Existence of 2-Factors in Tough Graphs without Forbidden Subgraphs

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Abstract

For a given graph R, a graph G is R-free if G does not contain R as an induced subgraph. It is known that every 2-tough graph with at least three vertices has a 2-factor. In graphs with restricted structures, it was shown that every $2K_2$-free $3/2$-tough graph with at least three vertices has a 2-factor, and the toughness bound $3/2$ is best possible. In viewing $2K_2$, the disjoint union of two edges, as a linear forest, in this paper, for any linear forest R on 5, 6, or 7 vertices, we find the sharp toughness bound t such that every t-tough R-free graph on at least three vertices has a 2-factor.

Keywords: 2-factor, toughness, forbidden subgraphs

1 Introduction

Let G be a simple, undirected graph and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. We denote the set of neighbors of a vertex $x \in V(G)$ by $N_G(x)$. The closed neighborhood of a vertex x in G, denoted by $N_G[x]$, is the set $\{x\} \cup N_G(x)$. For any subset $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S, $G - S$ denotes the subgraph $G[V(G) \setminus S]$, and $N_G(S) = \bigcup_{v \in S} N_G(v)$. Given disjoint subsets S and T of $V(G)$, we denote by $E_G(S,T)$ the set of edges which have one end vertex in S and the other end vertex in T, and let $e_G(S,T) = |E_G(S,T)|$. If $S = \{s\}$ is a singleton, we write $e_G(s,T)$ for $e_G(\{s\},T)$. If $H \subseteq G$ is a subgraph of G, and $T \subseteq V(G)$ with $T \cap V(H) = \emptyset$, we write $E_G(H,T)$ and $e_G(H,T)$ for notational simplicity.

For a given graph R, we say that G is R-free if there does not exist an induced copy of R in G. For integers a and b with $a \geq 0$ and $b \geq 1$, we denote by aP_b the graph consisting of a disjoint copies of the path P_b. When $a = 1$, $1P_b$ is just P_b, and when $a = 0$, $0P_b$ is the null graph. For two integers p and q, let $[p, q] = \{i \in \mathbb{Z} : p \leq i \leq q\}$.

Denote by $c(G)$ the number of components of G. Let $t \geq 0$ be a real number. We say a graph G is t-tough if for each cutset S of G we have $t \cdot c(G - S) \leq |S|$. The toughness of a graph G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough if G is non-complete and is defined to be ∞ if G is complete.

For an integer $k \geq 1$, a k-regular spanning subgraph is a *k-factor* of G. It is well known, according to a theorem by Enomoto, Jackson, Katerinis, and Saito [3] from 1998, that every k-tough graph with at least three vertices has a k-factor if $k|V(G)|$ is even and $|V(G)| \geq k + 1$. In terms of a sharp toughness bound, particular research interest has been taken when $k = 2$ for graphs with restricted structures. For example, it was shown that every 3/2-tough 5-chordal graph (graphs with no induced cycle of length at least 5) on at least three vertices has a 2-factor [1] and that every 3/2-tough $2K_2$-free graph on at least three vertices has a 2-factor [5]. The toughness bound 3/2 is best possible in both results.

A *linear forest* is a graph consisting of disjoint paths. In viewing $2K_2$ as a linear forest on 4 vertices and the result by Ota and Sanka [5] that every 3/2-tough $2K_2$-free graph on at least three vertices has a 2-factor, we investigate the existence of 2-factors in R-free graphs when R is a linear forest on 5, 6, or 7 vertices. These graphs R are listed below, where the unions are vertex disjoint unions.

1. P_5 $\quad P_4 \cup P_1$ $\quad P_3 \cup P_2$ $\quad P_3 \cup 2P_1$ $\quad 2P_2 \cup P_1$ $\quad P_2 \cup 3P_1$ $\quad 5P_1$;
2. P_6 $\quad P_5 \cup P_1$ $\quad P_4 \cup P_2$ $\quad P_4 \cup 2P_1$ $\quad 2P_3$ $\quad P_3 \cup P_2 \cup P_1$ $\quad P_3 \cup 3P_1$ $\quad 3P_2$ $\quad 2P_2 \cup 2P_1$ $\quad P_2 \cup 4P_1$ $\quad 6P_1$;
3. P_7 $\quad P_6 \cup P_1$ $\quad P_5 \cup P_2$ $\quad P_5 \cup 2P_1$ $\quad P_4 \cup P_3$ $\quad P_4 \cup P_2 \cup P_1$ $\quad P_4 \cup 3P_1$ $\quad 2P_3 \cup P_1$ $\quad P_3 \cup 2P_2$ $\quad P_3 \cup P_2 \cup 2P_1$ $\quad P_3 \cup 4P_1$ $\quad 3P_2 \cup P_1$ $\quad 2P_2 \cup 3P_1$ $\quad P_2 \cup 5P_1$ $\quad 7P_1$.

Our main results are the following:

Theorem 1. *Let $t > 0$ be a real number, R be any linear forest on 5 vertices, and G be a t-tough R-free graph on at least 3 vertices. Then G has a 2-factor provided that*

(1) $R \in \{P_4 \cup P_1, P_3 \cup 2P_1, P_2 \cup 3P_1\}$ and $t = 1$ unless

(a) $R = P_2 \cup 3P_1$, and $G \cong H_0$ or G contains H_1, H_2 or H_3 as a spanning subgraph such that $E(G) \setminus E(H_i) \subseteq E_G(S, V(G) \setminus (T \cup S))$ for each $i \in [1, 3]$, where H_i, S and T are defined in Figure 1.

(b) $R = P_3 \cup 2P_1$ and G contains H_1 as a spanning subgraph such that $E(G) \setminus E(H_1) \subseteq E_G(S, V(G) \setminus (T \cup S))$.

(2) $R = 5P_1$ and $t > 1$.

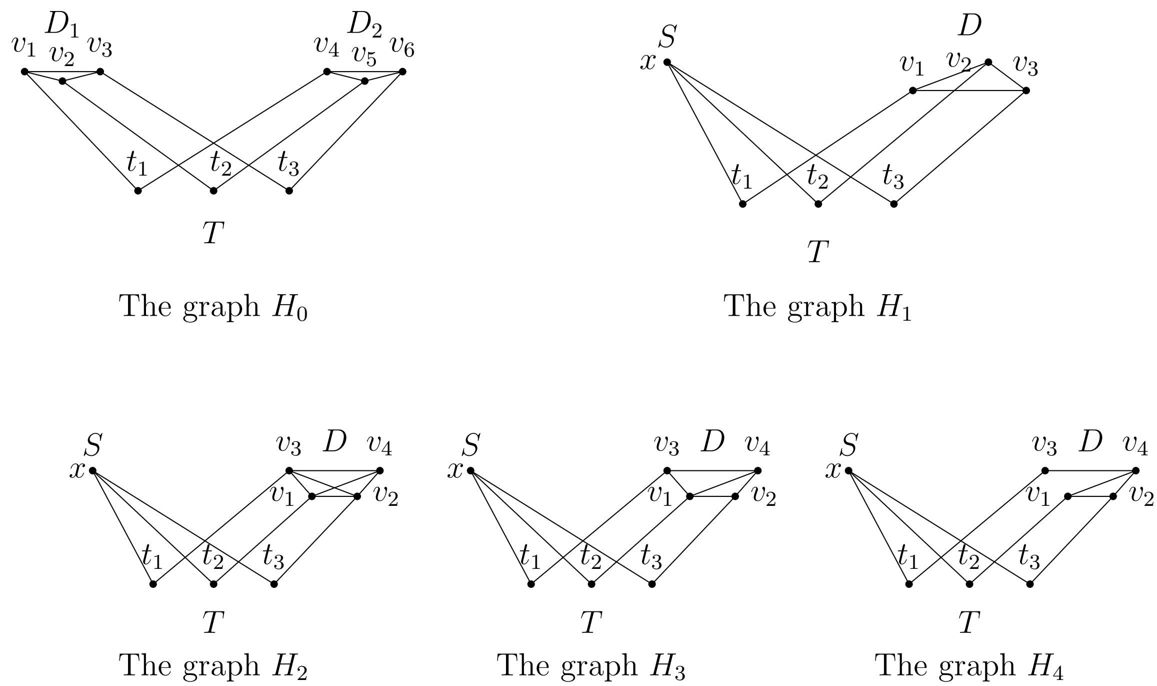


Figure 1: The four exceptional graphs for Theorem 1(1), where $S = \{x\}$ and $T = \{t_1, t_2, t_3\}$.

(3) $R \in \{P_5, P_3 \cup P_2, 2P_2 \cup P_1\}$ *and* $t = 3/2$.

Theorem 2. *Let $t > 0$ be a real number, R be any linear forest on 6 vertices, and G be a t-tough R-free graph on at least 3 vertices. Then G has a 2-factor provided that*

(1) $R \in \{P_4 \cup 2P_1, P_3 \cup 3P_1, P_2 \cup 4P_1, 6P_1\}$ *and* $t > 1$ *unless* $R = 6P_1$ *and G contains H_5 with $p = 5$ as a spanning subgraph such that* $E(G) \setminus E(H_5) \subseteq E_G(S, V(G) \setminus (T \cup S))$, *where H_5, S and T are defined in Figure 2.*

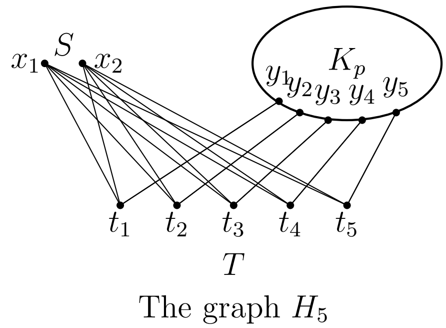
(2) $R \in \{P_6, P_5 \cup P_1, P_4 \cup P_2, 2P_3, P_3 \cup P_2 \cup P_1, 3P_2, 2P_2 \cup 2P_1\}$ *and* $t = 3/2$.

Theorem 3. *Let $t > 0$ be a real number, R be any linear forest on 7 vertices, and G be a t-tough R-free graph on at least 3 vertices. Then G has a 2-factor provided that*

(1) $R \in \{P_4 \cup 3P_1, P_3 \cup 4P_1, P_2 \cup 5P_1\}$ *and* $t > 1$ *unless*

  (a) when $R \neq P_4 \cup 3P_1$, G contains H_5 with $p = 5$ as a spanning subgraph such that $E(G) \setminus E(H_5) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S])$, *where H_5, S and T are defined in Figure 2.*

  (b) $R = P_2 \cup 5P_1$ and G contains one of $H_6, \ldots, H_{11}$ as a spanning subgraph such that $E(G) \setminus E(H_i) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S]) \cup E(G[V(G) \setminus (T \cup S)])$, *where H_i, S and T are defined in Figure 3 for each $i \in [6, 11]$.*



The graph H_5

Figure 2: The exceptional graph for Theorem 2(1), where $S = \{x_1, x_2\}$, $T = \{t_1, \ldots, t_5\}$, and $p = 5$.

