ON A COLORFUL PROBLEM BY DOL'NIKOV CONCERNING TRANSLATES OF CONVEX BODIES

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Abstract. In this note we study a conjecture by Jerónimo-Castro, Magazinov and Soberón which generalized a question posed by Dol'nikov. Let $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n$ be families of translates of a convex compact set $\mathcal{K}$ in the plane so that each two sets from distinct families intersect. We show that, for some j, $\bigcup_{i\neq j} \mathcal{F}_i$ can be pierced by at most 4 points. To do so, we use previous ideas from Gomez-Navarro and Roldán-Pensado together with an approximation result closely tied to the Banach-Mazur distance to the square.

1. Introduction

In 2011 Dol'nikov posed the following problem [MRB12, Problem 8].

Problem 1. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ be families of translates of a convex compact set $\mathcal{K}$ in the plane such that $A\cap B \neq \emptyset$ for each $A \in \mathcal{F}_i$, $B \in \mathcal{F}_j$ with $i \neq j$. Is it always true that some $\mathcal{F}_i$ has piercing number at most 3?*

The answer to this problem seems to be affirmative. The uncolored version (when $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$) was solved affirmatively by Karasev [Kar00], who later generalized it to higher dimensions [Kar08]. Jerónimo-Castro, Magazinov and Soberón [JCMS15] gave a positive answer to Problem 1 when $\mathcal{K}$ is either centrally symmetric or a triangle. They also stated the following stronger conjecture.

Conjecture 2. *For $n \geq 2$, let $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n$ be families of translates of a convex compact set $\mathcal{K}$ in the plane such that $A \cap B \neq \emptyset$ for each $A \in \mathcal{F}_i$, $B \in \mathcal{F}_j$ with $i \neq j$. Then there is some index j such that $\bigcup_{i\neq j} \mathcal{F}_i$ has piercing number at most 3.*

In the same paper they showed that this conjecture is true when $\mathcal{K}$ is an Euclidean disk.

Recently Gomez-Navarro and Roldán-Pensado proved that Problem 1 has a positive answer when $\mathcal{K}$ is either of constant width or is close to a Euclidean disk with respect to the Banach-Mazur distance [GNRP23]. They also showed that Dol'nikov's problem has a positive answer with 8 piercing points instead of 3 and that Conjecture 2 is true with 9 piercing points instead of 3.

The purpose of this paper is to prove Conjecture 2 with 4 piercing points instead of 3.

Theorem 3. *For $n \geq 2$, let $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n$ be families of translates of a convex compact set $\mathcal{K}$ in the plane such that $A \cap B \neq \emptyset$ for each $A \in \mathcal{F}_i$, $B \in \mathcal{F}_j$ with $i \neq j$. Then there is some index j such that $\bigcup_{i\neq j} \mathcal{F}_i$ has piercing number at most 4.*

The proof follows the ideas used to prove Theorem 2.3 from [GNRP23], together with an approximation result which is related to the Banach-Mazur distance to the square. The auxiliary results we require are stated in Section 2. Section 3 contains the proof of Theorem 3.

Key words and phrases. Colorful theorems; Piercing number; Banach-Mazur metric.

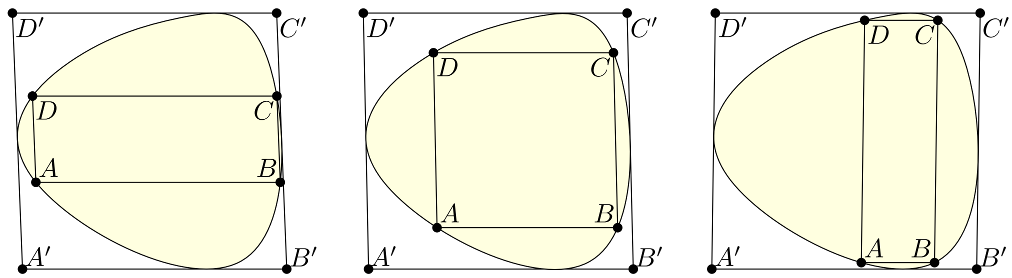

FIGURE 1. The parallelograms vary the ratio between their sides.

