# The most general structure of graphs with hamiltonian or hamiltonian connected square

Jan Ekstein* Herbert Fleischner †

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**Abstract**

On the basis of recent results on hamiltonicity, [4], and hamiltonian connectedness, [8], in the square of a 2-block, we determine the most general block-cutvertex structure a graph $G$ may have in order to guarantee that $G^2$ is hamiltonian, hamiltonian connected, respectively. Such an approach was already developed in [9] for hamiltonian total graphs.

**Keywords**: hamiltonian cycle, hamiltonian path, block-cutvertex graph, square of a graph

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## 1 Introduction and Preliminary Discussion

As for standard terminology and other terminology used in this paper, we refer to the book by Bondy and Murty, [2], and to the papers quoted in the references. Let $G$ be a connected graph. A *2-block* is a 2-connected graph or a block of $G$ containing more than two vertices. The square of a graph $G$,

*Department of Mathematics and European Centre of Excellence NTIS - New Technologies for the Information Society, Faculty of Applied Sciences, University of West Bohemia, Pilsen, Technická 8, 306 14 Plzeň, Czech Republic, EU
e-mail: `ekstein@kma.zcu.cz`.

†Institute of Logic and Computation, Algorithms and Complexity Group, Technical University of Vienna, Favoritenstrasse 9 - 11, 1040 Wien, Austria, EU
e-mail: `fleischner@ac.tuwien.ac.at`.

denoted $G^2$, is the graph obtained from $G$ by joining any two nonadjacent vertices which have a common neighbor, by an edge.

It was shown in 1970 and published in 1974 that the square of every 2-block contains a hamiltonian cycle, [6]. Key in proving this was the existence of EPS-graphs $S$ in connected bridgeless graphs $G$, where $S$ is the edge-disjoint union of a not necessarily connected eulerian subgraph $E$ and a linear forest $P$, and $S$ is connected and spans $G$, [5]. In subsequent papers [7], [9] the existence of various types of EPS-graphs was established. Their relevance was based on the fact that the total graph $T(G)$ of any connected graph $G$ other than $K_1$ is hamiltonian if and only if $G$ has an EPS-graph, [9]. This and the theory of EPS-graphs led to a description of the most general block-cutvertex graph $\mathrm{bc}(G)$ of a graph $G$ may have such that $T(G)$ is hamiltonian and if $\mathrm{bc}(G)$ does not have the corresponding structure, then exchanging certain 2-blocks in $G$ with some special 2-blocks yields a graph $G^*$ such that $\mathrm{bc}(G)$ and $\mathrm{bc}(G^*)$ are isomorphic but $T(G^*)$ is not hamiltonian, [9]. In dealing with hamiltonian cycles and hamiltonian paths by methods developed up to that point, it was shown in [7] that in the square of graphs hamiltonicity and vertex-pancyclicity are equivalent concepts, and so are hamiltonian connectedness and panconnectedness. In this context Theorem 3 stated below was established as a tool needed to prove the equivalences just mentioned.

However, in the course of time much shorter proofs of Fleischner's Theorem were developed [10], [13]; the same applies to Theorem 3 below, [12]. More recently, an algorithm yielding a hamiltonian cycle in the square of a 2-block in linear time, was developed, [1]. The methods developed in these much shorter proofs (including the algorithm just mentioned) do not seem to yield short proofs of Theorems 4 and 5 below, [4], [8]. These latter theorems are, on the other hand, instrumental in proving the central results of this paper, i.e., Theorems 1 and 2, and related algorithms.

Let $\mathrm{bc}(G)$ denote the *block-cutvertex graph* of $G$. Blocks corresponding to leaves of $\mathrm{bc}(G)$ are called *endblocks*, otherwise *innerblocks*. Note that a block in a graph $G$ is either a 2-block or a bridge of $G$. For each cutvertex $i$ of $G$, let $k_i$ be the number of 2-blocks of $G$ which include vertex $i$ and let $\mathrm{bn}(i)$ be the number of nontrivial bridges of $G$ which are incident with vertex $i$. In what follows a bridge is called nontrivial if it is not incident to a leaf.

Let $H$ be a subgraph of the graph $G$. We define $G-H := G-E(H)-\{v \in V(H) : d_H(v) = d_G(v)\}$.

In Theorem 1, we introduce an array $m_i(B)$ of numbers with an entry for each pair consisting of a cutvertex $i$ and a 2-block $B$ of $G$. We may think of this number $m_i(B)$ as the number of edges of $B$ incident with $i$ which are possibly contained in a hamiltonian cycle in $G^2$.

Statement of Theorem 1 describes the most general block-cutvertex structure a graph $G$ may have in order to guarantee that $G^2$ is hamiltonian using parameters $m_i(B)$ as in [9].

**Theorem 1.** *Let $G$ be a connected graph with at least three vertices. Let the 2-blocks of $G$ be labelled $B_1, B_2, ..., B_n$. Let the cutvertices of $G$ be labelled $1, 2, ..., s$. Suppose there is a labelling $m_i(B_t)$ for each $i \in \{1, 2, ..., s\}$ and each $t \in \{1, 2, ..., n\}$ such that the following conditions are fulfilled.*

*1) $0 \leq m_i(B_t) \leq 2$ for all $i$ and all 2-blocks $B_t$;*

*2) for 2-block $B_t$ $m_i(B_t) = 0$ if and only if cutvertex $i$ is not in $V(B_t)$;*

