On an f-coloring generalization of linear arboricity of multigraphs

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Abstract

Given a multigraph G and function $f:V(G)\to\mathbb{Z}_{\geq 2}$ on its vertices, a degree-f subgraph of G is a spanning subgraph in which every vertex v has degree at most f(v). The degree-f arboricity $a_f(G)$ of G is the minimum number of colors required to edge-color G into degree-f forests. At least for constant f, Truszczyński conjectured that $a_f(G)\le \max\{\Delta_f(G)+1, a(G)\}$ for every multigraph G, where $\Delta_f(G)=\max_{v\in V(G)}\lceil d(v)/f(v)\rceil$ and a(G) is the usual arboricity of G. This is a strong generalization of the Linear Arboricity Conjecture due to Akiyama, Exoo, and Harary. In this paper, we disprove Truszczyński's conjecture in a strong sense for general multigraphs. On the other hand, extending known results for linear arboricity, we prove that the conjecture holds for simple graphs with sufficiently large girth, and that it holds for all simple graphs asymptotically. More strongly, we prove these partial results in the setting of directed graphs, where the color classes are required to be analogously defined degree-f branchings.

Keywords: linear arboricity, pseudoarboricity, fractional arboricity, f-colorings, branchings

1 Introduction

In this writing, a multigraph may have parallel edges but no loops, unless otherwise stated. Given a multigraph G and a function $f:V(G)\to\mathbb{Z}_{\geq 1}$ on its vertex set, a degree-f subgraph (or an f-matching) is a spanning subgraph H of G such that every vertex $v\in V(G)$ has degree $d_H(v)\le f(v)$ in H. If every vertex v has degree exactly f(v), such a subgraph is commonly known as an f-factor. An f-coloring of G is an assignment of a color to every edge of G so that each color class is a degree-f subgraph. The f-chromatic index $\chi'_f(G)$ of G is the minimum number of colors required in an f-coloring of G. For a vertex subset $S\subseteq V(G)$, we let e(S) denote the number of edges in G with both endpoints in S, and we write $f(S)=\sum_{v\in S}f(v)$.

Hakimi and Kariv [22] introduced the notion of an f-coloring as a generalization of the case f = 1 of a proper edge-coloring, where $\chi'_1(G)$ is the usual chromatic index of G. This paper will study the similar problem of edge-coloring a multigraph into degree-f forests rather than degree-f subgraphs. This problem generalizes the most-studied cases $f = \infty$ (arboricity) and f = 2 (linear arboricity) to more general vertex weight functions.

The arboricity a(G) of a multigraph G is the minimum number of colors required to edge-color G so that every color class is a forest. A celebrated theorem of Nash-Williams [32] states that the arboricity of a multigraph G is given by

$$a(G) = \max_{S \subseteq V(G), |S| \ge 2} \left\lceil \frac{e(S)}{|S| - 1} \right\rceil.$$

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Given a function $f:V(G)\to\mathbb{Z}_{\geq 2}$, the degree-f arboricity $a_f(G)$ of multigraph G is the minimum number of colors required to edge-color G so that every color class is a degree-f forest. A degree-2 forest is more commonly known as a linear forest, and $a_2(G)=la(G)$ is called the linear arboricity of G. Unlike arboricity, determining the linear arboricity la(G) of a general multigraph G is NP-hard [33]. However, a conjecture known as the Linear Arboricity Conjecture, due to Akiyama, Exoo, and Harary [3], asserts that we can always determine the linear arboricity of simple graph to within an additive error of one. Observe that $la(G) \geq \lceil \Delta(G)/2 \rceil$ for every multigraph G because we require at least $\lceil d(v)/2 \rceil$ linear forests to cover the edges incident to vertex v. The following was conjectured in [3].

Conjecture 1 (Linear Arboricity Conjecture). For every simple graph G, we have $la(G) \leq \lceil (\Delta(G)+1)/2 \rceil$.

The Linear Arboricity Conjecture has been verified for many classes of simple graphs, including complete bipartite graphs [3], series-parallel graphs [43], planar graphs [42], and when $\Delta(G) \in \{3,4,5,6,8,10\}$ (see [3, 4, 14, 20]). Alon [5] proved that the Linear Arboricity Conjecture nearly holds for graphs with sufficiently large girth, and that it holds for all simple graphs asymptotically. Subsequent asymptotic improvements were given by Alon [6], by Ferber, Fox, and Jain [17], and by Lang and Postle [28], the latter of whom have given the currently best known asymptotic bound $la(G) \leq \Delta(G)/2 + O(\Delta(G)^{1/2}(\log \Delta(G))^{1/4})$. The Linear Arboricity Conjecture has also been proven for graphs of bounded sparsity (e.g., bounded degeneracy, treewidth, pseudoarboricity) when the maximum degree is sufficiently large (see [9, 11, 37, 40]).

There has not been as much work on the linear arboricity of multigraphs or on the degreef arboricity $a_f(G)$ for $f \neq 2$. Conjecture 1 does not extend to general multigraphs G since, for example, $la(G) = \Delta(G)$ when G consists of $\Delta(G)$ parallel edges between two vertices. Aït-djafer [2] generalized Conjecture 1 to the statement $la(G) \leq \lceil (\Delta(G) + \mu(G))/2 \rceil$ for every multigraph G, where $\mu(G)$ is the edge-multiplicity of G. She verified this when $\mu(G) \geq \Delta(G) - 2$, as well as when $\Delta(G)$ is close to a power of 2 and $\mu(G)$ is close to $\Delta(G)/2$. Caro and Roditty [10] proved that for constant functions f = t, every k-degenerate simple graph G satisfies $a_t(G) \leq \lceil (\Delta(G) + (t-1)k - 1)/t \rceil$.