2. Two auxiliary lemmas

Our proof is based on two lemmas. The first one is a special case of a theorem proved by Gomez-Navarro and Roldán-Pensado [GNRP23, Theorem 2.5(a)].

Lemma 4. *For $n \geq 2$, let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ be families of translates of a convex compact set $\mathcal{K}$ in the plane such that $A \cap B \neq \emptyset$ for each $A \in \mathcal{F}_i$, $B \in \mathcal{F}_j$ with $i \neq j$. If for every index j the family $\bigcup_{i \neq j} \mathcal{F}_i$ has piercing number larger than 3, then there is a line transversal to $\bigcup_i \mathcal{F}_i$.*

The tools behind the proof of Lemma 4 are the uncolored version of Problem 1 and the fact that a family of convex sets $\mathcal{F}$ on the plane has a transversal line in every direction if and only if $\mathcal{F}$ is pairwise intersecting. We refer the reader to [GNRP23] for the full details.

The second lemma essentially gives a way of approximating a convex body by a parallelogram. It implies that the Banach-Mazur distance from the square to any planar convex body is at most 2. This was already known [GLMP04, Theorem 5.5], however we require something slightly stronger. Given a convex body $\mathcal{K}$, it is known that the parallelogram P of maximal area contained in $\mathcal{K}$ satisfies that there is a translation of $2P$ that contains $\mathcal{K}$. We require a similar result where, instead of P having maximal area, the direction of one of the sides of P is fixed.

Lemma 5. *Let $\mathcal{K}$ be a convex body in the plane and let u be a fixed direction. Then, there is a parallelogram $P \subset \mathcal{K}$ such that one of the sides of P has direction u and there is a translated copy Q of $2P$ such that $\mathcal{K} \subset Q$.*

Proof. We may assume that $\mathcal{K}$ is smooth, as the general case follows from standard approximation arguments. Without loss of generality, the direction u is horizontal and the bottom and top horizontal supporting lines of $\mathcal{K}$ are $y = 0$ and $y = 1$, respectively.

The length $l(h)$ of the horizontal chord of $\mathcal{K}$ at height $h \in [0, 1]$ depends continuously on h and it is unimodular: $l(0) = 0$, then it increases until it attains some maximum m and then goes back to 0. Therefore, each $l \in [0, m)$ is attained exactly twice.

For every $l \in (0, m)$, let $ABCD$ be the inscribed parallelogram to $\mathcal{K}$ such that AB and CD are horizontal, and $AB = CD = l$. Let $A'B'C'D'$ be the parallelogram circumscribed around $\mathcal{K}$ such that the sides of $A'B'C'D'$ are parallel to the sides of $ABCD$. See Figure 1.

Note that $A'B'C'D'$ is homothetic to $ABCD$ if and only if $A'B'/B'C' = AB/BC$. Let α be the interior angle $\angle DAB$ and assume that $0 < r < R$ are real numbers such that there is a disk of radius r contained in $\mathcal{K}$ and $\mathcal{K}$ is contained in a disk of radius R. Then $B'C' = 1/\sin(\alpha)$ and $2r/\sin(\alpha) \leq A'B' \leq 2R/\sin(\alpha)$, therefore $2r \leq A'B'/B'C' < 2R$. However AB/BC tends to 0

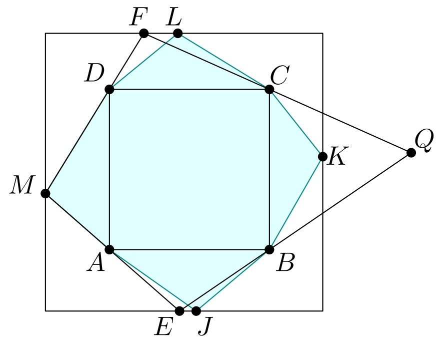

FIGURE 2. The points where $\mathcal{K}$ touches P and P', and the point Q.