(2) $R = 7P_1$ *and* $t > \frac{7}{6}$ *unless* G *contains* H_5 *with* $p = 5$ *as a spanning subgraph such that* $E(G) \setminus E(H_5) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S])$.

(3) $R \in \{P_7, P_6 \cup P_1, P_5 \cup P_2, P_5 \cup 2P_1, P_4 \cup P_2 \cup P_1, 2P_3 \cup P_1, P_4 \cup P_3, P_3 \cup 2P_2, P_3 \cup P_2 \cup 2P_1, 3P_2 \cup P_1, 2P_2 \cup 3P_1\}$ *and* $t = 3/2$.

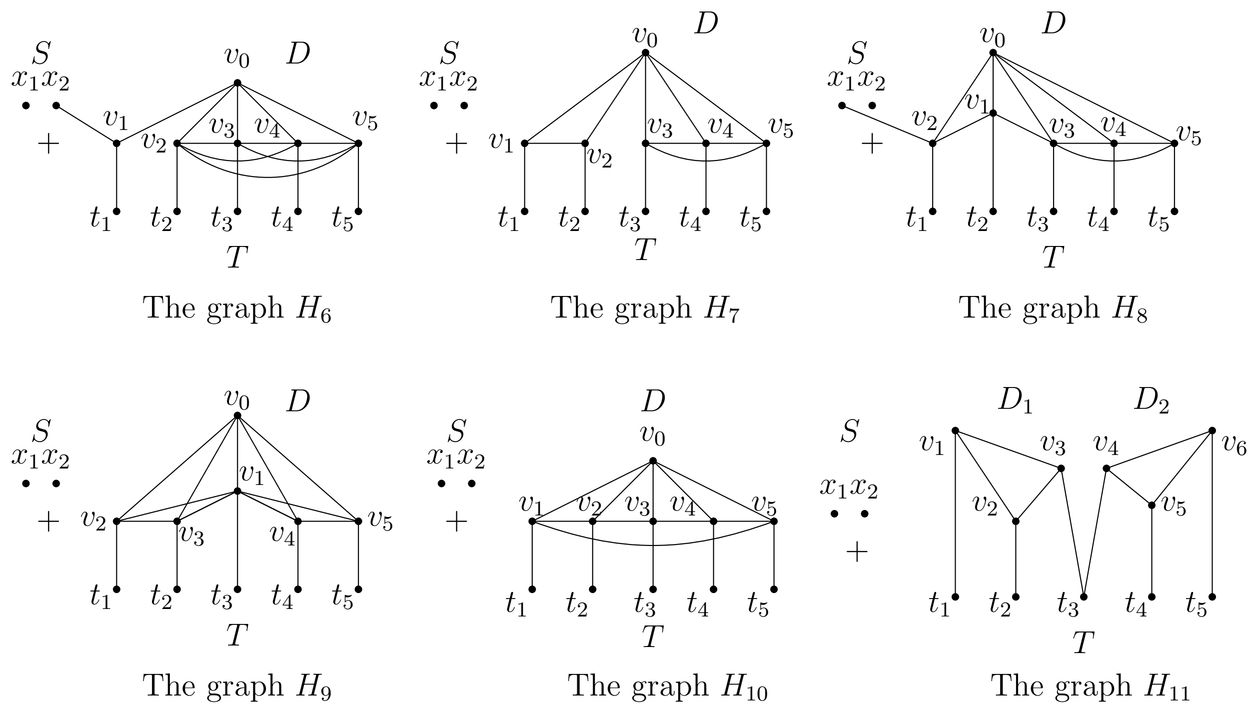


Figure 3: The five exceptional graphs for Theorem 3(1)(b), where $S = \{x_1, x_2\}$, $T = \{t_1, t_2, t_3, t_4, t_5\}$, and "+" represents the join of $H_i[S]$ and $H_i[T]$, $i \in [6, 11]$.

Remark 4 (Examples demonstrating sharp toughness bounds)**.** *The toughness bounds in Theorems 1 to 3 are all sharp.*

(1) *Theorem 1(1) when* $R \in \{P_4 \cup P_1, P_3 \cup 2P_1, P_2 \cup 3P_1\}$ *and* $t = 1$. *The graph showing that the toughness 1 is best possible is the complete bipartite* $K_{n-1,n}$ *for any integer* $n \geq 2$. *The graph* $K_{n,n-1}$ *is* P_4*-free and so is* R*-free, with* $\lim_{n\to\infty} \tau(K_{n,n-1}) = \lim_{n\to\infty} \frac{n-1}{n} = 1$, *but contains no 2-factor.*

(2) *Theorem 1(2), Theorem 2(1) and Theorem 3(1) and* $t > 1$. *The graph showing that the toughness is best possible is the graph* H_{12}, *which is constructed as below: let* $p \geq 3$, K_p *be a complete graph, and* $y_1, y_2, y_3 \in V(K_p)$ *be distinct,* $S = \{x\}$, *and* $T = \{t_1, t_2, t_3\}$, *then* H_{12} *is obtained from* K_p, S *and* T *by adding edges* $t_i x$ *and* $t_i y_i$ *for each* $i \in [1, 3]$. *See Figure 4 for a depiction. By inspection, the graph is* $5P_1$*-free and* $(P_4 \cup 2P_1)$*-free. So the graph is* R*-free for any* $R \in \{5P_1, P_4 \cup 2P_1, P_3 \cup 3P_1, P_2 \cup 4P_1, 6P_1, P_4 \cup 3P_1, P_3 \cup 4P_1, P_2 \cup 5P_1\}$. *For any given* $p \geq 3$, *the graph* H_{12} *does not contain a 2-factor, as any 2-factor has to contain the edges* t_1x, t_2x *and* t_3x. *We will show* $\tau(H_{12}) = 1$ *in the last section.*

(3) *For Theorem 1(3), Theorem 2(2) and Theorem 3(3) and* $t = \frac{3}{2}$: *note that all the graphs* R *in these cases contain* $2K_2$ *as an induced subgraph. Chvátal [2] constructed a sequence* $\{G_k\}_{k=1}^{\infty}$ *of split graphs (graphs whose vertex set can be partitioned into a clique and an independent set) having no 2-factors and* $\tau(G_k) = \frac{3k}{2k+1}$ *for each positive integer* k. *As the class of* $2K_2$*-free graphs is a superclass of split graphs,* $\frac{3}{2}$*-tough is the best possible toughness bound for a* $2K_2$*-free graph to have a 2-factor.*

(4) *Theorem 3(2) and* $t > \frac{7}{6}$. *The graph showing that the toughness is best possible is the graph* H_5 *with* $p \geq 6$, *which is constructed as below: let* $p \geq 5$, K_p *be a complete graph, and* $y_1, y_2, y_3, y_4, y_5 \in V(K_p)$ *be distinct,* $S = \{x_1, x_2\}$, *and* $T = \{t_1, t_2, t_3, t_4, t_5\}$. *Then* H_5 *is obtained from* K_p, S *and* T *by adding edges* $t_i x_j$ *and* $t_i y_i$ *for each* $i \in [1, 5]$ *and each* $j \in [1, 2]$. *See Figure 2 for a depiction. By inspection, the graph is* $7P_1$*-free. For any given* $p \geq 5$, *the graph* H_5 *does not contain a 2-factor, as any 2-factor has to contain at least three edges from one of* x_1 *and* x_2 *to at least three vertices of* T. *We will show* $\tau(H_5) = \frac{7}{6}$ *when* $p \geq 6$ *in the last section.*

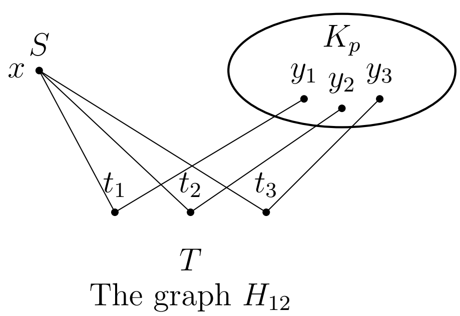


Figure 4: Sharpness example for Theorem 1(2), Theorem 2(1) and Theorem 3(1), where $S = \{x\}$ and $T = \{t_1, t_2, t_3\}$.

To supplement Theorems 1 to 3, we show that the exceptional graphs in Figures 1 to 3 satisfy the corresponding conditions below.

Theorem 5. *The following statements hold.*

(1) The graph H_i is $(P_2 \cup 3P_1)$-free, contains no 2-factor, and $\tau(H_i) = 1$ for each $i \in [0, 4]$, the graph H_1 is also $(P_3 \cup 2P_1)$-free.

(2) The graph H_i is $(P_2 \cup 5P_1)$-free and contains no 2-factor for each $i \in [5, 11]$, H_5 with $p = 5$ is $(P_3 \cup 4P_1)$-free and $6P_1$-free. Furthermore, $\tau(H_5) = \frac{6}{5}$ when $p = 5$ and $\tau(H_i) = \frac{7}{6}$ for each $i \in [6, 11]$.

We have explained that H_5 and H_{12} are R-free for the corresponding linear forests R and contain no 2-factor in Remark 4(2) and (4). The Theorem below is to verify the toughness of the graphs H_5 with $p \geq 6$ and H_{12}.

Theorem 6. *The following statements hold.*

(1) $\tau(H_5) = \frac{7}{6}$ when $p \geq 6$;

(2) $\tau(H_{12}) = 1$.

The remainder of this paper is organized as follows. In section 2, we introduce more notation and preliminary results on proving existence of 2-factors in graphs. In section 3, we prove Theorems 1-3. Theorems 5 and 6 are proved in the last section.

2 Preliminaries

One of the main proof ingredients of Theorems 1 to 3 is to apply Tutte's 2-factor Theorem. We start with some notation. Let S and T be disjoint subsets of vertices of a graph G, and D be a component of $G - (S \cup T)$. The component D is said to be an *odd component* (resp. *even component*) of $G - (S \cup T)$ if $e_G(D, T) \equiv 1 \pmod 2$ (resp. $e_G(D, T) \equiv 0 \pmod 2$). Let $h(S, T)$ be the number of all odd components of $G - (S \cup T)$. Define

$$\delta(S, T) = 2|S| - 2|T| + \sum_{y \in T} d_{G-S}(y) - h(S, T).$$

It is easy to see that $\delta(S, T) \equiv 0 \pmod 2$ for every S, $T \subseteq V(G)$ with $S \cap T = \emptyset$. We use the following criterion for the existence of a 2-factor, which is a restricted form of Tutte's f-factor Theorem.

Lemma 7 (Tutte [6]). *A graph G has a 2-factor if and only if $\delta(S, T) \geq 0$ for every S, $T \subseteq V(G)$ with $S \cap T = \emptyset$.*

An ordered pair (S, T), consisting of disjoint subsets of vertices S and T in a graph G, is called a *barrier* if $\delta(S, T) \leq -2$. By Lemma 7, if G does not have a 2-factor, then G has a barrier. In [4], a *biased barrier* of G is defined as a barrier (S, T) of G such that among all the barriers of G,

(1) $|S|$ is maximum; and

(2) subject to (1), $|T|$ is minimum.

