# ON THE LAPLACIAN SPECTRUM OF *k*-SYMMETRIC GRAPHS

SUNYO MOON AND HYUNGKEE YOO

ABSTRACT. For some positive integer k, if the finite cyclic group  $\mathbb{Z}_k$  can act freely on a graph G, then we say that G is k-symmetric. In 1985, Faria showed that the multiplicity of Laplacian eigenvalue 1 is greater than or equal to the difference between the number of pendant vertices and the number of quasi-pendant vertices. But if a graph has a pendant vertex, then it is at most 1-connected. In this paper, we investigate a class of 2-connected k-symmetric graphs with a Laplacian eigenvalue 1. We also identify a class of k-symmetric graphs in which all Laplacian eigenvalues are integers.

# 1. INTRODUCTION

A simple graph G = (V, E) is a combinatorial object consisting of a finite set V and a set E of unordered pairs of different elements of V. The elements of V and E are called the *vertices* and the *edges* of the graph G, respectively. For a given graph G, the vertex set and the edge set of G are denoted by V(G) and E(G), respectively.

Let G be a graph with enumerated vertices. The Laplacian matrix L(G) of G is defined as L(G) = D(G) - A(G), where D(G) is the diagonal matrix of vertex degrees and A(G) is the adjacency matrix of G. Thus the Laplacian matrix is symmetric. Note that the Laplacian matrix can be considered a positive-semidefinite quadratic form on the Hilbert space generated by V(G). Since the Laplacian matrix contains information on the structure of the graph, it has been studied importantly in various applied fields including artificial neural network research using graph shaped data [13, 14].

Let G be a graph with n vertices. For a square matrix M, we denote the characteristic polynomial of M by  $\mu(M, x)$ . A root of the characteristic polynomial of Laplacian matrix L(G) is called a *Laplacian eigenvalue* of G. Denote the all eigenvalues of L(G) by  $\lambda_n(G) \leq \lambda_{n-1}(G) \leq \cdots \leq \lambda_1(G)$ . It is well-known that  $\lambda_n(G) = 0$  and  $\lambda_1(G) \leq n$ . The multiset of Laplacian eigenvalues of G is called the *Laplacian spectrum* of G. The Laplacian spectrum of the complement graph  $\overline{G}$  of G is satisfying

$$0 = \lambda_n(\overline{G}) \le n - \lambda_1(G) \le \dots \le n - \lambda_{n-1}(G).$$

The Laplacian spectrum shows us several properties of the graph. For instance, Kirchhoff [15] proved that the number of spanning tree of a connected

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graph G with n vertices is  $\frac{1}{n}\lambda_1(G)\cdots\lambda_{n-1}(G)$ . Let  $m_G(\lambda)$  denote the multiplicity of  $\lambda$  as a Laplacian eigenvalue of G. Note that the multiplicity of 0 is equal to the number of connected components of G.

The connectivity  $\kappa(G)$  of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. A graph G is said to be *t*-connected if  $\kappa(G) \ge t$ . If a graph is *t*-connected, then it is (t-1)connected. Fiedler [9] proved that the second smallest Laplacian eigenvalue of G is less than or equal to  $\kappa(G)$ .

A pendant vertex of G is a vertex of degree 1. A quasi-pendant of G is a vertex adjacent to a pendant. We denote the number of pendants of G by p(G), and the number of quasi-pendant vertices by q(G). In [8], Faria showed that for any graph G,

$$m_G(1) \ge p(G) - q(G).$$

It implies that if p(G) is greater than q(G), then G has a Laplacian eigenvalue 1. Also, such graph G is at most 1-connected. In [1], Barik et al. found trees with a Laplacian eigenvalue 1 even though the right-hand side of the above inequality is 0. Since a tree has connectivity 1, we focus on 2-connected graph with a Laplacian eigenvalue 1.

The simplest way to obtain a 2-connected graph with a Laplacian eigenvalue 1 is the Cartesian product. The Cartesian product  $G \square H$  of graphs G and H is the graph with the vertex set  $V(G) \times V(H)$  such that two vertices (v, v') and (w, w') are adjacent if v = w and v' is adjacent to w' in H, or if v' = w' and v is adjacent to w in G. Fiedler [9] showed that the Laplacian eigenvalues of the Cartesian product  $G \square H$  are all possible sums of Laplacian eigenvalues of G and H. If either G or H has a Laplacian eigenvalue 1, then 1 is a Laplacian eigenvalue of  $G \square H$ . Špacapan [20] showed that the connectivity of  $G \square H$  is

$$\kappa(G \Box H) = \min\{\kappa(G)|H|, \kappa(H)|G|, \delta(G \Box H)\},\$$

where  $\delta(G \Box H)$  is the minimum degree of  $G \Box H$ . Remark that if G and H are connected graphs, then  $G \Box H$  is 2-connected. Thus we concentrate a 2-connected graph that does not decompose nontrivial graphs under the Cartesian product. If a graph does not admit the nontrivial Cartesian product decomposition, then the graph is called *prime* with respect to the Cartesian product. In this paper, we prove the following theorem.