(resp. ∞) as l tends to 0 (resp. m). This variation is continuous, so at some point $ABCD$ and $A'B'C'D'$ are homothetic.

By applying a linear transformation, we may assume that these parallelograms are squares and that $ABCD$ has unit side. Our result follows immediately if we manage to prove the following claim: if P is a unit square and P' is a square homothetic to P such that $P \subset \mathcal{K} \subset P'$, the vertices of P are in $\partial\mathcal{K}$ and $\mathcal{K}$ is internally tangent to the sides of P', then the homothety ratio between P and P' is at most 2. If this is the case, then $P \subset \mathcal{K}$, and for a translation Q of $2P$ we would have $\mathcal{K} \subset P' \subset Q$ as desired.

Recall that the vertices A, B, C, D of P lie in $\partial\mathcal{K}$ and let J, K, L, M be the points where the sides of P' touch $\partial\mathcal{K}$, as in Figure 3. These eight points form a convex polygon. Let E be the intersection of the line MA with the bottom side of P' and let F be the intersection of the MD with the top side of P'.

By convexity at the angle LDM, we have that F lies to the left of L. In turn, by convexity at the angle KCL we have that L lies to the left of line CB. Therefore, F (and analogously E) lies to the left of line CB. Then K lies below the line FC and above the line EB. We conclude that K lies to the left of the intersection Q of the lines FC and EB.

Proceeding by contradiction, we show that if the desired homothety ratio is larger than 2, then Q is strictly to the left of the side $B'C'$ of P', which is impossible since by the previous argument then K would not be on the side $B'C'$ of P'.

As in Figure 3, let ℓ be the horizontal line through M. Define a as the distance from D to ℓ and h as the distance from L to ℓ. Set X and Y as the intersections of the lines FC and EB with ℓ, respectively. Let X' be the intersection of the line AB and the vertical line through X and define Y' as the intersection of the line CD and the vertical line through Y.

Recall that P is a unit square, so $MX = MX/CD = h/(h-a)$. This shows that X depends on the vertical position of P but not on the horizontal position of P. The same is true for Y and consequently it is also true for X' and Y'. At this point, we may ignore the specific convex body $\mathcal{K}$ and study the possible diagrams we may obtain.

We claim that, if we translate P horizontally (i.e. in diagrams where P has the same vertical position), Q stays on the line $X'Y'$. Indeed, the lines BX', CY' and XY are horizontal, and thus projectively concurrent. The same is true for the lines BC, YY' and XX', since they are vertical. Therefore, by the dual of Pappus' theorem [Cox61] the lines BY, CX and $X'Y'$ are concurrent, which means that Q lies on the line $X'Y'$.

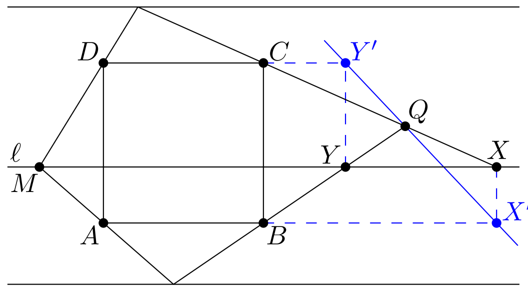

FIGURE 3. Auxiliary points and lines.

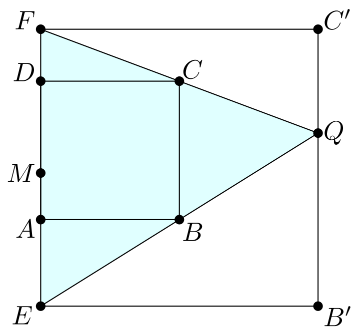

FIGURE 4. The extreme case when the homothety ratio is exactly 2.

Let us assume, that Q is above the line ℓ. Then X is to the right of Q and Y is to the left of Q. This implies that X' is below and to the right of Y'. As P is translated to the left, the point F also moves to the left and therefore the line XF intersects the line $X'Y'$ at a lower point. Since this point is Q and the slope of $X'Y'$ is not positive then Q moves to the right as P is translated to the left (i.e. among all possible diagrams where P has the same vertical position, the one where P is leftmost has rightmost Q). If Q is below ℓ the reasoning is analogous.

