# MONOCHROMATIC QUOTIENTS, PRODUCTS AND POLYNOMIAL SUMS IN THE RATIONALS 

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#### Abstract

Let $k, a \in \mathbb{N}$ and let $p_{1}, \cdots, p_{k} \in \mathbb{Q}[n]$ with zero constant term. We show that for any finite coloring of $\mathbb{Q}$, there are non-zero $x, y \in \mathbb{Q}$ such that there exists a color which contains a set of the form $$
\left\{x, \frac{x}{y^{a}}, x+p_{1}(y), \cdots, x+p_{k}(y)\right\}
$$ and there are non-zero $v, u \in \mathbb{Q}$ such that there exists a color which contains a set of the form $$
\left\{v, v \cdot u^{a}, v+p_{1}(u), \cdots, v+p_{k}(u)\right\} .
$$


## 1. Introduction

In the investigation of partition of sets, looking for Ramsey family on $\mathbb{N}$ and $\mathbb{Q}$ is a center topic. A Ramsey family $\mathcal{A}$ on $\mathbb{N}$ is a finite set of the maps from $\mathbb{N}^{i}$ to $\mathbb{Z}$ where $i \in \mathbb{N}$ such that for any finite coloring of $\mathbb{N}$, there exists $x \in \mathbb{N}^{i}$ such that $\{f(x): f \in \mathcal{A}\}$ is monochromatic. Similarily, a Ramsey family $\mathcal{A}$ on $\mathbb{Q}$ is a finite set of the maps from $\mathbb{Q}^{i}$ to $\mathbb{Q}$ where $i \in \mathbb{N}$ such that for any finite coloring of $\mathbb{Q}$, there exists $x \in \mathbb{Q}^{i}$ such that $\{f(x): f \in \mathcal{A}\}$ is monochromatic.

Naturally, we seek to search Ramsey family in $\mathbb{Z}\left[x_{1}, \cdots, x_{s}\right]$ and $\mathbb{Q}\left[x_{1}, \cdots, x_{s}\right]$ where $s \in \mathbb{N}$.

On $\mathbb{N}$, there are some results. I. Schur's theorem [11] states the family $\{(x, y) \mapsto$ $x,(x, y) \mapsto y,(x, y) \mapsto x+y\}$ is Ramsey on $\mathbb{N}$ and van der Waerden's thereom [12] states for any $k \in \mathbb{N}$, the family $\{(x, y) \mapsto x,(x, y) \mapsto x+y, \cdots,(x, y) \mapsto x+k y\}$ is Ramsey on $\mathbb{N}$. For general linear polynomials, R. Rado built a equivalent condition for a family of linear polynomials to be Ramsey on $\mathbb{N}$ in [9]. Based on it, we can verify that the family $\{x \mapsto x, x \mapsto x+3\}$ is not Ramsey on $\mathbb{N}$. For general polynomials, there are only a few results. Furstenberg-Sarközy theorem illustrates the family $\left\{(x, y) \mapsto x,(x, y) \mapsto x+y^{2}\right\}$ is Ramey on $\mathbb{N}$ (see [10]) and V. Bergelson extended it to $\left\{(x, y) \mapsto x,(x, y) \mapsto y,(x, y) \mapsto x+y^{2}\right\}$ in [1]. V. Bergelson and A. Leibman's polynomial extension of van der Waerden's thereom [4] declares that for any $k \in \mathbb{N}$, for any $p_{1}, \cdots, p_{k} \in \mathbb{Z}[n]$ with zero constant term, the family $\{(x, y) \mapsto$ $\left.x,(x, y) \mapsto x+p_{1}(y), \cdots,(x, y) \mapsto x+p_{k}(y)\right\}$ is Ramsey on $\mathbb{N}$. For aforementioned Ramsey familes on general polynomials, they do not contain polynomials $(x, y) \mapsto y$

[^0]and $(x, y) \mapsto x \cdot y$. For this, there exists a question which still lacks a complete answer .

Question 1.1. ([7, Question 3])Is the family $\{(x, y) \mapsto x,(x, y) \mapsto y,(x, y) \mapsto$ $x \cdot y,(x, y) \mapsto x+y\}$ Ramsey on $\mathbb{N}$ ?

