# Pairing strategies for the Maker-Breaker game on the hypercube with subcubes as winning sets 

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#### Abstract

We consider the Maker-Breaker positional game on the vertices of the $n$-dimensional hypercube $\{0,1\}^{n}$ with $k$-dimensional subcubes as winning sets. We describe a pairing strategy which allows Breaker to win if $n$ is a power of 4 and $k \geq n / 4+1$. Our results also imply that for all $n \geq 3$ there is a Breaker's win pairing strategy if $k \geq\left\lfloor\frac{3}{7} n\right\rfloor+1$.


## 1 Introduction

A positional game can be thought of as a generalization of Tic-Tac-Toe played on a hypergraph $(V, \mathcal{H})$ where the vertices can be considered the "board" on which the game is played, and the hyperedges can be considered the "winning sets." More formally, a positional game on $(V, \mathcal{H})$ is a two-player game where at every turn each player alternately occupies a previously unoccupied vertex from $V$. In a strong positional game, the first player to occupy all vertices of some hyperedge $A \in \mathcal{H}$ wins. If at the end of play no hyperedge is completely occupied by either player, that play is declared a draw. Normal $3 \times 3$ Tic-Tac-Toe is a strong positional game where the vertices of the hypergraph are the nine positions and the hyperedges are the eight winning lines. In a Maker-Breaker positional game, the first player, Maker, wins if Maker occupies all vertices of some hyperedge $A \in \mathcal{H}$, otherwise the second player, Breaker, wins. Therefore, by definition there are no draw plays in Maker-Breaker games. We say that a player $P$ has a winning strategy if no matter how the other player plays, player $P$ wins by following that strategy. It is well-known that in a finite Maker-Breaker game, exactly one player has a winning strategy. (For a nice introduction to positional games, please see [5], [6], and [22].)

Recall that the $n$-dimensional (Boolean) hypercube $Q_{n}$ is a bipartite graph whose vertex set is $\{0,1\}^{n}$ and whose edge set is the set of all pairs of vertices that differ in exactly one coordinate. A $k$-dimensional subcube of $Q_{n}$ is a subgraph of $Q_{n}$ that is isomorphic to $Q_{k}$. Let $\mathcal{Q}(n, k)$ denote the hypergraph whose vertex set is $\{0,1\}^{n}$ and whose hyperedge set is
the set of all $k$-dimensional subcubes of $Q_{n}$ (technically, each hyperedge is the set of vertices of some $k$-dimensional subcube). For example, $\mathcal{Q}(n, 1)=Q_{n}$.

In [23], Kruczek and Sundberg initiated the study of the Maker-Breaker game played on $\mathcal{Q}(n, k)$. Using a general Breaker's win criterion (Erdős-Selfridge theorem [16]) and Maker's win criterion (Theorem 1.2 [5]), they showed that Breaker has a winning strategy for the $\mathcal{Q}(n, k)$ game when $k \geq \log _{2}(n)+1$ and Maker has a winning strategy when $k \leq$ $\log _{2} \log _{2}(n)-1$. They also studied the specific question, "when can Breaker win by using a pairing strategy?" A pairing strategy for Breaker is a set $P$ of pairwise disjoint pairs of vertices in $Q_{n}$. Breaker uses $P$ as a strategy by playing as follows: each time Maker occupies a vertex $x$, if there is an unoccupied vertex $y$ such that $\{x, y\} \in P$, then Breaker immediately responds by occupying $y$; otherwise, Breaker occupies an arbitrary unoccupied vertex. This guarantees that Breaker occupies at least one vertex from each pair in $P$. If every $k$-dimensional subcube of $Q_{n}$ contains at least one pair from $P$, then Breaker wins, and we say $P$ is a Breaker's win pairing strategy. (We note that substantial work has been done on Maker-Breaker games on graphs where players occupy edges. However, in the $\mathcal{Q}(n, k)$ game, players occupy vertices. We are unaware of any papers, other than [23], which study the $\mathcal{Q}(n, k)$ game.) The goal of this paper is to improve upon the results in [23] pertaining to Breaker's win pairing strategies.

Let $p(n)$ be the smallest value of $k$ such that Breaker can win the positional game on $\mathcal{Q}(n, k)$ by using a pairing strategy. Proposition 9 in [14] implies that $p(n)>\ln (n)$. (Indeed, each pair can block at most $\binom{n-1}{k-1}$ subcubes and there are at most $2^{n-1}$ pairs. When $k=\lfloor\ln (n)\rfloor,\binom{n-1}{k-1} 2^{n-1}<\binom{n}{k} 2^{n-k}$, which is the total number of $k$-dimensional subcubes.) Kruczek and Sundberg [23] showed that $p(n) \leq n-3$. We improve on their result by proving the following:

Theorem 4 For each $n \geq 3$, there is a Breaker's win pairing strategy for $\mathcal{Q}\left(n,\left\lfloor\frac{3}{7} n\right\rfloor+1\right)$.
All pairs of vertices in the Breaker's win pairing strategies that we construct are edges; thus, our pairing strategies are matchings in $Q_{n}$.

The remainder of the paper is organized as follows. In Section 2, we give some basic definitions and explain the main techniques behind constructing our Breaker's win pairing strategies through an illustrative example. In Section 3, we state and prove a theorem which uses those techniques and can be used to show $p(n) \leq n / 3+1$ if $n=6 \cdot 4^{d}$ or $n=9 \cdot 4^{d}$ for some $d \geq 1$. In Section 4, we enhance the techniques from Section 3 to prove that $p(n) \leq n / 4+1$ when $n$ is a power of 4 . In Section 5 , we briefly discuss Breaker's win pairing strategies for specific values of $n$ and $k$, including the result that $p(n) \leq\left\lfloor\frac{3}{7} n\right\rfloor+1$ for all $n \geq 3$. In Section 6 , we prove $p(n) \leq n / 3+1$ when $n$ is a power of 3 . In Section 7 , we briefly mention how some of our results can be viewed as a variation of $d$-polychromatic edge colorings of $Q_{n}$.

## 2 The Basic Strategy

The basic idea of our technique is to "combine" a Breaker's win pairing strategy for $\mathcal{Q}(4,2)$ with one for $\mathcal{Q}(n, k)$ to create a Breaker's win pairing strategy for $\mathcal{Q}(4 n, b)$, where $b=$ $\max \{4 k-3, n+k\}$.

We represent each $k$-dimensional subcube of $Q_{n}$ by a list of $n$ symbols such that $k$ of the symbols are stars $(*)$ and each of the remaining $n-k$ symbols is 0 or 1 . For example, $(*, 0,1, *)$ represents the set

$$
\{0,1\} \times\{0\} \times\{1\} \times\{0,1\}=\{(0,0,1,0),(0,0,1,1),(1,0,1,0),(1,0,1,1)\}
$$

which is the vertex set of a 2-dimensional subcube of $Q_{4}$. We call 0-dimensional and 1dimensional subcubes vertices and edges, respectively. We abuse terminology and refer to $(*, 0,1, *)$ as a "vector" with four "coordinates."

The following set of edges (vectors) is a Breaker's win pairing strategy for $\mathcal{Q}(4,2)$ (verifying this is straightforward and left to the reader):

$$
\begin{aligned}
\operatorname{PS}(4,2)=\{ & (*, 0,0,0),(0, *, 1,0),(0,0, *, 1),(0,1,0, *) \\
& (*, 1,1,1),(1, *, 0,1),(1,1, *, 0),(1,0,1, *)\}
\end{aligned}
$$

We make use of the following properties in our proofs.

## Properties of $P S(4,2)$ :

1. For each pair of indices $1 \leq i<j \leq 4$ and each ordered pair $\left(b_{i}, b_{j}\right) \in\{0,1\}^{2}$, there is a vector in $P S(4,2)$ with $b_{i}$ (not a star) in coordinate $i$ and $b_{j}$ (not a star) in coordinate $j$, because every 2-dimensional subcube in $\mathcal{Q}(4,2)$ contains an edge from $\operatorname{PS}(4,2)$.
2. For each $j \in[4], P S(4,2)$ has exactly two vectors with a star in coordinate $j$ and those two vectors are complements of each other (where we consider 0 and 1 to be complements of each other and a star to be its own complement).

Given a Breaker's win pairing strategy $\mathbf{P}_{n}$ for $\mathcal{Q}(n, k)$, we combine $\mathbf{P}_{n}$, as explained below, with each edge in $P S(4,2)$ to obtain a set of edges $\operatorname{Bin} P S(4 \times n)$, which we later prove is a Breaker's win pairing strategy for $\mathcal{Q}(4 n, b)$ with $b=\max \{4 k-3, n+k\}$. Let $\mathbf{0}_{n}$ be the set of even parity vectors from $\{0,1\}^{n}$, i.e., the sum of the coordinates of each vector is even. Let $\mathbf{1}_{n}$ be the set of odd parity vectors from $\{0,1\}^{n}$. (When it is clear from the context, we drop the subscript $n$, and just write $\mathbf{0}$ or 1.) We take each edge in $P S(4,2)$ and replace each 0,1 , or $*$ with an element of $\mathbf{0}_{n}, \mathbf{1}_{n}$, or $\mathbf{P}_{n}$, respectively, to obtain an edge in $\operatorname{BinPS} S(4 \times n)$. For example, $(0,1,0, *) \in P S(4,2)$ yields the following set of edges in $\operatorname{BinPS}(4 \times n)$ :

$$
\begin{equation*}
\left(\mathbf{0}_{n} \times \mathbf{1}_{n} \times \mathbf{0}_{n} \times \mathbf{P}_{n}\right)=\left\{\left(\vec{x}_{1}, \vec{x}_{2}, \vec{x}_{3}, \vec{x}_{4}\right): \vec{x}_{1} \in \mathbf{0}_{n}, \vec{x}_{2} \in \mathbf{1}_{n}, \vec{x}_{3} \in \mathbf{0}_{n}, \vec{x}_{4} \in \mathbf{P}_{n}\right\} \tag{1}
\end{equation*}
$$

where $\left(\vec{x}_{1}, \vec{x}_{2}, \vec{x}_{3}, \vec{x}_{4}\right)$ represents an edge in $Q_{4 n}$, because $\vec{x}_{4}$ is an edge in $Q_{n}$ and $\vec{x}_{i}$ is a vertex in $Q_{n}$ for $i \in[3]$. We call the set in equation (1) a product-set. Doing this for each element of $P S(4,2)$, and taking the union of the resulting product-sets, gives the following set of edges:

$$
\begin{align*}
\operatorname{BinPS}(4 \times n)= & \left(\mathbf{P}_{n} \times \mathbf{0}_{n} \times \mathbf{0}_{n} \times \mathbf{0}_{n}\right) \cup\left(\mathbf{0}_{n} \times \mathbf{P}_{n} \times \mathbf{1}_{n} \times \mathbf{0}_{n}\right) \cup \\
& \left(\mathbf{P}_{n} \times \mathbf{1}_{n} \times \mathbf{1}_{n} \times \mathbf{1}_{n}\right) \cup\left(\mathbf{1}_{n} \times \mathbf{P}_{n} \times \mathbf{0}_{n} \times \mathbf{1}_{n}\right) \cup \\
& \left(\mathbf{0}_{n} \times \mathbf{0}_{n} \times \mathbf{P}_{n} \times \mathbf{1}_{n}\right) \cup\left(\mathbf{0}_{n} \times \mathbf{1}_{n} \times \mathbf{0}_{n} \times \mathbf{P}_{n}\right) \cup \\
& \left(\mathbf{1}_{n} \times \mathbf{1}_{n} \times \mathbf{P}_{n} \times \mathbf{0}_{n}\right) \cup\left(\mathbf{1}_{n} \times \mathbf{0}_{n} \times \mathbf{1}_{n} \times \mathbf{P}_{n}\right), \tag{2}
\end{align*}
$$

which we later prove is a Breaker's win pairing strategy for $\mathcal{Q}(4 n, b)$ with $b=\max \{4 k-$ $3, n+k\}$.

