Improved Lower Bounds on the Domination Number of Hypercubes and Binary Codes with Covering Radius One

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Abstract

A dominating set on an *n*-dimensional hypercube is equivalent to a binary covering code of length *n* and covering radius 1. It is still an open problem to determine the domination number $\gamma(Q_n)$ for $n \ge 10$ and $n \ne 2^k, 2^k - 1$ ($k \in \mathbb{N}$). When *n* is a multiple of 6, the best known lower bound is $\gamma(Q_n) \ge \frac{2^n}{n}$, given by Van Wee (1988). In this article, we present a new method using congruence properties due to Laurent Habsieger (1997) and obtain an improved lower bound $\gamma(Q_n) \ge \frac{(n-2)2^n}{n^2-2n-2}$ when *n* is a multiple of 6.

1. Introduction

Determining the domination number is an important optimization problem in graph theory, as well as an NP-complete problem in computational complexity theory [23]. The domination problem on hypercubes is equivalent to the covering code problem. Generic introductions to domination problems and covering codes can be found in [21][22].

The *n*-dimensional hypercube Q_n is defined recursively in terms of the cartesian product of graphs as follows,

$$Q_1 = K_2, \quad Q_n = K_2 \Box Q_{n-1}.$$
 (1.1)

Therefore, Q_n can also be defined as

$$V(Q_n) = 2^{\{1,2,\dots,n\}}, \ E(Q_n) = \{uv : u, v \in V(Q_n), u \subset v, \text{ and } |v \setminus u| = 1\}.$$
 (1.2)

To avoid confusion, we use small brackets to express vertices. For instance, the vertex $\{2,3,5\}$ is written as (2,3,5). Note that the vertex \emptyset is written as (0). Moreover, we define $(a_1,\ldots,a_i,0) \equiv (a_1,\ldots,a_i)$ to simplify some of our arguments.

Given $S \subseteq V(Q_n)$, we define the function g to express all different members in the union of the coordinates of the vertices in S. That is,

$$g(S) \coloneqq \bigcup_{v \in S} v. \tag{1.3}$$

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Moreover, let S[a] indicate the subset of S in which the number a is contained in the coordinates of every vertex. That is,

$$S[a] \coloneqq \{v : a \in v \text{ and } v \in S\}.$$

$$(1.4)$$

e.g. For $S = \{(1, 2, 3), (2, 5), (3, 5)\}, g(S) = \{1, 2, 3, 5\}$ and $S[5] = \{(2, 5), (3, 5)\}.$ For some $v \in V(Q_n)$ and $S \subseteq V(Q_n)$, we define the neighborhood of v and S as follows.

$$N_{i}[v] = \{u \in V(Q_{n}) : d(u, v) = i\}, N_{i}[S] = \bigcup_{v \in S} N_{i}[v].$$
$$N[v] = N_{0}[v] \cup N_{1}[v], N[S] = \bigcup_{v \in S} N[v].$$
(1.5)

Given $S \subseteq V(Q_n)$, if $N[S] = V(Q_n)$, then we call S a dominating set of Q_n . If there does not exist $S' \subseteq V(Q_n)$ such that |S'| < |S| and $N[S'] = V(Q_n)$, then we call S a minimum dominating set of Q_n , and call |S| the domination number of Q_n . In this article, we denote a given dominating set as D, and denote the domination number as $\gamma(Q_n)$. Tables 1 and 2 in the appendix give the latest results of the upper and lower bounds on $\gamma(Q_n)$. The summarization is due to Gerzson Kéri [2][3].

Definition 1. Given a dominating set D and $S \subseteq V(Q_n)$, we denote the excess of D on S by $\delta_S(D)$, which is defined as $\sum_{v \in S} (|N[v] \cap D| - 1)$. When D is clear, we briefly write $\delta_S \coloneqq \delta_S(D)$. Also, if $S = \{v\}$, then $\delta_v \coloneqq \delta_{\{v\}}$.

The term *excess* has been used in many previous works, such as [1][4]. We further define the symbols below. Likewise, the D can be omitted if it's clear.

$$V\delta^{x}(D) \coloneqq \{v : \delta_{v}(D) = x \text{ and } v \in V(Q_{n})\},\$$
$$V\delta(D) \coloneqq \bigcup_{x \ge 1} V\delta^{x}(D), \quad C(D) \coloneqq \bigcup_{x \ge 2} V\delta^{x}(D).$$
(1.6)

Previous studies came up with various congruence properties of $\delta_{N_i[v]}(D)$, which help to estimate $\delta_{V(Q_n)}(D)$, and thus obtained the lower bounds on $\gamma(Q_n)$ due to the relation $\delta_{V(Q_n)}(D) = (n+1)|D| - |V(Q_n)|$ from [4]. Theorem 1 is a segment of the properties given by Laurent Habsieger [1], which we will apply.

Theorem 1. (Habsieger) When n is a multiple of 6,

$$\delta_{N[v]}(D) \equiv 1 \pmod{2}, \text{ if } v \notin D. \tag{1.7}$$

$$\delta_{N[v]}(D) \equiv 0 \pmod{2}, \text{ if } v \in D.$$
(1.8)

$$\delta_{N_1[v]}(D) + \delta_{N_2[v]}(D) \equiv 0 \pmod{3}.$$
(1.9)

We then put forward the pivotal concept throughout this article, *surfeit*. Although it seems closely related to *excess*, we shall demonstrate that such further analyzation is enough to improve the known bounds.

Definition 2. Given a dominating set D and $S \subseteq V(Q_n)$, we denote the surfect of D on S by $\zeta_S(D)$, which is defined as $\sum_{v \in S \setminus D} (\delta_{N[v]}(D) - 1)$. When D is clear, we briefly write $\zeta_S \coloneqq \zeta_S(D)$.