The following properties of a biased barrier were derived in [4].

Lemma 8. *Let G be a graph without a 2-factor, and let (S, T) be a biased barrier of G. Then each of the following holds.*

(1) The set T is independent in G.

(2) If D is an even component with respect to (S, T), then $e_G(T, D) = 0$.

(3) If D is an odd component with respect to (S, T), then for any $y \in T$, $e_G(y, D) \leq 1$.

(4) If D is an odd component with respect to (S, T), then for any $x \in V(D)$, $e_G(x, T) \leq 1$.

Let G be a graph without a 2-factor and (S, T) be a barrier of G. For an integer $k \geq 0$, let $\mathcal{C}_{2k+1}$ denote the set of odd components D of $G - (S \cup T)$ such that $e_G(D, T) = 2k + 1$. The following result was proved as a claim in [4] but we include a short proof here for self-completeness.

Lemma 9. *Let G be a graph without a 2-factor, and let (S, T) be a biased barrier of G. Then $|T| \geq |S| + \sum_{k \geq 1} k|\mathcal{C}_{2k+1}| + 1$.*

Proof. Let $U = V(G) \setminus S$. Since (S, T) is a barrier,

$$\begin{aligned} \delta(S, T) &= 2|S| - 2|T| + \sum_{y \in T} d_{G-S}(y) - h(S, T) \\ &= 2|S| - 2|T| + \sum_{y \in T} d_{G-S}(y) - \sum_{k \geq 0} |\mathcal{C}_{2k+1}| \leq -2. \end{aligned}$$

By Lemma 8(1) and Lemma 8(2),

$$\sum_{y \in T} d_{G-S}(y) = \sum_{y \in T} e_G(y, U) = e_G(T, U) = \sum_{k \geq 0} (2k + 1)|\mathcal{C}_{2k+1}|.$$

Therefore, we have

$$-2 \geq 2|S| - 2|T| + \sum_{k \geq 0} (2k + 1)|\mathcal{C}_{2k+1}| - \sum_{k \geq 0} |\mathcal{C}_{2k+1}|,$$

which yields $|T| \geq |S| + \sum_{k \geq 1} k|\mathcal{C}_{2k+1}| + 1$. $\square$

We use the following lemmas in our proof.

Lemma 10. *Let $t \geq 1$, G be a t-tough graph on at least three vertices containing no 2-factor, and (S, T) be a barrier of G. Then the following statements hold.*

(1) If $\mathcal{C}_1 \neq \emptyset$, then $|S|+1 \geq 2t$. Consequently, $S=\emptyset$ implies $\mathcal{C}_1=\emptyset$, and $|S|=1$ implies $\mathcal{C}_1=\emptyset$ when $t>1$.

(2) $\bigcup_{k\geq 1}\mathcal{C}_{2k+1} \neq \emptyset$.

Proof. Since G is 1-tough and thus is 2-connected, $d_G(y) \geq 2$ for every $y \in T$. This together with Lemma 8(1)-(3) implies $|S| + \sum_{k\geq 0}|\mathcal{C}_{2k+1}| \geq 2$.

For the first part of (1), suppose to the contrary that $|S|+1 < 2t$. Let $D \in \mathcal{C}_1$ and $y \in V(T)$ be adjacent in G to some vertex $v \in V(D)$. As $e_G(D,T) = e_G(D,y) = 1$, $|S| + \sum_{k\geq 0}|\mathcal{C}_{2k+1}| \geq 2$. and $|T| \geq |S|+1$ by Lemma 9, we have $c(G-(S\cup\{y\})) \geq 2$ regardless of whether or not $S=\emptyset$. But $c(G-(S\cup\{y\})) \geq 2$ implies $\tau(G) < 2t/2 = t$, contradicting G being t-tough. The second part of (1) is a consequence of the first part.

For (2), suppose to the contrary that $\bigcup_{k\geq 1}\mathcal{C}_{2k+1} = \emptyset$. By Lemma 10(1), $|S| + |\mathcal{C}_1| \geq 2$ implies $|S| \geq 1$. Consequently, $|T| \geq 2$ by Lemma 9. As every component of $G-(S\cup T)$ in $\mathcal{C}_1$ is connected to exactly one vertex of T, S is a cutset of G with $c(G-S) \geq |T|$. However, $|T| \geq |S| + \sum_{k\geq 1} k|\mathcal{C}_{2k+1}| + 1 = |S|+1$, implying $\tau(G) < 1$, a contradiction. $\square$

A path P connecting two vertices u and v is called a (u,v)-*path*, and we write uPv or vPu in order to specify the two endvertices of P. Let uPv and xQy be two disjoint paths. If vx is an edge, we write $uPvxQy$ as the concatenation of P and Q through the edge vx. Let G be a graph without a 2-factor, and let (S,T) be a barrier of G. For $y \in T$, define

$$h(y) = |\{D: \ D \in \bigcup_{k\geq 1}\mathcal{C}_{2k+1} \quad \text{and} \quad e_G(y,D) \geq 1\}|.$$

Lemma 11. *Let G be a graph without a 2-factor, and let (S,T) be a biased barrier of G. Then the following statements hold.*

(1) If $|\bigcup_{k\geq 1}\mathcal{C}_{2k+1}| \geq 1$, then G contains an induced $P_4 \cup aP_1$, where $a = |T|-2$.

(2) If there exists $y_0 \in T$ with $h(y_0) \geq 2$, then for some integer $b \geq 7$, G contains an induced $P_b \cup aP_1$, where $a = |T|-3$. Furthermore, an induced $P_b \cup aP_1$ can be taken such that the vertices in aP_1 are from T and the path P_b has the form $y_1x_1^*P_1x_1y_0x_2P_2x_2^*y_2$, where $y_0,y_1,y_2 \in T$ and $x_1^*P_1x_1$ and $x_2^*P_2x_2$ are respectively contained in two distinct components from $\bigcup_{k\geq 1}\mathcal{C}_{2k+1}$ such that $e_G(x,T)=0$ for every internal vertex x from P_1 and P_2.

Proof. Lemma 8(1), (3) and (4) will be applied frequently in the arguments sometimes without mentioning it.

Let $D \in \bigcup_{k\geq 1}\mathcal{C}_{2k+1}$. The existence of D implies $|T| \geq 3$ and $|V(D)| \geq 3$ by Lemma 8(3) and (4). We claim that for a fixed vertex $x_1 \in V(D)$ such that $e_G(x_1,T)=1$, there exists distinct $x_2 \in V(D)$ and an induced (x_1,x_2)-path P in D with the following two properties: (a) $e_G(x_2,T)=1$, and (b) $e_G(x,T)=0$ for every $x \in V(P)\setminus\{x_1,x_2\}$. Note that the vertex x_1 exists by Lemma 8(4). Let $y_1 \in T$ be the vertex such that $e_G(x_1,T) = e_G(x_1,y_1) = 1$ and $W = N_G(T\setminus\{y_1\}) \cap V(D)$.

By Lemma 8(4), $x_1 \notin W$. Now in D, we find a shortest path P connecting x_1 and some vertex from W, say x_2. Then x_2 and P satisfy properties (a) and (b), respectively. Let $y_2 \in T$ such that $e_G(x_2, T) = e_G(x_2, y_2) = 1$. The vertex y_2 uniquely exists by the choice x_2 and Lemma 8(4). By Lemma 8(1) and (4), and the choice of P, we know that $y_1x_1Px_2y_2$ and $T \setminus \{y_1, y_2\}$ together contains an induced $P_4 \cup aP_1$. This proves (1).

We now prove (2). By Lemma 8(3), the existence of y_0 implies $|\bigcup_{k\geq 1} \mathcal{C}_{2k+1}| \geq 2$, which in turn gives $|T| \geq 3$ by Lemma 8(3) again. We let $D_1, D_2 \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$ be distinct such that $e_G(y_0, D_1) = 1$ and $e_G(y_0, D_2) = 1$. Let $x_i \in D_i$ such that $e_G(y_0, D_i) = e_G(y_0, x_i) = 1$. By the argument in the first paragraph above, we can find $x_i^* \in V(D_i) \setminus \{x_i\}$ and an (x_i, x_i^*)-path P_i in D_i for each $i \in \{1, 2\}$. By the choice of P_i and Lemma 8(4), there are unique $y_1, y_2 \in T \setminus \{y_0\}$ such that $x_i^*y_i \in E(G)$. If $y_1 \neq y_2$, by the choice of P_1 and P_2 and Lemma 8(1) and (4), we know that $y_1x_1^*P_1x_1y_0x_2P_2x_2^*y_2$ and $T \setminus \{y_0, y_1, y_2\}$ together contain an induced $P_b \cup aP_1$ for some integer $b \geq 7$. Thus we assume $y_1 = y_2$. Then the vertex y_1 can also play the role of y_0. Let $W = N_G(T \setminus \{y_0, y_1\}) \cap V(D_2)$. By Lemma 8(4), $x_2, x_2^* \notin W$. Now in D_2, we find a shortest path P_2^* connecting some vertex of $\{x_2, x_2^*\}$ and some vertex from W, say z. If P_2^* is an (x_2, z)-path, then $y_1x_1^*P_1x_1y_0x_2P_2^*z$ and $T \setminus \{y_0, y_1, y_2\}$ together contain an induced $P_b \cup aP_1$. If P_2^* is an (x_2^*, z)-path, then $y_0x_1P_1x_1^*y_1x_2^*P_2^*z$ and $T \setminus \{y_0, y_1, y_2\}$ together contain an induced $P_b \cup aP_1$. The second part of (2) is clear by the construction above. $\square$

Let G be a non-complete graph. A cutset S of $V(G)$ is a *toughset* of G if $\frac{|S|}{c(G-S)} = \tau(G)$.

Lemma 12. *If G is a connected graph and S is a toughset of G, then for every $x \in S$, x is adjacent in G to vertices from at least two components of $G - S$.*

Proof. Assume to the contrary that there exists $x \in S$ such that x is adjacent in G to vertices from at most one component of $G - S$. Then $c(G - (S \setminus \{x\})) = c(G - S) \geq 2$ and

$$\frac{|S \setminus \{x\}|}{c(G - (S \setminus \{x\}))} < \frac{|S|}{c(G - S)} = \tau(G),$$

contradicting G being $\tau(G)$-tough. $\square$

3 Proof of Theorems 1, 2, and 3

Let R be any linear forest on at most 7 vertices. If G is R-free, then G is also R^*-free for any supergraph R^* of R. To prove Theorems 1 to 3, we will show that the corresponding statements hold for a supergraph R^* of R, which simplifies the cases of possibilities of R. Let us first list the supergraphs that we will use.