This paper focuses on a strong generalization of the Linear Arboricity Conjecture due to Truszczyński [38]. For a multigraph G and function $f:V(G)\to\mathbb{Z}_{\geq 2}$, define the weighted maximum degree parameter

$$\Delta_f(G) = \max_{v \in V(G)} \left\lceil \frac{d(v)}{f(v)} \right\rceil.$$

Observe that $a_f(G) \ge \Delta_f(G)$ and $a_f(G) \ge a(G)$. Truszczyński conjectured the following when f is a constant function.

Conjecture 2. For every multigraph G and function $f: V(G) \to \mathbb{Z}_{\geq 2}$, we have

$$a_f(G) \le \max\{\Delta_f(G) + 1, a(G)\}.$$

More strongly, Truszczyński conjectured that for every multigraph G and integer $t \geq 2$, we have $a_t(G) = \max\{\lceil \Delta(G)/t \rceil, a(G)\}$ unless $a(G) = \Delta(G)/t$, in which case we have $a_t(G) \in \{\Delta(G)/t, \Delta(G)/t + 1\}$. He proved this when G is a complete multigraph with all edges having the same multiplicity, when G is a complete bipartite multigraph with all edges having the same

multiplicity, when the underlying simple graph of G is a forest, and when $t \ge \Delta(G) - a(G) + 1$. For the particular case f = 2 of linear arboricity, Conjecture 2 asserts that every multigraph G satisfies

$$la(G) \le \max \left\{ \left\lceil \Delta(G)/2 \right\rceil + 1, a(G) \right\}.$$

On the other hand, Nash-Williams' Theorem [32] for arboricity above implies that $a(G) \leq \lceil (\Delta(G) + 1)/2 \rceil$ for every simple graph G. This shows that Conjecture 2 is close to a generalization of the Linear Arboricity Conjecture, both to multigraphs and to other vertex weight functions f.

Our first main result is that Conjecture 2 is false in a strong sense for general multigraphs G, for any fixed constant function f = t.

Theorem 3. For every integer $t \geq 2$, there exists a constant $c_t > 1$ such that the following holds. For infinitely many integers $d \geq 2$, there exists a multigraph G such that $\max\{\Delta_t(G), a(G)\} = d$ and $a_t(G) \geq c_t d$.

We will prove Theorem 3 by exhibiting multigraphs G_t that have large fractional degree-t arboricity. We will then obtain the graph G of the theorem by replacing every edge in G_t with many parallel edges, which will make $a_t(G)$ and $\max\{\Delta_t(G), a(G)\}$ grow arbitrarily far apart by a constant factor $c_t > 1$. We will show that we can take the constant c_t in the theorem to satisfy $c_t \geq (4t+7)/(4t+6)$ for every integer $t \geq 2$, with slight improvements $c_2 \geq 9/8$, $c_3 \geq 15/14$, $c_4 \geq 21/20$ for $t \in \{2,3,4\}$.

The falsity of Conjecture 2 stands in contrast to other theorems and open conjectures on f-colorings. For example, in [40] this paper's author proved that a result analogous to Conjecture 2 holds for the degree-f pseudoarboricity of a multigraph. A pseudoforest is a multigraph where every component has at most one cycle (possibly a loop). The pseudoarboricity pa(G) of a multigraph G is the minimum number of colors required to edge-color G into pseudoforests. Analogous to Nash-Williams' Theorem, a theorem of Hakimi [23] states that the pseudoarboricity of a multigraph G (possibly with loops) is given by

$$pa(G) = \max_{S \subseteq V(G), |S| \ge 1} \left\lceil \frac{e(S)}{|S|} \right\rceil.$$

Given a multigraph G (possibly with loops) and function $f:V(G)\to\mathbb{Z}_{\geq 2}$, the degree-f pseudoarboricity $pa_f(G)$ of G is the minimum number of colors required to edge-color G into degree-f pseudoforests. It was shown in [40] that the degree-f pseudoarboricity has the exact formula

$$pa_f(G) = \max\{\Delta_f(G), pa(G)\}.$$

This result has a similar form to a conjecture on the f-chromatic index $\chi'_f(G)$ due to Nakano, Nishizeki, and Saito [30]. Their conjecture, in turn, is a generalization of the well-known Goldberg-Seymour Conjecture for the chromatic index [19, 34] (see also [12, 35]).

On the other hand, it was observed in [40] that the above formula for $pa_f(G)$ implies the following approximation of Conjecture 3.

Theorem 4. For every multigraph G and function $f: V(G) \to \mathbb{Z}_{\geq 2}$, we have

$$a_f(G) \le 2pa_{2f}(G) \le \max\{\Delta_f(G) + 1, 2pa(G)\}.$$

Note that $pa(G) \leq a(G) \leq 2pa(G)$ for every multigraph G. Thus, Theorem 2 shows that we cannot generally decrease 2pa(G) to a(G) in the above upper bound on $a_f(G)$. This raises the following question.

Question 5. For a given bounded function f, what is the optimal constant $c = c_f$ so that $a_f(G) \le (c + o(1)) \max\{\Delta_f(G), a(G)\}$ for every multigraph G? More generally, what pairs of constants c, c' are optimal so that $a_f(G) \le \max\{(c + o(1))\Delta_f(G), (c' + o(1))a(G)\}$ for every multigraph G?

Theorem 4 shows that the constant c_f of this question satisfies $c_f \leq 2$ for every function f. (This could have been derived from the simpler inequality $a_f(G) \leq 2pa_f(G)$.) Theorem 3 shows that generally $c_f > 1$, more precisely $c_f \geq (4t+7)/(4t+6)$ whenever f has maximum value t, with slight improvements for $t \in \{2, 3, 4\}$.

We conjecture that Truszczyński's Conjecture 2 holds when we restrict G to being a simple graph. Our next main results are support for this conjecture. Notice that if G is a simple graph that is $\Delta(G)$ -regular, then $a(G) = \lceil (\Delta(G)+1)/2 \rceil$ by Nash-Williams' Theorem, and then Conjecture 2 for simple graphs would be implied by the Linear Arboricity Conjecture. But for more general simple graphs G, a reduction to the Linear Arboricity Conjecture is not clear. One of our results is that Conjecture 2 nearly holds for simple graphs with sufficiently large girth.