We further define the symbols below. Likewise, the D can be omitted if it's clear.

$$V\zeta^{x}(D) \coloneqq \left\{ v : \delta_{N[v]}(D) = x + 1 \text{ and } v \in V(Q_{n}) \setminus D \right\}, \ V\zeta(D) \coloneqq \bigcup_{x \ge 1} V\zeta^{x}(D)$$
(1.10)

We look into the cases when n is a multiple of 6. By calculating $\zeta_{V(Q_n)}(D)$ using two different methods, we show that it leads to a contradiction if $\gamma(Q_n)$ is too small.

2. Generalities

All the arguments are considered in the cases when n is a multiple of 6.

Given an arbitrary value γ^* , we assume that there exists a dominating set D satisfying $|D| = \gamma^*$. We calculate $\zeta_{V(Q_n)}$ using two different methods, and write the value we obtain as ζ_{m1} and ζ_{m2} , respectively. A dominating set should lead to $\zeta_{m1} = \zeta_{m2}$. However, we will show that there must be $\zeta_{m2} > \zeta_{m1}$ when γ^* is too small, implying that such dominating set D cannot exist, so $\gamma(Q_n) > \gamma^*$.

For all $v \in V(Q_n) \setminus D$, we have $\delta_{N[v]} \geq 1$ by (1.7), so the first method to calculate $\zeta_{V(Q_n)}$ holds. Note that for all $u \in V\delta^x$, we have $|N[u] \setminus D| = n - x$, and $\sum_{x \in \mathbb{N}} x |V\delta^x| = \delta_{V(Q_n)}$. The value obtained this way is written as ζ_{m1} :

$$\zeta_{V(Q_n)} = \sum_{x \in \mathbb{N}} \sum_{u \in V\delta^x} x(n-x) - |V(Q_n) \setminus D|$$

= $\sum_{x \in \mathbb{N}} x(n-x)|V\delta^x| - 2^n + |D|$
= $(n-1)\delta_{V(Q_n)} - 2^n + |D| - \sum_{x \in \mathbb{N}} x(x-1)|V\delta^x| =: \zeta_{m1}.$ (2.1)

 ζ_{m1} attains its maximum when $C = \emptyset$. We write this value as ζ_{max} .

$$\zeta_{m1} \leq \zeta_{\max} \coloneqq (n-1)\delta_{V(Q_n)} - 2^n + |D|.$$

$$(2.2)$$

Let us consider another method to calculate $\zeta_{V(Q_n)}$. The value obtained this way is written as ζ_{m2} :

$$\zeta_{V(Q_n)} = \sum_{i \ge 0} \left(2i |V\zeta^{2i}| + (2i+1)|V\zeta^{2i+1}| \right) = \sum_{i \ge 1} 2i |V\zeta^{2i}| \rightleftharpoons \zeta_{m2}.$$
 (2.3)

Note that by (1.7) we have $V\zeta^{2i+1} = \emptyset$.

Lemma 1. For all $u \in V\delta^1$, if $d(u, C) \ge 3$, then $|N[u] \cap V\zeta| \ge 3$.

Proof. Assume without loss of generality that u = (0), $N[u] \cap D = \{(1), (a)\}$, where $a \in \{0, 2, 3, ..., n\}$, then $\{(0), (1, a)\} \subset V\delta^1$. For convenience we assume that $a \neq 0$, since the case a = 0 can be dealt with similarly. Let $S = V\delta \cap (N_1[u] \cup N_2[u]) \setminus \{(1, a)\}$. Applying (1.7) and (1.8) on the vertices in $N_1[u]$, we have the following:

For all
$$k \in \{1, 2, \dots, n\}$$
, there is $|S[k]| \equiv 0 \pmod{2}$. (2.4)

Moreover, by applying (1.9) on u, we have $|S| \equiv 2 \pmod{3}$. In particular, $|S| \ge 5$, otherwise there exists $k \in \{1, 2, ..., n\}$ such that |S[k]| = 1, contradicting (2.4).

Suppose that $|N[u] \cap V\zeta| \leq 2$, $N[u] \cap V\zeta \subseteq \{(b), (c)\}$ where $b, c \in \{0, 2, 3, ..., n\} \setminus \{(a)\}$, then $g(S) \subseteq \{1, a, b, c\}$. So let $T = \{(1), (a), (b), (c), (1, b), (1, c), (a, b), (a, c), (b, c)\}$, then $S \subseteq T$. If b = 0, then $S = T \setminus \{(0)\} = \{(1), (a), (c), (1, c), (a, c)\}$, contradicting (2.4). Hence $b, c \neq 0, (0) \notin N[u] \cap V\zeta, |S \cap N_1[u]| = 0$, $S = T \cap N_2[u] = \{(1, b), (1, c), (a, b), (a, c), (b, c)\}$, but this still contradicts (2.4). Therefore, $|N[u] \cap V\zeta| \geq 3$.

In other words, for each $u \in V\delta^1$, there must be $d(u, C) \leq 2$ if $|N[u] \cap V\zeta| \leq 2$. In Lemma 2 we will show that $|N[u] \cap V\zeta| \leq 2$ gives us more rigorous conditions. To simplify our arguments, given $v \in C$, we divide $(N_1[v] \cup N_2[v]) \cap V\delta^1$ into the following vertex sets:

$$T_{1}(v) \coloneqq (N_{1}[v] \setminus D) \cap V\delta^{1};$$

$$T_{2}(v) \coloneqq \left\{ u \in N_{2}[v] \cap V\delta^{1} : |N[u] \cap N[v] \cap D| = 0 \right\};$$

$$T_{3}(v) \coloneqq \left\{ u \in N_{2}[v] \cap V\delta^{1} : |N[u] \cap N[v] \cap D| = 2 \right\};$$

$$T_{4}(v) \coloneqq \left\{ u \in N_{2}[v] \cap V\delta^{1} : |N[u] \cap N[v] \cap D| = 1 \right\}; \text{ and}$$

$$T_{5}(v) \coloneqq (N_{1}[v] \cap D) \cap V\delta^{1}.$$

$$(2.5)$$

Lemma 2. For all $u \in V\delta^1$, if $|N[u] \cap V\zeta| \leq 2$, then $|N[u] \cap V\zeta| = 2$, and the following four claims hold.