*3) for 2-block $B_t$, $m_i(B_t) \geq bn(i)$, if cutvertex $i \in V(B_t)$;*

*4) $bn(i) \leq 2$ for all $i \in \{1, 2, ..., s\}$;*

*5) $\sum_{i=1}^{s} m_i(B_t) \leq 4$ for each 2-block $B_t$ of $G$ and, if $m_i(B_t) = 2$ for some $i$, then $\sum_{i=1}^{s} m_i(B_t) \leq 3$; and*

*6) $\sum_{t=1}^{n} m_i(B_t) \geq 2k_i + bn(i) - 2$ for each $i \in \{1, 2, ..., s\}$.*

*Then $G^2$ is hamiltonian.*

*Moreover, if the labelling $m_i(B_t)$ satisfying conditions 1), 2) and 3) is given and at least one of conditions 4), 5), 6) is violated by some $G$, then there exists a class of graphs $G'$ with non-hamiltonian square but $bc(G')$ and $bc(G)$ are isomorphic.*

Also, we obtain a similar result for hamiltonian connectedness (Theorem 2). Quite surprisingly, its formulation is much simpler than that of Theorem 1.

**Theorem 2.** *Let $G$ be a connected graph such that the following conditions are fulfilled:*

*1) there is no nontrivial bridge of $G$;*

*2) every block contains at most 2 cutvertices.*

*Then $G^2$ is hamiltonian connected.*

*Moreover,*

- *if a graph $G$ contains a nontrivial bridge, then $G^2$ is not hamiltonian connected;*

· *if $G$ contains a block containing more than 2 cutvertices, then there is a graph $G'$ such that $bc(G)$ and $bc(G')$ are isomorphic but $(G')^2$ is not hamiltonian connected.*

A fundamental result regarding hamiltonicity in the square of a 2-block is the following theorem.

**Theorem 3.** **[7]** *Suppose $v$ and $w$ are two arbitrarily chosen vertices of a 2-block $G$. Then $G^2$ contains a hamiltonian cycle $C$ such that the edges of $C$ incident to $v$ are in $G$ and at least one of the edges of $C$ incident to $w$ is in $G$. Furthermore, if $v$ and $w$ are adjacent in $G$, then these are three different edges.*

The hamiltonian theme in the square of a 2-block has been recently revisited ([3], [4], [8]), yielding the following results which are essential for this paper.

A graph $G$ is said to have the $\mathcal{H}_k$ *property* if for any given vertices $x_1, ..., x_k$ there is a hamiltonian cycle in $G^2$ containing distinct edges $x_1y_1, ..., x_ky_k$ of $G$.

**Theorem 4.** **[4]** *Given a 2-block $G$ on at least 4 vertices, then $G$ has the $\mathcal{H}_4$ property, and there are 2-blocks of arbitrary order greater than 4 without the $\mathcal{H}_5$ property.*

By a *$uv$-path* we mean a path from $u$ to $v$ in $G$. If a $uv$-path is hamiltonian, we call it a *$uv$-hamiltonian path*. Let $A = \{x_1, x_2, ..., x_k\}$ be a set of $k \geq 3$ distinct vertices in $G$. An $x_1x_2$-hamiltonian path in $G^2$ which contains $k-2$ distinct edges $x_iy_i \in E(G), i = 3, ..., k$, is said to be $\mathcal{F}_k$. A graph $G$ is said to have the $\mathcal{F}_k$ *property* if, for any set $A = \{x_1, x_2, ..., x_k\} \subseteq V(G)$, there is an $\mathcal{F}_k$ $x_1x_2$-hamiltonian path in $G^2$.

**Theorem 5.** **[8]** *Every 2-block on at least 4 vertices has the $\mathcal{F}_4$ property.*

A graph $G$ is said to have the *strong $\mathcal{F}_3$ property* if, for any set of 3 vertices $\{x_1, x_2, x_3\}$ in $G$, there is an $x_1x_2$-hamiltonian path in $G^2$ containing distinct edges $x_3z_3, x_iz_i \in E(G)$ for a given $i \in \{1, 2\}$. Such an $x_1x_2$-hamiltonian path in $G^2$ is called a strong $\mathcal{F}_3$ $x_1x_2$-hamiltonian path.

**Theorem 6.** **[8]** *Every 2-block has the strong $\mathcal{F}_3$ property.*

**Theorem 7.** **[8]** *Let $G$ be a 2-connected graph and let $x, y$ be two vertices in $G$. Then $G^2$ has an $xy$-hamiltonian path $P(x, y)$ such that*

*(i) $xz \in E(G) \cap E(P(x, y))$ for some $z \in V(G)$, and*

*(ii) either $yw \in E(G) \cap E(P(x, y))$ for some $w \in V(G)$, or else $P(x, y)$ contains an edge $uv$ for some vertices $u, v \in N(y)$.*

# 2 Proofs and algorithms

PROOF OF THEOREM 1

*Proof.* Set $P_0 = G - \cup_{t=1}^{n} B_t$. Then every component of $P_0$ is a tree. Since by 4) $\mathrm{bn}(i) \leq 2$ every component of $P_0$ is even a caterpillar.

For every caterpillar $T$ of $P_0$ except $T = K_2$ we have the following observation which can be proved easily.

*Observation: Let $T$ be a caterpillar with at least three vertices and $P = x_1 x_2 ... x_m$ be some longest path in $T$. Then $T^2$ contains a hamiltonian cycle containing edges $x_1x_2, x_{m-1}x_m$ and different edges $u_jv_j$, where $u_j, v_j \in N_G(x_j)$ for $j = 2, 3, ..., m-1$.*

See Figure 1 for illustration in which for $x_3$ we have $u_3 = x_2$ and $v_3 = x_4$.