(1) $P_4 \cup 3P_1$ is a supergraph of the following graphs: $P_4 \cup 2P_1, P_3 \cup 3P_1$, and $P_2 \cup 4P_1$;

(2) $6P_1$ is a supergraph of $5P_1$;

(3) $P_3 \cup 2P_2$ is a supergraph of $3P_2$;

(4) $P_7 \cup 2P_1$ is a supergraph of the following graphs:

   (a) $P_5, P_3 \cup P_2, 2P_2 \cup P_1$;

   (b) $P_6, P_5 \cup P_1, P_4 \cup P_2, 2P_3, P_3 \cup P_2 \cup P_1, 2P_2 \cup 2P_1$;

   (c) $P_7, P_6 \cup P_1, P_5 \cup 2P_1, P_4 \cup P_2 \cup P_1, 2P_3 \cup P_1, P_3 \cup P_2 \cup 2P_1, 2P_2 \cup 3P_1$.

Those supergraphs above together with the graphs R listed below cover all the 33 R graphs described in Theorems 1 to 3. Theorems 1 to 3 are then consequences of the theorem below.

Theorem 13. *Let $t > 0$ be a real number, R be a linear forest, and G be a t-tough R-free graph on at least 3 vertices. Then G has a 2-factor provided that*

(1) $R \in \{P_4 \cup P_1, P_3 \cup 2P_1, P_2 \cup 3P_1\}$ and $t = 1$ unless

   (a) $R = P_2 \cup 3P_1$, and $G \cong H_0$ or G contains H_1, H_2, H_3 or H_4 as a spanning subgraph such that $E(G) \setminus E(H_i) \subseteq E_G(S, V(G) \setminus (T \cup S))$ for each $i \in [1, 3]$, where H_i, S and T are defined in Figure 1.

   (b) $R = P_3 \cup 2P_1$ and G contains H_1 as a spanning subgraph such that $E(G) \setminus E(H_1) \subseteq E_G(S, V(G) \setminus (T \cup S))$.

(2) $R \in \{P_4 \cup 3P_1, P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1\}$ and $t > 1$ unless

   (a) when $R \neq P_4 \cup 3P_1$, G contains H_5 with $p = 5$ as a spanning subgraph such that $E(G) \setminus E(H_5) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S])$, where H_5, S and T are defined in Figure 2.

   (b) $R = P_2 \cup 5P_1$ and G contains one of $H_6, \ldots, H_{11}$ as a spanning subgraph such that $E(G) \setminus E(H_i) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S]) \cup E(G[V(G) \setminus (T \cup S)])$, where H_i, S and T are defined in Figure 3 for each $i \in [6, 11]$.

(3) $R = 7P_1$ and $t > \frac{7}{6}$ unless G contains H_5 with $p = 5$ as a spanning subgraph such that $E(G) \setminus E(H_5) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S])$.

(4) $R \in \{P_7 \cup 2P_1, P_5 \cup P_2, P_4 \cup P_3, P_3 \cup 2P_2, 3P_2 \cup P_1\}$ and $t = 3/2$.

Proof. Assume by contradiction that G does not have a 2-factor. By Lemma 7, G has a barrier. We choose (S, T) to be a biased barrier. Thus (S, T) and G satisfy all the properties listed in Lemma 8. These properties will be used frequently even without further mentioning sometimes. By Lemma 9,

$$|T| \geq |S| + \sum_{k \geq 1} k|\mathcal{C}_{2k+1}| + 1. \tag{1}$$

Since $t \geq 1$, by Lemma 10(2), we know that

$$\bigcup_{k \geq 1} \mathcal{C}_{2k+1} \neq \emptyset. \tag{2}$$

This implies $|T| \geq 3$ and so G contains an induced $P_4 \cup P_1$ by Lemma 11 (1). Thus we assume $R \neq P_4 \cup P_1$ in the rest of the proof.

Claim 1. $R \notin \{P_3 \cup 2P_1, P_2 \cup 3P_1\}$ *unless G falls under one of the exceptional cases as in (a) and (b) of Theorem 13(1).*

Proof. Assume instead that $R \in \{P_3 \cup 2P_1, P_2 \cup 3P_1\}$. Thus $t = 1$. We may assume that G does not fall under any of the exceptional cases as in (a) and (b) of Theorem 13 (1).

It must be the case that $|T| = 3$, as otherwise G contains an induced $P_4 \cup 2P_1$ by Lemma 11(1), and so contains an induced R. By Equation (1), we have $|\bigcup_{k\geq 1} \mathcal{C}_{2k+1}| + |S| \leq 2$. By Lemma 10(1), we have that $\mathcal{C}_0 = \emptyset$ if $S = \emptyset$. Since G is 1-tough and so $\delta(G) \geq 2$, Lemma 8(1)-(3) implies that $|\bigcup_{k\geq 1} \mathcal{C}_{2k+1}| + |S| = 2$. By (2), we have the two cases below.

Case 1: $|\bigcup_{k\geq 1} \mathcal{C}_{2k+1}| = 2$ and $S = \emptyset$.

Let $D_1, D_2 \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$ be the two odd components of $G - (S \cup T)$. Since $|T| = 3$, Lemma 8(3) implies that $e_G(D_i, T) = 3$ for each $i \in [1, 2]$. Let $y \in T$ and $x \in V(D_1)$ such that $xy \in E(G)$. We let x_1 be a neighbor of x from D_1. Then yxx_1 is an induced P_3 by Lemma 8(3). Let $y_1 \in T \setminus \{y\}$ such that $y_1x_1 \notin E(G)$, which is possible as $|T| = 3$ and $e_G(x_1, T) \leq 1$ by Lemma 8(4). We now let $x_2 \in V(D_2)$ such that $e_G(x_2, \{y, y_1\}) = 0$, which is again possible as $|N_G(T) \cap V(D_2)| = 3$ and each vertex of D_2 is adjacent in G to at most one vertex of T. However, yxx_1, y_1 and x_2 together form an induced copy of $P_3 \cup 2P_1$. Therefore, we assume $R = P_2 \cup 3P_1$.

We first claim that $|V(D_i)| = 3$ for each $i \in [1, 2]$. Otherwise, say $|V(D_2)| \geq 4$. Let $y \in T$ and $x \in V(D_1)$ such that $xy \in E(G)$. Take $x_1 \in V(D_2)$ such that $e_G(x_1, T) = 0$, which exists as $|N_G(T) \cap V(D_2)| = 3$. Then xy, x_1 and $T \setminus \{y\}$ together form an induced copy of $P_2 \cup 3P_1$, giving a contradiction. We next claim that $D_i = K_3$ for each $i \in [1, 2]$. Otherwise, say $D_1 \neq K_3$. As D_1 is connected, it follows that $D_1 = P_3$. If also $D_2 \neq K_3$ and so $D_2 = P_3$, then deleting the two vertices of degree 2 from both D_1 and D_2 gives three components (note that each vertex of T is adjacent in G to one vertex of D_1 and one vertex of D_2), showing that $\tau(G) \leq 2/3 < 1$. Thus $D_2 = K_3$. We let $x_1, x_2 \in V(D_1)$ be nonadjacent, $y_1, y_2 \in T$ such that $e_G(x_i, y_i) = 1$ for each $i \in [1, 2]$, and $z_1, z_2 \in V(D_2)$ such that $e_G(y_i, z_i) = 1$ for each $i \in [1, 2]$. Let $y \in T \setminus \{y_1, y_2\}$. Then z_1z_2, y, x_1 and x_2 together form an induced copy of $P_2 \cup 3P_1$, giving a contradiction.

Thus $|V(D_i)| = 3$ and $D_i = K_3$ for each $i \in [1, 2]$. However, this implies that $G \cong H_0$.

Case 2: $|\bigcup_{k\geq 1} \mathcal{C}_{2k+1}| = 1$ and $|S| = 1$.

Let $D \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$ be the odd component of $G - (S \cup T)$. Assume first that $R = P_3 \cup 2P_1$. Then we have $|V(D)| = 3$. Otherwise, $|V(D)| \geq 4$. Let $x \in V(D)$ such that $e_G(x, T) = 0$ and P be a shortest path of D from x to a vertex, say $x_1 \in V(D) \cap N_G(T)$. Let $y \in T$ such that $e_G(x_1, y) = 1$. Then xPx_1y and $T \setminus \{y\}$ form an induced copy of R, a contradiction.

Since G does not contain H_1 as a spanning subgraph such that $E(G)\setminus E(H_1) \subseteq E_G(S, V(G)\setminus(T\cup S))$, it follows that $D \neq K_3$. As D is connected, it follows that $D = P_3$. Now deleting the vertex in S together with the degree 2 vertex of D produces three components, showing that $\tau(G) \leq 2/3 < 1$.

Therefore, we assume now that $R = P_2 \cup 3P_1$. Since G does not contain H_1 as a spanning subgraph such that $E(G) \setminus E(H_1) \subseteq E_G(S, V(G) \setminus (T \cup S))$, the argument for the case $R = P_3 \cup 2P_1$ above implies that $|V(D)| \geq 4$. We claim that $|V(D)| = 4$. If $|V(D)| \geq 5$, we let $x_1, x_2 \in V(D) \setminus N_G(T)$ be any two distinct vertices. If $x_1x_2 \in E(G)$, then x_1x_2 together with T form an induced copy of R, a contradiction. Thus $V(D) \setminus N_G(T)$ is an independent set in G. However, $c(G - (S \cup (N_G(T) \cap V(D)))) = |T| + |V(D) \setminus N_G(T)| \geq 5$, implying that $\tau(G) \leq 4/5 < 1$.

Thus $|V(D)| = 4$. Let $x \in V(D)$ such that $e_G(x, T) = 0$. Since G does not contain H_i as a spanning subgraph such that $E(G) \setminus E(H_i) \subseteq E_G(S, V(G) \setminus (T \cup S))$ for each $i \in [2, 4]$, it follows that either $d_D(x) \leq 2$ or $d_D(x) = 3$ and $D = K_{1,3}$. If $d_D(x) = 3$, then as $D = K_{1,3}$, we have $c(G - (S \cup \{x\})) = 3$, implying $\tau(G) \leq 2/3 < 1$. Thus $d_D(x) \leq 2$. Let $V(D) = \{x, x_1, x_2, x_3\}$ and assume $xx_1 \notin E(D)$. Then $c(G - (S \cup \{x_2, x_3\})) = 4$, implying $\tau(G) \leq 3/4 < 1$. The proof of Case 2 is complete. $\square$

Thus by Claim 1 and the fact that $R \neq P_4 \cup P_1$, we can assume $R \notin \{P_4 \cup P_1, P_3 \cup 2P_1, P_2 \cup 3P_1\}$ from this point on. Therefore we have $t > 1$. This implies that G is 3-connected and so $\delta(G) \geq 3$. Thus $|S| + |\bigcup_{k\geq 0} \mathcal{C}_{2k+1}| \geq 3$ by Lemma 8(1)-(4).

Claim 2. *$|T| \geq 5$.*

Proof. Equation (2) implies $|T| \geq 3$. Assume to the contrary that $|T| \leq 4$. We consider the following two cases.

Case 1: $|T| = 3$.

