# Improved Lower Bounds on the Domination Number of Hypercubes and Binary Codes with Covering Radius One 

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#### Abstract

A dominating set on an $n$-dimensional hypercube is equivalent to a binary covering code of length $n$ and covering radius 1 . It is still an open problem to determine the domination number $\gamma\left(Q_{n}\right)$ for $n \geq 10$ and $n \neq 2^{k}, 2^{k}-1(k \in \mathbb{N})$. When $n$ is a multiple of 6 , the best known lower bound is $\gamma\left(Q_{n}\right) \geq \frac{2^{n}}{n}$, given by Van Wee (1988). In this article, we present a new method using congruence properties due to Laurent Habsieger (1997) and obtain an improved lower bound $\gamma\left(Q_{n}\right) \geq \frac{(n-2) 2^{n}}{n^{2}-2 n-2}$ when $n$ is a multiple of 6 .


## 1. Introduction

Determining the domination number is an important optimization problem in graph theory, as well as an NP-complete problem in computational complexity theory [23]. The domination problem on hypercubes is equivalent to the covering code problem. Generic introductions to domination problems and covering codes can be found in [21][22].

The $n$-dimensional hypercube $Q_{n}$ is defined recursively in terms of the cartesian product of graphs as follows,

$$
\begin{equation*}
Q_{1}=K_{2}, \quad Q_{n}=K_{2} \square Q_{n-1} . \tag{1.1}
\end{equation*}
$$

Therefore, $Q_{n}$ can also be defined as

$$
\begin{equation*}
V\left(Q_{n}\right)=2^{\{1,2, \ldots, n\}}, E\left(Q_{n}\right)=\left\{u v: u, v \in V\left(Q_{n}\right), u \subset v, \text { and }|v \backslash u|=1\right\} . \tag{1.2}
\end{equation*}
$$

To avoid confusion, we use small brackets to express vertices. For instance, the vertex $\{2,3,5\}$ is written as $(2,3,5)$. Note that the vertex $\emptyset$ is written as (0). Moreover, we define $\left(a_{1}, \ldots, a_{i}, 0\right) \equiv\left(a_{1}, \ldots, a_{i}\right)$ to simplify some of our arguments.

Given $S \subseteq V\left(Q_{n}\right)$, we define the function $g$ to express all different members in the union of the coordinates of the vertices in S . That is,

$$
\begin{equation*}
g(S):=\bigcup_{v \in S} v \tag{1.3}
\end{equation*}
$$

[^0]Moreover, let $S[a]$ indicate the subset of S in which the number $a$ is contained in the coordinates of every vertex. That is,

$$
\begin{equation*}
S[a]:=\{v: a \in v \text { and } v \in S\} . \tag{1.4}
\end{equation*}
$$

e.g. For $S=\{(1,2,3),(2,5),(3,5)\}, g(S)=\{1,2,3,5\}$ and $S[5]=\{(2,5),(3,5)\}$.

For some $v \in V\left(Q_{n}\right)$ and $S \subseteq V\left(Q_{n}\right)$, we define the neighborhood of $v$ and $S$ as follows.

$$
\begin{gather*}
N_{i}[v]=\left\{u \in V\left(Q_{n}\right): d(u, v)=i\right\}, N_{i}[S]=\bigcup_{v \in S} N_{i}[v] . \\
N[v]=N_{0}[v] \cup N_{1}[v], N[S]=\bigcup_{v \in S} N[v] . \tag{1.5}
\end{gather*}
$$

Given $S \subseteq V\left(Q_{n}\right)$, if $N[S]=V\left(Q_{n}\right)$, then we call $S$ a dominating set of $Q_{n}$. If there does not exist $S^{\prime} \subseteq V\left(Q_{n}\right)$ such that $\left|S^{\prime}\right|<|S|$ and $N\left[S^{\prime}\right]=V\left(Q_{n}\right)$, then we call $S$ a minimum dominating set of $Q_{n}$, and call $|S|$ the domination number of $Q_{n}$. In this article, we denote a given dominating set as $D$, and denote the domination number as $\gamma\left(Q_{n}\right)$. Tables 1 and 2 in the appendix give the latest results of the upper and lower bounds on $\gamma\left(Q_{n}\right)$. The summarization is due to Gerzson Kéri [2][3].

Definition 1. Given a dominating set $D$ and $S \subseteq V\left(Q_{n}\right)$, we denote the excess of $D$ on $S$ by $\delta_{S}(D)$, which is defined as $\sum_{v \in S}(|N[v] \cap D|-1)$. When $D$ is clear, we briefly write $\delta_{S}:=\delta_{S}(D)$. Also, if $S=\{v\}$, then $\delta_{v}:=\delta_{\{v\}}$.

The term excess has been used in many previous works, such as [1][4]. We further define the symbols below. Likewise, the $D$ can be omitted if it's clear.

$$
\begin{align*}
V \delta^{x}(D) & :=\left\{v: \delta_{v}(D)=x \text { and } v \in V\left(Q_{n}\right)\right\}, \\
V \delta(D) & :=\bigcup_{x \geq 1} V \delta^{x}(D), \quad C(D):=\bigcup_{x \geq 2} V \delta^{x}(D) . \tag{1.6}
\end{align*}
$$

Previous studies came up with various congruence properties of $\delta_{N_{i}[v]}(D)$, which help to estimate $\delta_{V\left(Q_{n}\right)}(D)$, and thus obtained the lower bounds on $\gamma\left(Q_{n}\right)$ due to the relation $\delta_{V\left(Q_{n}\right)}(D)=(n+1)|D|-\left|V\left(Q_{n}\right)\right|$ from [4]. Theorem 1 is a segment of the properties given by Laurent Habsieger [1], which we will apply.

