.5}$$

$$B_{2n-1}^+(x,y) = (1-x)^{n-1} (1-y)^{n-1} Q_{2n-1}^+\left(\frac{x}{1-x}, \frac{y}{1-y}\right). \tag{4.6}$$

Proof. For even index we have

$$Q_{2n}^+(x,y) = \sum_{\sigma\in\mathcal{B}_{2n}^+} \sum_{S\subseteq[2n]} \sum_{\substack{\mathrm{Des}_0(\sigma)\subseteq S_e\\ \mathrm{Des}_1(\sigma)\subseteq S_o}} x^{|S_o|} y^{|S_e|}. \tag{4.7}$$

Now, for any fixed $\sigma \in \mathcal{B}_{2n}^+$, writing $T_0 = S_e \setminus \mathrm{Des}_0(\sigma)$ and $T_1 = S_e \setminus \mathrm{Des}_1(\sigma)$, then $|S_e| = \mathrm{des}_0(\sigma) + |T_0|$ and $|S_o| = \mathrm{des}_1(\sigma) + |T_1|$; hence the inner double sum at the right-hand side of (4.7) is a sum over the pairs (T_0,T_1) such that $T_0 \subseteq \mathrm{E}[2n]$ and $T_1 \subseteq \mathrm{O}[2n]$, and thus equal to

$$y^{\mathrm{des}_0(\sigma)} x^{\mathrm{des}_1(\sigma)} (1+y)^{n-1-\mathrm{des}_0(\sigma)} (1+x)^{n-\mathrm{des}_1(\sigma)}. \tag{4.8}$$

Therefore

$$Q_{2n}^{+}(x,y)=(1+x)^{n}(1+y)^{n-1}B_{2n}^{+}\left(\frac{x}{1+x},\frac{y}{1+y}\right), \tag{4.9}$$

which is equivelent to (4.5).

For odd index, a similar reasoning can be applied with regard to the sum

$$Q_{2n-1}^{+}(x,y)=\sum_{\sigma\in\mathcal{B}_{2n-1}^{+}}\sum_{S\subseteq[2n-1]}\sum_{\substack{\mathrm{Des}_0(\sigma)\subseteq S_e\\ \mathrm{Des}_1(\sigma)\subseteq S_o}} x^{|S_o|}y^{|S_e|} \tag{4.10}$$

and leads to the formula

$$Q_{2n-1}^{+}(x,y)=(1+x)^{n-1}(1+y)^{n-1}B_{2n-1}^{+}\left(\frac{x}{1+x},\frac{y}{1+y}\right). \tag{4.11}$$

which is equivalent to (4.6). $\square$

Lemma 4.3. *We have*

$$G:=\sum_{n\geq 1}Q_{2n}^{+}(x,0)\frac{t^{2n}}{(2n)!}=\cosh(t)-1+\frac{x\sinh(t)\sinh(2t)}{1-x(\cosh(2t)-1)}, \tag{4.12}$$

and

$$H:=\sum_{n\geq 1}\sum_{S\subseteq \mathrm{O}[2n]}\binom{2n}{\mathrm{co}(S)}2^{2n}x^{|S|}\frac{t^{2n}}{(2n)!}=\cosh(2t)-1+\frac{x\sinh^2(2t)}{1-x(\cosh(2t)-1)}. \tag{4.13}$$

Proof. By definition, if $S=\{s_1,s_2,\ldots,s_{l-1}\}_<\subseteq \mathrm{O}[2n]$, let $\gamma_1=s_1$, $\gamma_i=s_i-s_{i-1}$ for $i=2,\ldots,l$ with $s_l=2n-1$, then γ_1 is odd and γ_i are even for $i=2,\ldots,l$. Therefore