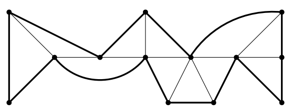

Figure 1: Hamiltonian cycle in a caterpillar for $m = 7$ (bold edges)

Every 2-block $B_t$ contains a hamitonian cycle in $(B_t)^2$ which is one of two types depending on labellings $m_i(B_t)$:

Let $m_i(B_t) \neq 2$ for every $i = 1, 2, ..., s$. If $B_t \cong C_3$, then we set $C_t = B_t$. Otherwise for at most 4 cutvertices $a, b, c, d$ it holds that $m_j(B_t) = 1$ for $j = a, b, c, d$ by condition 5). By Theorem 4, $(B_t)^2$ has a hamiltonian cycle $C_t$ containing 4 different edges $aa', bb', cc', dd'$ of $B_t$.

If $m_i(B_t) = 2$ for some $i \in \{1, 2, ..., s\}$, then at most one cutvertex $a$ has $m_a(B_t) = 1$ by condition 5). By Theorem 3, $(B_t)^2$ has a hamiltonian cycle $C_t$ containing 3 different edges $ii', ii'', aa'$ of $B_t$.

The union of hamiltonian cycles $C_t$ in $(B_t)^2$, for $t = 1, 2, ..., n$, hamiltonian cycles in the square of each catepillar (nontrivial component of $P_0$) and trivial components of $P_0$ is a connected spanning subgraph $S$ of $G^2$.

We construct a hamiltonian cycle $C$ in $G^2$ from $S$ repeating step by step the following procedure for every cutvertex $i$ of $G$ with $m_i(B) \geq 1$ for some 2-block $B$.

If $i$ does not exist, then $n = 0$ and $G = P_0$ is a caterpillar. Hence $S$ is a hamiltonian cycle in $G^2$. Otherwise we join all hamiltonian cycles from $S$ containing $i$ together with trivial components of $P_0$ containing $i$ to one cycle in the following way.

First assume that $\mathrm{bn}(i) = 0$.

By condition 6) we have $\sum_{t=1}^{n} m_i(B_t) \geq 2k_i - 2$. Without loss of generality for $k_i > 1$ we may assume that $m_i(B_1) \geq 1$, $m_i(B_2) \geq 1$ and $m_i(B_3) = m_i(B_4) = \cdots = m_i(B_{k_i}) = 2$, where $m_i(B_t)$ corresponds to the number of edges of $B_t$ incident to $i$ in $C_t$. If $k_i = 1$, then by condition 2) we have $m_i(B_1) \geq 1$.

We find a cycle $C^i$ on $\cup_{r=1}^{k_i} V(C_r) \cup L$, where $L$ is the set of all leaves incident to $i$, by appropriately replacing edges of $C_r \cap B_r$, $r = 1, 2, ..., k_i$, incident to $i$ (guaranteed by definition of $m_i(B_t)$) with edges of $G^2$ joining vertices in different $C_r$ adjacent to $i$ and leaves adjacent to $i$. Note that we preserve properties given by $m_j(B_t)$ for all $j \neq i$.

Now assume that $\mathrm{bn}(i) = 1$.

By condition 6) we have $\sum_{t=1}^{n} m_i(B_t) \geq 2k_i + 1 - 2 = 2k_i - 1$. Without loss of generality we may assume that $m_i(B_1) \geq 1$ and $m_i(B_2) = m_i(B_3) = \cdots = m_i(B_{k_i}) = 2$, where $m_i(B_t)$ corresponds to the number of edges of $B_t$ incident to $i$ in $C_t$. Let $T$ be the component of $P_0$ containing $i$.

If $T = K_2 = ii'$, where $i'$ is also a cutvertex of $G$ with $m_{i'}(B) \geq 1$ ($T$ is a trivial component of $P_0$), then we find a cycle $C^i$ on $\cup_{r=1}^{k_i} V(C_r) \cup V(T)$ containing the edge $ii'$ by appropriately replacing edges of $C_r \cap B_r$, $r = 1, 2, ..., k_i$, incident to $i$ (guaranteed by definition of $m_i(B_t)$) with edges of $G^2$ joining $i'$ and vertices in different $C_r$ adjacent to $i$. Also here we preserve properties given by $m_j(B_t)$ for all $j \neq i$.

If $T$ is a nontrivial component of $P_0$, then $T^2$ contains a hamiltonian cycle $C_T$ containing end-edges of any fixed longest path $P$ in $T$ (we choose end-edges containing cutvertices of $G$ with $m_i(B_t) \geq 1$) - see Observation above. Again we find a cycle $C^i$ on $\cup_{r=1}^{k_i} V(C_r) \cup V(C_T)$ by appropriately replacing edges of $C_r \cap B_r$, $r = 1, 2, ..., k_i$, incident to $i$ (guaranteed by definition of $m_i(B_t)$) and the end-edge $ii^*$ of $P$ with edges of $G^2$ joining $i^*$ and vertices in different $C_r$ adjacent to $i$. Again we preserve properties given by $m_j(B_t)$ for all $j \neq i$ and by $C_T$.

Finally assume that $\mathrm{bn}(i) = 2$.

By condition 6) we have $\sum_{t=1}^{n} m_i(B_t) \geq 2k_i + 2 - 2 = 2k_i$. It follows necessarily that $m_i(B_1) = m_i(B_2) = \cdots = m_i(B_{k_i}) = 2$, where $m_i(B_t)$ corresponds to the number of edges of $B_t$ incident to $i$ in $C_t$.