Since $|S| + |\bigcup_{k\geq 0} \mathcal{C}_{2k+1}| \geq 3$, we already have a contradiction to Equation (1) if $\mathcal{C}_1 = \emptyset$. Thus $\mathcal{C}_1 \neq \emptyset$, which gives $|S| \geq 2$ by Lemma 10(1). However, we again get a contradiction to Equation (1) as $\bigcup_{k\geq 1} \mathcal{C}_{2k+1} \neq \emptyset$ by Equation (2).

Case 2: $|T| = 4$.

By Lemma 8 (3), we know that $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 2$. First assume $|S| \leq 1$. Then $\mathcal{C}_1 = \emptyset$ by Lemma 10 (1). By Lemma 8, there are at least $3|T| = 12$ edges going from T to vertices in S and components in $\bigcup_{k\geq 1} \mathcal{C}_{2k+1}$. As $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 2$, it follows that $|\mathcal{C}_3| \geq 4$ if $|S| = 0$ and $|\mathcal{C}_3| \geq 3$ if $|S| = 1$, contradicting Equation (1).

Next, assume $|S| \geq 2$. By Equations (1) and (2), we have $|S| = 2$. Let D be the single component in $\mathcal{C}_3$. Define W_D to be a set of 2 vertices in D which are all adjacent in G to some vertex from T. Then $S \cup W_D$ is a cutset in G such that $|S \cup W_D| = 4$ and $c(G - (S \cup W_D)) \geq |T| = 4$, contradicting

$\tau(G) \geq t > 1$. $\square$

By Claim 2 and Lemma 11 (1), we see that G contains an induced $R = P_4 \cup 3P_1$. Thus we may assume $R \notin \{P_4 \cup P_1, P_3 \cup 2P_1, P_2 \cup 3P_1, P_4 \cup 3P_1\}$ from this point on.

Claim 3. *$R \notin \{P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1, 7P_1\}$ unless G falls under the exceptional cases as in (a) and (b) of Theorem 13(2).*

Proof. We may assume that G does not fall under the exceptional cases as in (a) and (b) of Theorem 13(2). Thus we show that $R \notin \{P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1, 7P_1\}$.

Assume to the contrary that $R \in \{P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1, 7P_1\}$. By Lemma 11(1), G contains an induced $P_4 \cup aP_1$, where $a = |T| - 2$. If $a \geq 5$, then each of $P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1$, and $7P_1$ is an induced subgraph of $P_4 \cup aP_1$, a contradiction. Thus $a \leq 4$ and so $|T| \leq 6$. As $|T| \leq 6$, we have that $\bigcup_{k>2} \mathcal{C}_{2k+1} = \emptyset$ by Lemma 8 (3). Since G is more than 1-tough and so is 3-connected, we have $\delta(G) \geq 3$. By Claim 2, $|T| \geq 5$. Thus, we have two cases.

Case 1: $|T| = 5$.

As $|T| = 5$, we have $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 3$. We consider two cases regarding whether or not $|\mathcal{C}_3 \cup \mathcal{C}_5| \geq 2$.

Case 1.1: $|\mathcal{C}_3 \cup \mathcal{C}_5| = 1$.

Let $D \in \mathcal{C}_{2k+1} \subseteq \mathcal{C}_3 \cup \mathcal{C}_5$. By Equation (1), $5 \geq |S| + k + 1$, so $|S| \leq 4 - k$. If $k = 1$, let W_D be a set of $2k$ vertices (which exist by Lemma 8 (4)) from D which are adjacent in G to vertices from T. Then $S \cup W_D$ forms a cutset and we have

$$t \leq \frac{|S| + 2k}{5} \leq \frac{4 + k}{5} = \frac{5}{5} = 1,$$

contradicting $t > 1$. Thus we assume $k = 2$. We consider two subcases.

Case 1.1.1: $|V(D)| \geq 6$.

For $R = P_3 \cup 4P_1$, let $x \in V(D)$ such that $e_G(x, T) = 0$. Let P be a shortest path in D from x to a vertex, say x^* from $N_G(T) \cap V(D)$. Let $y^* \in T$ such that $e_G(x^*, y^*) = 1$. Then xPx^*y^* and $T \setminus \{y^*\}$ contain $P_3 \cup 4P_1$ as an induced subgraph. We consider next that $R = 6P_1$. Then T and the vertex of D that is not adjacent in G to any vertex from T for an induced $6P_1$, giving a contradiction. For $R = 7P_1$, let W_D be the set of $2k+1$ vertices (which exist by Lemma 8(4)) from D which are adjacent in G to vertices from T. Then $S \cup W_D$ forms a cutset and we have

$$t \leq \frac{|S| + 2k + 1}{|T| + 1} \leq \frac{4 + k + 1}{6} = \frac{7}{6},$$

giving a contradiction to $t > 7/6$.

Lastly, we consider $R = P_2 \cup 5P_1$. For any $x \in V(D)$ such that $e_G(x, T) = 0$, it must be the case that x is adjacent in G to every vertex from $N_G(T) \cap V(D)$. Otherwise, let $x^* \in N_G(T) \cap V(D)$

such that $xx^* \notin E(G)$. Let $y^* \in T$ such that $e_G(x^*, y^*) = 1$. Then x^*y^* and $(T \setminus \{y^*\}) \cup \{x\}$ contain $P_2 \cup 5P_1$ as an induced subgraph. Furthermore, if $|V(D)| - |N_G(T) \cap V(D)| \geq 2$, then $V(D) \setminus (N_G(T) \cap V(D))$ is an independent set in G. Otherwise, an edge with both endvertices from $V(D) \setminus (N_G(T) \cap V(D))$ together with T induces $P_2 \cup 5P_1$. Thus if $|V(D)| \geq 7$, let W_D be the set of $2k+1$ vertices (which exist by Lemma 8(4)) from D which are adjacent in G to vertices from T. Then $S \cup W_D$ forms a cutset and we have

$$t \leq \frac{|S|+5}{|T|+2} \leq \frac{7}{7},$$

giving a contradiction to $t > 1$. Thus $|V(D)| = 6$. Let $x \in V(D)$ be the vertex such that $e_G(x, T) = 0$. Then it must be the case that $D - x$ has at most two components. Otherwise, we have $t \leq \frac{|S \cup \{x\}|}{3} = 1$.

Assume first that $c(D-x) = 2$. Let D_1 and D_2 be the two components of $D-x$, and assume further that $|V(D_1)| \leq |V(D_2)|$. Then as $|V(D-x)| = 5$, we have two possibilities: either $|V(D_1)| = 1$ and $|V(D_2)| = 4$ or $|V(D_1)| = 2$ and $|V(D_2)| = 3$. Since $\delta(G) \geq 3$, if $|V(D_1)| = 1$, then the vertex from D_1 must be adjacent in G to at least one vertex from S. When $|V(D_2)| = 4$ and $D_2 \neq K_4$, then D_2 has a cutset W of size 2 such that $c(D_2 - W) = 2$. Then $S \cup W \cup \{x\}$ is a cutset of G such that $c(G - (S \cup W \cup \{x\})) = 5$, showing that $t \leq 1$. Thus $D_2 = K_4$. However, this shows that G contains H_6 as a spanning subgraph. When $|V(D_2)| = 3$ and $D_2 \neq K_3$, then D_2 has a cutvertex x^*. Then $S \cup \{x, x^*\}$ is a cutset of G such that $c(G - (S \cup \{x, x^*\})) = 4$, showing that $t \leq \frac{4}{4} = 1$. Thus $D_2 = K_3$; however, this shows that G contains H_7 as a spanning subgraph.

Assume then that $c(D-x) = 1$. Let $D^* = D - x$. If $\delta(D^*) \geq 3$, then D^* is Hamiltonian and so G contains H_{10} as a spanning subgraph. Thus we assume $\delta(D^*) \leq 2$.

Assume first that D^* has a cutvertex x^*. Then $c(D^* - x) = 2$: as if $c(D^* - x) \geq 3$, then $c(G - (S \cup \{x, x^*\})) \geq 4$, implying $t \leq 1$. Let D_1^* and D_2^* be the two components of $D^* - x^*$, and assume further that $|V(D_1^*)| \leq |V(D_2^*)|$. Then as $|V(D^* - x^*)| = 4$, we have two possibilities: either $|V(D_1^*)| = 1$ and $|V(D_2^*)| = 3$ or $|V(D_1^*)| = 2$ and $|V(D_2^*)| = 2$. Since $\delta(G) \geq 3$, if $|V(D_1^*)| = 1$, then the vertex from D_1^* must be adjacent in G to at least one vertex from S. When $|V(D_2^*)| = 3$ and $D_2^* \neq K_3$, then D_2^* has a cutvertex x^{**}. Then $S \cup \{x, x^*, x^{**}\}$ is a cutset of G such that $c(G - (S \cup \{x, x^*, x^{**}\})) = 5$, showing that $t \leq 1$. Thus $D_2^* = K_3$. The vertex x^* is a cutvertex of D^* and so is adjacent in D^* to a vertex of D_1^* and a vertex of D_2^*. However, this shows that G contains H_8 as a spanning subgraph. When $|V(D_2^*)| = 2$, as G does not contain H_8 or H_9 as a spanning subgraph, x^* is adjacent in G to exactly one vertex, say x_1^*, of D_1^* and to exactly one vertex, say x_2^*, of D_2^*. Then $S \cup \{x, x_1^*, x_2^*\}$ is a cutset of G whose removal produces 5 components, showing that $\tau(G) \leq 1$.

Assume then that D^* is 2-connected. As $\delta(D^*) \leq 2$, D^* has a minimum cutset W of size 2. If $c(D^* - W) = 3$, then we have $c(G - (S \cup W \cup \{x\})) = 5$, showing that $t \leq 1$. Thus $c(D^* - W) = 2$. Then by analyzing the connection in D^* between W and the two components of $D^* - W$, we see that D^* contains C_5 as a spanning subgraph, showing that G contains H_{10} as a spanning subgraph.

Case 1.1.2: $|V(D)| = 5$.

Since G does not contain H_5 as a spanning subgraph, we have $D \neq K_5$. As $D \neq K_5$, D has a cutset W_D of size at most 3 such that each component of $D - W_D$ is a single vertex. Then

$$t \leq \frac{|S| + |W_D|}{|T|} \leq \frac{4-2+3}{5} = 1,$$

a contradiction.

Case 1.2: $|\mathcal{C}_3 \cup \mathcal{C}_5| \geq 2$.

By Equation (1), we have

$$4 \geq |S| + \sum_{k \geq 1} k|\mathcal{C}_{2k+1}|.$$

So one of the following holds:

1. $S = \emptyset$ and either $|\mathcal{C}_5| \leq 2$, $|\mathcal{C}_5| \leq 1$ and $|\mathcal{C}_3| \leq 2$, or $|\mathcal{C}_3| \leq 4$. In this case, $\mathcal{C}_1 = \emptyset$ by Lemma 10(1). Thus by Lemma 8(3), we have $e_G(T, V(G) \setminus T) \leq 12 < 3|T| = 15$.