Each subcube $S$ of $Q_{4 n}$ can be represented as a length $4 n$ vector with entries from $\{0,1, *\}$. We partition the coordinates of that vector into four bins, where bin $j$ contains coordinates $(j-1) n+1$ through $j n$. For each subcube $S$ of $Q_{4 n}$ and each $j \in[4]$, let $\left.S\right|_{\text {bin } j}$ denote $S$ restricted to bin $j$, i.e., if $S=\left(x_{1}, \ldots, x_{4 n}\right)$, then $\left.S\right|_{\operatorname{bin} j}=\left(x_{(j-1) n+1}, \ldots, x_{j n}\right)$. For an edge $\vec{e} \in \operatorname{BinPS}(4 \times n)$, the sets $\mathbf{0}_{n}, \mathbf{1}_{n}, \mathbf{P}_{n}$ determine the possible values for $\left.\vec{e}\right|_{\text {bin } j}$; thus, we call $\mathbf{0}_{n}, \mathbf{1}_{n}, \mathbf{P}_{n}$ bin-sets.

We say that a subcube $S_{1}$ blocks a subcube $S_{2}$, if $S_{1} \subseteq S_{2}$. For example, the edge $\vec{e}=(1,0,1, *)$ blocks the subcube $(*, 0,1, *)$ since $(1,0,1, *) \subset(*, 0,1, *)$. Moreover, $\vec{e}$ blocks exactly three 2-dimensional subcubes of $Q_{4}$, namely $(*, 0,1, *),(1, *, 1, *)$, and $(1,0, *, *)$. We say that a set of subcubes $T$ blocks a subcube $S$, if $T$ contains a subcube that blocks $S$. For example, since $Q_{n}$ is a bipartite graph with partite sets $\mathbf{0}_{n}$ and $\mathbf{1}_{n}$, for each edge $\vec{e}$, there exist vertices $\vec{v}_{0} \in \mathbf{0}_{n}$ and $\vec{v}_{1} \in \mathbf{1}_{n}$ such that $\vec{v}_{0}$ blocks $\vec{e}$ and $\vec{v}_{1}$ blocks $\vec{e}$. Thus, $\mathbf{0}_{n}$ blocks every 1-dimensional subcube of $Q_{n}$, as does $\mathbf{1}_{n}$.

Example 1: We can form $\operatorname{BinP} S(4 \times 1)$ by using $\mathbf{0}_{1}=\{0\}, \mathbf{1}_{1}=\{1\}, \mathbf{P}_{1}=\{*\}$, where * is the edge $\{0,1\}$, thus, $\mathbf{P}_{1}$ is a Breaker's win pairing strategy for $\mathcal{Q}(1,1)$. In this case, $b=\max \{4(1)-3,1+1\}=2$, and $\operatorname{BinPS}(4 \times 1)$ is precisely $P S(4,2)$.
Example 2: We can form $\operatorname{BinPS}(4 \times 2)$ by using $\mathbf{0}_{2}=\{(0,0),(1,1)\}, \mathbf{1}_{2}=\{(0,1),(1,0)\}$, $\mathbf{P}_{2}=\{(*, 0)\}$, where $\mathbf{P}_{2}$ is a Breaker's win pairing strategy for $\mathcal{Q}(2,2)$. In this case, $b=$ $\max \{4(2)-3,2+2\}=5$. Theorem 1 implies that $\operatorname{BinPS}(4 \times 2)$ is a Breaker's win pairing strategy for $\mathcal{Q}(8,5)$. (We directly justify this fact below.)

Each product-set has cardinality 8, for example,

$$
\begin{aligned}
\left(\mathbf{0}_{2} \times \mathbf{1}_{2} \times \mathbf{0}_{2} \times \mathbf{P}_{2}\right)=\{ & (0,0,0,1,0,0, *, 0),(1,1,0,1,0,0, *, 0) \\
& (0,0,0,1,1,1, *, 0),(1,1,0,1,1,1, *, 0) \\
& (0,0,1,0,0,0, *, 0),(1,1,1,0,0,0, *, 0) \\
& (0,0,1,0,1,1, *, 0),(1,1,1,0,1,1, *, 0)\}
\end{aligned}
$$

(we include extra spaces in the vectors to highlight the four bins) and $|\operatorname{Bin} P S(4 \times 2)|=64$.
Let us show that $\operatorname{Bin} P S(4 \times 2)$ is a Breaker's win pairing strategy for $\mathcal{Q}(8,5)$. Let $S$ be a 5 -dimensional subcube of $Q_{8}$. Recall that $\mathbf{0}_{2}$ and $\mathbf{1}_{2}$ each block any subcube of $Q_{2}$ with positive dimension, and $\mathbf{P}_{2}$ blocks the 2-dimensional subcube of $Q_{2}$.

Case 1: For each $j \in[4]$, the dimension of $\left.S\right|_{\text {bin } j}$ is positive. Then, for some $j_{1}, j_{2}, j_{3}, j_{4}$ with $\left\{j_{1}, j_{2}, j_{3}, j_{4}\right\}=[4],\left.S\right|_{\text {bin } j_{i}}$ has dimension 1 for $i \in[3]$ and $\left.S\right|_{\text {bin } j_{4}}$ has dimension 2 . We select a product-set that has $\mathbf{P}_{2}$ in bin $j_{4}$ because $\mathbf{P}_{2}$ is only guaranteed to block the 2-dimensional subcube of $Q_{2}$. In each of the other bins, the product-set has $\mathbf{0}_{2}$ or $\mathbf{1}_{2}$. This product-set blocks $S$. For example, if $S=(*, 1,0, *, *, *, 0, *)$, then $j_{4}=3$, and we can use $\left(\mathbf{0}_{2} \times \mathbf{0}_{2} \times \mathbf{P}_{2} \times \mathbf{1}_{2}\right)$ to block $S$. Indeed, $(1,1,0,0, *, 0,0,1) \in\left(\mathbf{0}_{2} \times \mathbf{0}_{2} \times \mathbf{P}_{2} \times \mathbf{1}_{2}\right)$ and $(1,1,0,0, *, 0,0,1) \subset S$. We could also use $\left(\mathbf{1}_{2} \times \mathbf{1}_{2} \times \mathbf{P}_{2} \times \mathbf{0}_{2}\right)$ to block $S$.

Case 2: Suppose $\left.S\right|_{\text {bin } j_{1}}$ has dimension 0 for some $j_{1} \in[4]$. Then for some $j_{2}, j_{3}, j_{4}$ such that $\left\{j_{1}, j_{2}, j_{3}, j_{4}\right\}=[4],\left.S\right|_{\text {bin } j_{2}}$ has dimension 1, while $\left.S\right|_{\text {bin } j_{3}}$ and $\left.S\right|_{\text {bin } j_{4}}$ both have dimension 2. Because $\mathbf{P}_{2}$ is only guaranteed to block the 2-dimensional subcube of $Q_{2}$, we must select a product-set that has $\mathbf{P}_{2}$ in bin $j_{3}$ or $j_{4}$. W.l.o.g., we will select a product-set
with $\mathbf{P}_{2}$ in bin $j_{4}$. By Property 2 of $P S(4,2)$, there is a product-set $A$ that has $\mathbf{P}_{2}$ in bin $j_{4}$ and has a bin-set $\left(\mathbf{0}_{2}\right.$ or $\left.\mathbf{1}_{2}\right)$ whose parity equals the parity of $\left.S\right|_{\text {bin } j_{1}}$ in bin $j_{1}$. We claim that $A$ blocks $S$. Bins $j_{2}$ and $j_{3}$ of $A$ each contain either $\mathbf{0}_{2}$ or $\mathbf{1}_{2}$, thus, $A$ blocks $S$. For example, if $S=(*, *, *, *, 1,0, *, 1)$, then $j_{1}=3$ and $j_{4} \in\{1,2\}$. Thus, we use the product-set $\left(\mathbf{P}_{2} \times \mathbf{1}_{2} \times \mathbf{1}_{2} \times \mathbf{1}_{2}\right)$ to block $S$ because it has $\mathbf{1}_{2}$ in bin 3 and $\mathbf{P}_{2}$ in bin 1 , or we could use $\left(\mathbf{0}_{2} \times \mathbf{P}_{2} \times \mathbf{1}_{2} \times \mathbf{0}_{2}\right)$ since it has $\mathbf{1}_{2}$ in bin 3 and $\mathbf{P}_{2}$ in bin 2.
Example 3: We can form $\operatorname{BinP} S(4 \times 3)$ by using $\mathbf{0}_{3}=\{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\}$, $\mathbf{1}_{3}=\{(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}, \mathbf{P}_{3}=\{(*, 0,0),(1, *, 1),(0,1, *)\}$, where $\mathbf{P}_{3}$ is a Breaker's win pairing strategy for $\mathcal{Q}(3,2)$. In this case, $b=\max \{4(2)-3,3+2\}=5$. Theorem 1 implies that $\operatorname{BinP} S(4 \times 3)$ is a Breaker's win pairing strategy for $\mathcal{Q}(12,5)$.

## 3 Main Theorem

In this section, we prove that $\operatorname{BinPS}(4 \times n)$ is a matching and that it blocks all subcubes of dimension $b=\max \{4 k-3, n+k\}$.

Observe that two subcubes of $Q_{n}$ are disjoint if and only if for some $j \in[n]$, one of the subcubes has 0 and the other has 1 in coordinate $j$ of their vector representations.

Lemma 1 For each $n \geq 1$, the set of edges $\operatorname{Bin} P S(4 \times n)$, as defined in equation (2), is a matching.

Proof of Lemma 1: Let $\vec{e}_{1}, \vec{e}_{2} \in \operatorname{BinPS}(4 \times n)$ satisfy $\vec{e}_{1} \neq \vec{e}_{2}$. Then $\left.\vec{e}_{1}\right|_{\operatorname{bin} j} \neq\left.\vec{e}_{2}\right|_{\text {bin } j}$ for some $j \in[4]$. Let $\left(x_{1}^{(i)}, \ldots, x_{4 n}^{(i)}\right)$ be the vector representation of $\vec{e}_{i}$ for $i \in\{1,2\}$.

Case 1: Suppose $\left.\vec{e}_{1}\right|_{\text {bin } j}$ and $\left.\vec{e}_{2}\right|_{\text {bin } j}$ are 0 -dimensional. Since $\left.\vec{e}_{1}\right|_{\text {bin } j} \neq\left.\vec{e}_{2}\right|_{\text {bin } j}$, then clearly there is some $k \in[n]$ such that $\left\{x_{(j-1) n+k}^{(1)}, x_{(j-1) n+k}^{(2)}\right\}=\{0,1\}$, thus, $\vec{e}_{1}$ and $\vec{e}_{2}$ are disjoint.