$$\begin{aligned}
G&=\sum_{n\geq 1}\sum_{S\subseteq \mathrm{O}[2n]}\binom{2n}{\mathrm{co}(S)}_{B^+}x^{|S|}\frac{t^{2n}}{(2n)!}\\
&=\sum_{n\geq 1}\left(\sum_{\gamma}\frac{1}{\gamma_1!}\cdots\frac{1}{\gamma_l!}x^{l-1}\right)2^{2n-\gamma_1}t^{2n}\\
&=\sum_{i\geq 1}\frac{t^{2i}}{(2i)!}+\sum_{l\geq 2}\left(\sum_{i\geq 1}\frac{(t)^{2i-1}}{(2i-1)!}\right)\left(x\sum_{i\geq 1}\frac{(2t)^{2i-1}}{(2i-1)!}\right)\left(x\sum_{i\geq 1}\frac{(2t)^{2i}}{(2i)!}\right)^{l-2}\\
&=\sum_{i\geq 1}\frac{t^{2i}}{(2i)!}+x\left(\sum_{i\geq 1}\frac{t^{2i-1}}{(2i-1)!}\right)\left(\sum_{i\geq 1}\frac{(2t)^{2i-1}}{(2i-1)!}\right)\frac{1}{1-x\sum_{i\geq 1}\frac{(2t)^{2i}}{(2i)!}},
\end{aligned} \tag{4.14}$$

which is the right-hand side of (4.12). Next,

$$\begin{aligned}
H &= \sum_{n\geq 1}\left(\sum_{\gamma}\frac{1}{\gamma_1!}\cdots\frac{1}{\gamma_l!}x^{l-1}\right)(2t)^{2n} \qquad (4.15)\\
&= \sum_{i\geq 1}\frac{(2t)^{2i}}{(2i)!}+\frac{1}{x}\sum_{l\geq 2}\left(x\sum_{i\geq 1}\frac{(2t)^{2i-1}}{(2i-1)!}\right)^2\left(x\sum_{i\geq 1}\frac{(2t)^{2i}}{(2i)!}\right)^{l-2}\\
&= \sum_{i\geq 1}\frac{(2t)^{2i}}{(2i)!}+x\left(\sum_{i\geq 1}\frac{(2t)^{2i-1}}{(2i-1)!}\right)^2\frac{1}{1-x\sum_{i\geq 1}\frac{(2t)^{2i}}{(2i)!}},
\end{aligned}$$

which is the right-hand of (4.13). □

Lemma 4.4. *We have*

$$F := \sum_{n\geq 1}\left(\sum_{S\subseteq \mathrm{O}[2n-1]}\binom{2n-1}{\mathrm{co}(S)}x^{|S|}\right)2^{2n-1}\frac{t^{2n-1}}{(2n-1)!}=\frac{\sinh(2t)}{1-x(\cosh(2t)-1)}, \qquad (4.16)$$

$$L := \sum_{n\geq 1}\left(\sum_{S\subseteq \mathrm{O}[2n-1]}\binom{2n-1}{\mathrm{co}(S)}_B x^{|S|}\right)\frac{t^{2n-1}}{(2n-1)!}=\frac{\sinh(t)}{1-x(\cosh(2t)-1)}. \qquad (4.17)$$

Proof. By definition, if $S=\{s_1,s_2,\ldots,s_{l-1}\}_<\subseteq \mathrm{O}[2n-1]$, let $\gamma_1=s_1$, $\gamma_i=s_i-s_{i-1}$ with $s_l=2n-1$, then γ_1 is odd and γ_i are even for $i=2,\ldots,l$. Therefore

$$\begin{aligned}
F &= \sum_{n\geq 1}\left(\sum_{\gamma}\frac{1}{\gamma_1!}\cdots\frac{1}{\gamma_l!}x^{l-1}\right)(2t)^{2n-1}\\
&= \sum_{l\geq 1}\left(\sum_{i\geq 1}\frac{(2t)^{2i-1}}{(2i-1)!}\right)\left(x\sum_{i\geq 1}\frac{(2t)^{2i}}{(2i)!}\right)^{l-1}\\
&= \left(\sum_{i\geq 1}\frac{(2t)^{2i-1}}{(2i-1)!}\right)\frac{1}{1-x\sum_{i\geq 1}\frac{(2t)^{(2i)}}{(2i)!}},
\end{aligned}$$

which equals the right-hand side of (4.16), besides

$$\begin{aligned}
L &= \sum_{n\geq 1}\left(\sum_{\gamma}\frac{1}{\gamma_1!}\cdots\frac{1}{\gamma_l!}x^{l-1}\right)2^{2n-1-\gamma_1}t^{2n-1}\\
&= \sum_{l\geq 1}\left(\sum_{i\geq 1}\frac{t^{2i-1}}{(2i-1)!}\right)\left(x\sum_{i\geq 1}\frac{(2t)^{2i}}{(2i)!}\right)^{l-1}\\
&= \left(\sum_{i\geq 1}\frac{t^{2i-1}}{(2i-1)!}\right)\frac{1}{1-x\sum_{i\geq 1}\frac{(2t)^{2i}}{(2i)!}},
\end{aligned}$$