Theorem 1. (Habsieger) When $n$ is a multiple of 6 ,

$$
\begin{gather*}
\delta_{N[v]}(D) \equiv 1(\bmod 2), \text { if } v \notin D .  \tag{1.7}\\
\delta_{N[v]}(D) \equiv 0(\bmod 2), \text { if } v \in D .  \tag{1.8}\\
\delta_{N_{1}[v]}(D)+\delta_{N_{2}[v]}(D) \equiv 0(\bmod 3) . \tag{1.9}
\end{gather*}
$$

We then put forward the pivotal concept throughout this article, surfeit. Although it seems closely related to excess, we shall demonstrate that such further analyzation is enough to improve the known bounds.
Definition 2. Given a dominating set $D$ and $S \subseteq V\left(Q_{n}\right)$, we denote the surfeit of $D$ on $S$ by $\zeta_{S}(D)$, which is defined as $\sum_{v \in S \backslash D}\left(\delta_{N[v]}(D)-1\right)$. When $D$ is clear, we briefly write $\zeta_{S}:=\zeta_{S}(D)$.

We further define the symbols below. Likewise, the $D$ can be omitted if it's clear.

$$
\begin{equation*}
V \zeta^{x}(D):=\left\{v: \delta_{N[v]}(D)=x+1 \text { and } v \in V\left(Q_{n}\right) \backslash D\right\}, V \zeta(D):=\bigcup_{x \geq 1} V \zeta^{x}(D) \tag{1.10}
\end{equation*}
$$

We look into the cases when $n$ is a multiple of 6 . By calculating $\zeta_{V\left(Q_{n}\right)}(D)$ using two different methods, we show that it leads to a contradiction if $\gamma\left(Q_{n}\right)$ is too small.

## 2. Generalities

All the arguments are considered in the cases when $n$ is a multiple of 6 .
Given an arbitrary value $\gamma^{*}$, we assume that there exists a dominating set $D$ satisfying $|D|=\gamma^{*}$. We calculate $\zeta_{V\left(Q_{n}\right)}$ using two different methods, and write the value we obtain as $\zeta_{m 1}$ and $\zeta_{m 2}$, respectively. A dominating set should lead to $\zeta_{m 1}=\zeta_{m 2}$. However, we will show that there must be $\zeta_{m 2}>\zeta_{m 1}$ when $\gamma^{*}$ is too small, implying that such dominating set $D$ cannot exist, so $\gamma\left(Q_{n}\right)>\gamma^{*}$.

For all $v \in V\left(Q_{n}\right) \backslash D$, we have $\delta_{N[v]} \geq 1$ by (1.7), so the first method to calculate $\zeta_{V\left(Q_{n}\right)}$ holds. Note that for all $u \in V \delta^{x}$, we have $|N[u] \backslash D|=n-x$, and $\sum_{x \in \mathbb{N}} x\left|V \delta^{x}\right|=\delta_{V\left(Q_{n}\right)}$. The value obtained this way is written as $\zeta_{m 1}$ :

$$
\begin{align*}
\zeta_{V\left(Q_{n}\right)} & =\sum_{x \in \mathbb{N}} \sum_{u \in V \delta^{x}} x(n-x)-\left|V\left(Q_{n}\right) \backslash D\right| \\
& =\sum_{x \in \mathbb{N}} x(n-x)\left|V \delta^{x}\right|-2^{n}+|D|  \tag{2.1}\\
& =(n-1) \delta_{V\left(Q_{n}\right)}-2^{n}+|D|-\sum_{x \in \mathbb{N}} x(x-1)\left|V \delta^{x}\right|=: \zeta_{m 1} .
\end{align*}
$$

$\zeta_{m 1}$ attains its maximum when $C=\emptyset$. We write this value as $\zeta_{\text {max }}$.

$$
\begin{equation*}
\zeta_{m 1} \leq \zeta_{\max }:=(n-1) \delta_{V\left(Q_{n}\right)}-2^{n}+|D| \tag{2.2}
\end{equation*}
$$

Let us consider another method to calculate $\zeta_{V\left(Q_{n}\right)}$. The value obtained this way is written as $\zeta_{m 2}$ :

$$
\begin{equation*}
\zeta_{V\left(Q_{n}\right)}=\sum_{i \geq 0}\left(2 i\left|V \zeta^{2 i}\right|+(2 i+1)\left|V \zeta^{2 i+1}\right|\right)=\sum_{i \geq 1} 2 i\left|V \zeta^{2 i}\right|=: \zeta_{m 2} . \tag{2.3}
\end{equation*}
$$

Note that by (1.7) we have $V \zeta^{2 i+1}=\emptyset$.

Lemma 1. For all $u \in V \delta^{1}$, if $d(u, C) \geq 3$, then $|N[u] \cap V \zeta| \geq 3$.
Proof. Assume without loss of generality that $u=(0), N[u] \cap D=\{(1),(a)\}$, where $a \in\{0,2,3, \ldots, n\}$, then $\{(0),(1, a)\} \subset V \delta^{1}$. For convenience we assume that $a \neq 0$, since the case $a=0$ can be dealt with similarly. Let $S=V \delta \cap\left(N_{1}[u] \cup N_{2}[u]\right) \backslash\{(1, a)\}$. Applying (1.7) and (1.8) on the vertices in $N_{1}[u]$, we have the following:

$$
\begin{equation*}
\text { For all } k \in\{1,2, \ldots, n\}, \text { there is }|S[k]| \equiv 0(\bmod 2) \tag{2.4}
\end{equation*}
$$

Moreover, by applying (1.9) on $u$, we have $|S| \equiv 2(\bmod 3)$. In particular, $|S| \geq 5$, otherwise there exists $k \in\{1,2, \ldots, n\}$ such that $|S[k]|=1$, contradicting (2.4).

Suppose that $|N[u] \cap V \zeta| \leq 2, N[u] \cap V \zeta \subseteq\{(b),(c)\}$ where $b, c \in\{0,2,3, \ldots, n\} \backslash\{(a)\}$, then $g(S) \subseteq\{1, a, b, c\}$. So let $T=\{(1),(a),(b),(c),(1, b),(1, c),(a, b),(a, c),(b, c)\}$, then $S \subseteq T$. If $b=0$, then $S=T \backslash\{(0)\}=\{(1),(a),(c),(1, c),(a, c)\}$, contradicting (2.4). Hence $b, c \neq 0,(0) \notin N[u] \cap V \zeta,\left|S \cap N_{1}[u]\right|=0, S=T \cap N_{2}[u]=\{(1, b),(1, c),(a, b),(a, c),(b, c)\}$, but this still contradicts (2.4). Therefore, $|N[u] \cap V \zeta| \geq 3$.