Theorem 6. Let G be a simple graph, let $f: V(G) \to \mathbb{Z}_{\geq 2}$ be a function, and let $d = \max\{\Delta_f(G), a(G)\}$. If G has girth $g \geq 4d$, then $a_f(G) \leq d+1$.

Our other result is that Conjecture 2 holds for all simple graphs asymptotically when the function f is bounded.

Theorem 7. For every integer $t \geq 2$, there exists a real constant $c_t > 0$ such that for every simple graph G and function $f: V(G) \to \mathbb{Z}_{\geq 2}$ with maximum value at most t, we have

$$a_f(G) \le d + c_t d^{3/4} (\log d)^{1/2},$$

where $d = \max\{\Delta_f(G), a(G)\}.$

In particular, the answer to Question 5 is $c_f = 1$ when we restrict to the class of simple graphs. This contrasts with the proof of Theorem 3, where the constructed multigraphs have arbitrarily many parallel edges.

The proofs of Theorem 6 and Theorem 7 will be extensions of Alon's [6] probabilistic proofs for the case f = 2 of linear arboricity. These proofs are also found in the book of Alon and Spencer [7]. As with Alon's proofs, it is more convenient to prove our results in the setting of directed graphs D. Thus, before our probabilistic proofs we will consider a directed version of a degree-f forest that we call a degree-f branching, and we will write a conjecture (Conjecture 12) analogous to Conjecture 2 on what we call the directed degree-f arboricity $\vec{a}_f(D)$. We will first prove our large-girth and asymptotic results for directed graphs, and from those we will deduce Theorem 6 and Theorem 7.

We organize the proofs in this paper as follows. In Section 2, we will prove Theorem 3. In Section 3, we will formulate a version of Conjecture 2 for directed multigraphs and then show how this directed version implies the undirected version. Finally, in Section 4 we will prove Theorem 6 and Theorem 7.

2 Proof of Theorem 3

In this section, we will prove Theorem 3 and thus disprove Conjecture 2. We will use a fractional relaxation of degree-f arboricity $a_f(G)$. Let G be a multigraph and let $f: V(G) \to \mathbb{Z}_{\geq 2}$ be a function. Let \mathcal{F} be the collection of edge-sets of degree-f forests in G. We have a variable y_F for

every edge-set $F \in \mathcal{F}$. The fractional degree-f arboricity $a_f^*(G)$ of a multigraph G is the optimal value of the following linear program.

$$\min \quad \sum_{F \in \mathcal{F}} y_F$$

$$(P_0) \quad \text{s.t.} \quad \sum_{F \in \mathcal{F} \colon e \in F} y_F \ge 1 \quad \forall e \in E(G),$$

$$y_F \ge 0 \quad \forall F \in \mathcal{F}.$$

Notice that if we add the integrality constraints $y_F \in \{0, 1\}$ for all $F \in \mathcal{F}$, then a feasible solution of (P_0) corresponds to a collection of degree-f forests that covers all of E(G). This implies that $a_f(G) \geq a_f^*(G)$.

We will work with a slightly simplified linear program. Let G' be the underlying simple graph of G, and let \mathcal{F}' be the collection of edge-sets of degree-f forests of G'. For every edge $e = uv \in E(G')$, let $\mu_e = \mu_G(u, v)$ denote the number of parallel edges between vertices u and v in G. It is easy to show that (P_0) has the same optimal value as the following linear program.

min
$$\sum_{F \in \mathcal{F}'} y_F$$

$$(P) \quad \text{s.t.} \quad \sum_{F \in \mathcal{F}': e \in F} y_F \ge \mu_e \quad \forall e \in E(G'),$$

$$y_F \ge 0 \quad \forall F \in \mathcal{F}'.$$

The dual of (P) is the following, where x_e is the dual variable associated with edge $e \in E(G')$.

$$\max \sum_{e \in E(G')} \mu_e x_e$$

$$(D) \quad \text{s.t.} \quad \sum_{e \in F} x_e \le 1 \quad \forall F \in \mathcal{F}',$$

$$x_e \ge 0 \quad \forall e \in E(G').$$

Feige, Ravi, and Singh [16] proved that the fractional linear arboricity $la^*(G) = a_2^*(G)$ of a d-regular simple graph satisfies $la^*(G) \leq d/2 + O(\sqrt{d})$. This is only slightly better than Lang and Postle's [28] more recent asymptotic upper bound for la(G), but their proof is simpler. Outside of their work, fractional linear arboricity has not been studied.

To prove Theorem 3, we will use the following easy observation. For a multigraph G and integer $m \ge 1$, let mG denote the multigraph obtained from G by replacing every edge of G by m parallel copies of that edge.

Observation 8. For every multigraph G, function $f:V(G)\to\mathbb{Z}_{\geq 2}$, and integer $m\geq 1$, we have

$$a_f(mG) \ge a_f^*(mG) = m \cdot a_f^*(G).$$

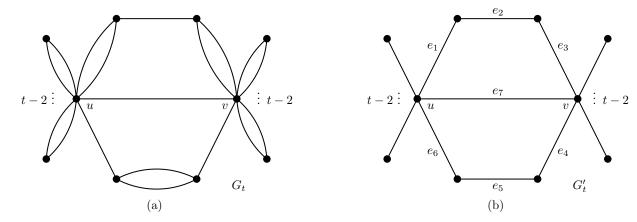


Figure 1

We also observe that $\Delta_f(mG) \leq m \cdot \Delta_f(G)$ and $a(mG) \leq m \cdot a(G)$. If we can find a multigraph G where $a_f^*(G) = c_f \cdot \max\{\Delta_f(G), a(G)\}$ and $c_f > 1$, then