Let $T$ be the nontrivial component of $P_0$ containing $i$. Note that $i$ is not an endvertex of $T$ because of $\mathrm{bn}(i) = 2$. Then $T^2$ contains a hamiltonian cycle $C_T$ containing end-edges of any fixed longest path in $T$ (we choose end-edges containing cutvertices of $G$ with $m_i(B_t) \geq 1$) and an edge $u_iv_i$ of $G^2$ where $u_i, v_i \in N_G(i)$ (see Observation above). We find a cycle $C^i$ on $\cup_{r=1}^{k_i} V(C_r) \cup V(C_T)$ by appropriately replacing edges of $C_r \cap B_r$, $r = 1, 2, ..., k_i$, incident to $i$ (guaranteed by definition of $m_i(B_t)$) and the edge $u_iv_i$ of $P$ with edges of $G^2$ joining $u_i, v_i$ and vertices in different $C_r$ adjacent to $i$ if $k_i > 1$. If, however, $k_i = 1$, then $u_i$ and $v_i$ are joined to the neighbors of $C_r \cap B_r$ in $N_G(i)$. Also here we preserve properties given by $m_j(B_t)$ for all $j \neq i$ and by $C_T$.

Now we choose next cutvertex $i$ with $m_i(B) \geq 1$ for some 2-block $B$ successively and we use all cycles formed in the previous steps instead of previously formed cycles. Note that we preserve all properties given by $m_j(B)$ for all $j \neq i$ in every case. We stop with the hamiltonian cycle in $G^2$ as required.

Now assume that there is no labelling satisfying conditions 1) - 6), that is, the labelling $m_i(B_t)$ satisfying conditions 1), 2) and 3) is given and at least one of conditions 4), 5), 6) is violated. We show that there exists a class of graphs $G'$ with non-hamiltonian square but $\mathrm{bc}(G')$ and $\mathrm{bc}(G)$ are isomorphic.

*Condition 4) does not hold.*

Hence $\mathrm{bn}(i) \geq 3$ for at least one $i \in \{1, 2, ..., s\}$. Clearly this is a class of graphs $G'$ such that the square of every such graph $G'$ does not contain a hamiltonian cycle (if we try to construct a hamiltonian cycle in the square, then the degree of the cutvertex $i$ is at least 3, a contradiction), e.g. see the graph in Figure 2 a), where $H_1$ is an arbitrary connected graphs, $H_2$, $H_3$, $H_4$ are arbitrary connected graphs with at least one edge each and $\mathrm{bn}(i) = 3$. Note that conditions 5) and 6) may hold.

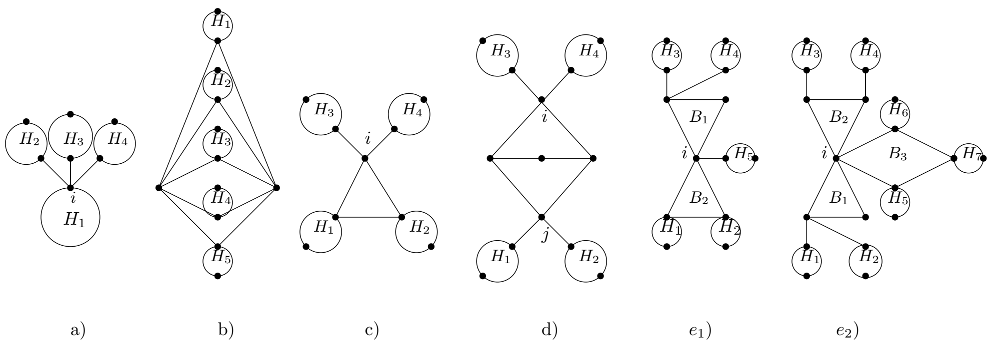


Figure 2: Graphs without hamiltonian square

*Condition 5) does not hold.*

Hence $\sum_{i=1}^{s} m_i(B) \geq 5$ for some 2-block $B$ and $m_i(B) < 2$ for all $i$ or $\sum_{j=1}^{s} m_j(B) \geq 4$ for some 2-block $B$ and $m_i(B) = 2$ for some $i \in \{1, 2, ..., s\}$.

First suppose that $k = \sum_{i=1}^{s} m_i(B) \geq 5$ for some 2-block $B$ of $G$ and $m_i(B) < 2$ for all $i$. Clearly $B$ has exactly $k$ cutvertices by condition 2). Then we exhange $B$ with $K_{2,k}$ where $k$ 2-valent vertices are cutvertices of $G$ and all other blocks with arbitrary blocks to get a class of graphs $G'$ such that $\mathrm{bc}(G')$ and $\mathrm{bc}(G)$ are isomorphic. The square of every such graph $G'$ does not contain a hamiltonian cycle (if we try to construct a hamiltonian cycle in the square, then the degree of at least one of the two $k$-valent vertices of $K_{2,k}$ is at least 3, a contradiction), e.g. see the graph in Figure 2 b), where $k = 5$ and $H_1, ..., H_5$ are arbitrary connected graphs with at least one edge each. Note that conditions 4), 6) and the second part of condition 5) may hold.

Now suppose that $\sum_{j=1}^{s} m_j(B) \geq 4$ for some 2-block $B$ and $m_i(B) = 2$ for some $i$. If $B$ contains at least 5 cutvertices of $G$, then we continue similarly as above. If $B$ contains $k$ cutvertices of $G$ where $2 \leq k \leq 4$, then without loss of generality we may assume that we tried to set the labelling $m_i(B_t)$ satisfying firstly condition 5) and subsequently condition 6). Hence $\mathrm{bn}(i) \geq 2$ and $\mathrm{bn}(j) \geq 2$ where $j$ is the second cutvertex of $G$ in $B$ if $k = 2$, otherwise we find a labelling $m_i(B_t)$ satisfying condition 5), a contradiction (see Algorithm 1 cases e) and f) below).