In other words, for each $u \in V \delta^{1}$, there must be $d(u, C) \leq 2$ if $|N[u] \cap V \zeta| \leq 2$. In Lemma 2 we will show that $|N[u] \cap V \zeta| \leq 2$ gives us more rigorous conditions. To simplify our arguments, given $v \in C$, we divide $\left(N_{1}[v] \cup N_{2}[v]\right) \cap V \delta^{1}$ into the following vertex sets:

$$
\begin{gather*}
T_{1}(v):=\left(N_{1}[v] \backslash D\right) \cap V \delta^{1} ; \\
T_{2}(v):=\left\{u \in N_{2}[v] \cap V \delta^{1}:|N[u] \cap N[v] \cap D|=0\right\} ; \\
T_{3}(v):=\left\{u \in N_{2}[v] \cap V \delta^{1}:|N[u] \cap N[v] \cap D|=2\right\} ;  \tag{2.5}\\
T_{4}(v):=\left\{u \in N_{2}[v] \cap V \delta^{1}:|N[u] \cap N[v] \cap D|=1\right\} ; \text { and } \\
T_{5}(v):=\left(N_{1}[v] \cap D\right) \cap V \delta^{1} .
\end{gather*}
$$

Lemma 2. For all $u \in V \delta^{1}$, if $|N[u] \cap V \zeta| \leq 2$, then $|N[u] \cap V \zeta|=2$, and the following four claims hold.

1. There exists $v \in C \backslash D$ such that $u \in T_{1}(v) \cup T_{2}(v)$, or there exists $v \in C \cap D$ such that $u \in T_{2}(v)$.
2. For all $v \in C \backslash D$ such that $u \in T_{1}(v)$, we have

$$
\begin{equation*}
\left|N_{1}[v] \cap V \delta\right| \leq 3 \tag{2.6}
\end{equation*}
$$

3. For all $v \in C$ such that $u \in T_{2}(v) \backslash D$, if we rename the coordinates so that $v=(0)$, $u=(a, b)$, then

$$
\begin{equation*}
(a),(b) \notin V \delta \text { and }\left|\left(N_{2}[v] \cap V \delta\right)[a]\right|,\left|\left(N_{2}[v] \cap V \delta\right)[b]\right| \leq 3 . \tag{2.7}
\end{equation*}
$$

4. For all $v \in C$ such that $u \in T_{2}(v) \cap D$, if we rename the coordinates so that $v=(0)$, $u=(a, b)$, then

$$
\begin{equation*}
\left|\left(N_{2}[v] \cap V \delta\right)[a]\right|,\left|\left(N_{2}[v] \cap V \delta\right)[b]\right| \leq 2 . \tag{2.8}
\end{equation*}
$$

Proof. Given $v \in C$, we will prove the following statements.
(A) If $u \in T_{1}(v), v \notin D$ and $|N[u] \cap V \zeta| \leq 2$, then $|N[u] \cap V \zeta|=2$ and (2.6) holds for $v$.
(B) If $u \in T_{2}(v) \backslash D$ and $|N[u] \cap V \zeta| \leq 2$, then $|N[u] \cap V \zeta|=2$ and (2.7) holds for $v$.
(C) If $u \in T_{2}(v) \cap D$ and $|N[u] \cap V \zeta| \leq 2$, then $|N[u] \cap V \zeta|=2$ and (2.8) holds for $v$.
(D) If $u \in T_{1}(v), v \in D$ and $|N[u] \cap V \zeta| \leq 2$, then $|N[u] \cap V \zeta|=2$ and there exists some $v^{\prime} \in C \backslash D$ such that $u \in T_{1}\left(v^{\prime}\right)$.
(E) If $u \in T_{3}(v) \cup T_{4}(v) \cup T_{5}(v)$ and $u \notin T_{1}\left(v^{\prime}\right) \cup T_{2}\left(v^{\prime}\right)$ for all $v^{\prime} \in C$, then $|N[u] \cap V \zeta| \geq 3$.
(A), (B), and (C) prove Claims 2, 3, 4 in Lemma 2. By Lemma 1 there exists $v_{0} \in C$ with $u \in \bigcup_{1 \leq i \leq 5} T_{i}\left(v_{0}\right)$, so Claim 1 follows by an application of (D) and (E) with $v=v_{0}$.
Below, (A) and (D) follow from Case 1, (B) and (C) follow from Case 2, while (E) follows from Cases 3 to 5 . We assume without loss of generality that $v=(0)$.

Case 1-(1) $u \in T_{1}(v)$ and $v \notin D$.
$\{u, v\} \subseteq N[u] \cap V \zeta$, so $|N[u] \cap V \zeta| \geq 2$. If equality holds, then $N[u] \cap V \zeta=\{u, v\}$. Let $u=(a) \notin D$. Since $u \in V \delta^{1}$, we have $|g(N[u] \cap D)|=3$. So if $\left|N_{1}[v] \cap V \delta\right| \geq 4$, then there exists $(k) \in N_{1}[v] \cap V \delta$ such that $k \notin g(N[u] \cap D),(a, k) \in N[u] \cap V \zeta$, which is a contradiction. Hence $\left|N_{1}[v] \cap V \delta\right| \leq 3$ and (A) is proved.

Case 1-(2) $u \in T_{1}(v)$ and $v \in D$.
If $\delta_{v} \geq 3$, then assume without loss of generality that (1), (2), (3) $\in D$ and that $u=$ (4). $u \notin C$, so $|\{(1,4),(2,4),(3,4)\} \cap D| \leq 1$. Note that $\{u,(1,4),(2,4),(3,4)\} \backslash D \subseteq N[u] \cap V \zeta$, so $|N[u] \cap V \zeta| \geq 3$. Therefore, if $|N[u] \cap V \zeta| \leq 2$, then $\delta_{v}=2$.