- 1. There exists $v \in C \setminus D$ such that $u \in T_1(v) \cup T_2(v)$, or there exists $v \in C \cap D$ such that $u \in T_2(v)$.
- 2. For all $v \in C \setminus D$ such that $u \in T_1(v)$, we have

$$|N_1[v] \cap V\delta| \le 3. \tag{2.6}$$

3. For all $v \in C$ such that $u \in T_2(v) \setminus D$, if we rename the coordinates so that v = (0), u = (a, b), then

$$(a), (b) \notin V\delta \text{ and } |(N_2[v] \cap V\delta)[a]|, |(N_2[v] \cap V\delta)[b]| \le 3.$$
(2.7)

4. For all $v \in C$ such that $u \in T_2(v) \cap D$, if we rename the coordinates so that v = (0), u = (a, b), then

$$|(N_2[v] \cap V\delta)[a]|, |(N_2[v] \cap V\delta)[b]| \le 2.$$
(2.8)

Proof. Given $v \in C$, we will prove the following statements.

- (A) If $u \in T_1(v)$, $v \notin D$ and $|N[u] \cap V\zeta| \leq 2$, then $|N[u] \cap V\zeta| = 2$ and (2.6) holds for v.
- (B) If $u \in T_2(v) \setminus D$ and $|N[u] \cap V\zeta| \leq 2$, then $|N[u] \cap V\zeta| = 2$ and (2.7) holds for v.
- (C) If $u \in T_2(v) \cap D$ and $|N[u] \cap V\zeta| \leq 2$, then $|N[u] \cap V\zeta| = 2$ and (2.8) holds for v.
- (D) If $u \in T_1(v)$, $v \in D$ and $|N[u] \cap V\zeta| \le 2$, then $|N[u] \cap V\zeta| = 2$ and there exists some $v' \in C \setminus D$ such that $u \in T_1(v')$.
- (E) If $u \in T_3(v) \cup T_4(v) \cup T_5(v)$ and $u \notin T_1(v') \cup T_2(v')$ for all $v' \in C$, then $|N[u] \cap V\zeta| \ge 3$.

(A), (B), and (C) prove Claims 2, 3, 4 in Lemma 2. By Lemma 1 there exists $v_0 \in C$ with $u \in \bigcup_{1 \leq i \leq 5} T_i(v_0)$, so Claim 1 follows by an application of (D) and (E) with $v = v_0$. Below, (A) and (D) follow from Case 1, (B) and (C) follow from Case 2, while (E) follows from Cases 3 to 5. We assume without loss of generality that v = (0).

Case 1-(1) $u \in T_1(v)$ and $v \notin D$.

 $\{u,v\} \subseteq N[u] \cap V\zeta$, so $|N[u] \cap V\zeta| \geq 2$. If equality holds, then $N[u] \cap V\zeta = \{u,v\}$. Let $u = (a) \notin D$. Since $u \in V\delta^1$, we have $|g(N[u] \cap D)| = 3$. So if $|N_1[v] \cap V\delta| \geq 4$, then there exists $(k) \in N_1[v] \cap V\delta$ such that $k \notin g(N[u] \cap D)$, $(a,k) \in N[u] \cap V\zeta$, which is a contradiction. Hence $|N_1[v] \cap V\delta| \leq 3$ and (A) is proved.

Case 1-(2) $u \in T_1(v)$ and $v \in D$.

If $\delta_v \geq 3$, then assume without loss of generality that (1), (2), (3) $\in D$ and that u = (4). $u \notin C$, so $|\{(1,4), (2,4), (3,4)\} \cap D| \leq 1$. Note that $\{u, (1,4), (2,4), (3,4)\} \setminus D \subseteq N[u] \cap V\zeta$, so $|N[u] \cap V\zeta| \geq 3$. Therefore, if $|N[u] \cap V\zeta| \leq 2$, then $\delta_v = 2$.

Let $N[v] \cap D = \{(0), (1), (2)\}$ and $u = (a) \notin D$. We have $(1), (2), (a) \in V\delta$, so for $w \in \{(1, a), (2, a)\}$, there is $w \notin N[u] \cap V\zeta$ if and only if $w \in D$. Thus, we assume that $\{(1, a), (2, a)\} \cap D = \{(1, a)\}, N[u] \cap V\zeta = \{u, (2, a)\}.$

There does not exist $k \in \{3, 4, \ldots, n\} \setminus \{a\}$ such that $(a, k) \in N[u] \cap V\zeta$. Therefore,

$$(N_1[u] \cup N_2[u]) \cap V\delta \subseteq \{(0), (1), (2), (1, a), (2, a), (1, 2, a)\}.$$
(*)

Applying (1.7) and (1.9) on u, we get

$$\delta_{(0)} + \delta_{(1)} + \delta_{(2)} + \delta_{(1,a)} + \delta_{(2,a)} + \delta_{(1,2,a)} \equiv 0 \pmod{3},\tag{2.9}$$

and

$$\delta_{(1,a)} + \delta_{(2,a)} \equiv 0 \pmod{2}.$$
 (2.10)

Moreover, $1 \le \delta_{(1,a)} \le 2$ since by (*) we know that $N[(1,a)] \cap D \subseteq \{(1), (1,a), (1,2,a)\}.$

If $\delta_{(1,a)} = 2$, then $(1, 2, a) \in D$ and thus $\delta_{(2,a)} \ge 1$. By (2.10) we know $\delta_{(2,a)} \ge 2$.