For $k = 3, 4$ we exhange $B$ with a cycle $C_k$ to get a class of graphs $G'$ such that $\mathrm{bc}(G')$ and $\mathrm{bc}(G)$ are isomorphic. The square of every such graph $G'$ does not contain a hamiltonian cycle (if we try to construct a hamiltonian cycle in the square, then the degree of the cutvertex $i$ is at least 3, a contradiction), e.g. see the graph in Figure 2 c), where $k = 3$ and $H_1, ..., H_4$ are arbitrary connected graphs with at least one edge each. Note that conditions 4), 6) and the first part of condition 5) may hold.

For $k = 2$, we exchange $B$ with $K_{2,3}$, where two of the three 2-valent vertices are $i$ and $j$, to get a class of graphs $G'$ such that $\mathrm{bc}(G')$ and $\mathrm{bc}(G)$ are isomorphic. The square of every such graph $G'$ does not contain a hamiltonian cycle (it is not possible to find a hamiltonian cycle in the square containing the third 2-valent vertex different from $i$, $j$, a contradiction), e.g. see the graph in Figure 2 d), where $H_1, ..., H_4$ are arbitrary connected graphs with at least one edge each. Note that conditions 4), 6) and the first part of condition 5) may hold.

*Condition 6) does not hold.*

Hence $\sum_{t=1}^{n} m_i(B_t) < 2k_i + \mathrm{bn}(i) - 2$ for some $i$ and consequently $m_i(B_t) = 1$ for at least $3 - \mathrm{bn}(i)$ 2-blocks containing $i$. Note that, clearly, $\mathrm{bn}(i) < 2$ with respect to condition 3).

Let $r$ be the number of 2-blocks with $m_i(B_t) = 1$. Each of these 2-blocks contains either exactly 2 cutvertices of $G$ or at least 3 cutvertices of $G$. Note that for 2-blocks containing only cutvertex $i$ we have $m_i(B_t) = 2$ (see Algorithm 1 case d) below). We exchange every 2-block containing exactly 2 cutvertices of $G$ with a cycle $C_3$ and every 2-block containing $k$ cutvertices of $G$, $k \geq 3$, with a cycle $C_k$. In the first case note that we assume without loss of generality that there is no labelling such that we switch values 1 and 2 for both cutvertices of this 2-block to get a permissible labelling (again see Algorithm 1 case e) below).

Since $r \geq 3 - \mathrm{bn}(i)$, by the exchanging 2-block mentioned above we get a class of graphs $G'$ such that $\mathrm{bc}(G')$ and $\mathrm{bc}(G)$ are isomorphic. The square of every such graph $G'$ does not contain a hamiltonian cycle (if we try to construct a hamiltonian cycle in the square, then the degree of the cutvertex $i$ is at least 3, a contradiction), e.g. see graphs in Figure 2 $e_1$) and $e_2$). For the graph in Figure 2 $e_1$) it holds that $r = 3 - \mathrm{bn}(i) = 3 - 1 = 2$, the 2-block $B_1$ has exactly 2 cutvertices of $G$, the 2-block $B_2$ has $k = 3$ cutvertices of $G$ (and hence $B_1$, $B_2$ are isomorphic to $C_3$) and $H_1, ..., H_5$ are arbitrary connected graphs with at least one edge. For the graph in Figure 2 $e_2$) it holds that $r = 3 - \mathrm{bn}(i) = 3 - 0 = 3$, the 2-block $B_1$ has exactly 2 cutvertices of $G$, the 2-block $B_2$ has $k = 3$ cutvertices of $G$, the 2-block $B_3$ has $k = 4$ cutvertices of $G$ (hence $B_1$, $B_2$ are isomorphic to $C_3$ and $B_3$ is isomorphic to $C_4$) and $H_1, ..., H_7$ are arbitrary connected graphs with at least one edge. Note that conditions 4) and 5) may hold.

This finishes the proof of Theorem 1. $\square$

If there is a graph $G$ such that every labelling $m_i(B_t)$ violates at least one of the conditions 4) - 6) of Theorem 1, then there is a graph $G'$ with $\mathrm{bc}(G') = \mathrm{bc}(G)$ such that $(G')^2$ is not hamiltonian as it has been shown in the proof of Theorem 1. On the other hand, if we are able to construct a labelling $m_i(B_t)$ satisfying conditions 1) - 6) using the following algorithm, then $G^2$ is hamiltonian as it has been shown in the proof of Theorem 1.

*ALGORITHM 1:*

Set $P_0 = G - \cup_{t=1}^{n} B_t$. If any component of $P_0$ is not a caterpillar, then $\mathrm{bn}(i) \geq 3$ for some $i \in \{1, 2, ..., s\}$ contradicting condition 4) in Theorem 1 and $G^2$ is not hamiltonian (e.g. see Figure 2 a)). STOP.

If $G = P_0$, then $G$ is a caterpillar, $n = 0$ and $G^2$ is hamiltonian (see Observation in the proof of Theorem 1) and $m_i(B_t)$ is not defined ($n = 0$). STOP.

If $G$ is a 2-block, $G^2$ is hamiltonian by Theorem 3 and $m_i(B_t)$ is not defined ($s = 0$ and $n = 1$). STOP.

We set $G_0 = G - P_0$ and $m_i(B_t) = 0$ if $i \notin V(B_t)$ for $i \in \{1, 2, ..., s\}$ and $t \in \{1, 2, ..., n\}$.