$$a_f(mG) \ge m \cdot a_f^*(G) \ge m \cdot c_f \cdot \max\{\Delta_f(G), a(G)\} \ge c_f \cdot \max\{\Delta_f(mG), a(mG)\},$$

which is the required inequality for Theorem 3. Therefore our goal is to find, for every constant function $f = t \ge 2$, a multigraph $G = G_t$ and constant $c_t > 1$ such that

$$a_t^*(G) \ge c_t \cdot \max\{\Delta_t(G), a(G)\}. \tag{1}$$

An example of such a multigraph G_t is shown in Figure 1(a). It consists of a 6-cycle that alternates in one and two parallel edges, an edge connecting two antipodal vertices u and v of the cycle, and t-2 parallel pairs to new vertices attached to each of u and v. The underlying simple graph G'_t of G_t is shown in Figure 1(b), with some edges labeled e_1, \ldots, e_7 . Consider the following solution x for (D) with respect to G'_t :

$$x_e = \begin{cases} 2/(2t+3) & \text{if } e \in \{e_1, e_3, e_7\}, \\ 1/(2t+3) & \text{otherwise.} \end{cases}$$

We will show that x is feasible, using the following claim.

Claim 9. In the simple graph G'_t , every degree-t forest has at most 2t+1 edges, and every degree-t forest containing each of e_1, e_3, e_7 at most 2t edges.

Proof. The simple graph G'_t has 6 + 2(t - 2) = 2t + 2 vertices, so every degree-t forest in G'_t has at most 2t + 1 edges. Suppose that degree-t forest F of G'_t contains e_1, e_3, e_7 . Then F does not contain e_2 , but it may contain e_5 . Besides e_1 and e_7 , F contains at most t - 2 additional edges incident to u; and besides e_3 and e_7 , F contains at most t - 2 additional edges incident to v. This implies that F has at most 3 + 1 + 2(t - 2) = 2t edges.

Fix a degree-t forest edge-set F in G'_t . If F contains at most two of e_1, e_3, e_7 , then by Claim 9 we have $|F| \leq 2t + 1$, so that

$$\sum_{e \in F} x_e \le 2 \cdot \frac{2}{2t+3} + (|F|-2) \cdot \frac{1}{2t+3} \le 1.$$

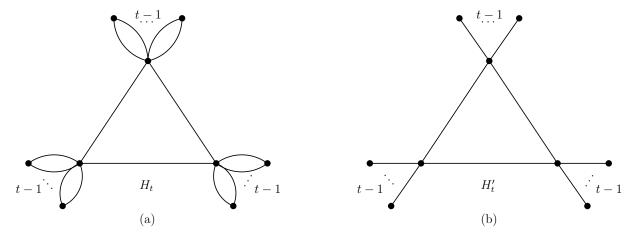


Figure 2

If F contains each of e_1, e_3, e_7 , then by Claim 9 we have $|F| \leq 2t$, so that

$$\sum_{e \in F} x_e = 3 \cdot \frac{2}{2t+3} + (|F|-3) \cdot \frac{1}{2t+3} \le 1.$$

Since also $x \geq 0$, this proves that x is feasible for (D).

We calculate objective value of x in (D) to be (4t+7)/(2t+3). Thus, by weak duality and the equivalence of (P) and (P_0) , we have that $a_t^*(G_t) \ge (4t+7)/(2t+3)$. (One could prove that x is in fact an optimal solution for (D), by exhibiting a feasible solution y for (P) with the same objective value, but this is not necessary for our proof.) On the other hand, we easily observe that $\Delta_t(G_t) = a(G_t) = 2$. Therefore,

$$a_t^*(G_t) \ge \frac{4t+7}{2t+3} = \frac{4t+7}{4t+6} \max\{\Delta_t(G_t), a(G_t)\},$$

which achieves inequality (1) above and thus proves Theorem 3 with $c_t \geq \frac{4t+7}{4t+6}$. (Note that $\Delta_t(mG_t) = a(mG_t) = 2m$ for every $m \geq 1$, so Theorem 3 applies to all even $d \geq 2$.)

For $t \leq 4$ we can improve this bound on c_t by replacing G_t with the multigraph H_t shown in Figure 2(a). The underlying simple graph H'_t of H_t is shown in Figure 2(b). One can show that a feasible solution for (D) with respect to H'_t is $x_e = 1/(3t-2)$ for every edge e, and that this x has objective value (6t-3)/(3t-2). Thus $a_t^*(H_t) \geq (6t-3)/(3t-2)$ while $\Delta_t(H_t) = a(H_t) = 2$, and this gives the bound $c_t \geq (6t-3)/(6t-4)$. This bound improves the one above for $t \leq 4$, giving $c_2 \geq 9/8$, $c_3 \geq 15/14$, and $c_4 \geq 21/20$.

The idea for constructing the multigraphs G_t and H_t above is to start with a suitable base graph and then to add certain gadgets to a subset of the vertices. These gadgets have the form of a number of parallel edges connecting to a new vertex. If the base graph has large fractional linear arboricity, then by adding a certain number of gadgets we will create a graph with large fractional degree-t arboricity, for any given $t \geq 2$. One can use this approach of adding gadgets to show that it is NP-complete to decide whether a given multigraph G has degree-f arboricity $a_f(G) = 2$, for any fixed function $f: V(G) \to \mathbb{Z}_{\geq 2}$, starting with the base case f = 2 due to Péroche [33]. Details are found in [41].

We observed the inequalities $a_f(G) \ge a_f^*(G) \ge \max\{\Delta_f(G), a(G)\}$ and showed that $a_f^*(G)$ and $\max\{\Delta_f(G), a(G)\}$ can be arbitrarily far apart. We leave open the question of whether $a_f(G)$ and $a_f^*(G)$ can be arbitrarily far apart.