If $\delta_{(1,a)} = 1$ and $(1,2) \notin D$, then by (*) we know $\delta_{(1)} = 2$, $\delta_{(2)} = 1$, and $\delta_{(1,2,a)} = 0$. Now that $\delta_{(0)} + \delta_{(1)} + \delta_{(2)} + \delta_{(1,a)} + \delta_{(1,2,a)} = 6$, and by (2.10) we know $\delta_{(2,a)} \ge 1$, so (2.9) suggests that $\delta_{(2,a)} \ge 3$.

If $\delta_{(1,a)} = 1$ and $(1,2) \in D$, then by (*) we know $\delta_{(1)} = 3$, $\delta_{(2)} = 2$, and $\delta_{(1,2,a)} = 1$. Now that $\delta_{(0)} + \delta_{(1)} + \delta_{(2)} + \delta_{(1,a)} + \delta_{(1,2,a)} = 9$, and by (2.10) we know $\delta_{(2,a)} \ge 1$, so (2.9) suggests that $\delta_{(2,a)} \ge 3$.

Therefore, there must be $(2, a) \in C \setminus D$ and $u \in T_1((2, a))$. This shows that there exists $v' \in C \setminus D$ such that $u \in T_1(v')$, and (D) is proved.

Case 2. $u \in T_2(v)$.

Let u = (a, b) where $(a), (b) \notin D$, then $\{(a), (b)\} \subseteq N[u] \cap V\zeta$, so $|N[u] \cap V\zeta| \geq 2$. Assume that equality holds. Consider the case $u \notin D$. We have $(a), (b) \notin V\delta$, for otherwise $u \in N[u] \cap V\zeta$, which is a contradiction. Also, $|g(N[u] \cap D)| = 4$, so if $|(N_2[v] \cap V\delta)[a]| \geq 4$, then there exists $w \in (N_2[v] \cap V\delta)[a], k \in w \setminus \{a\}$ such that

$$k \notin g(N[u] \cap D), \, (a, b, k) \in N[u] \cap V\zeta,$$

which is a contradiction. Likewise, we have $|(N_2[v] \cap V\delta)[b]| \leq 3$, so (B) is proved. The same argument can be applied to the case $u \in D$ and prove (C).

The following cases together prove (E).

Case 3. $u \in T_3(v)$ and $u \notin T_1(v') \cup T_2(v')$ for all $v' \in C$.

Let $(1), (2) \in D$ and u = (1, 2). We prove $|N[u] \cap V\zeta| \geq 3$ by contradiction, assuming $|N[u] \cap V\zeta| \leq 2$. If $(a) \in D$ for some $a \in \{0, 3, 4, \ldots, n\}$, then $(1, 2, a) \in N[u] \cap V\zeta$. This implies $\delta_v \leq 3$, and since such an a exists by $\delta_v \geq 2$, we have $|N[u] \cap V\zeta| \geq 1$.

If $|N[u] \cap V\zeta| = 1$, we let $N[u] \cap V\zeta = \{(1,2,a)\}$, where $a \in \{0,3,4,\ldots,n\}$, then $(N_1[u] \cup N_2[u]) \cap V\delta \subseteq \{(0), (1,a), (2,a)\}$ and $\delta_{(1,a)}, \delta_{(2,a)} \ge 1$. We know $\{(1,a), (2,a)\} \cap C \neq \emptyset$ by applying (1.9) on u. Let $(1,a) \in C$, then there exists $b \in \{0,3,4,\ldots,n\} \setminus \{a\}$ such that

$$(1, a, b) \in D, (1, b) \in V\delta, (1, 2, b) \in N[u] \cap V\zeta,$$

which is a contradiction. Therefore, $|N[u] \cap V\zeta| = 2$.

Now let $N[u] \cap V\zeta = \{(1, 2, 3), (1, 2, k)\}$, where $k \in \{0, 4, 5, \dots, n\}$, then

$$(N_1[u] \cup N_2[u]) \cap V\delta \subseteq \{(0), (1,3), (2,3), (1,k), (2,k), (1,2,3,k)\}.$$
(**)

Thus, $N[v] \cap D \subseteq \{(1), (2), (3), (k)\}$. We denote the excesses as $\delta_{(0)} =: o, \delta_{(1,3)} =: p, \delta_{(2,3)} =: q, \delta_{(1,k)} =: r, \delta_{(2,k)} =: s, \delta_{(1,2,3,k)} =: t$, respectively. By (1.7), (1.8) and (1.9) we

derive the following relations (note that $\delta_{(1,2)} = 1$):

$$o + p + r \equiv 1 \pmod{2}$$
, since $\delta_{(1)} + \delta_{N_1[(1)]} \equiv 0 \pmod{2}$; (I)

$$o + q + s \equiv 1 \pmod{2}$$
, since $\delta_{(2)} + \delta_{N_1[(2)]} \equiv 0 \pmod{2}$; (II)

$$p + q + t \equiv 0 \pmod{2}$$
, since $\delta_{(1,2,3)} + \delta_{N_1[(1,2,3)]} \equiv 1 \pmod{2}$; (III)

$$r + s + t \equiv 0 \pmod{2}$$
, since $\delta_{(1,2,k)} + \delta_{N_1[(1,2,k)]} \equiv 1 \pmod{2}$; (IV)

$$o + p + q + r + s + t \equiv 0 \pmod{3}$$
, since $\delta_{N_1[u]} + \delta_{N_2[u]} \equiv 0 \pmod{3}$. (V)

If o = 3, then $N[v] \cap D = \{(1), (2), (3), (k)\}$, and the following relations hold. Note that the restrictions leading to these results are due to (**).