START

We choose a 2-block $B$ containing at most 1 cutvertex of $G_0$. Note that $B$ is either a component of $G_0$ or an endblock of some component of $G_0$. If such endblock does not exist, we choose 2-block $B$ as a component of $G_0 - H$ or an endblock of $G_0 - H$ where $H$ is the union of all 2-blocks for which the labelling $m_i(B_t)$ is already set. Let $c_1, c_2, ..., c_k$ be all cutvertices of $G$ contained in $B$, $k \geq 1$.

a) If $k \geq 5$, then by condition 2) $m_{c_i}(B) \geq 1$ for $i = 1, 2, ..., k$. Hence condition 5) in Theorem 1 does not hold and $G^2$ may not be hamiltonian (e.g. see Figure 2 b)). STOP.

b) If $k \geq 3$ and $\mathrm{bn}(c_i) = 2$ for some $i \in \{1, 2, ..., k\}$ , then by condition 3) $m_{c_i}(B) = 2$ and by 2) $m_{c_j}(B) \geq 1$ for $j = 1, 2, ..., k$. Hence condition 5) in Theorem 1 does not hold and $G^2$ may not be hamiltonian (e.g. see Figure 2 c)). STOP.

c) If $k = 2$ and $\mathrm{bn}(c_1) = \mathrm{bn}(c_2) = 2$, then by condition 3) $m_{c_1}(B) = 2$ and $m_{c_2}(B) = 2$. Hence condition 5) in Theorem 1 does not hold and $G^2$ may not be hamiltonian (e.g. see Figure 2 d)). STOP.

d) If $k = 1$, then we set $m_{c_1}(B) = 2$ (we maximize values $m_i(B_t)$ with respect to condition 6) in Theorem 1). Note that, if the labelling $m_i(B_t)$ is set for all 2-blocks incident with $c_1$, then condition 6) holds for cutvertex $c_1$ with respect to the choice of $B$.

   If the labelling $m_i(B_t)$ is set for all 2-blocks of $G$, then the labelling $m_i(B_t)$ satisfies the conditions of Theorem 1 and $G^2$ is hamiltonian. STOP.

   Otherwise we go to START.

e) If $k = 2$ and $\mathrm{bn}(c_i) \leq 1$ for $i \in \{1, 2\}$, then we set $m_{c_1}(B)$ and $m_{c_2}(B)$ in the following way (without loss of generality $i = 1$).

   Let $\mathrm{bn}(c_2) = 2$. Then we set $m_{c_1}(B) = 1$ and $m_{c_2}(B) = 2$ with respect to conditions 2), 3) and 5).

   Let $\mathrm{bn}(c_2) \leq 1$. Then for at least one of $c_1$, $c_2$ it holds that $m_{c_j}(B_t)$ for $j \in \{1, 2\}$ is set for all 2-blocks $B_t$ except $B$ with respect to the choice of $B$ (again without loss of generality $j = 1$). We set $m_{c_1}(B) = 1$ and we verify condition 6) for $c_1$. If it holds, then we set $m_{c_2}(B) = 2$ (again we maximize values $m_i(B_t)$ with respect to condition 6)). If condition 6) for $c_1$ does not hold for $m_{c_1}(B) = 1$, then we set $m_{c_1}(B) = 2$ and $m_{c_2}(B) = 1$.

   Now in both cases we verify condition 6) for $c_1$ and $c_2$ if the labelling $m_{c_1}(B_t)$ and $m_{c_2}(B_t)$ is set for all 2-blocks $B_t$.

   If condition 6) does not hold in at least one case, then $G^2$ may not be hamiltonian (e.g. see Figure 2 $e_1$)). STOP.

   Hence suppose that condition 6) holds for $c_1$, $c_2$ if $m_{c_1}(B_t)$, $m_{c_2}(B_t)$ is set for all $B_t$, respectively.

   If the labelling $m_i(B_t)$ is set for all 2-blocks, then the labelling $m_i(B_t)$ satisfies the conditions of Theorem 1 and $G^2$ is hamiltonian. STOP.

   Otherwise we go to START.

f) If $k \in \{3,4\}$ and $\mathrm{bn}(c_i) \leq 1$, then we set $m_{c_i}(B) = 1$ for $i = 1, 2, ..., k$. We verify condition 6) for all $c_i$ if the labelling $m_{c_i}(B_t)$ is set for all 2-blocks $B_t$.

If condition 6) does not hold in at least one case, then $G^2$ may not be hamiltonian (e.g. see Figure 2 $e_2$)). STOP.

Hence suppose that condition 6) holds for all $c_i$, $i = 1, 2, ..., k$, for which $m_{c_i}(B_t)$ is set for all $B_t$.

If the labelling $m_i(B_t)$ is set for all 2-blocks, then the labelling $m_i(B_t)$ satisfies the conditions of Theorem 1 and $G^2$ is hamiltonian. STOP.

Otherwise we go to START.

PROOF OF THEOREM 2

*Proof.* Let $x, y \in V(G)$. First we prove that there exists an $xy$-hamiltonian path $P$ in $G^2$ if there is no nontrivial bridge of $G$ and every block contains at most 2 cutvertices.

(A) Suppose that $x$ and $y$ are in the same block $B$ of $G$. We proceed by induction on $n$, where $n$ is the number of blocks of $G$, $n \geq 1$.

For $n = 1$, clearly $G = B$. If $B = K_2 = xy$, then $G$ is also the $xy$-hamiltonian path in $G^2$ as required. If $B$ is a 2-block, then by Theorem 6, $G^2 = B^2$ contains an $xy$-hamiltonian path $P$ as required.

Now suppose that the statement of Theorem 2 is true for every graph with $n$ blocks and $G$ is a graph with $n+1$ blocks, $n \geq 1$. We distinguish 2 cases.