2. $|S| = 1$ and either $|\mathcal{C}_5| = 1$ and $|\mathcal{C}_3| = 1$ or $|\mathcal{C}_3| \leq 3$. In this case, again $\mathcal{C}_1 = \emptyset$ by Lemma 10(1). This implies there are a maximum of 14 edges incident to vertices in T, a contradiction.

3. $|S| = 2$ and $|\mathcal{C}_3| = 2$.

   Let $\mathcal{C}_3 = \{D_1, D_2\}$. Note that $|V(D_i)| \geq 3$ by Lemma 8(4) for each $i \in [1, 2]$. Since $|T| = 5$, there exists $y_0 \in T$ such that $e_G(y_0, D_i) = 1$ for each $i \in [1, 2]$. If $R = P_3 \cup 4P_1$, then T together with the two neighbors of y_0 from $V(D_1) \cup V(D_2)$ induce R. If $R = 6P_1$, then $T \setminus \{y_0\}$ together with the two neighbors of y_0 from $V(D_1) \cup V(D_2)$ gives an induced $6P_1$. If $R = 7P_1$, let $W_{D_i} \subseteq V(D_i) \setminus N_G(y_0)$ be the two vertices of D_i that are adjacent in G to vertices from T. Then $c(G - (S \cup W_{D_1} \cup W_{D_2} \cup \{y_0\})) = |T| - 1 + 2 = 6$. Thus $t \leq \frac{2+2+2+1}{6} = \frac{7}{6}$, contradicting $t > \frac{7}{6}$. Lastly, assume $R = P_2 \cup 5P_1$. If one of D_i has at least 4 vertices, say $|V(D_2)| \geq 4$, then let $x \in V(D_2)$ such that $e_G(x, T) = 0$, $x^* \in V(D_1)$ and $y^* \in T$ such that $e_G(x^*, y^*) = 1$. Then x^*y^* and $(T \setminus \{y^*\}) \cup \{x\}$ induce $P_2 \cup 5P_1$. Thus $|V(D_1)| = |V(D_2)| = 3$. If one of D_i, say $D_2 \neq K_3$, then D_2 has a cutvertex x. Let W be the set of any two vertices of D_1. Then $S \cup W \cup \{x\}$ is a cutset of G such that $c(G - (S \cup W \cup \{x\})) = 5$, showing that $t \leq \frac{5}{5} = 1$. Thus $D_1 = D_2 = K_3$. However, this shows that G contains H_{11} as a spanning subgraph.

Case 2: $|T| = 6$.

In this case, by Lemma 11(1), G has an induced $P_4 \cup 4P_1$, which contains each of $P_3 \cup 4P_1$, $P_2 \cup 5P_1$ and $6P_1$ as an induced subgraph. So we assume $R = 7P_1$ in this case and thus $t > \frac{7}{6}$.

Recall for $y \in T$, $h(y) = |\{D : \ D \in \bigcup_{k \geq 1} \mathcal{C}_{2k+1} \quad \text{and} \quad e_G(y, D) \geq 1\}|$. If there exists $y_0 \in T$ such that $h(y_0) \geq 2$, we let x_1, x_2 be the two neighbors of y_0 from the two corresponding components in $\bigcup_{k \geq 1} \mathcal{C}_{2k+1}$, respectively. Then $T \setminus \{y_0\}$ together with $\{x_1, x_2\}$ induces $7P_1$. Thus $h(y) \leq 1$ for each $y \in T$. This, together with $|T| = 6$, implies that we have either $|\mathcal{C}_3| \in \{1, 2\}$ and $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 2$ or $|\mathcal{C}_5| = 1$ and $\mathcal{C}_{2k+1} = \emptyset$ for any $1 \leq k \neq 2$.

If $|\mathcal{C}_3| = 1$ and $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 2$, then $|S| \leq 4$ by Equation (1). Let W be a set of two vertices from the component in $\mathcal{C}_3$ that are adjacent in G to vertices from T. Then $c(G - (S \cup W)) \geq 6$, indicating that $t \leq \frac{4+2}{6} < \frac{7}{6}$. For the other two cases, we have $|S| \leq 3$. If $|\mathcal{C}_3| = 2$ and $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 2$, let W be a set of four vertices, with two from one component in $\mathcal{C}_3$ and the other two from the other component in $\mathcal{C}_3$, which are adjacent in G to vertices from T. If $|\mathcal{C}_5| = 1$ and $\mathcal{C}_{2k+1} = \emptyset$ for any $1 \leq k \leq 2$, let W be a set of four vertices from the component in $\mathcal{C}_5$ that are adjacent in G to vertices from T. Then we have $c(G - (S \cup W)) \geq 6$, indicating that $t \leq \frac{3+4}{6} = \frac{7}{6}$. □

By Claim 3, we now assume that $R \in \{P_7 \cup 2P_1, P_5 \cup P_2, P_4 \cup P_3, P_3 \cup 2P_2, 3P_2 \cup P_1\}$ and $t = 3/2$.

Claim 4. *There exists $y \in T$ with $h(y) > 2$.*

Proof. Assume to the contrary that for every $y \in T$, we have $h(y) \leq 1$. Define the following partition of T:

$$\begin{aligned} T_0 &= \{y \in T : e_G(y, D) = 0 \text{ for all } D \in \bigcup_{k \geq 1} \mathcal{C}_{2k+1}\}, \\ T_1 &= \{y \in T : e_G(y, D) = 1 \text{ for some } D \in \bigcup_{k \geq 1} \mathcal{C}_{2k+1}\}. \end{aligned}$$

Note that $|T_1| = \sum_{k \geq 1} (2k+1)|\mathcal{C}_{2k+1}|$ by Lemma 8(3) and (4). For each $D \in \mathcal{C}_{2k+1}$ for some $k \geq 1$, we let W_D be a set of $2k$ vertices that each has in G a neighbor from T. As each $D - W_D$ is connected to exactly one vertex from T and each component from $\mathcal{C}_1$ is connected to exactly one vertex from T, it follows that

$$W = S \cup \bigcup_{D \in \bigcup_{k \geq 1} \mathcal{C}_{2k+1}} W_D$$

satisfies $c(G - W) \geq |T| \geq 5$, where $|T| \geq 5$ is by Claim 2.

By the toughness of G, we have

$$\begin{aligned} |S| + \sum_{k \geq 1} 2k|\mathcal{C}_{2k+1}| &= |W| \geq t|T| = t(|T_0| + |T_1|) \\ &= t\left(|T_0| + \sum_{k \geq 1} (2k+1)|\mathcal{C}_{2k+1}|\right). \end{aligned} \tag{3}$$

Since $t = 3/2$, the inequality above implies that $|S| \geq 3|T_0|/2 + \sum_{k \geq 1} (k + 3/2)|\mathcal{C}_{2k+1}|$. Thus

$$|S| + \sum_{k \geq 1} k|\mathcal{C}_{2k+1}| \geq 3|T_0|/2 + \sum_{k \geq 1} (2k + 3/2)|\mathcal{C}_{2k+1}| > |T_0| + \sum_{k \geq 1} (2k+1)|\mathcal{C}_{2k+1}| = |T|,$$

contradicting Equation (1). □

By Claim 4, there exists $y \in T$ such that $h(y) \geq 2$. Then as $|T| \geq 5$, by Lemma 11(2), G contains an induced $P_7 \cup 2P_1$. Thus we assume that $R \neq P_7 \cup 2P_1$. We assume first that $|\bigcup_{k \geq 1} \mathcal{C}_{2k+1}| \geq 3$ and let D_1, D_2, D_3 be three distinct odd components from $\bigcup_{k \geq 1} \mathcal{C}_{2k+1}$. Let $y_0 \in T$ such that $h(y_0) \geq 2$.

We assume, without loss of generality, that $e_G(y_0, D_1) = e_G(y_0, D_2) = 1$. By Lemma 11(2), G contains an induced $P_b \cup aP_1$, where $b \geq 7$ and $a = |T| - 3$, and the graph $P_b \cup aP_1$ can be chosen such that the vertices in aP_1 are from T and the path P_b has the form $y_1x_1^*P_1x_1y_0x_2P_2x_2^*y_2$, where $y_0, y_1, y_2 \in T$ and $x_1^*P_1x_1$ and $x_2^*P_2x_2$ are respectively contained in D_1 and D_2 such that $e_G(x, T) = 0$ for every internal vertex x from P_1 and P_2. If one of y_1 and y_2, say y_1 has a neighbor z_1 from $V(D_3)$, then $z_1y_1x_1^*P_1x_1y_0x_2P_2x_2^*y_2$ and $T \setminus \{y_0, y_1, y_2\}$ induce $P_8 \cup 2P_1$, which contains each of $P_5 \cup P_2$, $P_4 \cup P_3$, and $3P_2 \cup P_1$ as an induced subgraph. Let $z_2 \in V(D_3)$ be a neighbor of z_1. Then $z_2z_1y_1x_1^*P_1x_1y_0x_2P_2x_2^*y_2$ contains an induced $P_3 \cup 2P_2$ whether $e_G(z_2, \{y_0, y_2\}) = 0$ or 1. Thus we assume $e_G(y_i, D_3) = 0$ for each $i \in [1, 2]$ and so we can find $y_3 \in T \setminus \{y_0, y_1, y_2\}$ and $z \in V(D_3)$ such that $y_3z \in E(G)$. Then $y_1x_1^*P_1x_1y_0x_2P_2x_2^*y_2$ and zy_3 contains an induced $P_7 \cup P_2$, which contains each of $P_5 \cup P_2$, $P_3 \cup 2P_2$ and $3P_2 \cup P_1$ as an induced subgraph. We are only left to consider $R = P_4 \cup P_3$. As $e_G(y_i, D_3) = 0$ for each $i \in [1, 2]$, we can find distinct $y_3, y_4 \in T \setminus \{y_0, y_1, y_2\}$ and distinct $z_1, z_2 \in V(D_3)$ such that $y_3z_1, y_4z_2 \in E(G)$. We let P be a shortest path in D_3 connecting z_1 and z_2. If $e_G(y_0, V(P)) = 0$, then $y_3z_1Pz_2y_4$ and $y_1x_1^*P_1x_1y_0x_2P_2x_2^*y_2$ contains an induced $P_4 \cup P_3$. Thus $e_G(y_0, V(P)) = 1$. This in particular, implies that $|V(P)| \geq 3$. Then $y_3z_1Pz_2y_4$ and $y_1x_1^*P_1x_1$ together contain an induced $P_4 \cup P_3$.