$$\begin{split} 1 &\leq p \leq 2, \text{ since } N[(1,3)] \cap D \subseteq \{(1),(3),(1,3,k)\} \text{ and } (1),(3) \in D; \\ 1 &\leq q \leq 2, \text{ since } N[(2,3)] \cap D \subseteq \{(2),(3),(2,3,k)\} \text{ and } (2),(3) \in D; \\ 1 &\leq r \leq 2, \text{ since } N[(1,k)] \cap D \subseteq \{(1),(k),(1,3,k)\} \text{ and } (1),(k) \in D; \\ 1 &\leq s \leq 2, \text{ since } N[(2,k)] \cap D \subseteq \{(2),(k),(2,3,k)\} \text{ and } (2),(k) \in D; \\ t \leq 1, \text{ otherwise } u \in T_1((1,2,3,k)) \cup T_2((1,2,3,k)). \end{split}$$

Now if o + p + q + r + s + t = 12, then p = q = r = s = 2 and t = 1, contradicting (III), so the only possibility is o + p + q + r + s + t = 9. However, this implies that (p + q + t) + (r + s + t) - t = 6. By (III) and (IV) we know t = 0, and together with (I) and (II) we know that p, q, r, s have the same parity, which is impossible. Therefore, $|N[u] \cap V\zeta| \geq 3$.

If o = 2, then we assume without loss of generality that $N[v] \cap D \subseteq \{(1), (2), (k)\}$. This time we obtain $p \leq 1$, $q \leq 1$, $1 \leq r \leq 2$, $1 \leq s \leq 2$, $t \leq 1$. If o + p + q + r + s + t = 9, then p = q = 1, r = s = 2, t = 1, contradicting (III), so the only possibility left is o + p + q + r + s + t = 6. This implies that (p + q + t) + (r + s + t) - t = 4. By (III) and (IV) we know t = 0, and p, q as well as r, s have the same parity. Thus p = q and r = s, implying p + r = 2, which contradicts (I). Therefore, $|N[u] \cap V\zeta| \geq 3$.

Case 4. $u \in T_5(v)$ and $u \notin T_1(v') \cup T_2(v')$ for all $v' \in C$.

If $v \in D$, then we can prove $|N[u] \cap V\zeta| \ge 3$ using the same method as in Case 3. We sketch our arguments in a simplified version, for they are highly similar to those in Case 3:

Let $(1), (2) \in D$ and u = (1). We prove our claim by contradiction, assuming that $|N[u] \cap V\zeta| \leq 2$. Like in case 3 we see $\delta_v \leq 3$ and $|N[u] \cap V\zeta| = 2$. Assume without loss of generality that $N[u] \cap V\zeta = \{(1, 2), (1, 3)\}$, then

$$(N_1[u] \cup N_2[u]) \cap V\delta \subseteq \{(0), (2), (3), (1,2), (1,3), (1,2,3)\}.$$
(***)

We denote the excesses as $\delta_{(0)} \eqqcolon o$, $\delta_{(2)} \eqqcolon p$, $\delta_{(3)} \eqqcolon q$, $\delta_{(1,2)} \eqqcolon r$, $\delta_{(1,3)} \eqqcolon s$, $\delta_{(1,2,3)} \eqqcolon t$,

respectively. By (1.7), (1.8), and (1.9) we derive the following relations (note that $\delta_{(1)} = 1$):

$$o + r + s \equiv 1 \pmod{2}$$
, since $\delta_{(1)} + \delta_{N_1[(1)]} \equiv 0 \pmod{2}$; (VI)

$$o + p + q \equiv 1 \pmod{2}$$
, since $\delta_{(0)} + \delta_{N_1[(0)]} \equiv 0 \pmod{2}$; (VII)

$$p + r + t \equiv 0 \pmod{2}$$
, since $\delta_{(1,2)} + \delta_{N_1[(1,2)]} \equiv 1 \pmod{2}$; (VIII)

$$q + s + t \equiv 0 \pmod{2}$$
, since $\delta_{(1,3)} + \delta_{N_1[(1,3)]} \equiv 1 \pmod{2}$; (IX)

$$o + p + q + r + s + t \equiv 0 \pmod{3}$$
, since $\delta_{N_1[u]} + \delta_{N_2[u]} \equiv 0 \pmod{3}$. (X)

If o = 3, then $N[v] \cap D = \{(0), (1), (2), (3)\}$, and the following relations hold. Note that the restrictions leading to these results are due to (***).

$$1 \leq p \leq 2, \text{ since } N[(2)] \cap D \subseteq \{(0), (2), (2,3)\} \text{ and } (0), (2) \in D; \\ 1 \leq q \leq 2, \text{ since } N[(3)] \cap D \subseteq \{(0), (3), (2,3)\} \text{ and } (0), (3) \in D; \\ 1 \leq r \leq 2, \text{ since } N[(1,2)] \cap D \subseteq \{(1), (2), (1,2,3)\} \text{ and } (1), (2) \in D; \\ 1 \leq s \leq 2, \text{ since } N[(1,3)] \cap D \subseteq \{(1), (3), (1,2,3)\} \text{ and } (1), (3) \in D; \\ t \leq 1, \text{ otherwise } u \in T_2((1,2,3)).$$

We have o + p + q + r + s + t = 9 or 12. The latter contradicts (VIII), while the former suggests that (p + r + t) + (q + s + t) - t = 6, and using (VI) to (IX) we know t = 0 and p, q, r, s have the same parity, which is impossible.

If o = 2, then $N[v] \cap D \subseteq \{(0), (1), (2)\}$. This time we obtain $1 \leq p \leq 2$, $q \leq 1$, $1 \leq r \leq 2$, $s \leq 1$, $t \leq 1$, so we have o + p + q + r + s + t = 6 or 9, but again we can easily lead to contradictions using (VI) to (X). Therefore, $|N[u] \cap V\zeta| \geq 3$.