Let $N[v] \cap D=\{(0),(1),(2)\}$ and $u=(a) \notin D$. We have $(1),(2),(a) \in V \delta$, so for $w \in\{(1, a),(2, a)\}$, there is $w \notin N[u] \cap V \zeta$ if and only if $w \in D$. Thus, we assume that $\{(1, a),(2, a)\} \cap D=\{(1, a)\}, N[u] \cap V \zeta=\{u,(2, a)\}$.

There does not exist $k \in\{3,4, \ldots, n\} \backslash\{a\}$ such that $(a, k) \in N[u] \cap V \zeta$. Therefore,

$$
\begin{equation*}
\left(N_{1}[u] \cup N_{2}[u]\right) \cap V \delta \subseteq\{(0),(1),(2),(1, a),(2, a),(1,2, a)\} . \tag{}
\end{equation*}
$$

Applying (1.7) and (1.9) on $u$, we get

$$
\begin{equation*}
\delta_{(0)}+\delta_{(1)}+\delta_{(2)}+\delta_{(1, a)}+\delta_{(2, a)}+\delta_{(1,2, a)} \equiv 0(\bmod 3), \tag{2.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\delta_{(1, a)}+\delta_{(2, a)} \equiv 0(\bmod 2) . \tag{2.10}
\end{equation*}
$$

Moreover, $1 \leq \delta_{(1, a)} \leq 2$ since by $\left(^{*}\right)$ we know that $N[(1, a)] \cap D \subseteq\{(1),(1, a),(1,2, a)\}$.

If $\delta_{(1, a)}=2$, then $(1,2, a) \in D$ and thus $\delta_{(2, a)} \geq 1$. By (2.10) we know $\delta_{(2, a)} \geq 2$.
If $\delta_{(1, a)}=1$ and $(1,2) \notin D$, then by $\left({ }^{*}\right)$ we know $\delta_{(1)}=2, \delta_{(2)}=1$, and $\delta_{(1,2, a)}=0$. Now that $\delta_{(0)}+\delta_{(1)}+\delta_{(2)}+\delta_{(1, a)}+\delta_{(1,2, a)}=6$, and by (2.10) we know $\delta_{(2, a)} \geq 1$, so (2.9) suggests that $\delta_{(2, a)} \geq 3$.

If $\delta_{(1, a)}=1$ and $(1,2) \in D$, then by $\left(^{*}\right)$ we know $\delta_{(1)}=3, \delta_{(2)}=2$, and $\delta_{(1,2, a)}=1$. Now that $\delta_{(0)}+\delta_{(1)}+\delta_{(2)}+\delta_{(1, a)}+\delta_{(1,2, a)}=9$, and by (2.10) we know $\delta_{(2, a)} \geq 1$, so (2.9) suggests that $\delta_{(2, a)} \geq 3$.

Therefore, there must be $(2, a) \in C \backslash D$ and $u \in T_{1}((2, a))$. This shows that there exists $v^{\prime} \in C \backslash D$ such that $u \in T_{1}\left(v^{\prime}\right)$, and ( D ) is proved.

Case 2. $u \in T_{2}(v)$.
Let $u=(a, b)$ where $(a),(b) \notin D$, then $\{(a),(b)\} \subseteq N[u] \cap V \zeta$, so $|N[u] \cap V \zeta| \geq 2$. Assume that equality holds. Consider the case $u \notin D$. We have $(a),(b) \notin V \delta$, for otherwise $u \in N[u] \cap V \zeta$, which is a contradiction. Also, $|g(N[u] \cap D)|=4$, so if $\left|\left(N_{2}[v] \cap V \delta\right)[a]\right| \geq 4$, then there exists $w \in\left(N_{2}[v] \cap V \delta\right)[a], k \in w \backslash\{a\}$ such that

$$
k \notin g(N[u] \cap D),(a, b, k) \in N[u] \cap V \zeta,
$$

which is a contradiction. Likewise, we have $\left|\left(N_{2}[v] \cap V \delta\right)[b]\right| \leq 3$, so (B) is proved. The same argument can be applied to the case $u \in D$ and prove (C).

The following cases together prove (E).
Case 3. $u \in T_{3}(v)$ and $u \notin T_{1}\left(v^{\prime}\right) \cup T_{2}\left(v^{\prime}\right)$ for all $v^{\prime} \in C$.
Let (1), (2) $\in D$ and $u=(1,2)$. We prove $|N[u] \cap V \zeta| \geq 3$ by contradiction, assuming $|N[u] \cap V \zeta| \leq 2$. If $(a) \in D$ for some $a \in\{0,3,4, \ldots, n\}$, then $(1,2, a) \in N[u] \cap V \zeta$. This implies $\delta_{v} \leq 3$, and since such an $a$ exists by $\delta_{v} \geq 2$, we have $|N[u] \cap V \zeta| \geq 1$.

If $|N[u] \cap V \zeta|=1$, we let $N[u] \cap V \zeta=\{(1,2, a)\}$, where $a \in\{0,3,4, \ldots, n\}$, then $\left(N_{1}[u] \cup N_{2}[u]\right) \cap V \delta \subseteq\{(0),(1, a),(2, a)\}$ and $\delta_{(1, a)}, \delta_{(2, a)} \geq 1$. We know $\{(1, a),(2, a)\} \cap C \neq \emptyset$ by applying (1.9) on $u$. Let $(1, a) \in C$, then there exists $b \in\{0,3,4, \ldots, n\} \backslash\{a\}$ such that

$$
(1, a, b) \in D,(1, b) \in V \delta,(1,2, b) \in N[u] \cap V \zeta,
$$

which is a contradiction. Therefore, $|N[u] \cap V \zeta|=2$.
Now let $N[u] \cap V \zeta=\{(1,2,3),(1,2, k)\}$, where $k \in\{0,4,5, \ldots, n\}$, then

$$
\begin{equation*}
\left(N_{1}[u] \cup N_{2}[u]\right) \cap V \delta \subseteq\{(0),(1,3),(2,3),(1, k),(2, k),(1,2,3, k)\} \tag{**}
\end{equation*}
$$