3 A directed version

We have proven that Conjecture 2 is false for general multigraphs, but we believe the conjecture holds when we restrict to simple graphs G. In Section 4, we will prove Theorem 6 and Theorem 7 supporting this conjecture. To make it easier to prove these theorems, in this section we introduce an analogue of degree-f arboricity for directed multigraphs and formulate a directed version of Conjecture 2. This reformulated conjecture (Conjecture 12) is a generalization of the Directed Linear Arboricity Conjecture (Conjecture 11) due to Nakayama and Péroche [31].

For a directed multigraph D, let $\Delta^-(D)$ denote the maximum indegree $d^-(v)$ among vertices v of D, let $\Delta^+(D)$ denote the maximum outdegree, and let \overline{D} denote the underlying undirected multigraph. Two arcs are said to be *parallel* if they have the same head and tail, and *anti-parallel* if the head of one is the tail of the other and vice versa. The *arc-multiplicity* $\mu(D)$ is the maximum number of parallel arcs in D. We will call D a directed graph if $\mu(D) = 1$, that is, if D has no parallel arcs (but it may have anti-parallel arcs).

A directed graph B is a branching if every vertex v of B has indegree $d_B^-(v) \leq 1$ and its underlying undirected graph \overline{B} is a forest. An arborescence is a branching whose underlying undirected graph is a tree. For a directed multigraph D, the directed arboricity $\vec{a}(D)$ of D is the minimum number of colors required to color the arcs of D so that every color class is a branching. Observe that $\vec{a}(D) \geq \Delta^-(D)$ and $\vec{a}(D) \geq a(\overline{D})$. Using Edmonds' celebrated theorem on packing arborescences [13, 29], Frank [18] proved the following formula for directed arboricity $\vec{a}(D)$.

Theorem 10. For every directed multigraph D, we have

$$\vec{a}(D) = \max\left\{\Delta^{-}(D), a(\overline{D})\right\}.$$

Frank also showed via Nash-Williams' Theorem [32] that every directed multigraph D satisfies $a(\overline{D}) \leq \Delta^{-}(D) + \mu(D)$, so Theorem 10 implies that $\vec{a}(D) \leq \Delta^{-}(D) + \mu(D)$. In particular, if D is a directed graph then

$$\Delta^{-}(D) \le \vec{a}(D) \le \Delta^{-}(D) + 1,$$

a result also noted by Kareyan [27].

Theorem 10 has some resemblance to Conjecture 2. We introduce an f-coloring version of branchings to connect the statements rigorously. Let D be a directed multigraph, and let $f:V(D) \to \mathbb{Z}_{\geq 2}$ be a function. A directed subgraph B of D is a degree-f branching if it is a branching where every vertex v has outdegree $d_B^+(v) \le f(v) - 1$. Notice that the underlying undirected graph \overline{B} of a degree-f branching B is a degree-f forest. The directed degree-f arboricity $\overline{d}_f(D)$ of directed multigraph D is the minimum number of colors required to color the arcs of D so that every color class is a degree-f branching.

For the case f=2, a degree-2 branching is also called as a directed linear forest, and the directed degree-2 arboricity $\vec{a}_2(D) = \vec{l}a(D)$ is called the directed linear arboricity of D. Noting that $\vec{l}a(D) \geq \Delta^-(D)$ and $\vec{l}a(D) \geq \Delta^+(D)$ for every directed multigraph D, Nakayama and Péroche [31]

formulated the *Directed Linear Arboricity Conjecture* for directed graphs D:

$$\vec{la}(D) \le \max\left\{\Delta^-(D), \Delta^+(D)\right\} + 1.$$

Nakayama and Péroche proved that this conjecture holds if D is acyclic, if $\Delta^-(D)$, $\Delta^+(D) \leq 2$ and $|V(D)| \geq 4$, and if D is one of certain symmetric directed graphs G^* . (Here, G^* is obtained from the undirected graph G by replacing each edge by a pair of anti-parallel arcs.) However, He, Li, Bai, and Sun [26] later showed that their conjecture does not hold for the complete symmetric directed graphs K_3^* and K_5^* : $\Delta^-(K_n^*) = \Delta^+(K_n^*) = n-1$ for all n but $l\vec{a}(K_n^*) = n+1$ for $n \in \{3,5\}$. Still, they believe that these two directed graphs are the only counterexamples, leading to the following updated version of Nakayama and Péroche's conjecture.

Conjecture 11 (Directed Linear Arboricity Conjecture). For every directed graph D, we have

$$\vec{la}(D) \le \max \{ \Delta^{-}(D), \Delta^{+}(D) \} + 1,$$

except for $D = K_3^*, K_5^*$, in which case $\vec{la}(D) = \max \{\Delta^-(D), \Delta^+(D)\} + 2$.

Now we consider a generalization of Conjecture 11 for general vertex weight functions f and multigraphs D. As before, $\vec{a}_f(D) \geq \Delta^-(D)$ and $\vec{a}_f(D) \geq a(\overline{D})$. Now we also see that $\vec{a}_f(D) \geq \Delta^+_{f-1}(D) = \max_{v \in V(D)} \left\lceil \frac{d^+(v)}{f(v)-1} \right\rceil$. Based on previously written theorems and conjectures (ignoring Theorem 3), it is natural to conjecture the following.