**Theorem 1.1.** For any positive integer n, there is a 2-connected prime graph G with respect to the Cartesian product,

$$m_G(1) \ge n.$$

Meanwhile, integral spectra of Laplacian matrix or adjacency matrix are studied in various application fields including physics and chemistry [3, 4, 5]. A graph with integral Laplacian spectrum is called a *Laplacian integral* graph. If a graph does not include the path  $P_4$  as an induced subgraph, then it is called a *cograph*. In [19], Merris showed that a cograph is a Laplacian integral graph. Many researchers [7, 10, 16, 17] have investigated infinitely many classes of Laplacian integral graphs that are not cographs. In section 5, we introduce a new graph C(n, m) for some positive integers n and m. The graph C(n, m) is obtained by connecting several set of vertices for m parallel copies of *n*-complete graph  $K_n$  to the corresponding vertex of  $\overline{K}_n$ . Later, we examine that if  $n \ge 2$ , then  $\mathcal{C}(n, m)$  has the path  $P_4$  as the induced subgraph, that is, it is not a cograph. In this paper, we also prove the following theorem.

**Theorem 1.2.** There are infinitely many pairs of positive integers n and m, which make C(n,m) a Laplacian integral graph.

This paper is organized as follows. In Section 2, We provide some linear algebra results needed for proof of main theorems. In Section 3, We define the k-symmetric graph by relaxing the condition of symmetric graph, and examine its properties. In Section 4 and Section 5, we prove the main theorems and related properties.

## 2. Preliminaries

In this section, we introduce some definitions and properties that will be used in this paper. The set of all  $m \times n$  matrices over a field  $\mathbb{F}$  is denoted by  $M_{m \times n}(\mathbb{F})$ . Denote  $M_{n \times n}(\mathbb{F})$  by  $M_n(\mathbb{F})$ . We denote by  $I_n$  and  $J_n$  the  $n \times n$ identity matrix and the  $n \times n$  matrix whose entries are ones. Also,  $1_n$  is the *n*-vector of all ones.

Let  $A \in M_n(\mathbb{F})$  be a block matrix of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where  $A_{11} \in M_m(\mathbb{F})$ ,  $A_{12} \in M_{m \times (n-m)}(\mathbb{F})$ ,  $A_{21} \in M_{(n-m) \times m}(\mathbb{F})$  and  $A_{22} \in M_{n-m}(\mathbb{F})$ . It is well known that if  $A_{22}$  is invertible, then det  $A = \det A_{22} \det(A_{11} - A_{12}A_{22}^{-1}A_{21})$  (see [12, Chapter 0]).

For two matrices  $A = (a_{ij}) \in M_{m \times n}(\mathbb{F})$  and  $B \in M_{p \times q}(\mathbb{F})$ , the Kronecker product of A and B, denoted by  $A \otimes B$ , is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

We state some basic properties of the Kronecker product (for more details, see [11, Chapter 4]):

- (a)  $A \otimes (B + C) = A \otimes B + A \otimes C$ .
- (b)  $(B+C) \otimes A = B \otimes A + C \otimes A$ .
- (c)  $(A \otimes B)(C \otimes D) = AC \otimes BD$ .
- (d) If  $A \in M_m(\mathbb{F})$  and  $B \in M_n(\mathbb{F})$  are invertible, then  $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ .
- (e)  $\det(A \otimes B) = (\det A)^n (\det B)^m$  for  $A \in M_m(\mathbb{F})$  and  $B \in M_n(\mathbb{F})$ .

A matrix  $T \in M_n(\mathbb{F})$  of the form

$$T = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-(n-2)} \\ a_2 & a_1 & a_0 & \cdots & a_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix}$$

is called a *Toeplitz matrix*. In [18], the authors gave a Toeplitz matrix inversion formula.

**Theorem 2.1** ([18], Theorem 1). Let  $T = (a_{i-j})_{i,j=1}^n$  be a Toeplitz matrix and let  $f = (0, a_{n-1}-a_{-1}, \dots, a_2-a_{-(n-2)}, a_1-a_{-(n-1)})^T$  and  $e_1 = (1, 0, \dots, 0)^T$ . If each of the systems of equations Tx = f,  $Ty = e_1$  is solvable,  $x = (x_1, x_2, \dots, x_n)^T$ ,  $y = (y_1, y_2, \dots, y_n)^T$ , then

,

(a) *T* is invertible;  
(b) 
$$T^{-1} = T_1 U_1 + T_2 U_2$$
, where  
 $T_1 = \begin{pmatrix} y_1 & y_n & \cdots & y_2 \\ y_2 & y_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & y_n \\ y_n & \cdots & y_2 & y_1 \end{pmatrix}$ ,  $U_1 = \begin{pmatrix} 1 & -x_n & \cdots & -x_2 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -x_n \\ 0 & \cdots & 0 & 1 \end{pmatrix}$   
 $T_2 = \begin{pmatrix} x_1 & x_n & \cdots & x_2 \\ x_2 & x_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_n \\ x_n & \cdots & x_2 & x_1 \end{pmatrix}$ , and  $U_2 = \begin{pmatrix} 0 & y_n & \cdots & y_2 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & y_n \\ 0 & \cdots & 0 & 0 \end{pmatrix}$ 

**Corollary 2.2.** Let  $aI_n + bJ_n$  be a matrix in  $M_n(\mathbb{F})$ . Then

- (a)  $\det(aI_n + bJ_n) = a^{n-1}(a+nb).$
- (b) If  $aI_n + bJ_n$  is invertible, then its inverse matrix is

$$\frac{1}{a(a+nb)}((a+nb)I_n-bJ_n).$$

*Proof.* (a) It is easy to check that

$$\det \begin{pmatrix} a+b & b & b & \cdots & b \\ b & a+b & b & \cdots & b \\ b & b & a+b & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a+b \end{pmatrix} = \det \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & 2b & 3b & \cdots & a+nb \end{pmatrix}.$$

Hence the determinant of  $aI_n + bJ_n$  is  $a^{n-1}(a+nb)$ . (b) Note that the matrix  $aI_n + bJ_n$  is Toeplitz. Let

$$x = (0, \dots, 0)^T$$
 and  $y = \left(\frac{a+nb-b}{a(a+nb)}, \frac{-b}{a(a+nb)}, \dots, \frac{-b}{a(a+nb)}\right)^T$ .