For the question, J. Moreira answered it under leaving out polynomial $(x, y) \mapsto y$ in $[8$, Corollary 1.5]. If we consider the question on $\mathbb{Q}$, the fact that $(\mathbb{Q} \backslash\{0\}, \cdot)$ is a group makes it easier than one in $\mathbb{N}$. Recently, M. Bowen and M. Sabok [6, Theorem 1.1] showed that for any $k \in \mathbb{N}$, the family $\{(x, y) \mapsto x,(x, y) \mapsto$ $y,(x, y) \mapsto x \cdot y,(x, y) \mapsto x+y, \cdots,(x, y) \mapsto x+k y\}$ is Ramsey on $\mathbb{Q}$. For general polynomials, J. Moreira 's theorem [8, Theorem 1.4] guarantees that for any $k \in \mathbb{N}$, for any $p_{1}, \cdots, p_{k} \in \mathbb{Z}[n]$ with zero constant term, the family $\{(x, y) \mapsto x,(x, y) \mapsto$ $\left.x \cdot y,(x, y) \mapsto x+p_{1}(y), \cdots,(x, y) \mapsto x+p_{k}(y)\right\}$ is Ramsey on $\mathbb{N}$. Clearly, it is Ramsey on $\mathbb{Q}$. Our main result is to extend J. Moreira's family to a wider case on $\mathbb{Q}$ and it reflects the symmetry between multiplication and division on $\mathbb{Q}$. Specific statements are as follows.
Theorem 1.2. Let $k, a \in \mathbb{N}$ and let $p_{1}, \cdots, p_{k} \in \mathbb{Q}[t]$ with zero constant term. For any finite coloring of $\mathbb{Q}$, then
(1) there are non-zero $y \in \mathbb{Q}$ and an infinite subset $A$ of $\mathbb{Q} \backslash\{0\}$ such that

$$
A \cup\left(y^{-a} \cdot A\right) \cup\left(A+\left\{p_{i}(y): 1 \leq i \leq k\right\}\right)
$$

is monochromatic;
(2) there are non-zero $u \in \mathbb{Q}$ and an infinite subset $B$ of $\mathbb{Q} \backslash\{0\}$ such that

$$
B \cup\left(u^{a} \cdot B\right) \cup\left(B+\left\{p_{i}(u): 1 \leq i \leq k\right\}\right)
$$

is monochromatic.
The proof of Theorem 1.2 is based on three ingredients. The first ingredient(see Theorem 2.5) is the multiple recurrence for polynomial mapping, built by V. Bergelson and A. Leibman in [5], which helps us to build a van der Waerden-type result for piecewise syndetic subsets of $(\mathbb{Q},+)$. The second ingredient(see Lemma 2.4), established by M. Bowen and M. Sabok in [6], seeks to localize multiplicatively thick subsets according to the certain finite coloring of $\mathbb{Q}$. The third ingredient(see Theorem 2.2) is the partition regularity of piecewise syndetic subsets of $(\mathbb{Q},+)$.

The organization of the paper is as follows. In section 2, we recall some large subsets and multiple recurrence for polynomial mappings and bulid a van der Waerdentype result. In section 3, we prove the Theorem 1.2.

## 2. Preliminaries

2.1. Some large subsets. At first, we state the definitions of some large subsets. Before this, we introduce some notations. Let $S$ be a non-empty set. Let $\mathcal{F}(S)$ denote all finite subsets of $S$ and $\mathcal{F}^{*}(S)$ denote all finite non-empty subsets of $S$.

Definition 2.1. Let $G$ be an infinite, countable, abelian group and let $A \subset G$.
(a) $A$ is thick if and only if for any $F \in \mathcal{F}^{*}(G)$, there is some $x \in G$ such that $F x \subset A$;
(b) $A$ is syndetic if and only if there exists $F \in \mathcal{F}^{*}(G)$ such that $F A=G$;
(c) $A$ is piecewise syndetic if and only if there exists $F \in \mathcal{F}^{*}(G)$ such that $F A$ is thick;
(d) let $r \in \mathbb{N}$, $A$ is $\boldsymbol{I} \boldsymbol{P}_{\boldsymbol{r}}$ if and only if there exist $s_{1}, \cdots, s_{r} \in G$ such that $F P\left(\left\{s_{i}\right\}_{i=1}^{r}\right)=\left\{\prod_{i \in \alpha} s_{i}: \alpha \in \mathcal{F}^{*}(\{1, \cdots, r\})\right\} \subset A ;$
(e) let $r \in \mathbb{N}$, $A$ is $\boldsymbol{I} \boldsymbol{P}_{r}^{*}$ if and only if $A$ has non-empty intersection with any $I P_{r}$ subset of $G$.