Thus $|\cup_{k\geq 1} \mathcal{C}_{2k+1}| = 2$. Let $D_1, D_2 \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$ be the two components. Define the following partition of T:

$$\begin{aligned} T_0 &= \{y \in T : e_G(y, D_1) = e_G(y, D_2) = 0\}, \\ T_{11} &= \{y \in T : e_G(y, D_1) = 1 \text{ and } e_G(y, D_2) = 0\}, \\ T_{12} &= \{y \in T : e_G(y, D_1) = 0 \text{ and } e_G(y, D_2) = 1\}, \\ T_2 &= \{y \in T : e_G(y, D_1) = e_G(y, D_2) = 1\}. \end{aligned}$$

We have either $T_2 = \emptyset$ or $T_2 \neq \emptyset$. First suppose $T_2 = \emptyset$. Define the following vertex sets:

$$W_1 = N_G(T_{11}) \cap V(D_1) \quad \text{and} \quad W_2 = N_G(T_{12}) \cap V(D_2).$$

Then $|W_1| = |T_{11}| = 2k_1 + 1$ and $|W_2| = |T_{12}| = 2k_2 + 1$, where we assume $e_G(T, D_1) = 2k_1 + 1$ and $e_G(T, D_2) = 2k_2 + 1$ for some integers k_1 and k_2. Then $W = S \cup W_1 \cup W_2$ is a cutset of G with $c(G - W) \geq |T|$. By toughness, $|W| \geq \frac{3}{2}|T| = |T| + \frac{1}{2}|T|$. Since $|T| = |T_0| + |T_{11}| + |T_{12}|$, this gives us

$$\begin{aligned} |W| &\geq |T| + \frac{1}{2}|T_0| + \frac{1}{2}(|T_{11}| + |T_{12}|) \\ &= |T| + \frac{1}{2}|T_0| + \frac{1}{2}(2k_1 + 1 + 2k_2 + 1) \\ &= |T| + \frac{1}{2}|T_0| + k_1 + k_2 + 1. \end{aligned}$$

Thus $|W| = |S| + |W_1| + |W_2| = |S| + 2k_1 + 2k_2 + 2 \geq |T| + \frac{1}{2}|T_0| + k_1 + k_2 + 1$, which implies $|S| + k_1 + k_2 + 1 \geq |T| + \frac{1}{2}|T_0|$. Hence, by Equation (1), we have $|T| \geq |T| + \frac{1}{2}|T_0|$, giving a contradiction.

So we may assume $T_2 \neq \emptyset$. Now define the following vertex sets:

$$W_1 = N_G(T_{11}) \cap V(D_1), \quad W_2 = N_G(T_{12}) \cap V(D_2), \quad \text{and} \quad W_3 = N(T_2) \cap (V(D_1) \cup V(D_2)).$$

We have that $|W_1| = |T_{11}|$, $|W_2| = |T_{12}|$, and $|W_3| = 2|T_2|$. Now let $W = S \cup W_1 \cup W_2 \cup W_3$. Then W is a cutset of G with $c(G - W) \geq |T_0| + |T_{11}| + |T_{12}| + 1$ since $T_2 \neq \emptyset$. By toughness, $|W| \geq \frac{3}{2}(|T_0| + |T_{11}| + |T_{12}| + 1)$. Since $|W| = |S| + |W_1| + |W_2| + |W_3| = |S| + |T_{11}| + |T_{12}| + 2|T_2|$, we have $|S| + |T_{11}| + |T_{12}| + 2|T_2| \geq \frac{3}{2}|T_0| + \frac{3}{2}|T_{11}| + \frac{3}{2}|T_{12}| + \frac{3}{2}$. This implies

$$|S| \geq \frac{3}{2}|T_0| + \frac{1}{2}|T_{11}| + \frac{1}{2}|T_{12}| + 1.$$

Thus,

$$|S| + k_1 + k_2 \geq \frac{3}{2}|T_0| + \frac{1}{2}|T_{11}| + \frac{1}{2}|T_{12}| + 1 + k_1 + k_2. \tag{4}$$

We have that either $T_{11} \cup T_{12} \cup T_0 = \emptyset$ or $T_{11} \cup T_{12} \cup T_0 \neq \emptyset$. First suppose $T_{11} \cup T_{12} \cup T_0 = \emptyset$. Then $|T| = |T_2| = \frac{1}{2}(2k_1 + 1 + 2k_2 + 1) = k_1 + k_2 + 1$. Thus $|S| + k_1 + k_2 \geq |T|$, showing a contradiction to Equation (1).

So we may assume $T_{11} \cup T_{12} \cup T_0 \neq \emptyset$. Then

$$\begin{aligned}
|T| &= |T_0| + (2k_1 + 1 + 2k_2 + 1 - |T_2|) \\
&= |T_0| + (2k_1 + 2k_2 + 2) - \frac{1}{2}(2k_1 + 1 + 2k_2 + 1 - |T_{11}| - |T_{12}|) \\
&= |T_0| + \frac{1}{2}(2k_1 + 2k_2 + 2) + \frac{1}{2}|T_{11}| + \frac{1}{2}|T_{12}| \\
&= |T_0| + k_1 + k_2 + 1 + \frac{1}{2}|T_{11}| + \frac{1}{2}|T_{12}|.
\end{aligned}$$

Using the size of T and (4), we get $|S| + k_1 + k_2 \geq |T|$, showing a contradiction to Equation (1).

The proof of Theorem 13 is now finished. □

4 Proof of Theorems 5 and 6

Recall that for a graph G, $\alpha(G)$, the independence number of G, is the size of a largest independent set in G.

Proof of Theorem 5. For each $i \in [0, 11]$, H_i does not contain a 2-factor by Theorem 7. Thus to finish proving Theorem 13, we are only left to show the three claims below.

Claim 5. *The graph H_i is $(P_2 \cup 3P_1)$-free, H_1 is $(P_3 \cup 2P_1)$-free, and $\tau(H_i) = 1$ for each $i \in [0, 4]$.*

Proof. We first show that H_i is $(P_2 \cup 3P_1)$-free for each $i \in [0, 4]$. We only show this for H_0, as the proofs for H_i for $i \in [1, 4]$ are similar. In H_0, there are two types of edges xy: $x, y \in V(D_j)$ or $x \in V(D_j)$ and $y \in V(T)$, where $j \in [1, 2]$. Without loss of generality first consider the edge $v_1v_2 \in E(D_1)$ and the subgraph $F_1 = H_0 - (N_{H_0}[v_1] \cup N_{H_0}[v_2])$. We see $\alpha(F_1) = 2$. Now, without loss of generality, consider the edge v_1t_1 and the subgraph $F_2 = H_0 - (N_{H_0}[v_1] \cup N_{H_0}[t_1])$. We see $\alpha(F_2) = 2$. In either case, $P_2 \cup 3P_1$ cannot exist as an induced subgraph in H_0. Thus H_0 is $(P_2 \cup 3P_1)$-free.

Then we show that H_1 is $(P_3 \cup 2P_1)$-free. Two types of induced paths abc of length 3 exist: $a \in S, b \in T, c \in V(D)$ or $a \in T, b, c \in V(D)$. Without loss of generality, consider the path xt_1v_1 and the subgraph $F_1 = H_1 - (N_{H_1}[x] \cup N_{H_1}[t_1] \cup N_{H_1}[v_1])$. We see that F_1 is a null graph. Now, without loss of generality, consider the path $t_1v_1v_2$ and the subgraph $F_2 = H_1 - (N_{H_1}[t_1] \cup N_{H_1}[v_1] \cup N_{H_1}[v_2])$. We see $|V(F_2)| = 1$. In either case, $P_3 \cup 2P_1$ cannot exist as an induced subgraph in H_1. Thus H_1 is $(P_3 \cup 2P_1)$-free.

Let $i \in [0, 4]$. As $\delta(H_i) = 2$, $\tau(H_i) \leq 1$. It suffices to show $\tau(H_i) \geq 1$. Since H_i is 2-connected, we show that $c(H_i - W) \leq |W|$ for any $W \subseteq V(H_i)$ such that $|W| \geq 2$. If $|W| = 2$, by considering all the possible formations of W, we have $c(H_i - W) \leq |W|$. Thus we assume $|W| \geq 3$.

Assume by contradiction that there exists $W \subseteq V(H_i)$ with $|W| \geq 3$ and $c(H_i - W) \geq |W| + 1 \geq 4$. The size of a largest independent set of each H_0, H_2, H_3, and H_4 is 4, and of H_1 is 3. Since $c(H_i - W)$ is bounded above by the size of a largest independent set of H_i, we already obtain a contradiction if $i = 1$ or $|W| \geq 4$. So we assume $i \in \{0, 2, 3, 4\}$ and $|W| = 3$.

As $c(H_i - W) \geq 4$, for the graph H_0, we must have $\{v_1, v_2, v_3\} \cap W \neq \emptyset$ and $\{v_4, v_5, v_6\} \cap W \neq \emptyset$. As $|W| = 3$, we have either $W \cap T = \emptyset$ or $|W \cap T| = 1$. In either case, by checking all the possible formations of W, we get $c(H_0 - W) \leq 2$, contradicting the choice of W.

As $c(H_i - W) \geq 4$, for each $i \in [2, 4]$, we must have $x \in W$. Thus $t_j \notin W$ for $j \in [1, 3]$, as otherwise, $c(H_i - (W \setminus \{t_j\})) \geq 4$, contradicting the argument previously that $c(H_i - W^*) \leq 2$ for any $W^* \subseteq V(H_i)$ and $|W^*| \leq 2$. As $|W| = 3$, we then have $|W \cap \{v_1, v_2, v_3, v_4\}| = 2$. However, $c(H_i - W) \leq 3$ for $W = \{x, v_k, v_\ell\}$ for all distinct $k, \ell \in [1, 4]$. We again get a contradiction to the choice of W. $\square$

Claim 6. *The graph H_5 with $p = 5$ is $(P_3 \cup 4P_1)$-free, $(P_2 \cup 5P_1)$-free, and $6P_1$-free with $\tau(H_5) = \frac{6}{5}$.*

Proof. Let $p = 5$ and D be the odd component of $H_5 - (S \cup T)$. Note that $D = K_p = K_5$.

We first show that H_5 is $(P_3 \cup 4P_1)$-free. There are three types of induced paths xyz of length 3 in $H_5 : x \in S, y \in T, z \in V(D)$ or $x \in T, y, z \in V(D)$ or $x, z \in T, y \in S$. Without loss of generality, consider the path $x_1t_1y_1$ and the subgraph $F_1 = H_5 - (N_{H_5}[x_1] \cup N_{H_5}[t_1] \cup N_{H_5}[y_1])$. We see that F_1 is a null graph. Now consider the path $t_1y_1y_2$ and the subgraph $F_2 = H_5 - (N_{H_5}[t_1] \cup N_{H_5}[y_1] \cup N_{H_5}[y_2])$. We see $\alpha(F_2) = 3$. Finally consider the path $t_1x_1t_2$ and the subgraph $F_3 = H_5 - (N_{H_5}[t_1] \cup N_{H_5}[x_1] \cup N_{H_5}[t_2])$. We see $\alpha(F_3) = 3$. In any case, an induced copy of $P_3 \cup 4P_1$ cannot exist in H_5. Thus H_5 is $(P_3 \cup 4P_1)$-free.