If $v \notin D$, then let $u = (1) \in D$. By $\delta_v \ge 2$ we may assume $N[v] \cap D \supseteq \{(1), (2), (3)\}$. For $w \in \{(1, 2), (1, 3)\}$, we have $w \notin N[u] \cap V\zeta$ if and only if $w \in D$. Moreover, $(0) \in N[u] \cap V\zeta$. So if $|\{(1, 2), (1, 3)\} \cap D| = 0$, then $|N[u] \cap V\zeta| \ge 3$. If not, let $(1, 2) \in D$, then $(1, 2) \in C \cap D$ and $u \in T_5((1, 2))$, implying $|N[u] \cap V\zeta| \ge 3$.

Case 5. $u \in T_4(v)$ and $u \notin T_1(v') \cup T_2(v')$ for all $v' \in C$.

If $v \in D$, then we let $(1), (2) \in N[v] \cap D$, u = (1, a), where $(a) \notin D$. We have $\{(0), (1), (1, 2), (1, a)\} \subset V\delta$. Therefore, if $\{(1, a), (1, 2, a)\} \cap D = \emptyset$, then $\{(a), (1, a), (1, 2, a)\} \subseteq N[u] \cap V\zeta$; if $(1, 2, a) \in D$, then $(1, 2) \in C$ and $u \in T_3((1, 2))$; if $(1, a) \in D$, then $(1) \in C$ and $u \in T_5((1))$.

On the other hand, if $v \notin D$, then let $(1), (2), (3) \in N[v] \cap D$, u = (1, a), where $(a) \notin D$. We have $\{(1, 2), (1, 3)\} \subset V\delta$ and $(a) \in N[u] \cap V\zeta$. Thus, if $\{(1, 2, a), (1, 3, a)\} \cap D = \emptyset$, then $\{(a), (1, 2, a), (1, 3, a)\} \subseteq N[u] \cap V\zeta$; if $\{(1, 2, a), (1, 3, a)\} \cap D \neq \emptyset$, then there exists $w \in \{(1, 2), (1, 3)\} \cap C$ such that $u \in T_3(w)$.

By Case 3 and Case 4, every possible condition above implies that $|N[u] \cap V\zeta| \geq 3$. \Box

Given $v \in C$, we define $S_i(v) := \{u \in T_i(v) : |N[u] \cap V\zeta| = 2\}$. Lemma 3 gives an upper bound for $|S_1(v) \cup S_2(v)|$ when $v \notin D$ and an upper bound for $|S_2(v)|$ when $v \in D$.

Lemma 3. For $v \in C$, we have $|S_1(v) \cup S_2(v)| \leq \frac{3}{2}(n - \delta_v)$ if $v \notin D$, and $|S_2(v)| \leq \frac{3}{2}(n - \delta_v)$ if $v \in D$.

Proof. Assume without loss of generality that v = (0).

If $v \notin D$, then let $\{(1), (2), \dots, (\delta_v + 1)\} \subset D$. Define $A \coloneqq g(S_2(v) \cap D)$ and $B \coloneqq g(S_2(v)) \setminus A$. By (2.7) and (2.8) we know that

 $\forall k \in A$, we have $|S_2(v)[k]| \leq 2$; $\forall k \in B$, we have $|S_2(v)[k]| \leq 3$.

So $|S_2(v)| \le \frac{1}{2}(2|A| + 3|B|).$

By (2.7) we also know that for all $k \in g(S_1(v))$, we have $k \notin g(S_2(v) \setminus D)$, $k \notin B$. Hence, $|B| \leq n - \delta_v - 1 - |S_1(v)|$. Since $|A| + |B| \leq n - \delta_v - 1$, we have

$$|S_{1}(v) \cup S_{2}(v)| \leq |S_{1}(v)| + \frac{1}{2} \Big(2|A| + 3|B| \Big)$$

$$\leq |S_{1}(v)| + \frac{1}{2} \Big(2|S_{1}(v)| + 3(n - \delta_{v} - 1 - |S_{1}(v)|) \Big)$$

$$= |S_{1}(v)| + \frac{1}{2} \Big(3n - 3\delta_{v} - 3 - |S_{1}(v)| \Big).$$

By (2.6) we know that $|S_1(v)| \le 3$, so $|S_1(v) \cup S_2(v)| \le \frac{3}{2}(n - \delta_v)$.

On the other hand, if $v \in D$, then let $\{(0), (1), (2), \dots, (\delta_v)\} \subset D$. By (2.7), for all $k \in \{\delta_v + 1, \delta_v + 2, \dots, n\}$, we have $|S_2(v)[k]| \leq 3$. Therefore,

$$|S_2(v)| \le \frac{3}{2} |\{\delta_v + 1, \delta_v + 2, \dots, n\}| = \frac{3}{2}(n - \delta_v),$$

and Lemma 3 is proved.

By definition, $\sum_{i\geq 1}(2i+1)|V\zeta^{2i}| = \sum_{x\in\mathbb{N}}x\sum_{u\in V\delta^x}|N[u]\cap V\zeta|$, and we can now estimate its lower bound using the results in Lemma 3.

Lemma 4. For $n \ge 12$, the following inequality holds:

$$\sum_{i\geq 1} (2i+1)|V\zeta^{2i}|$$

$$\geq 3\delta_{V(Q_n)} - |V\delta^2| - 4.5|V\delta^{n-3}| - (n+1)|V\delta^{n-2}| - (2n-0.5)|V\delta^{n-1}| - 3n|V\delta^n|.$$

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Proof. By Claim 1 in Lemma 2, we have

$$\left\{u: |N[u] \cap V\zeta| - 3 < 0, u \in V\delta^{1}\right\} = \bigcup_{v \in C \setminus D} (S_{1}(v) \cup S_{2}(v)) \cup \bigcup_{v \in C \cap D} (S_{2}(v)).$$