The following results state some properties of the above large subsets.
Theorem 2.2. ([3, Theorem 2.5]) Let $G$ be an infinite, countable, abelian group and let $A, B \subset G$. If $A \cup B$ is piecewise syndetic, then $A$ or $B$ is piecewise syndetic.

Proposition 2.3. Let $G$ be an infinite, countable, abelian group and let $A \subset G$. If $A$ is thick, then $A \backslash F$ is thick where $F \in \mathcal{F}^{*}(G)$.

Proof. For any $H \in \mathcal{F}^{*}(G),\{x \in G: H x \cap F \neq \varnothing\}$ is finite or empty. Note that $\{x \in G: H x \subset A\}$ is thick. Then we can find $x \in G$ such that $H x \subset A \backslash F$. So $A \backslash F$ is still thick. This finishes the proof.

Next, we focus on specific group $\mathbb{Q}$. To avoid ambiguity, the thick subset of group $(\mathbb{Q},+)$ is called additively thick and the thick subset of group $(\mathbb{Q} \backslash\{0\}, \cdot)$ is called multiplicatively thick.

The following lemma provided in [6, Lemma 3.3] which plays a crucial role in the proof of our main result.
Lemma 2.4. Let $\mathbb{Q} \backslash\{0\}=\bigcup_{i=1}^{n} C_{i}$ be a finite coloring. There exist $k \in \mathbb{N}$, index sets $Y_{1}, \cdots, Y_{k} \subset\{1, \cdots, n\}$ and $F \in \mathcal{F}^{*}(\mathbb{Q} \backslash\{0\})$ such that
(a) for any $1 \leq l \leq k, \bigcup_{m \in Y_{l}} C_{m}$ is multiplicatively thick;
(b) for any $x \in \mathbb{Q} \backslash\{0\}$, there exists $1 \leq l \leq k$ such that for each $m \in Y_{l}$, one has $x \in F \cdot C_{m}$.
2.2. Polynomial mapping $\mathcal{F}(S) \rightarrow G$ and multiple recurrence. Let $S$ be a non-empty set. Let $G$ be an infinite, countable, torsion-free abelian group with identity element $e_{G}$. We reproduce the notation of polynomial mapping $\mathcal{F}(S) \rightarrow G$ which introduced by V. Bergelson and A. Leibman in [5, Section 1].

Let $\left\{g_{t}\right\}_{t \in T}$ be a collection of elements of $G$ indexed by a finite set $T$. We can define $\prod_{t \in T} g_{t}$. If $T$ is empty, we put $\prod_{t \in T} g_{t}=e_{G}$. Let $d \in \mathbb{N}$. We use $S^{d}$ to denote the produce $S \times \cdots \times S(d$ times $)$. Conventionally, we let $S^{0}=\{\varnothing\}$.

Let $d \in \mathbb{N} \cup\{0\}$. A monomial of degree $\boldsymbol{d}$ on $\boldsymbol{S}$ with values in $\boldsymbol{G}$ is a mapping $u: S^{d} \rightarrow G$. A monomial $u$ induces a monomial mapping $p_{u}: \mathcal{F}(S) \rightarrow G, \alpha \mapsto$ $\prod_{s \in \alpha^{d}} u(s)$.

A polynomial mapping $p: \mathcal{F}(S) \rightarrow G$ is the finite product of monomial mappings. The degree of $p(\operatorname{denoted} \operatorname{by} \operatorname{deg} p)$ is the minimum, taken over the set of all representations of $p$ as the product $p=\prod_{i=1}^{m} p_{u_{i}}$ of monomial mappings, of the maximum of the degree of monomial $u_{i}, 1 \leq i \leq m$.