Conjecture 12. For every directed multigraph D, we have

$$\vec{a}_f(D) \le \max \left\{ \Delta^-(D), \Delta^+_{f-1}(D), a(\overline{D}) \right\} + 1.$$

We will show that Conjecture 12 nearly implies Conjecture 2. Because Conjecture 2 is false for general multigraphs as we have shown, so is Conjecture 12. However, it could still be true for directed graphs (with no parallel arcs). Since a directed graph D satisfies $\Delta^-(D) \leq a(\overline{D}) \leq \Delta^-(D) + 1$ as noted above, Conjecture 12 for directed graphs is basically the statement that D satisfies

$$\vec{a}_f(D) \le \max \left\{ \Delta^-(D) + 1, \Delta^+_{f-1}(D) \right\} + 1.$$

In the case f=2, it is an easy observation that the Directed Linear Arboricity Conjecture (Conjecture 11) nearly implies the Linear Arboricity Conjecture (Conjecture 1): Given an undirected simple graph G, let D be a balanced orientation of G, meaning that every vertex v has both indegree and outdegree at most $\lceil d_G(v)/2 \rceil$ in D. (The existence of balanced orientations is an easy exercise from the theory of Euler tours or network flows, and it is also a special case of Theorem 13 below.) Then $\Delta^-(D), \Delta^+(D) \leq \lceil \Delta(G)/2 \rceil$, and so by Conjecture 11 (for $D \neq K_3^*, K_5^*$) we have $la(G) \leq la(D) \leq \max\{\Delta^-(D), \Delta^+(D)\} + 1 \leq \lceil \Delta(G)/2 \rceil + 1$, which is almost the Linear Arboricity Conjecture. To make this kind of reduction work for more general functions f, we will replace balanced orientations with the following more general orientation theorem due to Entringer and Tolman [15] (see also [40]).

Theorem 13 (Entringer-Tolman). Given a multigraph G (possibly with loops) and functions $g, h : V(G) \to \mathbb{Z}_{\geq 0}$, G can be oriented so that every vertex $v \in V(G)$ has indegree $d^-(v) \leq g(v)$ and outdegree $d^+(v) \leq h(v)$ if and only if

- (1) $d(v) \leq g(v) + h(v)$ for all $v \in V(G)$, and
- (2) $e(S) \le \min\{g(S), h(S)\}\$ for all $S \subseteq V(G)$.

Corollary 14. For every multigraph G and function $f: V(G) \to \mathbb{Z}_{\geq 2}$, if $\max\{\Delta_f(G), a(G)\} \leq d$ then G has an orientation D such that $\max\{\Delta^-(D), \Delta^+_{f-1}(D), a(\overline{D})\} \leq d$.

Proof. Define $g, h: V(G) \to \mathbb{Z}_{\geq 1}$ by g(v) = d and h(v) = d(f(v) - 1) for all $v \in V(G)$. Then $d_G(v) \leq d \cdot f(v) \leq g(v) + h(v)$ for all $v \in V(G)$, and $e(S) \leq d(|S| - 1) < d|S| = \min\{g(S), h(S)\}$ for all $S \subseteq V(G)$. By Theorem 13, G has an orientation D such that every vertex v has indegree at most d and outdegree at most d(f(v) - 1). That is, $\Delta^-(D) \leq d$ and $\Delta^+_{f-1}(D) \leq d$. We also see that $a(\overline{D}) = a(G) \leq d$.

Therefore, for any multigraph G, if $d = \max\{\Delta_f(G), a(G)\}$ and D is from Corollary 14, then Conjecture 12 implies that

$$a_f(G) \leq \vec{a}_f(D) \leq \max\{\Delta^-(D), \Delta^+_{f-1}(D), a(\overline{D})\} + 1 \leq d + 1 = \max\{\Delta_f(G), a(G)\} + 1,$$

which is almost Conjecture 2 as what we wanted to show. This reduction to directed graphs will be used in our proofs of Theorem 6 and Theorem 7.

We remark that Conjecture 12 can be viewed as a matroid problem, about how closely the covering number $\beta(M_1, M_2, M_3)$ of three matroids M_1, M_2, M_3 on ground set E(D) is determined by the covering number $\beta(M_1), \beta(M_2), \beta(M_3)$ of each of these matroids individually. We will not comment further on this perspective.

4 Large girth and asymptotics

In this section, we will prove Theorem 6 and Theorem 7. Specifically, we will first prove that Conjecture 12 on degree-f branchings holds for directed graphs with large directed girth, and then prove that it holds for all directed graphs asymptotically when the function f is bounded. Applying Corollary 14, these results imply the desired theorems. Recall that a directed graph D is taken to have no parallel arcs, but it may have anti-parallel arcs.

Our proofs are extensions of the probabilistic proofs of Alon [6], who proved such partial results for the Directed Linear Arboricity Conjecture (the case f=2) while improving and simplifying his original arguments in [5]. (See also Alon and Spencer [7].) Alon reduced to and wrote his proofs specifically for d-regular (directed) graphs, but we cannot do the same reduction for $f \neq 2$, so we write our proofs in the general non-regular setting. Recall that Conjecture 12 for a directed graph D is close to, but not exactly, the inequality

$$\vec{a}_f(D) \le \max \left\{ \Delta^-(D) + 1, \Delta^+_{f-1}(D) \right\} + 1.$$

4.1 Large girth

First we prove that Conjecture 12 holds for directed graphs with large directed girth. Here, the directed girth is the length of a shortest directed cycle. We will need Hakimi and Kariv's [22] f-coloring generalization of König's edge-coloring theorem, stated as follows.

Theorem 15. For every bipartite multigraph G and function $f:V(G)\to\mathbb{Z}_{\geq 1}$, we have $\chi'_f(G)=\Delta_f(G)$.

We will also need a result on independent transversals in graphs. Given a simple graph G and a partition $(V_i)_{i \in [k]}$ of its vertex set V(G), an independent transversal of G with respect to $(V_i)_{i \in [k]}$ is an independent set of G containing one vertex from each vertex class V_i . Aharoni, Alon, and Berger [1] proved the following result on independent transversals in line graphs.

Theorem 16. Let G be the line graph of a simple graph and let $(V_i)_{i \in [k]}$ be a partition of its vertex set V(G). If $|V_i| \geq \Delta(G) + 2$ for every i, then G has an independent transversal with respect to $(V_i)_{i \in [k]}$.

We now prove our result on directed graphs with large directed girth. The main differences in our proof compared to that of Alon [6] for the case f = 2 are that we use a different initial arc-coloring of the directed graph, and that we apply Theorem 16 to only an induced subgraph of our line graph rather than the entire line graph.