Then  $(aI_n + bJ_n)x = 0$  and  $(aI_n + bJ_n)y = e_1$ . By Theorem 2.1, the inverse of  $aI_n + bJ_n$  is

$$\frac{1}{a(a+nb)}((a+nb)I_n-bJ_n).$$

# 3. K-Symmetric graphs

Symmetry is an important property of graphs. We deal with graphs that has symmetric property. Let G be a graph. An *automorphism*  $\varphi$  of a graph G is a permutation of V(G) such that  $\varphi(v)$  and  $\varphi(w)$  are adjacent if and only if v and w are adjacent where v and w are vertices of G. The set of all automorphisms of G is called an *automorphism group* of G and denoted by Aut(G). A graph G is *symmetric* if Aut(G) acts transitively on both vertices of G and ordered pairs of adjacent vertices. This implies that G is *regular*, that is, all vertices have the same degree. However, it is a very difficult problem to determine whether a given graph is a symmetric graph. Thus we concentrate the cyclic part of Aut(G). In this section, we define k-symmetric graphs and give some their properties. Also, we construct a k-symmetric graph from other k-symmetric graphs.

**Definition 3.1.** Let k be a positive integer. A graph G is k-symmetric if there is a subgroup  $\mathcal{H}$  of Aut(G) such that  $\mathcal{H}$  is isomorphic to  $\mathbb{Z}_k$  and  $\mathcal{H}$  freely act on vertices. A generator of  $\mathcal{H}$  is called a k-symmetric automorphism.

The above definition tells us that all graphs are 1-symmetric because the trivial group freely acts on any graph. If a graph G with n vertices is n-symmetric, then the automorphism group  $\operatorname{Aut}(G)$  has a cyclic subgroup  $\mathcal{H}$  which transitively acts on vertices. Thus G is regular. However, the converse is not true even though G is a symmetric graph. Before examining this, we check the following proposition.

**Proposition 3.2.** Let G be a graph with n vertices. If G is n-symmetric, then either G or its complement  $\overline{G}$  have a Hamiltonian cycle.

Proof. Let G be an *n*-symmetric graph with *n* vertices, and let  $\varphi$  be an *n*-symmetric automorphism of G. Choose a vertex v. If v and  $\varphi(v)$  are adjacent, then  $\varphi^i(v)$  and  $\varphi^{i+1}(v)$  are also adjacent for any interger i. Since G is *n*-symmetric, the group generated by  $\varphi$  acts freely and transitively on V(G). Thus the sequence  $v, \varphi(v), \ldots, \varphi^n(v)$  induces a Hamiltonian cycle of G. Suppose that v and  $\varphi(v)$  are not adjacent in G. Then v and  $\varphi(v)$  are adjacent in  $\overline{G}$ . Hence the sequence of vertices induces a Hamiltonian cycle of  $\overline{G}$ .

For example, the Petersen graph in Figure 1 is 5-symmetric because the 5-fold rotation satisfies the 5-symmetric automorphism condition. The Petersen graph is a symmetric graph with 10 vertices. But since the Petersen graph is not Hamiltonian, it is not 10-symmetric. For any positive integer k, k-symmetric graphs are satisfying the following properties.

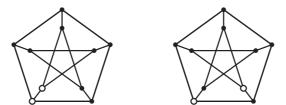


FIGURE 1. The Petersen graphs with different bases for the 5-fold rotation

**Proposition 3.3.** Let G be a k-symmetric graph for some integer k and let d be a divisor of k. Then G is a d-symmetric graph.

Proof. Let  $\varphi$  be a k-symmetric automorphism of G and let k = k'd for some integer k'. Define an automorphism  $\psi$  of G by  $\psi = \varphi^{k'}$ . Then  $\psi^d(v) = \varphi^{k'd}(v) = \varphi^k(v) = \operatorname{id}_G(v)$  for all  $v \in V(G)$ . The subgroup  $\langle \psi \rangle$  of Aut(G) is isomorphic to  $\mathbb{Z}_d$ . Thus G is a d-symmetric graph.  $\Box$  **Proposition 3.4.** Let  $G_1$  and  $G_2$  be k-symmetric graphs for some integer k. Then  $G_1 \cup G_2$  is k-symmetric graph.

*Proof.* Let  $\varphi_1$  and  $\varphi_2$  be k-symmetric automorphisms of  $G_1$  and  $G_2$ , respectively. Then the automorphism  $\varphi_1 + \varphi_2$  of  $G_1 \cup G_2$  is defined by

$$(\varphi_1 + \varphi_2)(v) = \begin{cases} \varphi_1(v), & \text{if } v \in V(G_1), \\ \varphi_2(v), & \text{if } v \in V(G_2) \end{cases}$$

Hence  $G_1 \cup G_2$  is a k-symmetric graph.