We then show that H_5 is $(P_2 \cup 5P_1)$-free. There are three types of edges xy in $H_5 : x \in S, y \in T$ or $x \in T, y \in V(D)$ or $x, y \in V(D)$. Without loss of generality, consider the edge x_1t_1 and the subgraph $F_1 = H_5 - (N_{H_5}[x_1] \cup N_{H_5}[t_1])$. We see $|V(F_1)| = 4$. Now consider the edge t_1y_1 and the subgraph $F_2 = H_5 - (N_{H_5}[t_1] \cup N_{H_5}[y_1])$. We see $|V(F_2)| = 4$. Finally, consider the edge y_1y_2 and the subgraph $F_3 = H_5 - (N_GH5[y_1] \cup N_{H_5}[y_2])$. We see $\alpha(F_3) = 3$. In any case, no induced copy of $P_2 \cup 5P_1$ can exist in H_5. Thus H_5 is $(P_2 \cup 5P_1)$-free.

We lastly show that H_5 is $6P_1$-free. There are three types of vertices x in $H_5 : x \in S, x \in T$, or

$x \in V(D)$. Without loss of generality, consider the vertex x_1 and the subgraph $F_1 = H_5 - N_{H_5}[x_1]$. We see $\alpha(F_1) = 1$. Now consider the vertex t_1 and the subgraph $F_2 = H_5 - N_{H_5}[t_1]$. We see $\alpha(F_2) = 4$. Finally, consider the vertex y_1 and the subgraph $F_3 = H_5 - N_{H_5}[y_1]$. We see $\alpha(F_3) = 4$. In any case, no induced copy of $6P_1$ can exist in H_5. Thus H_5 is $6P_1$-free.

We now show that $\tau(H_5) = \frac{6}{5}$. Let W be a toughset of H_5. Then $S \subseteq W$. Otherwise, by the structure of H_5, we have $c(H_5 - W) \leq 3$ and $|W| \geq 5$. As $S \subseteq W$ and the only neighbor of each vertex of T in $H_5 - S$ is contained in a clique of H_5, we have $T \cap W = \emptyset$. Since $c(H_5 - W) \geq 2$, it follows that $W \cap V(D) \neq \emptyset$. Then $c(H_5 - W) = |W \cap V(D)|$ if $|W \cap V(D)| \leq 3$ or $|W \cap V(D)| = 5$, and $c(H_5 - W) = |W \cap V(D)| + 1$ if $|W \cap V(D)| = 4$. The smallest ratio of $\frac{|W|}{c(H_5 - W)}$ is $\frac{6}{5}$, which happens when $|W \cap V(D)| = 4$. $\square$

Claim 7. *The graph H_i is $(P_2 \cup 5P_1)$-free with $\tau(H_i) = \frac{7}{6}$ for each $i \in [6, 11]$.*

Proof. We show first that each H_i is $(P_2 \cup 5P_1)$-free. We do this only for the graph H_6, as the proofs for the rest graphs are similar. For any edge $ab \in E(H_6)$, we see $\alpha(H_6 - (N_{H_6}[a] \cup N_{H_6}[b])) \leq 4$. Thus no induced copy of $(P_2 \cup 5P_1)$ can exist in H_6. Thus H_6 is $(P_2 \cup 5P_1)$-free.

We next show that $\tau(H_i) = \frac{7}{6}$ for each $i \in [6, 10]$. We have $c(H_i - (S \cup \{v_1, \dots, v_5\})) = 6$, implying $\tau(H_i) \leq \frac{7}{6}$. Suppose $\tau(H_i) < \frac{7}{6}$. Let W be a toughset of H_i. As each H_i is 3-connected, we have $|W| \geq 3$. Thus $c(H_i - W) \geq 3$. We have that either $S \subseteq W$ or $S \not\subseteq W$. Suppose the latter. Then we have $S \cap V(H_i - W) \neq \emptyset$. Then all vertices in $T \setminus W$ are contained in the same component as the one which contains $S \setminus W$. Since $c(H_i - W) \geq 3$, by the structure of H_i, it follows that we have either $T \subseteq W$ or $\{v_1, \dots, v_5\} \subseteq W$. In either case, we have $c(H_i - W) \leq 3$, implying $\frac{|W|}{c(H_i - W)} \geq \frac{5}{3} > \frac{7}{6}$, a contradiction. So $S \subseteq W$. By Lemma 12, $t_j \notin W$ for all $j \in [1, 5]$. Thus each $t_j \in V(H_i - W)$. Now either $v_0 \in W$ or $v_0 \notin W$. Suppose $v_0 \in W$, then we cannot have all $v_j \in W$ without violating Lemma 12. In this case, the minimum ratio $\frac{|W|}{c(H_i - W)}$ occurs when $|W \cap \{v_1, v_2, v_3, v_4, v_5\}| = 3$. This implies $\frac{|W|}{c(H_i - W} \geq \frac{6}{5} > \frac{7}{6}$, a contradiction. Thus $v_0 \notin W$ and we must have $v_0 \in V(H_i - W)$. This implies $\{v_1 \dots v_5\} \subseteq W$ and $\frac{|W|}{c(H_i - W)} = \frac{7}{6}$, a contradiction. Thus $\tau(H_i) = \frac{7}{6}$ for each $i \in [6, 10]$.

Lastly we show $\tau(H_{11}) = \frac{7}{6}$. We have $c(H_{11} - (S \cup \{v_1, v_2, t_3, v_4, v_5\})) = 6$, implying $\tau(H_{11}) \leq \frac{7}{6}$. Suppose $\tau(H_{11}) < \frac{7}{6}$. Let W be a tough set of H_{11}. As H_{11} is 3-connected, we have $|W| \geq 3$. Thus $c(H_{11} - W) \geq 3$. We have that either $S \subseteq W$ or $S \not\subseteq W$. Suppose the latter. Then we have $S \cap V(H_{11} - W) \neq \emptyset$. Then all vertices in $T \setminus W$ are contained in the same component as the one which contains $S \setminus W$. Since $c(H_{11} - W) \geq 3$, by the structure of H_{11}, it follows that $|W| \geq 5$ and $c(H_{11} - W) \leq 4$. This implies $\frac{|W|}{c(H_{11} - W)} \geq \frac{5}{4} > \frac{7}{6}$, a contradiction. So $S \subseteq W$. By Lemma 12, $t_i \notin W$ for $i \in \{1, 2, 4, 5\}$. Thus $t_i \in V(H_{11} - W)$ for $i \in \{1, 2, 4, 5\}$ and we must have $W \cap \{v_1, v_2, v_3, v_4, v_5, v_6, t_3\} \neq \emptyset$. If $t_3 \notin W$, then $\frac{|W|}{c(H_{11 - W})} \geq \frac{6}{5} > \frac{7}{6}$, a contradiction. Thus $t_3 \in W$. Then v_3 and v_4 are respectively in two distinct components of $H_{11} - W$ by Lemma 12. Thus $W \cap \{v_1, v_2, v_5, v_6\} \neq \emptyset$ as $c(H_{11} - W) \geq 3$. Furthermore, we have $c(H_{11} - W) = |W \cap \{v_1, v_2, v_5, v_6\}| + 2$. The smallest ratio of $\frac{|W|}{c(H_{11} - W)}$ is $\frac{7}{6}$, which happens when $\{v_1, v_2, v_5, v_6\} \subseteq W$. Again we get a contradiction to the assumption that $\tau(H_{11}) < \frac{7}{6}$. Thus $\tau(H_{11}) = \frac{7}{6}$. $\square$

The proof of Theorem 13 is complete. $\square$

Proof of Theorem 6. Let $p \geq 6$ and D be the odd component of $H_5 - (S \cup T)$. Note that $D = K_p$. Since $c(H_5 - (S \cup \{y_1, \ldots, y_5\})) = 6$, we have $\tau(H_5) \leq \frac{7}{6}$. We show $\tau(H_5) \geq \frac{7}{6}$. Let W be a toughset of H_5. Then either $S \subseteq W$ or $S \not\subseteq W$. Suppose the latter. Then we have $S \cap V(H_5 - W) \neq \emptyset$. Then all vertices in $T \setminus W$ are contained in the same component as the one containing $S \setminus W$. Since $c(H_5 - W) \geq 2$, by the structure of H_5, it follows that we have either $T \subseteq W$ or $\{y_1, \ldots, y_5\} \subseteq W$. In either case, we have $c(H_5 - W) \leq 3$, implying $\frac{|W|}{c(H_5 - W)} \geq \frac{5}{3} > \frac{7}{6}$. Now suppose $S \subseteq W$. By Lemma 12, $t_i \notin W$ for all i. Thus each $t_i \in V(H_5 - W)$. Furthermore, $c(H_5 - W) = |W \cap V(D)| + 1$. Since W is a cutset of G, we have $|W \cap V(D)| \geq 2$. The smallest ratio of $\frac{|W|}{c(H_5 - W)}$ is $\frac{7}{6}$, which happens when $|W \cap V(D)| = 5$.

For the graph H_{12}, we have $c(H_{12} - (S \cup \{y_1, y_2, y_3\})) = 4$, implying $\tau(H_{12}) \leq \frac{4}{4} = 1$. We show $\tau(H_{12}) \geq 1$. Let W be a toughset of H_{12}. Then either $S \subseteq W$ or $S \not\subseteq W$. Suppose the latter. Then we have $S \cap V(H_{12} - W) \neq \emptyset$. Then all vertices in $T \setminus W$ are contained in the same component as the one containing $S \setminus W$. Since $c(H_{12} - W) \geq 2$, by the structure of H_{12}, it follows that we have either $T \subseteq W$ or $\{y_1, y_2, y_3\} \subseteq W$. In either case, we have $c(H_{12} - W) \leq 2$, implying $\frac{|W|}{c(H_{12} - W)} \geq \frac{3}{2} > 1$. Now suppose $S \subseteq W$. By Lemma 12, $t_i \notin W$ for all i. Thus each $t_i \in V(H_{12} - W)$. This implies $|\{y_1, y_2, y_3\} \cap W| = 2$ or 3. In either case we see $\frac{|W|}{c(H_{12} - W)} = 1$. $\square$

References

[1] D. Bauer, G.Y. Katona, D. Kratsch, and H.J. Veldman. Chordality and 2-factors in tough graphs. *Discrete Applied Mathematics*, 99:323–329, 2000.

[2] V. Chvátal. Tough graphs and Hamiltonian circuits. *Discrete Math.*, 5:87–95, 1973.

[3] H. Enomoto, B. Jackson, P. Katerinis, and A. Saito. Toughness and the existence of k-factors. III. *Discrete Mathematics*, 189(1):277–282, 1998.

[4] J. Kanno and S. Shan. Vizing's 2-factor conjecture involving toughness and maximum degree conditions. *Electron. J. Comb.*, 26(2):P2.17, 2019.

[5] K. Ota and M. Sanka. Toughness, 2-factors and hamiltonian cycles in $2k_2$-free graphs. 2021.

[6] W.T. Tutte. A short proof of the factor theorem for finite graphs. *Canadian J. Math.*, 6:347–352, 1954.