Case 2: Suppose $\left.\vec{e}_{1}\right|_{\text {bin } j}$ and $\left.\vec{e}_{2}\right|_{\text {bin } j}$ are 1-dimensional. Then $\left.\vec{e}_{1}\right|_{\text {bin } j},\left.\vec{e}_{2}\right|_{\text {bin } j} \in \mathbf{P}_{n}$. Since $\mathbf{P}_{n}$ is a matching and $\left.\vec{e}_{1}\right|_{\text {bin } j} \neq\left.\vec{e}_{2}\right|_{\text {bin } j}$, there is some $k \in[n]$ such that $\left\{x_{(j-1) n+k}^{(1)}, x_{(j-1) n+k}^{(2)}\right\}=$ $\{0,1\}$, thus, $\vec{e}_{1}$ and $\vec{e}_{2}$ are disjoint.

Recall that $\operatorname{BinPS}(4 \times n)$ is the union of eight pairwise disjoint product sets, as defined in equation (2). If $\vec{e}_{1}$ and $\vec{e}_{2}$ are in the same product-set, then either Case 1 or Case 2 occurs. If they are in distinct product-sets (which correspond to different edges in $P S(4,2)$ ), then because $P S(4,2)$ is a matching in $Q_{4}$, we are guaranteed that Case 1 occurs for some (possibly other) $j \in[4]$.

Theorem 1 If there exists a Breaker's win pairing strategy for $\mathcal{Q}(n, k)$, then there exists a Breaker's win pairing strategy for $\mathcal{Q}(4 n, b)$, where $b=\max \{4 k-3, n+k\}$.

Proof of Theorem 1: Let $S$ be a subcube of $Q_{4 n}$. Let $i \in[4]$. Suppose that $\left.S\right|_{\text {bin } i}$ has dimension $c$. If $c \geq k$, then we say that $\left.S\right|_{\text {bin } i}$ is higher dimensional (HD).

We use $\operatorname{Bin} P S(4 \times n)$ as our pairing strategy, where $\mathbf{0}=\mathbf{0}_{n}, \mathbf{1}=\mathbf{1}_{n}$, and $\mathbf{P}$ is a Breaker's win pairing strategy for $\mathcal{Q}(n, k)$. Recall that $\mathbf{0}$ and $\mathbf{1}$ can each block any subcube of $Q_{n}$ with positive dimension, and $\mathbf{P}$ can block any HD subcube of $Q_{n}$.

Let $S$ be a $b$-dimensional subcube of $Q_{4 n}$. Since $b \geq 4 k-3>4(k-1),\left.S\right|_{\text {bin } j}$ is HD for some $j \in[4]$. Since $b \geq n+k$ and $k \geq 1,\left.S\right|_{\text {bin } j}$ has dimension 0 for at most two values of $j \in[4]$.

Case 1: Suppose that $\left.S\right|_{\text {bin } j}$ has positive dimension for each $j \in[4]$, and that $\left.S\right|_{\text {bin } i}$ is HD for some $i \in[4]$. Let $A$ be a product-set with $\mathbf{P}$ in bin $i$. Since $\left.S\right|_{\text {bin } i}$ is HD, $\mathbf{P}$ blocks $\left.S\right|_{\text {bin } i}$. In each of the other bins, $A$ has either $\mathbf{0}$ or $\mathbf{1}$. Since $\left.S\right|_{\text {bin } j}$ has positive dimension for all $j \neq i, \mathbf{0}$ and $\mathbf{1}$ can each block $\left.S\right|_{\text {bin } j}$ for all $j \neq i$. Thus, $A$ blocks $S$.

Case 2: Suppose instead that $\left.S\right|_{\text {bin } i}$ and $\left.S\right|_{\text {bin } j}$ both have dimension 0 for some pair $\{i, j\} \subset[4]$. Because of Property 1 of $P S(4,2)$, there is a product-set $A$ whose bin-sets in bins $i$ and $j$ match the parities of $\left.S\right|_{\text {bin } i}$ and $\left.S\right|_{\text {bin } j}$, respectively. There are $b$ coordinates in the vector representation of $S$ that are stars. Since $b \geq n+k,\left.S\right|_{\text {bin } \ell}$ is HD for each $\ell \in[4]-\{i, j\}$, and can be blocked by any bin-set $\mathbf{0}, \mathbf{1}$, or $\mathbf{P}$. Therefore, $A$ blocks $S$.

Case 3: Suppose that $j$ is the unique value in $\{1,2,3,4\}$ such that $\left.S\right|_{\text {bin } j}$ has dimension 0. Also suppose that $\left.S\right|_{\text {bin } i}$ is HD for some $i \in[4]$. By Property 2 of $P S(4,2)$, we can deduce that there is a product-set $A$ that has $\mathbf{P}$ in bin $i$ and a bin-set in bin $j$ whose parity matches that of $\left.S\right|_{\text {bin } j}$. In each of the other two bins, $A$ has either $\mathbf{0}$ or 1 . Since $\left.S\right|_{\text {bin } \ell}$ has positive dimension for each $\ell \in[4]-\{i, j\}, A$ blocks $S$.

In light of Lemma 1, we conclude that $\operatorname{BinPS}(4 \times n)$ is a Breaker's win pairing strategy for $\mathcal{Q}(4 n, b)$.

Corollary 1 Suppose there exists a Breaker's win pairing strategy for $\mathcal{Q}(n, k)$.
(a) If $k \geq n / 3+1$, then there exists a Breaker's win pairing strategy for $\mathcal{Q}(4 n, 4 k-3)$.
(b) If $k=\lfloor n / 3\rfloor+1$, then there exists a Breaker's win pairing strategy for $\mathcal{Q}(4 n,\lfloor 4 n / 3\rfloor+1)$.

## 4 Rotating the Pairing Strategies

Under certain conditions, we are able to improve upon Theorem 1 using a technique we call, "rotating pairing strategies."

To illustrate the idea, we construct a Breaker's win pairing strategy for $\mathcal{Q}(9,4)$ using the rotating pairing strategy idea. (The reader may choose to skip to Theorem 2.) For simplicity, we have chosen an example based on $\mathbf{P}_{3}=\{(*, 0,0),(1, *, 1),(0,1, *)\}$ (instead of $P S(4,2)$ ) which has only three bins, each of size three. We obtain four different Breaker's win pairing strategies for $\mathcal{Q}(3,2)$ which together partition the set of edges of $Q_{3}$, by using translations of $\mathbf{P}_{3}$, namely,

$$
\begin{aligned}
& \mathbf{P}^{(0)}=\{(*, 0,0),(1, *, 1),(0,1, *)\}=\mathbf{P}_{3}, \\
& \mathbf{P}^{(1)}=\{(*, 0,1),(0, *, 0),(1,1, *)\}=\mathbf{P}_{3}+(1,0,1), \\
& \mathbf{P}^{(2)}=\{(*, 1,0),(0, *, 1),(1,0, *)\}=\mathbf{P}_{3}+(1,1,0), \\
& \mathbf{P}^{(3)}=\{(*, 1,1),(1, *, 0),(0,0, *)\}=\mathbf{P}_{3}+(0,1,1) .
\end{aligned}
$$

(We will use the fact that $\mathbf{P}^{(0)} \cup \mathbf{P}^{(1)} \cup \mathbf{P}^{(2)} \cup \mathbf{P}^{(3)}$ contains every edge in $Q_{3}$.) We partition $\mathbf{0}_{3}$ and $\mathbf{1}_{3}$ into four sets each as follows:

$$
\mathbf{0}^{(0)}=\{(0,0,0)\}, \quad \mathbf{1}^{(0)}=\{(0,0,1)\}
$$

$$
\begin{array}{ll}
\mathbf{0}^{(1)}=\{(0,1,1)\}, & \mathbf{1}^{(1)}=\{(0,1,0)\}, \\
\mathbf{0}^{(2)}=\{(1,0,1)\}, & \mathbf{1}^{(2)}=\{(1,0,0)\}, \\
\mathbf{0}^{(3)}=\{(1,1,0)\}, & \mathbf{1}^{(3)}=\{(1,1,1)\} .
\end{array}
$$

(In higher dimensions, $\mathbf{0}^{(j)}$ and $\mathbf{1}^{(j)}$ will not be singletons.) Our pairing strategy is

$$
\operatorname{Bin} P S^{R}(3 \times 3)=(\mathbf{P} \times \mathbf{0} \times \mathbf{0})^{R} \cup(\mathbf{1} \times \mathbf{P} \times \mathbf{1})^{R} \cup(\mathbf{0} \times \mathbf{1} \times \mathbf{P})^{R}
$$

where we define the "rotating" product-sets as follows:

$$
\begin{aligned}
& (\mathbf{P} \times \mathbf{0} \times \mathbf{0})^{R}=\bigcup_{i, j}\left(\mathbf{P}^{(i+j)} \times \mathbf{0}^{(i)} \times \mathbf{0}^{(j)}\right), \\
& (\mathbf{1} \times \mathbf{P} \times \mathbf{1})^{R}=\bigcup_{i, j}\left(\mathbf{1}^{(j)} \times \mathbf{P}^{(i+j)} \times \mathbf{1}^{(i)}\right) \\
& (\mathbf{0} \times \mathbf{1} \times \mathbf{P})^{R}=\bigcup_{i, j}\left(\mathbf{0}^{(i)} \times \mathbf{1}^{(j)} \times \mathbf{P}^{(i+j)}\right)
\end{aligned}
$$

where $i, j$, and $i+j$ are all evaluated modulo 4 . Notice, for example, that $(\mathbf{0} \times \mathbf{1} \times \mathbf{P})^{R}$ is the union of $4^{2}$ sets, such as,

$$
\left(\mathbf{0}^{(2)} \times \mathbf{1}^{(3)} \times \mathbf{P}^{(1)}\right)=\{(1,0,1)\} \times\{(1,1,1)\} \times\{(*, 0,1),(0, *, 0),(1,1, *)\}
$$

and

$$
\left(\mathbf{0}^{(3)} \times \mathbf{1}^{(1)} \times \mathbf{P}^{(0)}\right)=\{(1,1,0)\} \times\{(0,1,0)\} \times\{(*, 0,0),(1, *, 1),(0,1, *)\} .
$$

We present an equivalent description of the "rotating" product-sets. For $\vec{x} \in \mathbf{0}^{(j)}$ or $\vec{x} \in \mathbf{1}^{(j)}$, let $\operatorname{Index}(\vec{x})=j$. Then we can write the "rotating" product-sets as
$(\mathbf{P} \times \mathbf{0} \times \mathbf{0})^{R}=\left\{(\vec{x}, \vec{y}, \vec{z}): \vec{x} \in \mathbf{P}^{(j)}, \vec{y} \in \mathbf{0}, \vec{z} \in \mathbf{0}\right.$, where $\left.j=\operatorname{Index}(\vec{y})+\operatorname{Index}(\vec{z}) \quad(\bmod 4)\right\}$,
$(1 \times \mathbf{P} \times 1)^{R}=\left\{(\vec{x}, \vec{y}, \vec{z}): \vec{x} \in \mathbf{1}, \vec{y} \in \mathbf{P}^{(j)}, \vec{z} \in \mathbf{1}\right.$, where $\left.j=\operatorname{Index}(\vec{x})+\operatorname{Index}(\vec{z})(\bmod 4)\right\}$, $(\mathbf{0} \times \mathbf{1} \times \mathbf{P})^{R}=\left\{(\vec{x}, \vec{y}, \vec{z}): \vec{x} \in \mathbf{0}, \vec{y} \in \mathbf{1}, \vec{z} \in \mathbf{P}^{(j)}\right.$, where $\left.j=\operatorname{Index}(\vec{x})+\operatorname{Index}(\vec{y})(\bmod 4)\right\}$.