Theorem 17. Let D be a directed graph, let $f: V(D) \to \mathbb{Z}_{\geq 2}$ be a function, and let $d = \max\{\Delta^-(D), \Delta^+_{f-1}(D)\}$. If D has directed girth $g \geq 4d$, then $\vec{a}_f(D) \leq d+1$.

Proof. Construct an auxiliary bipartite graph G with parts X and Y as follows: for every vertex v of D put a copy v_X in X and v_Y in Y, and for every arc e of D with tail u and head v put an edge in G between u_X and v_Y . Then v_X has degree $d_D^+(v)$ and v_Y has degree $d_D^-(v)$ in G, for every $v \in V(D)$. Define the function $g: V(G) \to \mathbb{Z}_{\geq 1}$ by $g(v_X) = f(v) - 1$ and $g(v_Y) = 1$ for all $v \in V(D)$. By Theorem 15,

$$\chi_g'(G) = \Delta_g(G) = \max \left\{ \max_{v \in V(D)} \left\lceil \frac{d_D^+(v)}{f(v) - 1} \right\rceil, \max_{v \in V(D)} d_D^-(v) \right\} = \max \left\{ \Delta_{f-1}^+(D), \Delta^-(D) \right\} = d.$$

Thus G can be edge-colored into d degree-g subgraphs. These subgraphs of G correspond to directed subgraphs B_1, \ldots, B_d of D with $d_{B_i}^-(v) \leq 1$ and $d_{B_i}^+(v) \leq f(v) - 1$ for all $1 \leq i \leq d$ and $v \in V(D)$. The B_i 's are thus directed degree-f pseudoforests, meaning that every vertex v of B_i has indegree at most one and the underlying undirected graph $\overline{B_i}$ is a degree-f pseudoforest.

Observe that if we remove one arc from every monochromatic directed cycle in D, the remaining color classes will be degree-f branchings. Let D' be the spanning directed subgraph of D that is the union all monochromatic directed cycles in D, let H be the line graph of $\overline{D'}$, and let $(V_i)_{i \in [k]}$ be the edge sets of the monochromatic directed cycles in D'. Then $(V_i)_{i \in [k]}$ is a partition of V(H), and by the directed girth condition we have $|V_i| \geq 4d$ for all $i \in [k]$. Since $\Delta(H) \leq 4d - 2$, $|V_i| \geq 4d = (4d-2)+2$ for every i, and H is the line graph of a simple graph (as D cannot have antiparallel arcs by the directed girth assumption), Theorem 16 implies that there is an independent transversal of H with respect to $(V_i)_{i \in [k]}$. But this means that there is a (directed) matching M of D' containing an arc from every monochromatic directed cycle in D. Then $M, B_1 \setminus M, \ldots, B_d \setminus M$ are all degree-f branchings, giving us an arc-coloring of D into d+1 degree-f branchings. \square

By Corollary 14, Theorem 17 implies the desired large-girth result on Conjecture 2.

Theorem 6. Let G be a simple graph, let $f: V(G) \to \mathbb{Z}_{\geq 2}$ be a function, and let $d = \max\{\Delta_f(G), a(G)\}$. If G has girth $g \geq 4d$, then $a_f(G) \leq d+1$.

Alon originally proved these large-girth results for the case f=2 under the girth condition $g \geq 100d$ [5], which he later improved to $g \geq 8ed$ [6]. He derived these girth conditions from earlier versions of Theorem 16 on independent transversals. Using the Lovász local lemma, Alon proved Theorem 16 for general graphs G (not just line graphs) under the conditions $|V_i| \geq 25\Delta(G)$ [5] and $|V_i| \geq 2e\Delta(G)$ [6], respectively. Haxell [25] subsequently improved these class size conditions to $|V_i| \geq 2\Delta(G)$. Haxell's bound is best possible if one does not assume that G is the line graph of a simple graph [36].

4.2 Asymptotics

Now we prove that Conjecture 12 holds asymptotically for all directed graphs D when the function f is bounded. Similar to Alon's [6] proof for the case f = 2, we show that every directed graph D can be decomposed into a specified number of directed subgraphs each with large directed girth and approximately the same maximum degree, and then we apply Theorem 17 to each of these directed subgraphs individually. We use the following slight modification of a lemma of Alon [6].

Lemma 18. Let D be a directed graph and let $f: V(D) \to \mathbb{Z}_{\geq 2}$ be a function. Suppose that $d = \max\{\Delta^-(D), \Delta^+_{f-1}(D)\}$ is sufficiently large compared to the maximum value of f, and let $k \leq d^{9/10}$ be a positive integer. Then there is a k-coloring of V(D) using the colors $0, 1, \ldots, k-1$ with the following property: for every vertex v and color i, the numbers

$$d^-(v,i) = |\{u \in V(D) : (u,v) \in E(D) \text{ and } u \text{ is colored } i\}|,$$

 $d^+(v,i) = |\{u \in V(D) : (v,u) \in E(D) \text{ and } u \text{ is colored } i\}|$

satisfy
$$d^-(v,i), \frac{d^+(v,i)}{f(v)-1} \le \frac{d}{k} + 3\sqrt{\frac{d \log d}{k}}.$$

Proof. Start by augmenting D to a directed graph D', adding auxiliary vertices and arcs so that every vertex $v \in V(D)$ has indegree $d_{D'}^-(v) = d$ and outdegree $d_{D'}^+(v) = d(f(v) - 1)$ in D' (not caring about the indegrees and outdegrees of the added vertices). Consider a random coloring of the vertices of D' with the colors $0, 1, \ldots, k-1$, where the color of every vertex chosen uniformly at random. For every vertex $v \in V(D)$ and color i, let $A_{v,i}^-$ be the event that $d_{D'}^-(v,i) > d/k + 3\sqrt{(d\log d)/k}$. Observe that $d_{D'}^-(v,i)$ is a binomial random variable with mean d/k. By a version of Chernoff's inequality (see Appendix A in [7]), we have that $\Pr[A_{v,i}^-] \leq 1/d^4$. Likewise, letting $A_{v,i}^+$ be the event that $d_{D'}^+(v,i)/(f(v)-1) > d/k + 3\sqrt{(d\log d)/k}$, we have that $\Pr[A_{v,i}^+] \leq 1/d^{4(f(v)-1)} \leq 1/d^4$.