Let  $\varphi$  be a k-symmetric automorphism of a graph G and let  $\mathrm{id}_H$  be the identity automorphism of a graph H. Then the automorphism  $\varphi \times \mathrm{id}_H$  of  $G \square H$  is k-symmetric. Thus we obtain the following proposition.

**Proposition 3.5.** Let G be a k-symmetric graphs for some integer k. For any graph H, the Cartesian product  $G \square H$  is k-symmetric graph.

Let G be a graph with a k-symmetric automorphism  $\varphi$ . Then  $\mathbb{Z}_k$  acts on V(G) as follows. For any  $i \in \mathbb{Z}_k$  and  $v \in V(G)$ , we define  $i \cdot v = \varphi^i(v)$ . For any vertex v, the orbit of v is denoted by  $\mathbb{Z}_k \cdot v$ . Let  $B_{\varphi}$  be a minimal subset of V(G) such that

$$\bigcup_{i=0}^{k-1} \varphi^i(B_{\varphi}) = V(G).$$

Alternatively,  $B_{\varphi}$  is a minimal subset of V(G) such that

$$\bigcup_{v \in B_{\varphi}} \mathbb{Z}_k \cdot v = V(G)$$

The set  $B_{\varphi}$  is called a *base* of  $\varphi$ . Since k choices are possible for each orbit,  $B_{\varphi}$  is not unique as drawn in Figure 1. Note that the size of the base  $B_{\varphi}$  is |V(G)|.

Now we introduce how to construct a k-symmetric graph from other ksymmetric graphs for any positive integer k. First we observe a graph join. Let  $H_1$  and  $H_2$  be graphs. The graph join  $H_1 \vee H_2$  of  $H_1$  and  $H_2$  is a graph obtained by joining each vertex of  $H_1$  to all vertices of  $H_2$ . Since every graph is 1-symmetric with respect to identity map, we can understand graph join  $H_1 \vee H_2$  as a join of the bases  $V(H_1)$  and  $V(H_2)$  of  $id_{H_1}$  and  $id_{H_2}$ . From this fact, we generalize graph join.

**Definition 3.6.** For  $i \in \{1,2\}$ , let  $G_i$  be a k-symmetric graph with a k-symmetric automorphism  $\varphi_i$ , and let  $B_i$  be a chosen base of  $\varphi_i$ . The k-symmetric join is a graph obtained by joining each vertex of  $\varphi_1^j(B_1)$  to all vertices of  $\varphi_2^j(B_2)$  for all  $j \in \mathbb{Z}_k$ . The k-symmetric join is denoted by  $(G_1, \varphi_1, B_1) \lor_k (G_2, \varphi_2, B_2)$ . If we choose arbitrary k-symmetric automorphisms and its bases of  $G_1$  and  $G_2$ , then the k-symmetric join is simply denoted by  $G_1 \lor_k G_2$ .

The k-symmetric join preserves the k-symmetry. Because the k-symmetric automorphism of  $G_1 \vee_k G_2$  is  $\varphi_1 + \varphi_2$  and its base is  $B_1 \cup B_2$ . Definition 3.6 derives that the graph join is the 1-symmetric join. Note that, n-symmetric joins are not unique even if the base of each  $G_i$  is unique. For instance, the Cartesian product of 5-cycle  $C_5$  with  $K_2$  and the Petersen graph are both 5-symmetric joins of two 5-cycles, but they are not isomorphic as drawn in Figure 2.

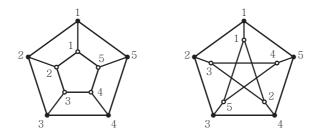


FIGURE 2. The 5-symmetric join of two 5-cycles is  $C_5 \square K_2$  if both automorphisms are (1, 2, 3, 4, 5), and the Petersen graph if automorphisms are (1, 2, 3, 4, 5) and (1, 3, 5, 2, 4)

## 4. 2-connected k-symmetric graphs with Laplacian eigenvalue 1

In this section, we prove Theorem 1.1. First we consider the multiplicity of an integral Laplacian eigenvalue. Recall that for a given graph G, the partition  $\pi = (V_1, V_2, \dots, V_k)$  of V(G) is an *equitable partition* if for all  $i, j \in$  $\{1, 2, \dots, k\}$  and for any  $v \in V_i$  the number  $d_{ij} = |N_G(v) \cap V_j|$  depends only on i and j. The  $k \times k$  matrix  $L^{\pi}(G) = (b_{ij})$  defined by

$$b_{ij} = \begin{cases} -d_{ij}, & \text{if } i \neq j, \\ \sum_{s=1}^k d_{is} - d_{ij}, & \text{if } i = j \end{cases}$$

is called the *divisor matrix* of G with respect to  $\pi$ .

**Lemma 4.1** ([2, 6]). Let G be a graph and let  $\pi = (V_1, \ldots, V_k)$  be an equitable partition of G with divisor matrix  $L^{\pi}(G)$ . Then each eigenvalue of  $L^{\pi}(G)$  is also an eigenvalue of L(G).

In the following theorem, we obtain the multiplicity of n of k-symmetric join of graphs where n is the size of a base.