Theorem 5 implies that $\operatorname{BinP} S^{R}(3 \times 3)$ is a Breaker's win pairing strategy for $\mathcal{Q}(9,4)$. In contrast,

$$
\operatorname{BinPS}(3 \times 3)=\left(\mathbf{P}_{3} \times \mathbf{0}_{3} \times \mathbf{0}_{3}\right) \cup\left(\mathbf{1}_{3} \times \mathbf{P}_{3} \times \mathbf{1}_{3}\right) \cup\left(\mathbf{0}_{3} \times \mathbf{1}_{3} \times \mathbf{P}_{3}\right),
$$

is a Breaker's win pairing strategy for $\mathcal{Q}(9,5)$ (proof omitted), but not for $\mathcal{Q}(9,4)$. For example, let $S$ be a 4 -dimensional subcube of $Q_{9}$ so that $\left.S\right|_{\text {bin } 1}$ is 0 -dimensional, $\left.S\right|_{\text {bin } 2}$ is 1dimensional, and $\left.S\right|_{\text {bin } 3}$ is 3 -dimensional. If $\left.S\right|_{\text {bin } 1}$ has even parity, then both $\operatorname{BinPS}(3 \times 3)$ and $\operatorname{BinP} S^{R}(3 \times 3)$ block $S$ using the product-sets $\left(\mathbf{0}_{3} \times \mathbf{1}_{3} \times \mathbf{P}_{3}\right)$ and $(\mathbf{0} \times \mathbf{1} \times \mathbf{P})^{R}$, respectively. (This is because $\mathbf{1}_{3}=\mathbf{1}$ blocks subcubes with positive dimension, and any subcube of $Q_{3}$ blocks $\left.S\right|_{\text {bin } 3}$ because it is 3 -dimensional.) However, if $\left.S\right|_{\text {bin } 1}$ has odd parity, then $(\mathbf{1} \times \mathbf{P} \times \mathbf{1})^{R}$ blocks $S$, but $\left(\mathbf{1}_{3} \times \mathbf{P}_{3} \times \mathbf{1}_{3}\right)$ might not (because $\left.S\right|_{\text {bin } 2}$ is 1-dimensional, but bin 2 contains $\left.\mathbf{P}_{3}\right)$. For example, suppose $S=(0,1,0, *, 1,0, *, *, *)$. Observe that $\left(\mathbf{1}^{(1)} \times \mathbf{P}^{(2)} \times \mathbf{1}^{(1)}\right) \subset(\mathbf{1} \times \mathbf{P} \times \mathbf{1})^{R}$, and $(0,1,0, *, 1,0,0,1,0) \in\left(\mathbf{1}^{(1)} \times \mathbf{P}^{(2)} \times \mathbf{1}^{(1)}\right)$, which blocks $S$. However, since $\left.S\right|_{\text {bin } 2}=(*, 1,0)$, which is not blocked by $\mathbf{P}_{3}=\{(*, 0,0),(1, *, 1),(0,1, *)\}$, $\left(\mathbf{1}_{3} \times \mathbf{P}_{3} \times \mathbf{1}_{3}\right)$ does not block $S$.

Theorem 2 Suppose there exists a set of matchings $\left\{\mathbf{P}^{(0)}, \ldots, \mathbf{P}^{(m-1)}\right\}$ such that each $\mathbf{P}^{(j)}$ is a Breaker's win pairing strategy for $\mathcal{Q}(n, k)$ and $\bigcup_{j} \mathbf{P}^{(j)}$ equals the set of edges of $Q_{n}$. Moreover, suppose that there is a partition of $\mathbf{0}_{n}\left(\right.$ of $\left.\mathbf{1}_{n}\right)$ of size $m$ such that every subcube of $Q_{n}$ of dimension $n-k+2$ contains at least one vertex from each of the sets in the partition of $\mathbf{0}_{n}$ (of $\mathbf{1}_{n}$ ). Then there exists a Breaker's win pairing strategy for $\mathcal{Q}(4 n, b)$, where $b=\max \{4 k-3, n+1\}$.

## Proof of Theorem 2:

Let $\left\{\mathbf{0}^{(0)}, \ldots \mathbf{0}^{(m-1)}\right\}$ and $\left\{\mathbf{1}^{(0)}, \ldots, \mathbf{1}^{(m-1)}\right\}$ be the given partitions of $\mathbf{0}_{n}$ and $\mathbf{1}_{n}$, respectively. For every $\vec{x} \in \mathbf{0}^{(j)} \cup \mathbf{1}^{(j)}$, we define $\operatorname{Index}(\vec{x})$ to be $j$. Let

$$
\begin{aligned}
\operatorname{BinP} P S^{R}(4 \times n)= & \\
& (\mathbf{P} \times \mathbf{0} \times \mathbf{0} \times \mathbf{0})^{R} \cup(\mathbf{0} \times \mathbf{P} \times \mathbf{1} \times \mathbf{0})^{R} \cup \\
& (\mathbf{P} \times \mathbf{1} \times \mathbf{1} \times \mathbf{1})^{R} \cup(\mathbf{1} \times \mathbf{P} \times \mathbf{0} \times \mathbf{1})^{R} \cup \\
& (\mathbf{0} \times \mathbf{0} \times \mathbf{P} \times \mathbf{1})^{R} \cup(\mathbf{0} \times \mathbf{1} \times \mathbf{0} \times \mathbf{P})^{R} \cup \\
& (\mathbf{1} \times \mathbf{1} \times \mathbf{P} \times \mathbf{0})^{R} \cup(\mathbf{1} \times \mathbf{0} \times \mathbf{1} \times \mathbf{P})^{R}
\end{aligned}
$$

where, for example,

$$
\begin{aligned}
(\mathbf{0} \times \mathbf{0} \times \mathbf{P} \times \mathbf{1})^{R}=\{(\vec{x}, \vec{y}, \vec{z}, \vec{w}): & \vec{x} \in \mathbf{0}, \vec{y} \in \mathbf{0}, \vec{z} \in \mathbf{P}^{(j)}, \vec{w} \in \mathbf{1} \\
& \text { where } j=\operatorname{Index}(\vec{x})+\operatorname{Index}(\vec{y})+\operatorname{Index}(\vec{w}) \quad(\bmod m)\} .
\end{aligned}
$$

Let $b=\max \{4 k-3, n+1\}$, and let $S$ be a $b$-dimensional subcube of $Q_{4 n}$. Suppose that $\left.S\right|_{\operatorname{bin} i}$ has dimension $c$ for some $i \in[4]$. If $1 \leq c \leq k-1$, then we say that $\left.S\right|_{\operatorname{bin} i}$ is lower dimensional ( $L D$ ). (Note: if $\left.S\right|_{\text {bin } i}$ has dimension 0, it is not LD.) Recall, if $c \geq k$, then $\left.S\right|_{\text {bin } i}$ is HD.

Since $b \geq 4 k-3,\left.S\right|_{\text {bin } i}$ is HD for at least one $i \in[4]$. Since $b \geq n+1,\left.S\right|_{\text {bin } i}$ has dimension 0 for at most two values of $i \in[4]$.

We can follow the proof of Theorem 1 for Case 1 and Case 3 in that proof. For Case 2 in that proof, if $\left.S\right|_{\text {bin } i_{1}}$ and $\left.S\right|_{\text {bin } i_{2}}$ both have dimension 0 , and $\left.S\right|_{\text {bin } i_{3}}$ and $\left.S\right|_{\text {bin } i_{4}}$ are both HD, where $\left\{i_{1}, i_{2}, i_{3}, i_{4}\right\}=[4]$, then we can follow the proof of Theorem 1. Therefore, we may assume that $\left.S\right|_{\text {bin } i_{1}}$ and $\left.S\right|_{\text {bin } i_{2}}$ both have dimension $0,\left.S\right|_{\text {bin } i_{3}}$ is LD, and $\left.S\right|_{\text {bin } i_{4}}$ is HD. Property 1 of $P S(4,2)$ implies that there is a product-set $A$ whose bin-sets in bins $i_{1}$ and $i_{2}$ match the parities of $\left.S\right|_{\text {bin } i_{1}}$ and $\left.S\right|_{\text {bin } i_{2}}$, respectively. If $A$ has $\mathbf{0}$ or $\mathbf{1}$ in bin $i_{3}$, then $A$ blocks $S$ because $\mathbf{0}$ and $\mathbf{1}$ each block subcubes with any positive dimension, and $\mathbf{P}$ blocks HD subcubes. Suppose instead that $A$ has $\mathbf{P}$ in bin $i_{3}$, and w.l.o.g., $A$ has $\mathbf{0}$ in bin $i_{4}$. Since $\bigcup_{j} \mathbf{P}^{(j)}$ equals the set of edges of $Q_{n}$ and $\left.S\right|_{\text {bin } i_{3}}$ has positive dimension, there is a bin-set $\mathbf{P}^{(j)}$ which contains an edge that blocks $\left.S\right|_{\text {bin } i_{3}}$. Let $c \in\{0, \ldots, m-1\}$ satisfy $\operatorname{Index}\left(\left.S\right|_{\text {bin } i_{1}}\right)+\operatorname{Index}\left(\left.S\right|_{\text {bin } i_{2}}\right)+c=j(\bmod m)$. Since, by assumption, every subcube of $Q_{n}$ of dimension $n-k+2$ contains at least one vertex from $\mathbf{0}^{(c)}$, there is a vertex $\vec{x} \in \mathbf{0}^{(c)}$ that blocks $\left.S\right|_{\text {bin } i_{4}}$ as long as $\left.S\right|_{\text {bin } i_{4}}$ has dimension at least $n-k+2$. Since $\left.S\right|_{\text {bin } i_{3}}$ is LD and the sum of the dimensions of $\left.S\right|_{\text {bin } i_{3}}$ and $\left.S\right|_{\text {bin } i_{4}}$ is $b \geq n+1,\left.S\right|_{\text {bin } i_{4}}$ has dimension at least $n+1-(k-1)=n-k+2$. Therefore, we can find an edge in $A$ that blocks $S$.

For Theorem 2 to be useful, we need there to exist, for some $\mathcal{Q}(n, k)$ with $k<n / 3+1$, a set of Breaker's win pairing strategies whose union is the set of edges of $Q_{n}$; for if $k \geq n / 3+1$,
then $4 k-3 \geq n+k$, and hence $b=4 k-3$ in both Theorems 1 and 2 . The existence of such sets of pairing strategies is proved in Theorem 3, which states that, for $d \geq 0$, the edges of $Q_{4^{d+1}}$ can be partitioned into $4^{d+1}$ Breaker's win pairing strategies for $\mathcal{Q}\left(4^{d+1}, 4^{d}+1\right)$. We prove Theorem 3 by induction on $d$. To introduce definitions and techniques we use in that proof, let us consider here the cases $d=0$ and $d=1$.