which is equal to the right-hand side of (4.17). □

Lemma 4.5. *We have*

$$\sum_{n\geq 1}Q^{+}_{2n}(x,y)\frac{t^{2n}}{(2n)!}=\frac{G}{1-yH}, \tag{4.18}$$

$$\sum_{n\geq 1}Q^{+}_{2n-1}(x,y)\frac{t^{2n-1}}{(2n-1)!}=L+\frac{yF\,G}{1-yH}. \tag{4.19}$$

Proof. The left hand side of (4.18) is

$$\begin{aligned}
&\sum_{n\geq 1}\left(\sum_{S\subseteq[2n]}\alpha^{+}_{2n}(S)y^{|S_e|}x^{|S_o|}\right)\frac{t^{2n}}{2n!}\\
&=\sum_{n\geq 1}\left[\sum_{S_1\subseteq \mathrm{O}[2m_1]}\binom{2m_1}{\mathrm{co}(S_1)}_B x^{|S_1|}\frac{t^{2m_1}}{2m_1!}y\right]\cdot\prod_{i=2}^{l}\left[\sum_{S_i\subseteq \mathrm{O}[2m_i]}\binom{2m_i}{\mathrm{co}(S_i)}2^{2m_i}x^{|S_i|}\frac{t^{2m_i}}{2m_i!}y\right]\\
&=\sum_{l\geq 1}y^{l-1}\cdot G\cdot H^{l-1},
\end{aligned}$$

which equals $\frac{G}{1-yH}$. The left-hand side of (4.19) is

$$\begin{aligned}
&\sum_{n\geq 1}\left[\sum_{S_1\subseteq \mathrm{O}[2m_1]}\binom{2m_1}{\mathrm{co}(S_1)}_B x^{|S_1|}\frac{t^{2m_1}}{2m_1!}y\right]\cdot\prod_{i=2}^{l}\left[\sum_{S_i\subseteq \mathrm{O}[2m_i]}\binom{2m_i}{\mathrm{co}(S_2)}2^{2m_i}x^{|S_i|}\frac{t^{2m_i}}{2m_i!}y\right]\\
&=L+yF\cdot G\sum_{l\geq 0}(yH)^{l},
\end{aligned}$$

which equals $L+\frac{yF\cdot G}{1-yH}$. □

Now, combining Lemma 4.2 and Lemma 4.5 we have

$$\sum_{n\geq 1} B^{+}_{2n}(x,y)\frac{t^{2n}}{(2n)!} = \frac{(\cosh(at)-1)(2x\cosh(at)+x+1)}{1+xy-(x+y)\cosh(2at)}, \tag{4.20}$$

$$\sum_{n\geq 1} B^{+}_{2n-1}(x,y)\frac{t^{2n-1}}{(2n-1)!} = \frac{\sinh(at)(x-1)(2\cosh(at)y-y-1)}{a(xy+1-(x+y)\cosh(2at))} \tag{4.21}$$

with $a^2=(1-x)(1-y)$. It follows from (4.2) that

$$\sum_{n\geq 1} B^{-}_{2n}(x,y)\frac{t^{2n}}{(2n)!} = \frac{y(\cosh(at)-1)(2\cosh(at)+x+1)}{1+xy-(x+y)\cosh(2at)}, \tag{4.22}$$

$$\sum_{n\geq 1} B^{-}_{2n-1}(x,y)\frac{t^{2n-1}}{(2n-1)!} = \frac{y\sinh(at)(x-1)(-2\cosh(at)+y+1)}{a(xy+1-(x+y)\cosh(2at))}. \tag{4.23}$$

Combining (4.20) with (4.22) and (4.21) with (4.23), we complete the proof of Theorem 1.2.

5. Concluding remarks

In [4] Carlitz and Scoville also considered the more general modulus $m>2$ for descents rather than parity, i.e., $m=2$. They obtained a general generating function. However, apart from $m=2$ the generating function is quite explicit only for certain special cases when $m=4$. For the q-analogue, there are some nice generating functions given by Kurşungöz and Yee [19]. It would be very interesting to have results in this direction.

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