Thus, $N[v] \cap D \subseteq\{(1),(2),(3),(k)\}$. We denote the excesses as $\delta_{(0)}=: o, \delta_{(1,3)}=: p$, $\delta_{(2,3)}=: q, \delta_{(1, k)}=: r, \delta_{(2, k)}=: s, \delta_{(1,2,3, k)}=: t$, respectively. By (1.7), (1.8) and (1.9) we
derive the following relations (note that $\delta_{(1,2)}=1$ ):

$$
\begin{align*}
& o+p+r \equiv 1(\bmod 2),  \tag{I}\\
& o+q+s \equiv 1(\operatorname{sod} 2),  \tag{II}\\
& o \text { since } \delta_{(1)}+\delta_{N_{1}[(1)]} \equiv 0(\bmod 2) ;  \tag{III}\\
& p+q+t \equiv 0(\bmod 2),  \tag{IV}\\
& r+s+\delta_{N_{1}[(2)]} \equiv 0(\bmod 2) ;  \tag{V}\\
& r+s \equiv 0(\bmod 2), \\
& \delta_{(1,2,3)}+\delta_{N_{1}[(1,2,3)]} \equiv 1(\bmod 2) ; \\
& o+p+q+r+s+t \equiv 0(\bmod 3), \\
& \delta_{(1,2, k)}+\delta_{N_{1}[(1,2, k)]} \equiv 1(\bmod 2) ; \\
& \delta_{N_{1}[u]}+\delta_{N_{2}[u]} \equiv 0(\bmod 3) .
\end{align*}
$$

If $o=3$, then $N[v] \cap D=\{(1),(2),(3),(k)\}$, and the following relations hold. Note that the restrictions leading to these results are due to $\left({ }^{* *}\right)$.

$$
\begin{aligned}
& 1 \leq p \leq 2, \text { since } N[(1,3)] \cap D \subseteq\{(1),(3),(1,3, k)\} \text { and }(1),(3) \in D ; \\
& 1 \leq q \leq 2, \text { since } N[(2,3)] \cap D \subseteq\{(2),(3),(2,3, k)\} \text { and }(2),(3) \in D ; \\
& 1 \leq r \leq 2, \text { since } N[(1, k)] \cap D \subseteq\{(1),(k),(1,3, k)\} \text { and }(1),(k) \in D ; \\
& 1 \leq s \leq 2 \text {, since } N[(2, k)] \cap D \subseteq\{(2),(k),(2,3, k)\} \text { and }(2),(k) \in D ; \\
& \quad t \leq 1 \text {, otherwise } u \in T_{1}((1,2,3, k)) \cup T_{2}((1,2,3, k)) .
\end{aligned}
$$

Now if $o+p+q+r+s+t=12$, then $p=q=r=s=2$ and $t=1$, contradicting (III), so the only possibility is $o+p+q+r+s+t=9$. However, this implies that $(p+q+t)+(r+s+t)-t=6$. By (III) and (IV) we know $t=0$, and together with (I) and (II) we know that $p, q, r, s$ have the same parity, which is impossible. Therefore, $|N[u] \cap V \zeta| \geq 3$.

If $o=2$, then we assume without loss of generality that $N[v] \cap D \subseteq\{(1),(2),(k)\}$. This time we obtain $p \leq 1, q \leq 1,1 \leq r \leq 2,1 \leq s \leq 2, t \leq 1$. If $o+p+q+r+s+t=9$, then $p=q=1, r=s=2, t=1$, contradicting (III), so the only possibility left is $o+p+q+r+s+t=6$. This implies that $(p+q+t)+(r+s+t)-t=4$. By (III) and (IV) we know $t=0$, and $p, q$ as well as $r, s$ have the same parity. Thus $p=q$ and $r=s$, implying $p+r=2$, which contradicts (I). Therefore, $|N[u] \cap V \zeta| \geq 3$.

Case 4. $u \in T_{5}(v)$ and $u \notin T_{1}\left(v^{\prime}\right) \cup T_{2}\left(v^{\prime}\right)$ for all $v^{\prime} \in C$.
If $v \in D$, then we can prove $|N[u] \cap V \zeta| \geq 3$ using the same method as in Case 3 . We sketch our arguments in a simplified version, for they are highly similar to those in Case 3:

Let (1), (2) $\in D$ and $u=(1)$. We prove our claim by contradiction, assuming that $|N[u] \cap V \zeta| \leq 2$. Like in case 3 we see $\delta_{v} \leq 3$ and $|N[u] \cap V \zeta|=2$. Assume without loss of generality that $N[u] \cap V \zeta=\{(1,2),(1,3)\}$, then

$$
\begin{equation*}
\left(N_{1}[u] \cup N_{2}[u]\right) \cap V \delta \subseteq\{(0),(2),(3),(1,2),(1,3),(1,2,3)\} . \tag{***}
\end{equation*}
$$

We denote the excesses as $\delta_{(0)}=: o, \delta_{(2)}=: p, \delta_{(3)}=: q, \delta_{(1,2)}=: r, \delta_{(1,3)}=: s, \delta_{(1,2,3)}=: t$,
respectively. By (1.7), (1.8), and (1.9) we derive the following relations (note that $\delta_{(1)}=1$ ):

$$
\begin{align*}
& o+r+s \equiv 1(\bmod 2),  \tag{VI}\\
& o+p+q \equiv 1(\operatorname{sod} 2),  \tag{VII}\\
& o \text { since } \delta_{(1)}+\delta_{N_{1}[(1)]} \equiv 0(\bmod 2) ; \\
& p+r+t \equiv 0(\bmod 2),  \tag{IX}\\
& \text { since } \delta_{(1,2)}+\delta_{N_{1}[(0)]} \equiv 0(\bmod 2) ;  \tag{X}\\
& q+s+t \equiv 0(\bmod 2), \\
& \text { since } \delta_{(1,3)}+\delta_{N_{1}[(1,3)]} \equiv 1(\bmod 2) ; \\
& o+p+q+r+s+t \equiv 0(\bmod 2) ;
\end{align*}
$$

If $o=3$, then $N[v] \cap D=\{(0),(1),(2),(3)\}$, and the following relations hold. Note that the restrictions leading to these results are due to $\left({ }^{* * *}\right)$.