We have seen that $P S(4,2)$ is a Breaker's win pairing strategy for the case $d=0$. To obtain four different Breaker's win pairing strategies for $\mathcal{Q}(4,2)$ which together partition the set of edges of $Q_{4}$, we use translations of $P S(4,2)$, namely,

$$
\begin{aligned}
& P S_{0}(4,2)=P S(4,2) \quad=\quad\{(*, 0,0,0),(0, *, 1,0),(0,0, *, 1),(0,1,0, *), \\
& (*, 1,1,1),(1, *, 0,1),(1,1, *, 0),(1,0,1, *)\}, \\
& P S_{1}(4,2)=P S(4,2)+(0,0,1,1)=\{(*, 0,1,1),(0, *, 0,1),(0,0, *, 0),(0,1,1, *), \\
& (*, 1,0,0),(1, *, 1,0),(1,1, *, 1),(1,0,0, *)\}, \\
& P S_{2}(4,2)=P S(4,2)+(0,1,0,1)=\{(*, 1,0,1),(0, *, 1,1),(0,1, *, 0),(0,0,0, *), \\
& (*, 0,1,0),(1, *, 0,0),(1,0, *, 1),(1,1,1, *)\}, \\
& P S_{3}(4,2)=P S(4,2)+(0,1,1,0)=\{(*, 1,1,0),(0, *, 0,0),(0,1, *, 1),(0,0,1, *), \\
& (*, 0,0,1),(1, *, 1,1),(1,0, *, 0),(1,1,0, *)\} \text {. }
\end{aligned}
$$

Since every translation is an automorphism, $P S_{j}(4,2)$, for $0 \leq j \leq 3$, is a Breaker's win pairing strategy for $\mathcal{Q}(4,2)$ that satisfies Properties 1 and 2 of $P S(4,2)$. Thus, we now have four Breaker's win pairing strategies from which to construct our rotating product-sets for $d>0$.

For the case $d=1$, we can obtain one Breaker's win pairing strategy for $\mathcal{Q}(16,5)$ by applying Theorem 2, with $\mathbf{P}^{(j)}=P S_{j}(4,2)$ for $0 \leq j \leq 3$ and using any partitions of $\mathbf{0}_{4}$ and $1_{4}$ with four nonempty parts each (since $n-k+2=4$ ). However, to obtain sixteen Breaker's win pairing strategies for $\mathcal{Q}(16,5)$ which together partition the set of edges of $Q_{16}$, Theorem 3 generalizes the idea of Theorem 2 by using "shifted" rotating product-sets, which we define below.

Suppose $\bigcup_{0}^{m-1} \mathbf{P}^{(j)}$ equals the set of edges in $Q_{n}$, with each $\mathbf{P}^{(j)}$ a Breaker's win pairing strategy for $\mathcal{Q}(n, k)$, and $\left\{\mathbf{0}^{(0)}, \ldots \mathbf{0}^{(m-1)}\right\}$ and $\left\{\mathbf{1}^{(0)}, \ldots, \mathbf{1}^{(m-1)}\right\}$ are partitions of $\mathbf{0}_{n}$ and $\mathbf{1}_{n}$, respectively. An example of a rotating product-set shifted by $s$, where $0 \leq s \leq m-1$, is

$$
\begin{aligned}
& (\mathbf{0} \times \mathbf{0} \times \mathbf{P} \times \mathbf{1})^{R(s)}=\left\{(\vec{x}, \vec{y}, \vec{z}, \vec{w}): \vec{x} \in \mathbf{0}, \vec{y} \in \mathbf{0}, \vec{z} \in \mathbf{P}^{(i)}, \vec{w} \in \mathbf{1}\right. \\
& \quad \text { where } i=s+\operatorname{Index}(\vec{x})+\operatorname{Index}(\vec{y})+\operatorname{Index}(\vec{w}) \quad(\bmod m)\} .
\end{aligned}
$$

Note that when $s=0$, we obtain $(\mathbf{0} \times \mathbf{0} \times \mathbf{P} \times \mathbf{1})^{R(0)}=(\mathbf{0} \times \mathbf{0} \times \mathbf{P} \times \mathbf{1})^{R}$.
For each $s \in\{0, \ldots, m-1\}$, we define four pairing strategies $\operatorname{Bin} P S_{0}^{R(s)}(4 \times n), \operatorname{Bin} P S_{1}^{R(s)}(4 \times$ $n)$, $\operatorname{Bin} P S_{2}^{R(s)}(4 \times n), \operatorname{Bin} P S_{3}^{R(s)}(4 \times n)$ based on $P S_{0}(4,2), P S_{1}(4,2), P S_{2}(4,2), P S_{3}(4,2)$, respectively. For example,

$$
\operatorname{BinP} S_{3}^{R(s)}(4 \times n)=
$$

$$
\begin{aligned}
& (\mathbf{P} \times \mathbf{1} \times \mathbf{1} \times \mathbf{0})^{R(s)} \cup(\mathbf{0} \times \mathbf{P} \times \mathbf{0} \times \mathbf{0})^{R(s)} \cup(\mathbf{0} \times \mathbf{1} \times \mathbf{P} \times \mathbf{1})^{R(s)} \cup(\mathbf{0} \times \mathbf{0} \times \mathbf{1} \times \mathbf{P})^{R(s)} \\
& \cup(\mathbf{P} \times \mathbf{0} \times \mathbf{0} \times \mathbf{1})^{R(s)} \cup(\mathbf{1} \times \mathbf{P} \times \mathbf{1} \times \mathbf{1})^{R(s)} \cup(\mathbf{1} \times \mathbf{0} \times \mathbf{P} \times \mathbf{0})^{R(s)} \cup(\mathbf{1} \times \mathbf{1} \times \mathbf{0} \times \mathbf{P})^{R(s)}
\end{aligned}
$$

Thus, the sixteen Breaker's win pairing strategies for $\mathcal{Q}(16,5)$ which result from Theorem 3 are $\operatorname{BinPS} S_{j}^{R(s)}(4 \times 4)$ for $0 \leq j \leq 3$ and $0 \leq s \leq 3$, where we use the bin-sets $\mathbf{0}_{4}, \mathbf{1}_{4}$ and $\mathbf{P}^{(j)}=P S_{j}(4,2)$ in the definition of each $\operatorname{BinP} S_{j}^{R(s)}(4 \times 4)$.

Theorem 3 requires the following lemma.
Lemma 2 For all $k \geq 1$, the sets $\mathbf{0}_{2^{k}}$ and $\mathbf{1}_{2^{k}}$ can be partitioned into subsets $A_{1}, \ldots, A_{2^{k}}$ and $B_{1}, \ldots, B_{2^{k}}$, respectively, so that every subcube of $Q_{2^{k}}$ of dimension $2^{k-1}+1$ contains a vertex in every $A_{j}$ and $B_{j}$.

Proof of Lemma 2: We proceed by induction on $k$. The case $k=1$ is trivial. Suppose for some $k \geq 1$ we have $A_{1}, \ldots, A_{2^{k}}$ and $B_{1}, \ldots, B_{2^{k}}$ as in the statement of the lemma. For each $\ell \in\left[2^{k}\right]$, let

$$
\begin{aligned}
G_{\ell} & =\bigcup_{i+j \equiv \ell}\left(A_{i} \times A_{j}\right) \\
H_{\ell} & =\bigcup_{i+j \equiv \ell}\left(B_{i} \times B_{j}\right) \\
I_{\ell} & =\bigcup_{i+j \equiv \ell}\left(A_{i} \times B_{j}\right) \\
J_{\ell} & =\bigcup_{i+j \equiv \ell}\left(B_{i} \times A_{j}\right)
\end{aligned}
$$

where all the equivalencies are taken modulo $2^{k}$. We will show that $\left\{G_{1}, \ldots, G_{2^{k}}, H_{1}, \ldots, H_{2^{k}}\right\}$ is the desired partition of $\mathbf{0}_{2^{k+1}}$. A similar proof shows that $\left\{I_{1}, \ldots, I_{2^{k}}, J_{1}, \ldots, J_{2^{k}}\right\}$ is the desired partition of $\mathbf{1}_{2^{k+1}}$.

We can write any vertex $v \in Q_{2^{k+1}}$ as $v=\left(v_{1}, v_{2}\right)$ with $v_{i} \in Q_{2^{k}}$. Let $v \in \mathbf{0}_{2^{k+1}}$; then $v_{1}$ and $v_{2}$ must have the same parity, so $\left(v_{1}, v_{2}\right)$ is in either $\left(A_{i} \times A_{j}\right)$ or ( $B_{i} \times B_{j}$ ) for some $i, j \in\left[2^{k}\right]$, and hence $v$ is in either $G_{\ell}$ or $H_{\ell}$, where $i+j \equiv \ell\left(\bmod 2^{k}\right)$. As these sets are pairwise disjoint, $\left\{G_{1}, \ldots, G_{2^{k}}, H_{1}, \ldots, H_{2^{k}}\right\}$ partitions $\mathbf{0}_{2^{k+1}}$.

Given a subcube $S$ of $Q_{2^{k+1}}$, divide its coordinates into two bins, with coordinates $1, \ldots, 2^{k}$ in bin 1 , and $2^{k}+1, \ldots, 2^{k+1}$ in bin 2 . If $S$ has dimension $2^{k}+1$, then $\left.S\right|_{\text {bin } 1}$ or $\left.S\right|_{\text {bin } 2}$, say $\left.S\right|_{\text {bin } 1}$, has dimension at least $2^{k-1}+1$. So $\left.S\right|_{\text {bin } 1}$ contains a vertex in every $A_{i}$ and $B_{i}$. And $\left.S\right|_{\text {bin } 2}$ has dimension at least one, so it contains a vertex in each of $\mathbf{0}_{2^{k}}$ and $1_{2^{k}}$, i.e., in some $A_{j}$ and some $B_{j^{\prime}}$. Now, for every $\ell \in\left[2^{k}\right]$ there exist $i$ and $i^{\prime}$ such that $i+j \equiv i^{\prime}+j^{\prime} \equiv \ell\left(\bmod 2^{k}\right)$. So $S$ contains a vertex in each of $\left(A_{i} \times A_{j}\right),\left(B_{i} \times A_{j}\right)$, $\left(A_{i^{\prime}} \times B_{j^{\prime}}\right)$, and $\left(B_{i^{\prime}} \times B_{j^{\prime}}\right)$. So $S$ contains a vertex in every $G_{\ell}$ and $H_{\ell}$.

Theorem 3 For each $d \geq 0$, there exist $4^{d+1}$ pairwise disjoint Breaker's win pairing strategies for $\mathcal{Q}\left(4^{d+1}, 4^{d}+1\right)$ with equal cardinalities which partition the set of edges of $Q_{4^{d+1}}$.