**Theorem 4.2.** Let  $G_1, \ldots, G_l$  be k-symmetric graphs for some k and let  $G = G_1 \vee_k \cdots \vee_k G_l$  be the k-symmetric join of  $G_1, \ldots, G_l$ . Let  $n = \frac{|V(G)|}{k}$ . Then

$$m_G(n) \ge l - 1.$$

*Proof.* The partition  $\pi = (V(G_1), \ldots, V(G_l))$  is an equitable partition of G. Then we have

$$L^{\pi}(G) = \begin{pmatrix} n - n_1 & -n_2 & \cdots & -n_l \\ -n_1 & n - n_2 & \cdots & -n_l \\ \vdots & \vdots & \ddots & \vdots \\ -n_1 & -n_2 & \cdots & n - n_l \end{pmatrix},$$

where  $n_i = \frac{|V(G_i)|}{k}$  for i = 1, ..., l. Since the characteristic polynomial of  $L^{\pi}(G)$  is  $\mu(L^{\pi}(G), x) = x(x-n)^{l-1}$ , by Lemma 4.1, we obtain

$$m_G(n) \ge l - 1.$$

If each  $G_i$  in the above theorem is *n*-symmetric graph with *n* vertices, then the size of a base of *G* is *l*.

**Corollary 4.3.** Let  $G_1, \ldots, G_l$  be n-symmetric graphs with n vertices. Then for any their n-symmetric join G,

$$m_G(l) \ge l - 1.$$

Let G be an n-symmetric graph with n vertices. Take an n-symmetric automorphism  $\varphi$  of G. Let G' be a graph that n-symmetric join of m copies of G along  $\varphi$ . Then since each base of the copy of G is a vertex, the base of G' induces the m complete graph  $K_m$ . Since G' is constructed by same n-symmetric automorphism, G' becomes the Cartesian product of G and  $K_m$ .

**Corollary 4.4.** Let G be n-symmetric graphs with n vertices. Then for any positive integer m,

$$m_{K_m \square G}(m) \ge m - 1.$$

By the Spacapan's result [20] about the connectivity of the Cartesian product in Section 1, we realize that for any positive integer m, there is a m-connected graph G with  $m_G(m) \ge m - 1$ .

Now consider a special case of k-symmetric join. For any  $i \in \{1, \ldots, l\}$ , let  $G_i$  be a k-symmetric graph for some positive integer k and let  $\varphi_i$  be an associated k-symmetric automorphism. Let  $B_i^1$  be a base of  $G_i$ , and let  $B_i^j = \varphi_i^j(B_i^1)$ . Racall that the union of  $G_1, \ldots, G_l$  is also k-symmetric with the k-symmetric automorphism  $\varphi_1 + \cdots + \varphi_l$  and the base  $B_1^1 \cup \cdots \cup B_l^1$ . Define a graph G by k-symmetric joining  $\overline{K}_k$  and  $G_1 \cup \cdots \cup G_l$ . Then the subgraph induced by a base of G has a cut-vertex as drawn in Figure 3 (a). From this fact, we can take an equitable partition  $\pi = (V_0, V_1, \ldots, V_l)$ where  $V_0 = V(\overline{K}_k)$  and  $V_i = V(G_i)$  for any  $i \in \{1, \ldots, l\}$  as drawn in Figure 3 (b). Remark that for any distinct i and i', there is no edge connecting two subgraphs  $G_i$  and  $G_{i'}$  in G. To prove Theorem 1.1, we need the following two theorems.

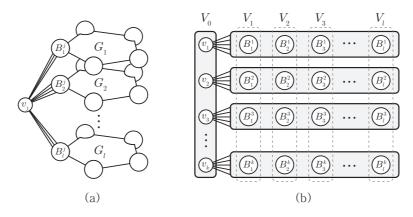


FIGURE 3. k-symmetric graph  $G = \overline{K}_k \lor_k (G_1 \cup \cdots \cup G_l)$ 

**Theorem 4.5.** Let  $G_1, \ldots, G_l$  be k-symmetric graphs for some positive integers l and k, and let  $G = \overline{K}_k \vee_k (G_1 \cup \cdots \cup G_l)$ . Then

$$m_G(1) \ge l - 1.$$

*Proof.* Suppose that  $G_1, \ldots, G_l$  and G are the graphs in the statement of the theorem. Let  $V_0$  be the vertices set of  $\overline{K}_k$  and let  $V_i$  be the vertices set of  $G_i$  for  $i = 1, \ldots, l$ . Our observation implies that the partition  $\pi = (V_0, V_1, \ldots, V_l)$  is an equitable partition of G. Then the divisor matrix  $L^{\pi}(G)$  is equal to

where  $n_i = \frac{|V_i|}{k}$  for i = 1, ..., l and  $n = \sum_{i=1}^l n_i$ . We can partition the matrix  $xI - L^{\pi}(G)$  into four blocks as

(x-n)	$n_1$	$n_2$		$n_l$ )	
1	x - 1	0		0	
1	0	x - 1		0	
:		÷	·.	÷	
1	0	0		x-1 /	

Then the characteristic polynomial of  $L^{\pi}(G)$  is

$$\mu(L^{\pi}(G), x) = (x-1)^{l} \left( (x-n) - \frac{1}{x-1} n \right)$$
$$= (x-1)^{l-1} ((x-n)(x-1) - n)$$
$$= (x-1)^{l-1} x (x - (n+1)).$$

By Lemma 4.1, we obtain  $m_G(1) \ge l - 1$ .