$$
\begin{aligned}
& 1 \leq p \leq 2, \text { since } N[(2)] \cap D \subseteq\{(0),(2),(2,3)\} \text { and }(0),(2) \in D ; \\
& 1 \leq q \leq 2, \text { since } N[(3)] \cap D \subseteq\{(0),(3),(2,3)\} \text { and }(0),(3) \in D ; \\
& 1 \leq r \leq 2 \text {, since } N[(1,2)] \cap D \subseteq\{(1),(2),(1,2,3)\} \text { and }(1),(2) \in D ; \\
& 1 \leq s \leq 2 \text {, since } N[(1,3)] \cap D \subseteq\{(1),(3),(1,2,3)\} \text { and }(1),(3) \in D ; \\
& \quad t \leq 1 \text {, otherwise } u \in T_{2}((1,2,3)) .
\end{aligned}
$$

We have $o+p+q+r+s+t=9$ or 12. The latter contradicts (VIII), while the former suggests that $(p+r+t)+(q+s+t)-t=6$, and using (VI) to (IX) we know $t=0$ and $p, q, r, s$ have the same parity, which is impossible.

If $o=2$, then $N[v] \cap D \subseteq\{(0),(1),(2)\}$. This time we obtain $1 \leq p \leq 2, q \leq 1$, $1 \leq r \leq 2, s \leq 1, t \leq 1$, so we have $o+p+q+r+s+t=6$ or 9 , but again we can easily lead to contradictions using (VI) to (X). Therefore, $|N[u] \cap V \zeta| \geq 3$.

If $v \notin D$, then let $u=(1) \in D$. By $\delta_{v} \geq 2$ we may assume $N[v] \cap D \supseteq\{(1),(2),(3)\}$. For $w \in\{(1,2),(1,3)\}$, we have $w \notin N[u] \cap V \zeta$ if and only if $w \in D$. Moreover, $(0) \in N[u] \cap V \zeta$. So if $|\{(1,2),(1,3)\} \cap D|=0$, then $|N[u] \cap V \zeta| \geq 3$. If not, let $(1,2) \in D$, then $(1,2) \in C \cap D$ and $u \in T_{5}((1,2))$, implying $|N[u] \cap V \zeta| \geq 3$.

Case 5. $u \in T_{4}(v)$ and $u \notin T_{1}\left(v^{\prime}\right) \cup T_{2}\left(v^{\prime}\right)$ for all $v^{\prime} \in C$.
If $v \in D$, then we let $(1),(2) \in N[v] \cap D, u=(1, a)$, where $(a) \notin D$.
We have $\{(0),(1),(1,2),(1, a)\} \subset V \delta$. Therefore, if $\{(1, a),(1,2, a)\} \cap D=\emptyset$, then $\{(a),(1, a),(1,2, a)\} \subseteq N[u] \cap V \zeta$; if $(1,2, a) \in D$, then $(1,2) \in C$ and $u \in T_{3}((1,2))$; if $(1, a) \in D$, then $(1) \in C$ and $u \in T_{5}((1))$.

On the other hand, if $v \notin D$, then let (1), (2), (3) $\in N[v] \cap D, u=(1, a)$, where $(a) \notin D$. We have $\{(1,2),(1,3)\} \subset V \delta$ and $(a) \in N[u] \cap V \zeta$. Thus, if $\{(1,2, a),(1,3, a)\} \cap D=\emptyset$, then $\{(a),(1,2, a),(1,3, a)\} \subseteq N[u] \cap V \zeta$; if $\{(1,2, a),(1,3, a)\} \cap D \neq \emptyset$, then there exists $w \in\{(1,2),(1,3)\} \cap C$ such that $u \in T_{3}(w)$.

By Case 3 and Case 4, every possible condition above implies that $|N[u] \cap V \zeta| \geq 3$.

Given $v \in C$, we define $S_{i}(v):=\left\{u \in T_{i}(v):|N[u] \cap V \zeta|=2\right\}$. Lemma 3 gives an upper bound for $\left|S_{1}(v) \cup S_{2}(v)\right|$ when $v \notin D$ and an upper bound for $\left|S_{2}(v)\right|$ when $v \in D$.

Lemma 3. For $v \in C$, we have $\left|S_{1}(v) \cup S_{2}(v)\right| \leq \frac{3}{2}\left(n-\delta_{v}\right)$ if $v \notin D$, and $\left|S_{2}(v)\right| \leq \frac{3}{2}\left(n-\delta_{v}\right)$ if $v \in D$.

Proof. Assume without loss of generality that $v=(0)$.
If $v \notin D$, then let $\left\{(1),(2), \ldots,\left(\delta_{v}+1\right)\right\} \subset D$.
Define $A:=g\left(S_{2}(v) \cap D\right)$ and $B:=g\left(S_{2}(v)\right) \backslash A$. By (2.7) and (2.8) we know that

$$
\forall k \in A \text {, we have }\left|S_{2}(v)[k]\right| \leq 2 ; \forall k \in B, \text { we have }\left|S_{2}(v)[k]\right| \leq 3
$$

So $\left|S_{2}(v)\right| \leq \frac{1}{2}(2|A|+3|B|)$.
By (2.7) we also know that for all $k \in g\left(S_{1}(v)\right)$, we have $k \notin g\left(S_{2}(v) \backslash D\right), k \notin B$. Hence, $|B| \leq n-\delta_{v}-1-\left|S_{1}(v)\right|$. Since $|A|+|B| \leq n-\delta_{v}-1$, we have

$$
\begin{aligned}
\left|S_{1}(v) \cup S_{2}(v)\right| & \leq\left|S_{1}(v)\right|+\frac{1}{2}(2|A|+3|B|) \\
& \leq\left|S_{1}(v)\right|+\frac{1}{2}\left(2\left|S_{1}(v)\right|+3\left(n-\delta_{v}-1-\left|S_{1}(v)\right|\right)\right) \\
& =\left|S_{1}(v)\right|+\frac{1}{2}\left(3 n-3 \delta_{v}-3-\left|S_{1}(v)\right|\right)
\end{aligned}
$$

By (2.6) we know that $\left|S_{1}(v)\right| \leq 3$, so $\left|S_{1}(v) \cup S_{2}(v)\right| \leq \frac{3}{2}\left(n-\delta_{v}\right)$.
On the other hand, if $v \in D$, then let $\left\{(0),(1),(2), \ldots,\left(\delta_{v}\right)\right\} \subset D$.
By (2.7), for all $k \in\left\{\delta_{v}+1, \delta_{v}+2, \ldots, n\right\}$, we have $\left|S_{2}(v)[k]\right| \leq 3$. Therefore,

$$
\left|S_{2}(v)\right| \leq \frac{3}{2}\left|\left\{\delta_{v}+1, \delta_{v}+2, \ldots, n\right\}\right|=\frac{3}{2}\left(n-\delta_{v}\right),
$$

and Lemma 3 is proved.