Proof of Theorem 3: We proceed by induction on $d$. The Breaker's win pairing strategies $P S_{j}(4,2)$ for $0 \leq j \leq 3$ handle the case $d=0$. Let $d \geq 1$. By the inductive hypothesis, there exist $4^{d}$ disjoint Breaker's win pairing strategies $\mathbf{P}^{(0)}, \ldots, \mathbf{P}^{\left(4^{d}-1\right)}$ for $\mathcal{Q}(n, k)=\mathcal{Q}\left(4^{d}, 4^{d-1}+\right.$ 1) with equal cardinalities which partition the set of edges of $Q_{4^{d}}$. We will show that $\operatorname{BinP} S_{j}^{R(s)}(4 \times n)$ is a Breaker's win pairing strategy for $\mathcal{Q}\left(4^{d+1}, 4^{d}+1\right)$ for $0 \leq j \leq 3$ and $0 \leq s \leq 4^{d}-1$, where we use $\mathbf{0}_{4^{d}}, \mathbf{1}_{4^{d}}$, and the Breaker's win pairing strategies $\mathbf{P}^{(i)}$ from the inductive hypothesis in the definitions of the product-sets. Moreover, we will show that the Breaker's win pairing strategies $\operatorname{BinP} S_{j}^{R(s)}(4 \times n)$ form a partition of the edges of $Q_{4^{d+1}}$. To do so, we follow the proof of Theorem 2, with $m=n=4^{d}$, and with one minor change, as follows.

By Lemma 2, there is a partition of $\mathbf{0}_{4^{d}}$ (of $\mathbf{1}_{4^{d}}$ ) of size $m$ such that every subcube of $Q_{n}$ of dimension $\frac{1}{2} 4^{d}+1$ contains at least one vertex from each of the sets in the partition of $\mathbf{0}_{4^{d}}$ (of $\mathbf{1}_{4^{d}}$ ). Since $n-k+2=\frac{3}{4} 4^{d}+1>\frac{1}{2} 4^{d}+1$, the hypotheses for Theorem 2 are satisfied. We can substitute $\operatorname{BinP} S_{j}^{R(s)}(4 \times n)$ for $\operatorname{BinP} S^{R}(4 \times n)$ throughout the proof of Theorem 2 and reach the conclusion that $\operatorname{BinP} S_{j}^{R(s)}(4 \times n)$ is a Breaker's win pairing strategy for $\mathcal{Q}(4 n, n+1)=$ $\mathcal{Q}\left(4^{d+1}, 4^{d}+1\right)$ (since $\left.\max (4 k-3, n+1)=n+1=4 k-3\right)$. The only minor change we make is to write "let $c \in\{0, \ldots, m-1\}$ satisfy $\left(s+\operatorname{Index}\left(\left.S\right|_{\text {bin } i_{1}}\right)+\operatorname{Index}\left(\left.S\right|_{\text {bin } i_{2}}\right)+c\right)=j$ $(\bmod m) . "$

It remains to show that the sets $\operatorname{BinP} S_{j}^{R(s)}(4 \times n)$ form a partition of $E\left(Q_{4^{d+1}}\right)$. Let $E\left(Q_{4^{d+1}}\right)$ be the set of edges of $Q_{4^{d+1}}$. We will first show that

$$
E\left(Q_{4^{d+1}}\right) \subseteq \bigcup_{j, s} \operatorname{BinP} S_{j}^{R(s)}(4 \times n)
$$

which implies $\bigcup_{j, s} \operatorname{BinP} S_{j}^{R(s)}(4 \times n)=E\left(Q_{4^{d+1}}\right)$.
Let $S \in E\left(Q_{4^{d+1}}\right)$. Suppose that $\left.S\right|_{\text {bin } i_{1}},\left.S\right|_{\text {bin } i_{2}},\left.S\right|_{\text {bin } i_{3}}$ are all vertices in $Q_{4^{d}}$, and $\left.S\right|_{\text {bin } i_{4}}$ is an edge in $E\left(Q_{4^{d}}\right)$. Let $\vec{x}$ be the edge in $E\left(Q_{4}\right)$ such that coordinate $i_{\ell}$ of $\vec{x}$ matches the parity of $\left.S\right|_{\text {bin } i_{\ell}}$ for $1 \leq \ell \leq 3$, and coordinate $i_{4}$ of $\vec{x}$ is a star. Since $P S_{0}(4,2), P S_{1}(4,2)$, $P S_{2}(4,2), P S_{3}(4,2)$ partition $E\left(Q_{4}\right), \vec{x} \in P S_{j}(4,2)$ for some $0 \leq j \leq 3$.

Since $\mathbf{P}^{(0)}, \ldots, \mathbf{P}^{\left(4^{d}-1\right)}$ partition $E\left(Q_{4^{d}}\right),\left.S\right|_{\text {bin } i_{4}} \in \mathbf{P}^{(t)}$ for some $0 \leq t \leq 4^{d}-1$. Let $s \in\left\{0, \ldots, 4^{d}-1\right\}$ satisfy

$$
s+\operatorname{Index}\left(\left.S\right|_{\operatorname{bin} i_{1}}\right)+\operatorname{Index}\left(\left.S\right|_{\operatorname{bin} i_{2}}\right)+\operatorname{Index}\left(\left.S\right|_{\text {bin } i_{3}}\right)=t \quad\left(\bmod 4^{d}\right)
$$

Let $A_{s}$ be the product-set from $\operatorname{BinP} S_{j}^{R(s)}(4 \times n)$ which corresponds to $\vec{x}$. Then $S \in A_{s}$ because of how $\vec{x}$ was chosen. Therefore, $E\left(Q_{4^{d+1}}\right) \subseteq \bigcup_{j, s} \operatorname{BinP} S_{j}^{R(s)}(4 \times n)$, as desired.

It remains to show that the sets $\operatorname{BinP} S_{j}^{R(s)}(4 \times n)$ with $0 \leq j \leq 3$ and $0 \leq s \leq$ $4^{d}-1$ are pairwise disjoint. Since every $\mathbf{P}^{(i)}$ has the same cardinality and $\mathbf{P}^{(0)}, \ldots, \mathbf{P}^{\left(4^{d}-1\right)}$ partition $E\left(Q_{4^{d}}\right)$, which has cardinality $4^{d}\left(2^{4^{d}-1}\right),\left|\mathbf{P}^{(i)}\right|=2^{4^{d}-1}$ for $0 \leq i \leq 4^{d}-1$. Each $\operatorname{BinP} S_{j}^{R(s)}(4 \times n)$ is the union of eight product-sets. Since $|\mathbf{0}|=|\mathbf{1}|=2^{4^{d}-1}$, each product-set has cardinality $\left(2^{4^{d}-1}\right)^{3}\left(2^{4^{d}-1}\right)$. Thus, $\left|\operatorname{BinP} S_{j}^{R(s)}(4 \times n)\right| \leq 8\left(2^{4^{d}-1}\right)^{4}=2^{4^{d+1}-1}$, and

$$
\left|\bigcup_{j, s} \operatorname{BinP} S_{j}^{R(s)}(4 \times n)\right| \leq \sum_{j, s}\left|\operatorname{BinP} S_{j}^{R(s)}(4 \times n)\right| \leq 4^{d+1} 2^{4^{d+1}-1}=\left|E\left(Q_{4^{d+1}}\right)\right|
$$

Therefore the inequalities must in fact be equalities, and hence the Breaker's win pairing strategies $\operatorname{BinP} S_{j}^{R(s)}(4 \times n)$ form a partition of $E\left(Q_{4^{d+1}}\right)$.

## 5 Pairing Strategies for Specific Values of $n$ and $k$

Both Theorems 1 and 2 require the existence of a Breaker's win pairing strategy for a game played on the vertices of $Q_{n}$ to construct a Breaker's win pairing strategy for a game played on the vertices of $Q_{4 n}$. The following two lemmas allow us to construct Breaker's win pairing strategies for games played on the vertices of $Q_{d}$ where $d$ is not divisible by 4 .

Lemma 3 ([23]) If there is a Breaker's win pairing strategy for the Maker-Breaker game played on $\mathcal{Q}(n, k)$, then there is a Breaker's win pairing strategy for the Maker-Breaker game played on $\mathcal{Q}(n+1, k+1)$.

Lemma 4 If there exists a matching which is a Breaker's win pairing strategy for the MakerBreaker game played on $\mathcal{Q}(N, k)$, then there is a matching which is a Breaker's win pairing strategy for the Maker-Breaker game played on $\mathcal{Q}(n, k)$ for all $n \leq N$.

Both lemmas are fairly easy to justify. For a full proof of Lemma 3, see [23]. To understand the idea behind Lemma 4, for example, observe that there is a natural correspondence between the set of $k$-dimensional subcubes of $Q_{n}$ and the set of $k$-dimensional subcubes of $Q_{N}$ whose last $N-n$ coordinates are fixed at 0 . The set of edges from our Breaker's win pairing strategy which blocks those $k$-dimensional subcubes must also have their last $N-n$ coordinates fixed at 0 . If we truncate each of those edges after their $n^{\text {th }}$ coordinate, we will obtain a Breaker's win pairing strategy for the set of $k$-dimensional subcubes of $Q_{n}$.

So far we have exhibited Breaker's win pairing strategies for $\mathcal{Q}(3,2), \mathcal{Q}(4,2)$ and $\mathcal{Q}(9,4)$. Let us provide a Breaker's win pairing strategy for $\mathcal{Q}(6,3)$ in order to help us construct Breaker's win pairing strategies for other values of $n$ and $k$.

To construct a Breaker's win pairing strategy for $\mathcal{Q}(6,3)$, we will use sets of edges that resemble cyclic permutations. For example, let

$$
\begin{aligned}
\langle(*, 0,1,0,0,0)\rangle=\{ & (*, 0,1,0,0,0) \\
& (0, *, 0,1,0,0) \\
& (0,0, *, 0,1,0) \\
& (0,0,0, *, 0,1) \\
& (1,0,0,0, *, 0) \\
& (0,1,0,0,0, *)\} .
\end{aligned}
$$

Then,

$$
\begin{equation*}
\langle(*, 0,1,0,0,0)\rangle \cup\langle(*, 1,0,1,1,1)\rangle \cup\langle(*, 0,1,1,0,0)\rangle \cup\langle(*, 1,0,0,1,1)\rangle \tag{3}
\end{equation*}
$$

is a Breaker's win pairing strategy for $\mathcal{Q}(6,3)$ consisting of 24 edges (verified by computer).
If we start with our Breaker's win pairing strategy for $\mathcal{Q}(6,3)$ and repeated apply Corollary 1 (b), then we obtain a Breaker's win pairing strategy for $\mathcal{Q}\left(6 \cdot 4^{n}, 2 \cdot 4^{n}+1\right)$ for all $n \geq 0$. Likewise, if we start with our Breaker's win pairing strategy for $\mathcal{Q}(9,4)$ and repeated apply

Corollary 1 (b), then we obtain a Breaker's win pairing strategy for $\mathcal{Q}\left(9 \cdot 4^{n}, 3 \cdot 4^{n}+1\right)$ for all $n \geq 0$. Theorem 3 states that there is a Breaker's win pairing strategy for $\mathcal{Q}\left(4^{n+1}, 4^{n}+1\right)$ for all $n \geq 0$.