- $B$ has exactly one cutvertex $c$.

  Without loss of generality we assume that $x \neq c$. If $B$ is a 2-block, then by Theorem 6, $B^2$ contains an $xy$-hamiltonian path $P_B$ containing an edge $cy'$ where $y'$ is a neighbor of $c$ in $B$. Note that $y' = x$ or $c = y$ is possible. If $B = K_2$, then $B = xy = y'c$ and $P_B = xy$ is an $xc$-hamiltonian path in $B^2$. By the induction hypothesis $(G - B)^2$ contains a $cc'$-hamiltonian path $P_G$ where $c'$ is a neighbor of $c$ in $G - B$. Then $P = P_B \cup P_G - cy' + y'c'$ is an $xy$-hamiltonian path in $G^2$ as required.

- $B$ has two cutvertices $c_1$, $c_2$.

  We denote by $G_1, G_2$ the two components of $G-B$ such that $c_i \in V(G_i)$ and let $c_i'$ be a neighbor of $c_i$ in $G_i$, $i = 1, 2$. By the induction hypothesis $(G_i)^2$ contains a $c_i c_i'$-hamiltonian path $P_{G_i}$, $i = 1, 2$.

  a) $c_i \notin \{x, y\}$ ($x$ and $y$ are not cutvertices).
     By Theorem 5, $B^2$ contains an $xy$-hamiltonian path $P_B$ containing the edges $c_i z_i$ where $z_i$ is a neighbor of $c_i$ in $B$, $i = 1, 2$. Note that $z_i \in \{x, y\}$ is possible.

  b) Up to symmetry $c_1 = x$ and $c_2 \neq y$ (either $x$ or $y$ is a cutvertex of $G$).
     By Theorem 6, $B^2$ contains an $xy$-hamiltonian path $P_B$ containing the edges $c_i z_i$ where $z_i$ is a neighbor of $c_i$ in $B$, $i = 1, 2$. Note that $z_1 = c_2$ or $z_2 = y$ is possible.

  c) $c_1 = x$ and $c_2 = y$ (similarly $c_1 = y$ and $c_2 = x$).
     By Theorem 7, $B^2$ contains an $xy$-hamiltonian path $P_B$ containing either the edges $c_i z_i$ where $z_i$ is a neighbor of $c_i$ in $B$, $i = 1, 2$, or the edges $c_1 z_1$, $uv$ where $z_1$ is a neighbor of $c_1$ in $B$ and $u, v$ are neighbors of $c_2$ in $B$.

  In all cases except the case c), if $uv$ is the edge of $P_B$,

  $$P = P_{G_1} \cup P_B \cup P_{G_2} - \{c_1 z_1, c_2 z_2\} \cup \{c_1' z_1, c_2' z_2\}$$

  is an $xy$-hamiltonian path in $G^2$ as required.

  It remains to find an $xy$-hamiltonian path in $G^2$ if $uv$ is the edge of $P_B$.

  If $G_2 = K_2 = c_2 c_2'$, then

  $$P = P_{G_1} \cup P_B - \{c_1 z_1, uv, c_2 c_2'\} \cup \{c_1' z_1, c_2' u, c_2' v\}$$

  is an $xy$-hamiltonian path in $G^2$ as required.

  If $G_2 \neq K_2$, then we prove that $(G_2)^2$ contains a hamiltonian cycle $C$ containing edges $c_2 u_2$, $c_2 v_2$ of $G_2$. Let $B_1, B_2, ..., B_k$ be all 2-blocks of $G_2$ containing $c_2$. By Theorem 3, for $i = 1, 2, ..., k$, $(B_i)^2$ contains a hamiltonian cycle $C_i'$ containing three different edges $c_2 u_2^i$, $c_2 v_2^i$, $y_i y_i'$ of $B_i$ where $y_i$ is the second cutvertex of $G_2$ in $B_i$ if it exists.

If $y_i$ exists, then we denote by $H_i$ a component of $G_2-(B_i-y_i)$ containing $y_i$. By the induction hypothesis $(H_i)^2$ contains a $y_id_i$-hamiltonian path $P_i$ where $d_i$ is a neighbor of $y_i$ in $H_i$. Then we set $C_i = C'_i \cup P_i - y_iy'_i + y'_id_i$. If $y_i$ does not exist, then we set $C_i = C'_i$.

Let $T$ be the set of all leaves of $G_2$ adjacent to $c_2$. Then we find a cycle $C$ on $\cup_{i=1}^{k} V(C_i) \cup T$ by appropriately replacing edges $c_2u_2^i$, $c_2v_2^i$ with edges of $G^2$ joining $u_2^i$, $v_2^i$ in different $C_i$ and leaves adjacent to $c_2$ (similarly as in the proof of Theorem 1) such that we preserve two edges ($c_2u_2^i$, $c_2v_2^i$ or $c_2l_1$, $c_2l_2$ where $l_1, l_2$ are two leaves of $G_2$ adjacent to $c_2$) as $c_2u_2$, $c_2v_2$.

Now

$$P = P_{G_1} \cup P_B \cup C - \{c_1z_1, uv, c_2u_2, c_2v_2\} \cup \{c'_1z_1, u_2u, v_2v\}$$

is an $xy$-hamiltonian path in $G^2$ as required.

(B) Suppose that $x$ and $y$ are in different blocks of $G$.