By definition, $\sum_{i \geq 1}(2 i+1)\left|V \zeta^{2 i}\right|=\sum_{x \in \mathbb{N}} x \sum_{u \in V \delta^{x}}|N[u] \cap V \zeta|$, and we can now estimate its lower bound using the results in Lemma 3.

Lemma 4. For $n \geq 12$, the following inequality holds:

$$
\begin{aligned}
& \sum_{i \geq 1}(2 i+1)\left|V \zeta^{2 i}\right| \\
& \geq 3 \delta_{V\left(Q_{n}\right)}-\left|V \delta^{2}\right|-4.5\left|V \delta^{n-3}\right|-(n+1)\left|V \delta^{n-2}\right|-(2 n-0.5)\left|V \delta^{n-1}\right|-3 n\left|V \delta^{n}\right|
\end{aligned}
$$

Proof. By Claim 1 in Lemma 2, we have

$$
\left\{u:|N[u] \cap V \zeta|-3<0, u \in V \delta^{1}\right\}=\bigcup_{v \in C \backslash D}\left(S_{1}(v) \cup S_{2}(v)\right) \cup \bigcup_{v \in C \cap D}\left(S_{2}(v)\right) .
$$

Therefore,

$$
\begin{aligned}
& \sum_{i \geq 1}(2 i+1)\left|V \zeta^{2 i}\right| \\
& =\sum_{x \in \mathbb{N}} x \sum_{u \in V \delta^{x}}|N[u] \cap V \zeta| \\
& =3 \delta_{V\left(Q_{n}\right)}+\sum_{x \in \mathbb{N}} x \sum_{u \in V \delta^{x}}(|N[u] \cap V \zeta|-3) \\
& =3 \delta_{V\left(Q_{n}\right)}+\sum_{v \in C} \delta_{v}(|N[v] \cap V \zeta|-3)+\sum_{u \in V \delta^{1}}(|N[u] \cap V \zeta|-3) \\
& \geq 3 \delta_{V\left(Q_{n}\right)}+\sum_{v \in C} \delta_{v}\left(n-\delta_{v}-3\right)+\sum_{v \in C \backslash D u \in S_{1}(v) \cup S_{2}(v)}(|N[u] \cap V \zeta|-3) \\
& \quad+\sum_{v \in C \cap D} \sum_{u \in S_{2}(v)}(|N[u] \cap V \zeta|-3) \\
& =3 \delta_{V\left(Q_{n}\right)}+\sum_{v \in C \backslash D}\left(\delta_{v}\left(n-\delta_{v}-3\right)-\left|S_{1}(v) \cup S_{2}(v)\right|\right)+\sum_{v \in C \cap D}\left(\delta_{v}\left(n-\delta_{v}-3\right)-\left|S_{2}(v)\right|\right) \\
& \geq 3 \delta_{V\left(Q_{n}\right)}+\sum_{v \in C}\left(\delta_{v}\left(n-\delta_{v}-3\right)-\frac{3}{2}\left(n-\delta_{v}\right)\right) .
\end{aligned}
$$

Note that the last inequality is due to Lemma 3. A short calculation shows that for $n \geq 12$,

$$
\delta_{v}\left(n-\delta_{v}-3\right)-\frac{3}{2}\left(n-\delta_{v}\right) \geq\left\{\begin{array}{l}
0, \text { if } 3 \leq \delta_{v} \leq n-4, \text { or } \delta_{v}=2 \text { and } n \geq 18 ; \\
-1, \text { if } \delta_{v}=2 \text { and } n=12 ; \\
-4.5, \text { if } \delta_{v}=n-3 ; \\
-n-1, \text { if } \delta_{v}=n-2 ; \\
-2 n+0.5, \text { if } \delta_{v}=n-1 ; \\
-3 n, \text { if } \delta_{v}=n,
\end{array}\right.
$$

and Lemma 4 follows.

Finally, we can estimate $\zeta_{m 2}-\zeta_{m 1}$.

Lemma 5. When $n \geq 12, \zeta_{m 2}-\zeta_{m 1} \geq 2 \delta_{V\left(Q_{n}\right)}-\zeta_{\max }$.
Proof. By (2.3) and Lemma 4, we have

$$
\begin{aligned}
& \zeta_{m 2}-2 \delta_{V\left(Q_{n}\right)} \\
& =\frac{1}{3}\left(\sum_{i \geq 1}\left(6 i\left|V \zeta^{2 i}\right|\right)-6 \delta_{V\left(Q_{n}\right)}\right) \\
& =\frac{1}{3}\left(\sum_{i \geq 1}(2 i-2)\left|V \zeta^{2 i}\right|+2 \sum_{i \geq 1}(2 i+1)\left|V \zeta^{2 i}\right|-6 \delta_{V\left(Q_{n}\right)}\right) \\
& \geq \frac{1}{3}\left(\sum_{i \geq 1}(2 i-2)\left|V \zeta^{2 i}\right|-2\left|V \delta^{2}\right|-9\left|V \delta^{n-3}\right|-(2 n+2)\left|V \delta^{n-2}\right|\right. \\
& \left.\quad-(4 n-1)\left|V \delta^{n-1}\right|-6 n\left|V \delta^{n}\right|\right) .
\end{aligned}
$$

By (2.1) and (2.2) we have

$$
\begin{aligned}
& \zeta_{m 1}-\zeta_{\max } \\
& =-\sum_{x \in \mathbb{N}} x(x-1)\left|V \delta^{x}\right| \\
& =-2\left|V \delta^{2}\right|-6\left|V \delta^{3}\right|-\ldots-(n-3)(n-4)\left|V \delta^{n-3}\right|-(n-2)(n-3)\left|V \delta^{n-2}\right| \\
& \quad-(n-1)(n-2)\left|V \delta^{n-1}\right|-n(n-1)\left|V \delta^{n}\right| .
\end{aligned}
$$

Therefore, $\zeta_{m 2}-2 \delta_{V\left(Q_{n}\right)} \geq \zeta_{m 1}-\zeta_{\max }$ when $n \geq 12$ and Lemma 5 follows.