For each $n \geq 0$, there remain three intervals for which we have not yet described a Breaker's win pairing strategy, namely, for games played on the vertices of $Q_{d}$ where $4^{n+1}<$ $d<6 \cdot 4^{n}$, or $6 \cdot 4^{n}<d<9 \cdot 4^{n}$, or $9 \cdot 4^{n}<d<4^{n+2}$. To establish the existence of Breaker's win pairing strategies for these values of $d$, we can use Lemmas 3 and 4. We use the same approach for each interval. Specifically, for an interval of the form $N_{1}<d<N_{2}$, we have a Breaker's win pairing strategy for $\mathcal{Q}\left(N_{1}, k_{1}\right)$ and $\mathcal{Q}\left(N_{2}, k_{2}\right)$. When $d=N_{1}+j$ for $1 \leq j \leq 4^{n}$, we use Lemma 3 and our Breaker's win pairing strategy for $\mathcal{Q}\left(N_{1}, k_{1}\right)$ to obtain a Breaker's win pairing strategy for $\mathcal{Q}\left(N_{1}+j, k_{1}+j\right)$. When $d=N_{1}+j$ for $4^{n}+1 \leq j<N_{2}$, we use Lemma 4 and our Breaker's win pairing strategy for $\mathcal{Q}\left(N_{2}, k_{2}\right)$ to obtain a Breaker's win pairing strategy for $\mathcal{Q}\left(N_{1}+j, k_{2}\right)$.

After applying this technique to all three intervals, we obtain Breaker's win pairing strategies for

$$
\begin{array}{ll}
\mathcal{Q}\left(4^{n+1}+0 \cdot 4^{n}+j, 1 \cdot 4^{n}+1+j\right), & 1 \leq j \leq 4^{n}, \\
\mathcal{Q}\left(4^{n+1}+1 \cdot 4^{n}+j, 2 \cdot 4^{n}+1\right), & 1 \leq j \leq 4^{n}, \\
\mathcal{Q}\left(4^{n+1}+2 \cdot 4^{n}+j, 2 \cdot 4^{n}+1+j\right), & 1 \leq j \leq 4^{n}, \\
\mathcal{Q}\left(4^{n+1}+3 \cdot 4^{n}+j, 3 \cdot 4^{n}+1\right), & 1 \leq j \leq 2 \cdot 4^{n}, \\
\mathcal{Q}\left(4^{n+1}+5 \cdot 4^{n}+j, 3 \cdot 4^{n}+1+j\right), & 1 \leq j \leq 4^{n}, \\
\mathcal{Q}\left(4^{n+1}+6 \cdot 4^{n}+j, 4 \cdot 4^{n}+1\right) & 1 \leq j \leq 6 \cdot 4^{n} .
\end{array}
$$

Using the results stated in this section, we have established the existence of a non-trivial Breaker's win pairing strategy for $\mathcal{Q}(N, K)$ for each $N \geq 3$. When $N=4^{n+1}$, we have that $K$ is $N / 4+1$. When $N=6 \cdot 4^{n}$ or $N=9 \cdot 4^{n}$, we have that $K$ is $N / 3+1$. We can ask the following question. What is the largest value that the ratio $K / N$ attains using the results above? We observe that as $N$ increases from $4^{n+1}$ to $5 \cdot 4^{n}$, the ratio $K / N$ increases from $\frac{1}{4}+\frac{1}{N}$ to $\frac{2}{5}+\frac{1}{N}$. As $N$ increases from $5 \cdot 4^{n}$ to $6 \cdot 4^{n}, K / N$ decreases from $\frac{2}{5}+\frac{1}{N}$ to $\frac{1}{3}+\frac{1}{N}$. As $N$ increases from $6 \cdot 4^{n}$ to $7 \cdot 4^{n}, K / N$ increases from $\frac{1}{3}+\frac{1}{N}$ to $\frac{3}{7}+\frac{1}{N}$. As $N$ increases from $7 \cdot 4^{n}$ to $9 \cdot 4^{n}, K / N$ decreases from $\frac{3}{7}+\frac{1}{N}$ to $\frac{1}{3}+\frac{1}{N}$. As $N$ increases from $9 \cdot 4^{n}$ to $10 \cdot 4^{n}, K / N$ increases from $\frac{1}{3}+\frac{1}{N}$ to $\frac{2}{5}+\frac{1}{N}$. As $N$ increases from $10 \cdot 4^{n}$ to $4^{n+2}, K / N$ decreases from $\frac{2}{5}+\frac{1}{N}$ to $\frac{1}{4}+\frac{1}{N}$. The largest value $K / N$ achieves is $\frac{3}{7}+\frac{1}{N}$, when $N=7 \cdot 4^{n}$ and $K=3 \cdot 4^{n}+1$. One can check that for each $N \geq 3$, there is a Breaker's win pairing strategy for $K=\left\lfloor\frac{3}{7} N\right\rfloor+1$. Thus, we have the following theorem:

Theorem 4 For each $N \geq 3$, there is a Breaker's win pairing strategy for $\mathcal{Q}\left(N,\left\lfloor\frac{3}{7} N\right\rfloor+1\right)$.
We present the values of $N$ and $K$ corresponding to the (locally) minimum and (locally)
maximum values achieved by $K / N$ in the following table.

| $N$ | $K$ | (local) $\max / \mathrm{min}$ |
| :---: | :---: | :---: |
| $4^{n}$ | $N / 4+1$ | $\min$ |
| $5 \cdot 4^{n}$ | $(2 / 5) N+1$ | $\max$ |
| $6 \cdot 4^{n}$ | $N / 3+1$ | $\min$ |
| $7 \cdot 4^{n}$ | $(3 / 7) N+1$ | $\max$ |
| $9 \cdot 4^{n}$ | $N / 3+1$ | $\min$ |
| $10 \cdot 4^{n}$ | $(2 / 5) N+1$ | $\max$ |

## 6 Extra Results

In Lemma 5, we state a generalization of Lemma 2. Lemma 5 is used to prove Theorem 5, but it is also interesting in its own right. Theorem 5 provides additional pairing strategies that are not covered in Section 5.

Lemma 5 For all $n \geq 1$ and all $c \geq 2$, the sets $\mathbf{0}_{c^{n}}$ and $\mathbf{1}_{c^{n}}$ can each be partitioned into subsets $A_{1}, \ldots, A_{\left(2^{c-1}\right)^{n}}$ and $B_{1}, \ldots, B_{\left(2^{c-1}\right)^{n}}$, respectively, so that every subcube $S$ of $Q_{c^{n}}$ of dimension $c^{n}-c^{n-1}+1$ contains a vertex from each $A_{j}$ and each $B_{j}$.

Proof of Lemma 5: We proceed by induction on $n$. The case $n=1$ is trivial. Suppose for some $n \geq 1$ we have $A_{1}, \ldots, A_{\left(2^{c-1}\right)^{n}}$ and $B_{1}, \ldots, B_{\left(2^{c-1}\right)^{n}}$ as in the statement of the lemma.

For each vector $\left(b_{1}, \ldots, b_{c}\right) \in\{0,1\}^{c}$ and each vector of indices $\left(i_{1}, \ldots, i_{c}\right) \in\left[\left(2^{c-1}\right)^{n}\right]^{c}$, define the set $\left(D_{i_{1}} \times \cdots \times D_{i_{c}}\right)$ where $D_{i_{j}}=A_{i_{j}}$ if $b_{j}=0$ and $D_{i_{j}}=B_{i_{j}}$ if $b_{j}=1$. For example, if $c=3$ and $n=1$, we could have $(1,1,0) \in\{0,1\}^{3}$ and $(4,1,2) \in[4]^{3}$ which result in the set $\left(B_{4} \times B_{1} \times A_{2}\right)$. Then for each $\vec{b} \in \mathbf{0}_{c}$ and each $\ell \in\left[\left(2^{c-1}\right)^{n}\right]$, we define the set

$$
A_{(\vec{b}, \ell)}=\bigcup_{i_{1}+\cdots+i_{c} \equiv \ell}\left(D_{i_{1}} \times \cdots \times D_{i_{c}}\right),
$$

where the congruence is taken modulo $\left(2^{c-1}\right)^{n}$. For example, in the case $c=3$, we define $4^{n+1}$ sets

$$
\begin{array}{ll}
A_{((0,0,0), \ell)}=\bigcup_{i_{1}+i_{2}+i_{3} \equiv \ell}\left(A_{i_{1}} \times A_{i_{2}} \times A_{i_{3}}\right), & 1 \leq \ell \leq 4^{n}, \\
A_{((0,1,1), \ell)}=\bigcup_{i_{1}+i_{2}+i_{3} \equiv \ell}\left(A_{i_{1}} \times B_{i_{2}} \times B_{i_{3}}\right), & 1 \leq \ell \leq 4^{n}, \\
A_{((1,0,1), \ell)}=\bigcup_{i_{1}+i_{2}+i_{3} \equiv \ell}\left(B_{i_{1}} \times A_{i_{2}} \times B_{i_{3}}\right), & 1 \leq \ell \leq 4^{n}, \\
A_{((1,1,0), \ell)}=\bigcup_{i_{1}+i_{2}+i_{3} \equiv \ell}\left(B_{i_{1}} \times B_{i_{2}} \times A_{i_{3}}\right), & 1 \leq \ell \leq 4^{n},
\end{array}
$$

where the congruences are taken modulo $4^{n}$.

For each $\vec{b} \in \mathbf{1}_{c}$ and each $\ell \in\left[\left(2^{c-1}\right)^{n}\right]$, we define the sets $B_{(\vec{b}, \ell)}$ similarly. We show that the $\left(2^{c-1}\right)^{n+1}$ sets $A_{(\vec{b}, \ell)}$ form the desired partition of $\mathbf{0}_{c^{n+1}}$. A similar proof shows that the $\left(2^{c-1}\right)^{n+1}$ sets $B_{(\vec{b}, \ell)}$ form the desired partition of $\mathbf{1}_{c^{n+1}}$.

Let $\vec{x} \in \mathbf{0}_{c^{n+1}}$. We partition the coordinates of $\vec{x}$ into $c$ bins of size $c^{n}$ and write $\vec{x}=$ $\left(\vec{x}_{1}, \ldots, \vec{x}_{c}\right)$, where $\vec{x}_{j}$ is $\vec{x}$ restricted to bin $j$. Let $b_{j}$ be the parity of $\vec{x}_{j}$ for each $j \in[c]$. Since $\vec{x} \in \mathbf{0}_{c^{n+1}},\left(b_{1}, \ldots, b_{c}\right) \in \mathbf{0}_{c}$. Since $\bigcup A_{i}=\mathbf{0}_{c^{n}}$ and $\bigcup B_{i}=\mathbf{1}_{c^{n}}$, there exists an $\ell \in\left[\left(2^{c-1}\right)^{n}\right]$ such that $\vec{x} \in A_{\left(\left(b_{1}, \ldots, b_{c}\right), \ell\right)}$. As the sets $A_{(\vec{b}, \ell)}$ are pairwise disjoint, they partition $\mathbf{0}_{c^{n+1}}$.

Let $S$ be a subcube of $Q_{c^{n+1}}$ of dimension $c^{n+1}-c^{n}+1$. Partition the coordinates of $S$ into $c$ bins each of size $c^{n}$. Since $c^{n+1}-c^{n}+1=(c-1) c^{n}+1,\left.S\right|_{\text {bin } j}$ has dimension at least 1 for every $j \in[c]$. Since $c^{n+1}-c^{n}+1=c\left(c^{n}-c^{n-1}\right)+1,\left.S\right|_{\text {bin } j}$ has dimension at least $c^{n}-c^{n-1}+1$ for some $j \in[c]$. W.l.o.g., $\left.S\right|_{\text {bin } c}$ has dimension at least $c^{n}-c^{n-1}+1$. Thus, $\left.S\right|_{\text {bin } c}$ contains a vertex in every $A_{i}$ and $B_{i}$. For each $j \in[c-1]$, since $\left.S\right|_{\text {bin } j}$ has dimension at least $1,\left.S\right|_{\text {bin } j}$ contains a vertex in each of $\mathbf{0}_{c^{n}}$ and $\mathbf{1}_{c^{n}}$. Thus, there exist two sequences, $k_{1}, k_{2}, \ldots, k_{c-1}$ and $m_{1}, m_{2}, \ldots, m_{c-1}$ such that $k_{i}, m_{i} \in\left[\left(2^{c-1}\right)^{n}\right]$ for each $i \in[c-1]$ and $\left.S\right|_{\text {bin } j}$ contains a vertex from $A_{k_{j}}$ and $B_{m_{j}}$ for each $j \in[c-1]$.