Let $P_G$ be any $xy$-path in $G$ and $c \in V(P_G) \setminus \{x, y\}$ be a cutvertex of $G$. Let $K$ be the component of $G - c$ containing $x$, $G_y = G - V(K)$ and $G_x = G - G_y$. Clearly $G_x \cup G_y = G$ and $G_x \cap G_y = c$. If $G_x$, $G_y$ are isomorphic to $K_2$, then we set $P_x = G_x$, $P_y = G_y$, respectively. If $G_x$, $G_y$ are 2-blocks, then $(G_x)^2$, $(G_y)^2$ contains an $xc$-hamiltonian path $P_x$, a $cy$-hamiltonian path $P_y$ by Theorem 6, respectively. We proceed by induction on $n$, where $n$ is the number of blocks of $G$, $n \geq 2$.

First assume that $G$ has exactly 2 blocks. Hence $G_x$, $G_y$ are isomorphic to $K_2$ or 2-blocks and $P = P_x \cup P_y$ is an $xy$-hamiltonian path in $G^2$ as required.

Now suppose that the statement of Theorem 2 is true for every graph with $n$ blocks and $G$ is a graph with $n+1$ blocks, $n \geq 2$. If $G_x$, $G_y$ is not a block, then by the induction hypothesis $(G_x)^2$, $(G_y)^2$ contains an $xc$-hamiltonian path $P_x$, a $cy$-hamiltonian path $P_y$, respectively. Then $P = P_x \cup P_y$ is an $xy$-hamiltonian path in $G^2$ as required.

Now it remains to prove that if there is a nontrivial bridge of $G$, then $G^2$ is not hamiltonian connected and if $G$ contains a block containing more than 2 cutvertices, then there is a graph $G'$ such that $\mathrm{bc}(G)$ and $\mathrm{bc}(G')$ are isomorphic but $(G')^2$ is not hamiltonian connected.

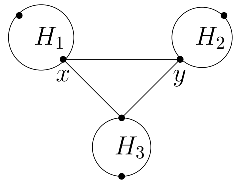


Figure 3: Graphs without $xy$-hamiltonian path in the square

Clearly, if there exists a nontrivial bridge $xy$ in $G$, then there is no $xy$-hamiltonian path in $G^2$ and $G^2$ is not hamiltonian connected.

Finally assume that $G$ contains a block $B$ containing $r$ cutvertices, where $r > 2$. Then we exhange $B$ with a cycle $C_r$ and all other blocks with arbitrary blocks to get a class of graphs $G'$ such that $\mathrm{bc}(G')$ and $\mathrm{bc}(G)$ are isomorphic. Clearly the square of every such graph $G'$ does not contain a hamiltonian path between arbitrary two cutvertices of $G'$ in $C_r$ and hence $(G')^2$ is not hamiltonian connected, e.g. with Figure 3, where $r = 3$ and $H_1, H_2, H_3$ are arbitrary connected graphs with at least one edge. □

Similarly as for Theorem 1 we state the following algorithm to verify conditions of Theorem 2.

*ALGORITHM 2:*

Let $G' = G - S$ where $S$ is the set of all endblocks of $G$. Let $\mathrm{cvn}_G(B)$ be the number of cutvertices of $G$ in $B$.

START

Find an endblock $B$ of $G'$.

- If $B$ is a bridge of $G'$, then $B$ is a nontrivial bridge of $G$ and $G^2$ is not hamiltonian connected. STOP.
- Let $B$ be a 2-block.
  - If $\mathrm{cvn}_G(B) > 2$, then $G^2$ may not be hamiltonian connected (e.g. see Figure 3). STOP.
  - If $\mathrm{cvn}_G(B) \leq 2$, then $G' := G' - B$.
    * If $G' = \emptyset$, then $G^2$ is hamiltonian connected. STOP.
    * If $G' \neq \emptyset$, then go to START.

In both algorithms in this paper, determining blocks and especially endblocks and bridges, cutvertices, block-cutvertex graphs, and the parameters $\mathrm{bn}(i)$, $\mathrm{cvn}_G(B)$ can be determined in polynomial time.

As a consequence, polynomial running time in Algorithm 2 is guaranteed. For, determining (potentially) not being Hamiltonian connected, can be determined instantly once a nontrivial bridge, a block with more than 2 cutvertices has been found. And deleting an endblock reduces the size of $G'$ linearly.

Now consider the running time of Algorithm 1. The first decision to be made is whether $P_0$ is a forest of caterpillars – this can be done in linear time. After that, at every step 'one chooses a 2-block $B$ as a component of $G_0 - H$ or an endblock of $G_0 - H$ where $H$ is the union of all 2-blocks for which the labelling $m_i(B_t)$ is already set'. Clearly, identifying such $B$ can be done in linear time. The same applies to working through the cases for defining the various values of $m_i(B)$.

Summarizing, it follows that both algorithms run in polynomial time. We note however, that these algorithms can only decide the existence or potential non-existence of hamiltonian cycles or hamiltonian paths in the square of graphs under consideration; they do not construct any such cycle or path.

# 3 Conclusion

The main results of this paper are Theorem 1 and Theorem 2. As we mention in Introduction Fleischner in [7] proved that in the square of graphs hamiltonicity and vertex-pancyclicity are equivalent concepts, and so are hamiltonian connectedness and panconnectedness. Hence we proved in fact that for graphs satisfying assumptions of Theorem 1, Theorem 2 the square of these graphs is vertex-pancyclic, panconnected, respectively.

As an easy corollary of Theorem 2 we get the following result.

**Corollary 8.** *Let $G$ be a block-chain. Then $G^2$ is panconnected if and only if every innerblock of $G$ is a 2-block.*

Moreover Corollary 8 is also the answer to Problem 1 stated by Chia et al. in [11] that for a graph $G$ with only two cutvertices it is true that $G^2$ is panconnected if and only if the unique block containing the two cutvertices is not the complete graph on two vertices.

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