Therefore,

$$\begin{split} \sum_{i \ge 1} (2i+1) |V\zeta^{2i}| \\ &= \sum_{x \in \mathbb{N}} x \sum_{u \in V\delta^x} |N[u] \cap V\zeta| \\ &= 3\delta_{V(Q_n)} + \sum_{x \in \mathbb{N}} x \sum_{u \in V\delta^x} \left(|N[u] \cap V\zeta| - 3 \right) \\ &= 3\delta_{V(Q_n)} + \sum_{v \in C} \delta_v \left(|N[v] \cap V\zeta| - 3 \right) + \sum_{u \in V\delta^1} \left(|N[u] \cap V\zeta| - 3 \right) \\ &\ge 3\delta_{V(Q_n)} + \sum_{v \in C} \delta_v (n - \delta_v - 3) + \sum_{v \in C \setminus D} \sum_{u \in S_1(v) \cup S_2(v)} \left(|N[u] \cap V\zeta| - 3 \right) \\ &+ \sum_{v \in C \cap D} \sum_{u \in S_2(v)} \left(|N[u] \cap V\zeta| - 3 \right) \\ &= 3\delta_{V(Q_n)} + \sum_{v \in C \setminus D} \left(\delta_v (n - \delta_v - 3) - |S_1(v) \cup S_2(v)| \right) + \sum_{v \in C \cap D} \left(\delta_v (n - \delta_v - 3) - |S_2(v)| \right) \\ &\ge 3\delta_{V(Q_n)} + \sum_{v \in C} \left(\delta_v (n - \delta_v - 3) - \frac{3}{2} (n - \delta_v) \right). \end{split}$$

Note that the last inequality is due to Lemma 3. A short calculation shows that for $n \ge 12$,

$$\delta_{v}(n-\delta_{v}-3) - \frac{3}{2}(n-\delta_{v}) \geq \begin{cases} 0, \text{ if } 3 \leq \delta_{v} \leq n-4, \text{ or } \delta_{v} = 2 \text{ and } n \geq 18; \\ -1, \text{ if } \delta_{v} = 2 \text{ and } n = 12; \\ -4.5, \text{ if } \delta_{v} = n-3; \\ -n-1, \text{ if } \delta_{v} = n-2; \\ -2n+0.5, \text{ if } \delta_{v} = n-1; \\ -3n, \text{ if } \delta_{v} = n, \end{cases}$$

and Lemma 4 follows.

Finally, we can estimate $\zeta_{m2} - \zeta_{m1}$.

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Lemma 5. When $n \ge 12$, $\zeta_{m2} - \zeta_{m1} \ge 2\delta_{V(Q_n)} - \zeta_{max}$.

Proof. By (2.3) and Lemma 4, we have

$$\begin{split} \zeta_{m2} &- 2\delta_{V(Q_n)} \\ &= \frac{1}{3} \left(\sum_{i \ge 1} (6i|V\zeta^{2i}|) - 6\delta_{V(Q_n)} \right) \\ &= \frac{1}{3} \left(\sum_{i \ge 1} (2i-2)|V\zeta^{2i}| + 2\sum_{i \ge 1} (2i+1)|V\zeta^{2i}| - 6\delta_{V(Q_n)} \right) \\ &\ge \frac{1}{3} \left(\sum_{i \ge 1} (2i-2)|V\zeta^{2i}| - 2|V\delta^2| - 9|V\delta^{n-3}| - (2n+2)|V\delta^{n-2}| \\ &- (4n-1)|V\delta^{n-1}| - 6n|V\delta^n| \right). \end{split}$$

By (2.1) and (2.2) we have

$$\begin{aligned} \zeta_{m1} &- \zeta_{\max} \\ &= -\sum_{x \in \mathbb{N}} x(x-1) |V \delta^x| \\ &= -2 |V \delta^2| - 6 |V \delta^3| - \ldots - (n-3)(n-4) |V \delta^{n-3}| - (n-2)(n-3) |V \delta^{n-2}| \\ &- (n-1)(n-2) |V \delta^{n-1}| - n(n-1) |V \delta^n|. \end{aligned}$$

Therefore, $\zeta_{m2} - 2\delta_{V(Q_n)} \ge \zeta_{m1} - \zeta_{\max}$ when $n \ge 12$ and Lemma 5 follows.

Theorem 2. If $n \equiv 0 \pmod{6}$, then $\gamma(Q_n) \ge \frac{(n-2)2^n}{n^2 - 2n - 2}$.

Proof. Theorem 2 holds true for n = 6 by $\gamma(Q_6) = 12$, so assume $n \ge 12$. Consider a minimum dominating set of Q_n . We have

$$\delta_{V(Q_n)} = (n+1)\gamma(Q_n) - 2^n, \ \zeta_{\max} = (n-1)\delta_{V(Q_n)} - 2^n + \gamma(Q_n) = n^2\gamma(Q_n) - n2^n.$$

By Lemma 5, $\zeta_{m2} - \zeta_{m1} \ge 2\delta_{V(Q_n)} - \zeta_{\max}$, so there must be $\zeta_{\max} - 2\delta_{V(Q_n)} \ge 0$,

$$(n^2 - 2n - 2)\gamma(Q_n) - (n - 2)2^n \ge 0, \ \gamma(Q_n) \ge \frac{(n - 2)2^n}{n^2 - 2n - 2}.$$

Corollary 1. $\gamma(Q_{12}) \ge 348$, $\gamma(Q_{18}) \ge 14666$, $\gamma(Q_{24}) \ge 701709$, $\gamma(Q_{30}) \ge 35876816$.

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3. Conclusion

Tables 1 and 2 in the appendix are due to Gerzson Kéri [2][3]. When n is a multiple of 6, previously the best known result was $\gamma(Q_n) \geq \frac{2^n}{n}$, given by van Wee [4]. Our lower bound is higher, and several improvements are listed in Corollary 1.