**Theorem 4.6.** Let  $G_1, \ldots, G_l$  be connected k-symmetric graphs for some integers  $l, k \geq 2$ . Then the graph  $G = \overline{K}_k \lor_k (G_1 \cup \cdots \cup G_l)$  is a 2-connected prime graph with respect to Cartesian product.

*Proof.* First we prove that the graph  $G = \overline{K}_k \vee_k (G_1 \cup \cdots \cup G_l)$  is 2-connected. Suppose that there is a cut-vertex v of G in  $\overline{K}_k$ . Since  $k \geq 2$ , there is another vertex w in  $\overline{K}_k$ . Since  $l \geq 2$ , there are two independent paths  $P_1$  and  $P_2$  from v to w passing through  $G_1$  and  $G_2$ , respectively. This implies that v lies on the cycle  $P_1 \cup P_2$ , and hence v is not a cut-vertex. Next we suppose that a cut-vertex of G is not lying on  $\overline{K}_k$ . Without loss of generality, assume that a cut-vertex v is in  $G_1$ . Since  $l, k \geq 2$  and G is connected, there is a cycle containing v in G. It follows that v is not a cut-vertex. Therefore, Gis 2-connected.

Now, we show that the graph G is prime with respect to the Cartesian product. It is well-known that if two edges of a nontrivial Cartesian product are incident, then they are included in a subgraph  $C_4$  of the Cartesian product. For any vertex u of G in  $\overline{K}_k$ , two incident edges e and f of u such that the endpoints of e and f are contained in different graphs  $G_i$  and  $G_j$ 

for some integers i and j. But there is no  $C_4$  including e and f. Thus we deduce that G is prime.

Theorems 4.5 and 4.6 imply Theorem 1.1. Note that, if one of the graphs  $G_1, \ldots, G_l$  is disconnected, then there is a counterexample. For example, if  $G_1$  is  $\overline{K}_k$ , then  $G = \overline{K}_k \vee_k (G_1 \cup \cdots \cup G_l)$  has a cut vertex.

# 5. LAPLACIAN INTEGRAL GRAPHS

In this section, we discuss k-symmetric graphs with integral Laplacian spectrum. Let n and m be positive integers. Since the n-complete graph  $K_n$ is n-symmetric, the disjoint union of m copies of  $K_n$ , denoted by  $mK_n$ , is also n-symmetric. We consider the n-symmetric join of  $\overline{K}_n$  and  $mK_n$ . Denote the graph  $\overline{K}_n \vee_n mK_n$  by  $\mathcal{C}(n,m)$ . Now we observe that a graph  $\mathcal{C}(n,m)$  is not a cograph for  $n \geq 2$ . Let v and w be vertices in  $\overline{K}_n$ . Then there are two vertices in  $K_n$  which are adjacent to v and w, respectively. Thus the graph  $\mathcal{C}(n,m)$  contains the path  $P_4$  as an induced subgraph. We will show that a graph  $\mathcal{C}(n,m)$  is Laplacian integral for some positive integers n and m. In the following theorem, we give the characteristic polynomial of  $L(\mathcal{C}(n,m))$ .

**Theorem 5.1.** Let n and m be positive integers. Then the characteristic polynomial of  $L(\mathcal{C}(n,m))$  is

$$x(x-1)^{m-1}(x-(n+1))^{(m-1)(n-1)}(x-(m+1))(x^2-(m+n+1)x+mn)^{n-1}.$$

*Proof.* The Laplacian matrix of  $\mathcal{C}(n,m)$  is

$$L(\mathcal{C}(n,m)) = \begin{pmatrix} mI_n + L(\overline{K}_n) & -1_m^T \otimes I_n \\ -1_m \otimes I_n & I_m \otimes (I_n + L(K_n)) \end{pmatrix}.$$

We consider  $xI_{n(m+1)} - L(\mathcal{C}(n,m))$  as a matrix over the field of rational functions  $\mathbb{C}(x)$ . Then the characteristic polynomial of  $L(\mathcal{C}(n,m))$  is

$$\mu(L(\mathcal{C}(n,m)),x) = \det(xI_{n(m+1)} - L(\mathcal{C}(n,m)))$$

$$= \det\begin{pmatrix} (x-m)I_n & 1_m^T \otimes I_n \\ 1_m \otimes I_n & I_m \otimes ((x-1)I_n - L(K_n)) \end{pmatrix}$$

$$= \det(I_m \otimes ((x-1)I_n - L(K_n))) \det((x-m)I_n \\ - (1_m^T \otimes I_n)(I_m \otimes ((x-1)I_n - L(K_n)))^{-1}(1_m \otimes I_n))$$

$$= \det((x-1)I_n - L(K_n))^m \det((x-m)I_n \\ - m((x-1)I_n - L(K_n))^{-1}).$$

Since det $(xI_n - L(K_n)) = \mu(L(K_n), x)$ , we obtain det $((x - 1)L_n - L(K_n))^m - \mu(L(K_n))^m = 1)^m$ 

$$= (x-1)^m (x-(n+1))^{m(n-1)}.$$

Now, we compute the determinant of  $(x - m)I_n - m((x - 1)I_n - L(K_n))^{-1}$ . By Corollary 2.2 (b), we have

$$((x-1)I_n - L(K_n))^{-1} = ((x-(n+1))I_n + J_n)^{-1}$$
  
=  $\frac{1}{(x-1)(x-(n+1))}((x-1)I_n - J_n).$ 

This implies that

$$\det \left( (x-m)I_n - m((x-1)I_n - L(K_n))^{-1} \right)$$

$$= \frac{\det \left( (x-m)(x-1)(x-(n+1))I_n - m((x-1)I_n - J_n) \right)}{(x-1)^n (x-(n+1))^n}$$

$$= \frac{\det \left( (x^3 - (m+n+2)x^2 + (mn+m+n+1)x - mn)I_n + mJ_n \right)}{(x-1)^n (x-(n+1))^n}$$