The following theorem states the multiple recurrence phenomena for such polynomials.

Theorem 2.5. ([5, Theorem 4.1]) Let $G$ be an infinite, countable, torsion-free abelian group of automorphisms of a compact metric space $(X, \rho)$. For any $k, d \in \mathbb{N}$ and any $\epsilon>0$, there exists $N \in \mathbb{N}$ such that if $S$ is a set with cardinality $\geq N$ and $p_{1}, \cdots, p_{k}$ : $\mathcal{F}(S) \rightarrow G$ are polynomial mappings with $\operatorname{deg} p_{i} \leq d, p_{i}(\varnothing)=I d_{X}, 1 \leq i \leq k$, then there exist $x \in X$ and $\alpha \in \mathcal{F}^{*}(S)$ such that for each $1 \leq i \leq k, \rho\left(x, p_{i}(\alpha) x\right)<\epsilon$.

Remark 2.6. Assume that $X$ is minimal with respect to the action of $G$, that is, $X$ does not contain proper non-empty closed $G$-invariant proper subsets. By [5, Proof of Theorem 4.1], the set of the points $x \in X$ satisfy the requirements of the theorem is dense in $X$.
2.3. A van der Waerden-type result. Based on the Theorem 2.5, we have the following result.

Proposition 2.7. Let $G$ be an infinite, countable, torsion-free abelian group with identity element $e_{G}$ and $A$ be a piecewise syndetic subset of $G$. For any $k, d \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that if $S$ is a set with cardinality $\geq N$ and $p_{1}, \cdots, p_{k}$ : $\mathcal{F}(S) \rightarrow G$ are polynomial mappings with $\operatorname{deg} p_{i} \leq d, p_{i}(\varnothing)=e_{G}, 1 \leq i \leq k$, then there exists $\alpha \in \mathcal{F}^{*}(S)$ such that $A \cap p_{1}(\alpha)^{-1} A \cap \cdots \cap p_{k}(\alpha)^{-1} A$ is piecewise syndetic.

Proof. Let $\Omega=\{0,1\}^{G}$. The element of $\Omega$ can be written as $\mathbf{w}=(w(g))_{g \in G}$. Specially, let $\mathbf{0}$ denote element $\mathbf{y}$ with $y(g)=0$ for any $g \in G$. Since $G$ is countable, we can write $G$ as $\left\{g_{1}, g_{2}, \cdots, g_{n}, \cdots\right\}$. Define a metric $\rho$ on $\Omega$ by

$$
\rho(\mathbf{w}, \mathbf{u})=\frac{1}{\min \left\{i \in \mathbb{N}: w\left(g_{i}\right) \neq u\left(g_{i}\right)\right\}}
$$

for any $\mathbf{w}, \mathbf{u} \in \Omega$. Then $(\Omega, \rho)$ is a compact metric space. $G$ can act on $(\Omega, \rho)$ by $(g \mathbf{w})(h)=\mathbf{w}(g h)$ for any $\mathbf{w} \in \Omega, g, h \in G$.

Define $\mathbf{v} \in \Omega$ by $v(g)=1_{A}(g)$ for any $g \in G$. Let $X=\overline{\{g \mathbf{v}: g \in G\}}$. Then we have the following claim. Its proof will be provided in the last paragraph.

Claim 1. There exists $\boldsymbol{O} \neq \boldsymbol{x} \in X$ such that $Y$ is minimal with respect to the action of $G$ where $Y=\overline{\{g \boldsymbol{x}: g \in G\}}$.

Based on this claim, we can reach the conclusion. Let $U=\left\{\mathbf{w} \in \Omega: w\left(e_{G}\right)=\right.$ 1\}. Apply Theorem 2.5 to $(Y, \rho), k, d$ and $\frac{1}{2^{m}}$ where $g_{m}=e_{G}$, then there exists $N \in \mathbb{N}$ such that if $S$ is a set with cardinality $\geq N$ and $p_{1}, \cdots, p_{k}: \mathcal{F}(S) \rightarrow G$ are polynomial mappings with $\operatorname{deg} p_{i} \leq d, p_{i}(\varnothing)=I d_{Y}, 1 \leq i \leq k$, then there exist $\mathbf{z} \in Y \cap U$ and $\alpha \in \mathcal{F}^{*}(S)$ such that for each $1 \leq i \leq k, \mathbf{z}\left(e_{G}\right)=\mathbf{z}\left(p_{i}(\alpha)\right)$.