Let $\left(b_{1}, \ldots, b_{c}\right) \in \mathbf{0}_{c}$ and let $\ell \in\left[\left(2^{c-1}\right)^{n}\right]$. Let $E=\left\{j \in[c-1]: b_{j}=0\right\}$ and let $F=\left\{j \in[c-1]: b_{j}=1\right\}$. For each $j \in[c-1]$, let $D_{j}=A_{k_{j}}$ if $j \in E$ and $D_{j}=B_{m_{j}}$ if $j \in F$, where $A_{k_{j}}$ and $B_{m_{j}}$ are as defined above. Let $i \in\left[\left(2^{c-1}\right)^{n}\right]$ satisfy $i+\sum_{j \in E} k_{j}+\sum_{j \in F} m_{j} \equiv \ell$ $\left(\bmod \left(2^{c-1}\right)^{n}\right)$. If $b_{c}=0$, let $D_{c}=A_{i}$, otherwise, let $D_{c}=B_{i}$. Since $\left.S\right|_{\text {bin } j}$ contains a vertex from $D_{j}$ for each $j \in[c], S$ contains a vertex from $\left(D_{1} \times \cdots \times D_{c}\right)$. Thus, $S$ contains a vertex from $A_{(\vec{b}, \ell)}$ for all $\vec{b} \in \mathbf{0}_{c}$ and all $\ell \in\left[\left(2^{c-1}\right)^{n}\right]$.

Theorem 5 For each $d \geq 0$, there exist $4^{d+1}$ disjoint Breaker's win pairing strategies for $\mathcal{Q}\left(3^{d+1}, 3^{d}+1\right)$ with equal cardinalities which partition the set of edges of $Q_{3^{d+1}}$.

The proof of Theorem 5 (which we omit) is very similar to the proof of Theorem 3, except we use the following pairing strategies, which use rotating product-sets shifted by $s$ :

$$
\begin{aligned}
& \operatorname{Bin} P S_{0}^{R(s)}(3 \times n)=(\mathbf{P} \times \mathbf{0} \times \mathbf{0})^{R(s)} \cup(\mathbf{1} \times \mathbf{P} \times \mathbf{1})^{R(s)} \cup(\mathbf{0} \times \mathbf{1} \times \mathbf{P})^{R(s)}, \\
& \operatorname{Bin} P S_{1}^{R(s)}(3 \times n)=(\mathbf{P} \times \mathbf{0} \times \mathbf{1})^{R(s)} \cup(\mathbf{0} \times \mathbf{P} \times \mathbf{0})^{R(s)} \cup(\mathbf{1} \times \mathbf{1} \times \mathbf{P})^{R(s)}, \\
& \operatorname{Bin} P S_{2}^{R(s)}(3 \times n)=(\mathbf{P} \times \mathbf{1} \times \mathbf{0})^{R(s)} \cup(\mathbf{0} \times \mathbf{P} \times \mathbf{1})^{R(s)} \cup(\mathbf{1} \times \mathbf{0} \times \mathbf{P})^{R(s)}, \\
& \operatorname{Bin} P S_{3}^{R(s)}(3 \times n)=(\mathbf{P} \times \mathbf{1} \times \mathbf{1})^{R(s)} \cup(\mathbf{1} \times \mathbf{P} \times \mathbf{0})^{R(s)} \cup(\mathbf{0} \times \mathbf{0} \times \mathbf{P})^{R(s)},
\end{aligned}
$$

where, for example,

$$
\begin{aligned}
(\mathbf{0} \times \mathbf{1} \times \mathbf{P})^{R(s)}=\{(\vec{x}, \vec{y}, \vec{z}): & \vec{x} \in \mathbf{0}, \vec{y} \in \mathbf{1}, \vec{z} \in \mathbf{P}^{(i)}, \\
& \text { where } i=s+\operatorname{Index}(\vec{x})+\operatorname{Index}(\vec{y})(\bmod m)\},
\end{aligned}
$$

where we assume that we have $m=4^{d}$ matchings $\mathbf{P}^{(i)}$ (of equal cardinality) which partition the edges of $Q_{3^{d}}$ and each $\mathbf{P}^{(i)}$ is a Breaker's win pairing strategy for $\mathcal{Q}\left(3^{d}, 3^{d-1}+1\right)$ in order to produce $4^{d+1}$ Breaker's win pairing strategies for $\mathcal{Q}\left(3^{d+1}, 3^{d}+1\right)$.

We use $\mathbf{P}^{(0)}, \mathbf{P}^{(1)}, \mathbf{P}^{(2)}, \mathbf{P}^{(3)}$ from the beginning of Section 4 for the case $d=0$.

## 7 Conclusion

Let $p(n)$ be the smallest value of $k$ such that Breaker wins the positional game on $\mathcal{Q}(n, k)$ by using a pairing strategy. We have proven the following upper bounds. If $n \in\left\{4^{d+1}: d \in \mathbb{N}\right\}$, then $p(n) \leq \frac{n}{4}+1$. If $n \in\left\{3^{d+1}: d \in \mathbb{N}\right\} \cup\left\{6 \cdot 4^{d}: d \in \mathbb{N}\right\} \cup\left\{9 \cdot 4^{d}: d \in \mathbb{N}\right\}$, then $p(n) \leq \frac{n}{3}+1$. In general, for all $n \geq 3, p(n) \leq \frac{3}{7} n+1$. To obtain a lower bound on $p(n)$, we cite Proposition 9 in [14], which implies that $p(n)>\ln (n)$. Thus, there is a large gap between the upper and lower bounds on $p(n)$ for most values of $n$. It would be interesting to improve any of these bounds. With regards to small specific values of $n$, because Maker has a winning strategy for $\mathcal{Q}(5,2)$ (see [23]) and $\mathcal{Q}(2,1)$, we know that $p(3)=p(4)=2$ and $p(5)=p(6)=3$. It would be nice to also determine the exact values of, say, $p(7)$ and $p(8)$.

We note that there is no direct analogue to Theorems 3 and 5 for $\mathcal{Q}\left(c^{d+1}, c^{d}+1\right)$ for $c \geq 5$ using our proof method. Indeed, Theorems 3 and 5 rely on the Breaker's win pairing strategies for $\mathcal{Q}(4,2)$ and $\mathcal{Q}(3,2)$ in order to create $\operatorname{BinP} S_{j}^{R(s)}(4 \times n)$ and $\operatorname{BinP} S_{j}^{R(s)}(3 \times n)$. Since Maker has a winning strategy for $\mathcal{Q}(c, 2)$ for all $c \geq 5$, there are no Breaker's win pairing strategies for $\mathcal{Q}(c, 2)$ from which we would create the product-sets for $\operatorname{Bin} P S_{j}^{R(s)}(c \times n)$ for all $c \geq 5$.

As a final note, we mention that some of our results can be viewed as being related to a Turán-type problem on $Q_{n}$. Let ex $(G, H)$ be the maximum number of edges in a subgraph of $G$ which does not contain a copy of $H$. In [15], Erdős discussed some problems that he believed deserved more attention, including determining $\operatorname{ex}\left(Q_{n}, C_{4}\right)$, which he conjectured to be $\left(\frac{1}{2}+o(1)\right)\left|E\left(Q_{n}\right)\right|$. Much work has been done related to determining ex $\left(Q_{n}, C_{2 t}\right)$, see for example, [2], [3], [4], [7], [8], [9], [11], [12], [13], [17, 18], [27].

In [1], Alon, Krech, and Szabó change the focus to studying ex $\left(Q_{n}, Q_{d}\right)$. In particular, let $c(n, d)$ be the minimum number of edges that must be deleted from $Q_{n}$ so that no copy of $Q_{d}$ remains, and let $c_{d}=\lim _{n \rightarrow \infty} c(n, d) /\left|E\left(Q_{n}\right)\right|$. (For a study of $c(n, d)$ in a computer science context, see [20].) In their approach, Alon, Krech, and Szabó used a "Ramsey-type framework," which involved studying d-polychromatic colorings of the edges of $Q_{n}$, i.e., colorings in which every $d$-dimensional subcube of $Q_{n}$ contains an edge from every color class. They define $p c(n, d)$ to be the largest integer $p$ such that there exists a $d$-polychromatic coloring of the edges of $Q_{n}$ in $p$ colors, and $p_{d}=\lim _{n \rightarrow \infty} p c(n, d)$. They also define higher-dimensional analogues, where the definition of $p c^{(\ell)}(n, d)$ is based on coloring each $\ell$-dimensional subcube of $Q_{n}$ so that each $d$-dimensional subcube contains an $\ell$-dimensional subcube of each color. Thus, $p c(n, d)$ is the special case $\ell=1$. They proved upper and lower bounds for $p_{d}$ for all $d \geq 1$ and that $p_{d}^{(0)}=d+1$ for all $d \geq 0$. In [24], Offner proved that $p_{d}$ equals the lower bound given by Alon, Krech, and Szabó. Much work related to polychromatic colorings on the hypercube has been done, for example, [10], [19], [21], [25], and [26].

We note that Theorems 5 and 3 provide a $\left(3^{d}+1\right)$-polychromatic proper coloring of $Q_{3^{d+1}}$ and a $\left(4^{d}+1\right)$-polychromatic proper coloring of $Q_{4^{d+1}}$ for all $d \geq 0$, both using $4^{d+1}$ colors, i.e., each color class forms a matching. It would be interesting to determine for which values of $n$ and $d$ there exists a $d$-polychromatic proper coloring of $Q_{n}$.

We also note that Lemma 5 provides a $\left(c^{n}-c^{n-1}+1\right)$-polychromatic coloring of the vertices of $Q_{c^{n}}$ using $\left(2^{c-1}\right)^{n}$ colors and only vertices from $\mathbf{0}_{c^{n}}$ (or $\mathbf{1}_{c^{n}}$ ). If we let $A_{1}, \ldots, A_{\left(2^{c-1}\right)^{n}}$ be the partition of $\mathbf{0}_{c^{n}}$ and $B_{1}, \ldots, B_{\left(2^{c-1}\right)^{n}}$ be the partition of $\mathbf{1}_{c^{n}}$, then $A_{1} \cup B_{1}, \ldots, A_{\left(2^{c-1}\right)^{n}} \cup B_{\left(2^{c-1}\right)^{n}}$
works as a sort of $\left(c^{n}-c^{n-1}+1\right)$-polychromatic double-coloring of the vertices of $Q_{c^{n}}$ using $\left(2^{c-1}\right)^{n}$ colors, i.e., every $\left(c^{n}-c^{n-1}+1\right)$-dimensional subcube contains two vertices from each color class. It could be interesting to ask for which values of $n, d$, and $p$ do there exist $d$-polychromatic double-colorings of $Q_{n}$ using $p$ colors.

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