Each of the events $A_{v,i}^-, A_{v,i}^+$ is mutually independent of the events $A_{u,j}^-, A_{u,j}^+$ for all the vertices $u \in V(D)$ that do not have a common neighbor with v in \overline{D} . Thus each of $A_{v,i}^-, A_{v,i}^+$ is mutually independent of all but at most $k(td)^2$ of the events $A_{u,j}^-, A_{u,j}^+$, where t is the maximum output of f. Since $e(1/d^4)(k(td)^2+1) < 1$ for d sufficiently large compared to t, by the symmetric Lovász local lemma (see Chapter 5 in [7]) no event $A_{v,i}^-$ or $A_{v,i}^+$ occurs. Thus there is a coloring of V(D') satisfying $d_{D'}^-(v,i), d_{D'}^+(v,i)/(f(v)-1) \le d/k + 3\sqrt{(d\log d)/k}$ for all $v \in V(D)$ and $0 \le i \le k-1$. Deleting the auxiliary vertices and arcs from D', this gives a desired coloring of V(D).

We can now prove the following asymptotic version of Conjecture 12 for directed graphs.

Theorem 19. For every integer $t \geq 2$, there exists a constant $c_t > 0$ such that for every directed graph D and function $f: V(D) \to \mathbb{Z}_{\geq 2}$ with maximum value at most t, we have

$$\vec{a}_f(D) \le d + c_t d^{3/4} (\log d)^{1/2}$$

where $d = \max\{\Delta^{-}(D), \Delta_{f-1}^{+}(D)\}.$

Proof. We may assume that d is sufficiently large compared to t wherever necessary. Pick a prime k satisfying $5d^{1/2} \leq k \leq 10d^{1/2}$. By Lemma 18, there exists a k-coloring ϕ of V(D) satisfying the stated inequalities. For each $0 \leq i \leq k-1$, let D_i be the spanning directed subgraph of D with arc set $E(D_i) = \{(u,v) \in E(D) : \phi(v) \equiv \phi(u) + i \pmod{k}\}$. The inequalities in Lemma 18 imply that $d_i = \max\{\Delta^-(D_i), \Delta^+_{f-1}(D_i)\} \leq d/k + 3\sqrt{(d\log d)/k}$ for every $0 \leq i \leq k-1$. Moreover, for $i \neq 0$ the length of every directed cycle in D_i is divisible by k, and thus D_i has directed girth $g_i \geq k \geq 4d_i$ (using that $k \geq 5d^{1/2}$ and d is sufficiently large). By Theorem 17, we deduce that $\vec{a}_f(D_i) \leq d_i + 1 \leq d/k + 3\sqrt{(d\log d)/k} + 1$ for every $1 \leq i \leq k-1$. For D_0 we only use the trivial inequality $\vec{a}_f(D_0) \leq 2d_0 \leq 2d/k + 6\sqrt{(d\log d)/k}$, obtained by first using Theorem 15 to arc-color D_0 into d_0 directed degree-f pseudoforests, and then trivially arc-coloring each of these directed degree-f pseudoforests into 2 degree-f branchings. These inequalities together with $5d^{1/2} \leq k \leq 10d^{1/2}$ give us that

$$\vec{a}_f(D) \le (k-1) \left(\frac{d}{k} + 3\sqrt{\frac{d \log d}{k}} + 1 \right) + \left(\frac{2d}{k} + 6\sqrt{\frac{d \log d}{k}} \right) \le d + c_t d^{3/4} (\log d)^{1/2},$$

for some constant c_t depending on t (since we assumed d is large compared to t).

By Corollary 14, Theorem 19 implies the desired asymptotic version of Conjecture 2 for simple graphs.

Theorem 7. For every integer $t \geq 2$, there exists a real constant $c_t > 0$ such that for every simple graph G and function $f: V(G) \to \mathbb{Z}_{\geq 2}$ with maximum value at most t, we have

$$a_f(G) \le d + c_t d^{3/4} (\log d)^{1/2},$$

where $d = \max\{\Delta_f(G), a(G)\}.$

We conclude with a few remarks. Alon [6] stated that the lower order term in Theorem 19 (and thus also in Theorem 7) can be improved to $c'd^{2/3}(\log d)^{1/3}$ when f=2, by using "recursion" instead of a naive arc-coloring of D_0 , as well as changing some parameters. This kind of modification could perhaps also work for more general vertex weight functions f. Alon, Teague, and Wormald [8] later recovered the same lower order term stated by Alon, in the undirected case, using a different and in some ways simpler method. Instead of using a large girth result like Theorem 17 or Theorem 6 above, they use a variant of the classical and easier result that the complete graph K_{2n} can be decomposed into n Hamiltonian paths. However, their proof method does not appear to generalize as readily as Alon's for general functions f.

Alon's [5] original asymptotic proof of the Linear Arboricity Conjecture involved coloring the edges of the graph rather than the vertices as we did above. This original approach resulted in a worse error term, but Alon noted without proof that this approach also gives the desired asymptotics more generally for multigraphs G with bounded edge-multiplicity $\mu(G)$. One can check that this

approach also generalizes well to other vertex weight functions f on the vertices, when one works in the directed setting. That is, the asymptotic upper bound $a_f(G) \leq (1 + o(1)) \max\{\Delta_f(G), a(G)\}$ that we proved above holds more generally for multigraphs G with bounded edge-multiplicity. In particular, the optimal constant in Question 5 is $c_f = 1$ more generally for such multigraphs.

Acknowledgments

I would like to thank Nathan Benedetto Proença for aiding my understanding and exposition of the linear programming in Section 2. I would also like to thank Penny Haxell for the helpful edits.

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