By Corollary 2.2 (a), we have

$$\det\left((x^3 - (m+n+2)x^2 + (mn+m+n+1)x - mn)I_n + mJ_n\right)$$
  
=  $x(x - (m+1))(x - (n+1))(x - 1)^{n-1}(x^2 - (m+n+1)x + mn)^{n-1}.$ 

Hence the determinant of  $(x-m)I_n - m((x-1)I_n - L(K_n))^{-1}$  is

$$\frac{x(x-(m+1))(x^2-(m+n+1)x+mn)^{n-1}}{(x-(n+1))^{n-1}(x-1)}$$

Therefore the characteristic polynomial of  $L(\mathcal{C}(n,m))$  is

$$x(x-1)^{m-1}(x-(n+1))^{(m-1)(n-1)}(x-(m+1))(x^2-(m+n+1)x+mn)^{n-1}.$$

The following corollary induces Theorem 1.2.

## **Corollary 5.2.** Let n, m, k, and l be positive integers with $l \neq 1$ . Then

- (a) If C(n,m) is Laplacian integral, then C(m,n) is also Laplacian integral.
- (b) A graph C(kl, (k+1)(l-1)) is Laplacian integral.
- (c) A graph  $C(k^2 + k, k^2 + k)$  is regular Laplacian integral.

*Proof.* (a) It is obvious by Theorem 5.1.

(1)

(b) If the quadratic  $x^2 - (m + n + 1)x + mn$  has two integer roots, then  $\mathcal{C}(m,n)$  is Laplacian integral, by Theorem 5.1. Let k, l, r and s be positive integers with n = kl and m = rs. Suppose that kr and ls are roots of the quadratic. Then, by Vieta's formulas, we have kr + ls = rs + kl + 1, that is,

$$(s-k)r - (s-k)l + 1 = 0$$

If s = k then it is a contradiction. If  $s - k \neq 0$ , then  $r - l + \frac{1}{s-k} = 0$ . Since r and l are integers, s - k must be 1. Plugging s = k + 1 into the equation (1), we have r = l - 1. Since m is a positive integer, lis not equal to 1. Thus C(kl, (k+1)(l-1)) is Laplacian integral for any positive integers k and  $l \neq 1$ .

(c) If m = n, then  $\mathcal{C}(n, m)$  is regular. By (b), a graph  $\mathcal{C}(k^2 + k, k^2 + k)$  is regular Laplacian integral graph.

Now, we consider the *n*-complete graph  $K_n$  as a *k*-symmetric graph for some divisor *k* of *n*. Note that a base of  $K_n$  as a *k*-symmetric graph is not unique, but the *k*-symmetric join of  $\overline{K}_k$  and  $mK_n$  is unique up to isomorphism. We denote by  $\mathcal{C}(n,k,m)$  the graph  $\overline{K}_k \vee_k mK_n$ . In the similar way to the proof of Theorem 5.1, we get the characteristic polynomial of  $L(\mathcal{C}(n,k,m))$ .

**Theorem 5.3.** Let n and m be positive integers. Let k be a divisor of n and let d = n/k. Then the characteristic polynomial of  $L(\mathcal{C}(n,k,m))$  is

$$x(x-1)^{m-1}(x-(n+1))^{m(n-1)-k+1}(x-(md+1))(x^2-(md+n+1)x+mdn)^{k-1}.$$

*Proof.* The Laplacian matrix of  $\mathcal{C}(n,k,m)$  is

$$L(\mathcal{C}(n,k,m)) = \begin{pmatrix} mdI_k + L(\overline{K}_k) & -\mathbf{1}_m^T \otimes (I_k \otimes \mathbf{1}_d^T) \\ -\mathbf{1}_m \otimes (I_k \otimes \mathbf{1}_d) & I_m \otimes (I_n + L(K_n)) \end{pmatrix}.$$

Then the characteristic polynomial of  $L(\mathcal{C}(n,k,m))$  is

$$\mu(L(\mathcal{C}(n,k,m)),x) = \det\left(xI_{n(m+1)} - L(\mathcal{C}(n,k,m))\right).$$

Consider  $xI_{n(m+1)} - L(\mathcal{C}(n,k,m))$  as a matrix over the field of rational functions  $\mathbb{C}(x)$ . Then

$$\det \left(xI_{n(m+1)} - L(\mathcal{C}(n,k,m))\right)$$

$$= \det \left(\begin{array}{cc} (x - md)I_k & \mathbf{1}_m^T \otimes (I_k \otimes \mathbf{1}_d^T) \\ \mathbf{1}_m \otimes (I_k \otimes \mathbf{1}_d) & I_m \otimes ((x - 1)I_n - L(K_n)) \end{array}\right)$$

$$= \det \left(I_m \otimes ((x - 1)I_n - L(K_n))\right) \det \left((x - md)I_k \\ - (\mathbf{1}_m^T \otimes (I_k \otimes \mathbf{1}_d^T))(I_m \otimes ((x - 1)I_n - L(K_n)))^{-1}(\mathbf{1}_m \otimes (I_k \otimes \mathbf{1}_d))$$

$$= \det \left((x - 1)I_n - L(K_n)\right)^m \det \left((x - md)I_k \\ - m\left(I_k \otimes \mathbf{1}_d^T\right)((x - 1)I_n - L(K_n))^{-1}(I_k \otimes \mathbf{1}_d)\right).$$