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References

- L. Habsieger, Binary codes with covering radius one: Some new lower bounds, Discrete Math. 176 (1997) 664–671.
- [2] G. Kéri, Tables for bounds on covering codes, 21 Nov. 2011. http://old.sztaki.hu/~keri/codes/index.htm
- [3] G. Kéri, Lefedő kódok kérdései: Optimális vagy optimálishoz közeli lefedő kódok szerkesztésének módszertana, története, egyéb vonatkozásai, adatbázisa, 1st ed. Beau Bassin: GlobeEdit (2017) 66-69.
- G. J. M. Van Wee, Improved sphere bounds on the covering radius of codes, IEEE Trans. Infrom. Theory 34 (1988) 237-245.
- [5] O. Taussky and J. Todd, Covering theorems for groups, Ann. Soc. Polon. Math., 21 (1948), 303–305.
- [6] R. G. Stanton and J. G. Kalbfleisch, Covering problems for dichotomized matchings, Aequationes Math., 1 (1968), 94–103.
- [7] J. G. Kalbfleisch and R. G. Stanton, A combinatorial theorem of matching, J. London Math. Soc. (1), 44 (1969), 60–64; and (2), 1 (1969), 398.
- [8] P. R. J. Östergård and U. Blass, On the size of optimal binary codes of length 9 and covering radius 1, IEEE Trans. Inform. Theory, 47 (2001), 2556–2557.
- [9] L. T. Wille, Improved binary code coverings by simulated annealing, Congr. Numer., 73 (1990), 53–58.
- [10] L. T. Wille, New binary covering codes obtained by simulated annealing, IEEE Trans. Inform. Theory, 42 (1996), 300–302.
- [11] R. Bertolo, P. R. J. Östergård and W. D. Weakley, An updated table of binary/ternary mixed covering codes, J. Combin. Des., 12 (2004), 157–176.
- [12] U. Blass and S. Litsyn, Several new lower bounds on the size of codes with covering radius one, IEEE Trans. Inform. Theory, 44 (1998), 1998–2002.
- [13] G. D. Cohen, A. C. Lobstein, and N. J. A. Sloane, Further results on the covering radius of codes, IEEE Trans. Inform. Theory, 32 (1986), 680–694.
- [14] P. R. J. Östergård and W. D. Weakley, Constructing covering codes with given automorphism, Designs, Codes and Cryptography, 16 (1999), 65–73.

- [15] W. Haas, Lower Bounds for Binary Codes of Covering Radius One, IEEE Trans. Inform. Theory, 53 (2007), 2880-2881.
- [16] L. Habsieger and A. Plagne, New lower bounds for covering codes, Discrete Math., 222 (2000), 125–149.
- [17] D. Li and W. Chen, New lower bounds for binary covering codes, IEEE Trans. Inform. Theory, 40 (1994), 1122–1129.
- [18] G. Kéri, Types of superregular matrices and the number of n-arcs and complete n-arcs in PG(r, q), Journal of Combinatorial Designs, 14 (2006), 363-390 és 16 (2008), 262.
- [19] P. R. J. Östergård and M. K. Kaikkonen, New upper bounds for binary covering codes, Discrete Math., 178 (1998), 165–179.
- [20] A. Plagne, A remark on Haas' method, Discrete Math., 309 (2009), 3318-3322
- [21] T. W. Haynes, S. Hedetniemi, P. Slater, Fundamentals of Domination in Graphs, CRC press, 2013.
- [22] G. Cohen, I. Honkala, S. Litsyn, A. Lobstein, Covering Codes, Elsevier, 1997.
- [23] Michael R. Garey, David S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman, p. 190, problem GT2.

Appendix	
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n	bounds of	on $\gamma(Q_n)$	n	bounds on $\gamma(Q_n)$		n	bounds o	$\operatorname{on} \gamma(Q_n)$
1	1	1	12	342(i)	380(e)	23	352827(<i>l</i>)	393216(<i>o</i>)
2	2	2	13	598(h)	704(j)	24	699051(<i>i</i>)	786432(r)
3	2	2	14	1172(k)	1408(r)	25	1298238(h)	1556480(<i>p</i>)
4	4	4	15	2048(c)	2048(c)	26	2581111(<i>i</i>)	3112960(<i>r</i>)
5	7(a)	7(a)	16	4096(i)	4096(r)	27	4794174(q)	5767168(<i>p</i>)
6	12(b)	12(b)	17	7419(<i>l</i>)	8192(<i>r</i>)	28	9587084(<i>m</i>)	11534336(r)
7	16(c)	16(c)	18	14564(i)	16384(<i>r</i>)	29	17997161(<i>l</i>)	23068672(r)
8	32(b)	32(r)	19	26309(m)	31744(n)	30	35791395(<i>i</i>)	46137344(<i>r</i>)
9	62(d)	62(e)	20	52618(m)	63488(r)	31	67108864(<i>c</i>)	67108864(<i>c</i>)
10	107(<i>f</i>)	120(e)	21	96125(h)	122880(<i>o</i>)	32	134217728(<i>i</i>)	134217728(r)
11	180(g)	192(h)	22	190651(i)	245760(r)	33	253523901(h)	268435456(r)

Table 1: the latest results of the bounds on $\gamma(Q_n)$

a	Taussky, Todd, 1948 [5]	j	Östergård, Weakly, 1999 [14]
b	Stanton, Kalbfleisch, 1968,1969 [6][7]	k	Habsieger, 1997 [1]
c	perfect code	l	Haas, 2007-2008 [15]
d	Östergård, Blass, 2001 [8]	m	Habsieger, Plagne, 2000 [16]
e	Wille, 1990, 1996 [9][10]	n	Li, Chen, 1994 [17]
f	Bertolo, Östergård, Weakley, 2004 [11]	0	Kéri, 2006 [18]
g	Blass, Litsyn, 1998 [12]	p	Östergård, Kaikkonen, 1998 [19]
h	Cohen, Lobstein, Sloane, 1986 [13]	q	Plagne, 2009 [20]
i	van Wee,1988 [4]	r	$\gamma(Q_{n+1}) \le 2\gamma(Q_n)$

Table 2: References for Table 1