If Q lies on ℓ then $X'Y'$ is vertical and Q does not move horizontally as P is translated left.

Hence, we only need to prove that Q lies strictly to the left of the side $B'C'$ of P' in the limit case when the square P has its left side contained in the left side of P', as in Figure 4.

In this case, if the homothety ratio is exactly 2, then CQB and FQE are homothetic triangles from Q, so the distance from Q to CB is half the distance from Q to FE. This means that Q lies exactly on the side $B'C'$ of P'. Then, if the homothety ratio is larger than 2, then the distance from Q to CB is less than half the distance from Q to FE, and thus Q is strictly to the left of side $B'C'$. This implies that K is also strictly to the left of the side $B'C'$ of P', which is the desired contradiction to our original assumption that the homothety ratio was larger than 2. We conclude that the homothety ratio of both squares is at most 2, as desired. □

3. PROOF OF THEOREM 3

Once we have the Lemmas from the previous section, the proof of Theorem 3 is simple. We start by using Lemma 4. If for some index j it happens that $\bigcup_{i\neq j}\mathcal{F}_i$ has piercing number at most

3, then we are done. Otherwise, there is a line ℓ transversal to $\bigcup_i \mathcal{F}_i$ with direction, say, u. By Lemma 5, there is a paralellogram $P \subset \mathcal{K}$ such that one of the sides of P has direction u and there is a translated copy R of $2P$ such that $\mathcal{K} \subset R$. Let v be the direction of the other side of the paralellogram P.

By projecting the convex bodies in the sets $\mathcal{F}_i$ to a line m orthogonal to v, we obtain collections $\mathcal{I}_1, \dots, \mathcal{I}_n$ of intervals on m such that $I \cap I' \neq \emptyset$ for each $I \in \mathcal{I}_i$, $I' \in \mathcal{I}_j$ with $i \neq j$. What follows is a common generalization of the colorful Helly theorem (see e.g. [Bár21]). If every two intervals in $\bigcup_i \mathcal{I}_i$ intersect, then by Helly's theorem there is a point common to all of them. If not, there are two of these intervals, say I and I', that are disjoint. These intervals must then belong to the same family I_j and therefore any interval I^* not in this family must intersect both I and I'. Thus, I^* contains any point separating I from I'. In both cases there is an index j such that the elements of $\bigcup_{i\neq j} \mathcal{I}_i$ have a point in common. By lifting this point in direction v, we obtain a line transversal ℓ' to $\bigcup_{i\neq j} \mathcal{F}_i$ with direction v. This implies that every translate of $\mathcal{K}$ in $\bigcup_{i\neq j} \mathcal{F}_i$ intersects both the line ℓ with direction u and the line ℓ' with direction v. We now exhibit four points that pierce all translates of $\mathcal{K}$ with this property.

Consider the sets

$$\mathbf{K} = \{x \in \mathbb{R}^2 : \mathcal{K} + x \text{ intersects both } \ell \text{ and } \ell'\} \text{ and}$$
$$\mathbf{R} = \{x \in \mathbb{R}^2 : R + x \text{ intersects both } \ell \text{ and } \ell'\}.$$

Note that the set $\mathbf{R}$ is congruent to R and, since P is a parallelogram, the set $-P$ is congruent to P. Hence, the set $\mathbf{R}$ can be covered with four copies of $-P$, say $-P + a$, $-P + b$, $-P + c$ and $-P + d$. Then the points a, b, c and d pierce any translate $\mathcal{K} + x$ that intersect both ℓ and ℓ'. Indeed, if $x \in \mathbf{K} \subset \mathbf{R}$ then x belongs to either $-P + a$, $-P + b$, $-P + c$ or $-P + d$. Without loss of generality we may assume that $x \in -P + a$ which implies that $a \in \mathcal{K} + x$. $\square$

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