Theorem 2. If $n \equiv 0(\bmod 6)$, then $\gamma\left(Q_{n}\right) \geq \frac{(n-2) 2^{n}}{n^{2}-2 n-2}$.
Proof. Theorem 2 holds true for $n=6$ by $\gamma\left(Q_{6}\right)=12$, so assume $n \geq 12$. Consider a minimum dominating set of $Q_{n}$. We have

$$
\delta_{V\left(Q_{n}\right)}=(n+1) \gamma\left(Q_{n}\right)-2^{n}, \zeta_{\max }=(n-1) \delta_{V\left(Q_{n}\right)}-2^{n}+\gamma\left(Q_{n}\right)=n^{2} \gamma\left(Q_{n}\right)-n 2^{n} .
$$

By Lemma $5, \zeta_{m 2}-\zeta_{m 1} \geq 2 \delta_{V\left(Q_{n}\right)}-\zeta_{\max }$, so there must be $\zeta_{\max }-2 \delta_{V\left(Q_{n}\right)} \geq 0$,

$$
\left(n^{2}-2 n-2\right) \gamma\left(Q_{n}\right)-(n-2) 2^{n} \geq 0, \gamma\left(Q_{n}\right) \geq \frac{(n-2) 2^{n}}{n^{2}-2 n-2}
$$

Corollary 1. $\gamma\left(Q_{12}\right) \geq 348, \gamma\left(Q_{18}\right) \geq 14666, \gamma\left(Q_{24}\right) \geq 701709, \gamma\left(Q_{30}\right) \geq 35876816$.

## 3. Conclusion

Tables 1 and 2 in the appendix are due to Gerzson Kéri [2][3]. When $n$ is a multiple of 6 , previously the best known result was $\gamma\left(Q_{n}\right) \geq \frac{2^{n}}{n}$, given by van Wee [4]. Our lower bound is higher, and several improvements are listed in Corollary 1.

## Acknowledgements

We thank Gerard Jennhwa Chang and Chun-Ju Lai for encouragements and comments. We are grateful to Gerzson Kéri, Patric Östergård and Jan-Christoph Schlage-Puchta for their useful correspondence. We would like to thank Laurent Habsieger, whose fine paper inspired us a lot. Lastly, we would like to thank the reviewers and the editors, who helped us to improve the quality of this manuscript.

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## Appendix

| $n$ | bounds on $\gamma\left(Q_{n}\right)$ |  | $n$ | bounds on $\gamma\left(Q_{n}\right)$ |  | $n$ | bounds on $\gamma\left(Q_{n}\right)$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 12 | $342(i)$ | $380(e)$ | 23 | $352827(l)$ | $393216(o)$ |
| 2 | 2 | 2 | 13 | $598(h)$ | $704(j)$ | 24 | $699051(i)$ | $786432(r)$ |
| 3 | 2 | 2 | 14 | $1172(k)$ | $1408(r)$ | 25 | $1298238(h)$ | $1556480(p)$ |
| 4 | 4 | 4 | 15 | $2048(c)$ | $2048(c)$ | 26 | $2581111(i)$ | $3112960(r)$ |
| 5 | $7(a)$ | $7(a)$ | 16 | $4096(i)$ | $4096(r)$ | 27 | $4794174(q)$ | $5767168(p)$ |
| 6 | $12(b)$ | $12(b)$ | 17 | $7419(l)$ | $8192(r)$ | 28 | $9587084(m)$ | $11534336(r)$ |
| 7 | $16(c)$ | $16(c)$ | 18 | $14564(i)$ | $16384(r)$ | 29 | $17997161(l)$ | $23068672(r)$ |
| 8 | $32(b)$ | $32(r)$ | 19 | $26309(m)$ | $31744(n)$ | 30 | $35791395(i)$ | $46137344(r)$ |
| 9 | $62(d)$ | $62(e)$ | 20 | $52618(m)$ | $63488(r)$ | 31 | $67108864(c)$ | $67108864(c)$ |
| 10 | $107(f)$ | $120(e)$ | 21 | $96125(h)$ | $122880(o)$ | 32 | $134217728(i)$ | $134217728(r)$ |
| 11 | $180(g)$ | $192(h)$ | 22 | $190651(i)$ | $245760(r)$ | 33 | $253523901(h)$ | $268435456(r)$ |

Table 1: the latest results of the bounds on $\gamma\left(Q_{n}\right)$

| $a$ | Taussky, Todd, 1948 [5] | $j$ | Östergård, Weakly, 1999 [14] |
| :---: | :---: | :---: | :---: |
| $b$ | Stanton, Kalbfleisch, 1968,1969 [6][7] | $k$ | Habsieger, 1997 [1] |
| $c$ | perfect code | $l$ | Haas, 2007-2008 [15] |
| $d$ | Östergård, Blass, 2001 [8] | $m$ | Habsieger, Plagne, 2000 [16] |
| $e$ | Wille, 1990, 1996 [9][10] | $n$ | Li, Chen, 1994 [17] |
| $f$ | Bertolo, Östergård, Weakley, 2004 [11] | $o$ | Kéri, 2006 [18] |
| $g$ | Blass, Litsyn, 1998 [12] | $p$ | Östergård, Kaikkonen, 1998 [19] |
| $h$ | Cohen, Lobstein, Sloane, 1986 [13] | $q$ | Plagne, 2009 [20] |
| $i$ | van Wee,1988 [4] | $r$ | $\gamma\left(Q_{n+1}\right) \leq 2 \gamma\left(Q_{n}\right)$ |

Table 2: References for Table 1


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