It is easily check that

$$\det ((x-1)I_n - L(K_n))^m = (x-1)^m (x-(n+1))^{m(n-1)}$$

Now, we compute det  $((x-md)I_k - m(I_k \otimes 1_d^T)((x-1)I_n - L(K_n))^{-1}(I_k \otimes 1_d))$ . By Corollary 2.2 (b), we have

$$((x-1)I_n - L(K_n))^{-1} = \frac{1}{(x-1)(x-(n+1))}((x-1)I_n - J_n).$$

Note that the matrix  $(x-1)I_n - J_n$  can be written in the Kronecker product form  $(x-1)I_k \otimes I_d - J_k \otimes J_d$ . It follows that

$$(I_k \otimes 1_d^T)((x-1)I_n - L(K_n))^{-1}(I_k \otimes 1_d)$$
  
=  $(x-1)^{-1}(x-(n+1))^{-1}(I_k \otimes 1_d^T)((x-1)I_k \otimes I_d - J_k \otimes J_d)(I_k \otimes 1_d)$   
=  $(x-1)^{-1}(x-(n+1))^{-1}(d(x-1)I_k - d^2J_k).$ 

Then we have

$$=\frac{\det\left((x-md)I_{k}-m(I_{k}\otimes 1_{d}^{T})((x-1)I_{n}+L(K_{n}))^{-1}(I_{k}\otimes 1_{d})\right)\right)}{\det\left((x-md)(x-1)(x-(n+1))I_{k}-md(x-1)I_{k}+md^{2}J_{k}\right)}{(x-1)^{k}(x-(n+1))^{k}}.$$

By Corollary 2.2 (a), we obtain

$$\det \left( (x - md)(x - 1)(x - (n + 1))I_k - md(x - 1)I_k + md^2 J_k \right)$$
  
=  $\det \left( (x^3 - (md + n + 2)x^2 + (md + 1)(n + 1)x - mdn)I_k + md^2 J_k \right)$   
=  $x(x - (md + 1))(x - (n + 1))(x - 1)^{k-1}(x^2 - (md + n + 1)x + mdn)^{k-1}.$ 

Hence the determinant of  $(x-md)I_k - m(I_k \otimes \mathbb{1}_d^T)((x-1)I_n - L(K_n))^{-1}(I_k \otimes \mathbb{1}_d)$  is

$$\frac{x(x-(md+1))(x^2-(md+n+1)x+mdn)^{k-1}}{(x-1)(x-(n+1))^{k-1}}.$$

Thus the characteristic polynomial of  $L(\mathcal{C}(n,k,m))$  is

$$x(x-1)^{m-1}(x-(n+1))^{m(n-1)-k+1}(x-(md+1))(x^2-(md+n+1)x+mdn)^{k-1}.$$

The next two corollaries tell us about the relation between Laplacian integral graphs  $\mathcal{C}(n,n,m)$  and  $\mathcal{C}(n,k,m')$  for some positive integers n, m, m' and k with  $k \mid n$ .

**Corollary 5.4.** Suppose that C(n, n, m) is Laplacian integral for some positive integers n and m. Let d be a divisor of n. If m is divisible by d, then  $C(n, \frac{n}{d}, \frac{m}{d})$  is Laplacian integral.

*Proof.* Suppose that C(n, n, m) is Laplacaian integral for some positive integers n and m. Then the polynomial  $x^2 - (m+n+1)x + mn$  in the characteristic polynomial of C(n, n, m) can be factored over the integers. Let d be a divisor of n. By Theorem 5.3, it is enough to show that the quadratic in the characteristic polynomial of  $C(n, \frac{n}{d}, \frac{m}{d})$  has integral roots. Since the quadratic is

$$x^{2} - \left(\frac{m}{d}dn + n + 1\right)x + \frac{m}{d}dn = x^{2} - (m + n + 1)x + mn,$$

the graph  $\mathcal{C}(n, \frac{n}{d}, \frac{m}{d})$  is Laplacian integral.

**Corollary 5.5.** Suppose that C(n, k, m) is Laplacian integral for some positive integers n, m and k with  $k \mid n$ . Let d = n/k. Then C(n, n, md) is Laplacian integral.

*Proof.* The proof is similar that of Theorem 5.4. Since the quadratic in the characteristic polynomial of C(n, n, md) is

$$x^2 - (md + n + 1)x + mdn,$$

it is easy to see that  $\mathcal{C}(n, n, md)$  is Laplacian integral.

#### DECLARATION OF COMPETING INTEREST

There is no competing interest.

#### S. MOON AND H. YOO

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Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea

Email address: symoon89@hanyang.ac.kr

Institute of Mathematical Sciences, Ewha Womans University, Seoul 03760, Korea

Email address: hyungkee@ewha.ac.kr