Let $V=\left\{\mathbf{w} \in \Omega: w\left(e_{G}\right)=w\left(p_{1}(\alpha)\right)=\cdots=w\left(p_{k}(\alpha)\right)=1\right\}$. Since $(Y, G)$ is minimal, the set $B=\{g \in G: g \mathbf{z} \in Y \cap V\}=\left\{a_{1}, a_{2}, \cdots, a_{n}, \cdots\right\}$ is a syndetic subset of $G$. Moreover, we can find a pairwise distinct sequence $\left\{h_{j}\right\}_{j \geq 1} \subset G$ such that for each $j \geq 1$, we have $h_{j} a_{1} \mathbf{v}, \cdots, h_{j} a_{j} \mathbf{v} \in V$. That is,

$$
A \cap p_{1}(\alpha)^{-1} A \cap \cdots \cap p_{k}(\alpha)^{-1} A \supset \bigcup_{j \geq 1}\left\{h_{j} a_{1}, \cdots, h_{j} a_{j}\right\} .
$$

Therefore, $A \cap p_{1}(\alpha)^{-1} A \cap \cdots \cap p_{k}(\alpha)^{-1} A$ is piecewise syndetic.
The rest of the proof is to verify Claim 1. Let $\left\{F_{n}\right\}_{n \geq 1}$ be a strictly increasing sequence of $\mathcal{F}^{*}(G)$ with $\bigcup_{n>1} F_{n}=G$. Since $A$ is piecewise syndetic, there exist $H \in \mathcal{F}^{*}(G)$ and sequence $\left\{a_{n}\right\}_{n \geq 1} \subset G$ such that $a_{n}^{-1} F_{n} \subset H^{-1} A$. Without loss of generality, we can say that $a_{n} \mathbf{v} \rightarrow \mathbf{u}$ as $n \rightarrow \infty$ where $\mathbf{u} \in X$. Let $C=\{g \in G$ : $\mathbf{u}(g)=1\}$. For any $h \in G$, there exist infinite $n \geq 1$ and $g \in H$ such that $g h \in a_{n} A$. That is, $g h \in C$. Clearly, $H^{-1} C=G$. So $C$ is syndetic. Let $Z=\overline{\{g \mathbf{u}: g \in G\}} \subset X$. Clearly, $\mathbf{0} \notin Z$. Take a non-empty closed $G$-invariant subset $Z^{\prime}$ of $Z$ such that $\left(Z^{\prime}, G\right)$ is minimal. Then any element of $Z^{\prime}$ can satisfy the requirments of Claim 1. This finishes the proof.

Based on the above, we can bulid a van der Waerden-type result for piecewise syndetic subsets of infinite, countable, torsion-free abelian group. Before specific statements, we introduce the degree for polynomials between general abelian groups.
Definition 2.8. ([2, Definition 7.7]) Let $G$ and $H$ be abelian groups. Given $d \in \mathbb{N}$, a map $p: G \rightarrow H$ is a polynomial of degree $\boldsymbol{d}$ if the application of any $d+1$ of the discrete difference operators $\delta_{g}, g \in G$ defined by $\left(\delta_{g} p\right)(x)=p(g x)(p(x))^{-1}$ for any $x \in G$, reduces $p$ to the constant map which takes identity element of $H$.

Proposition 2.9. Let $H$ be an infinite, countable, torsion-free abelian group with with identity element $e_{H}$ and $A$ be a piecewise syndetic subset of $H$. For any $k, d \in \mathbb{N}$, there exists $r \in \mathbb{N}$ such that for any infinite, countable, torsion-free abelian group $G$ with identity element $e_{G}$ and all polynomials $p_{1}, \cdots, p_{k}: G \rightarrow H$ of degree at most $d$ with $p_{i}\left(e_{G}\right)=e_{H}, 1 \leq i \leq k$, the set

$$
\left\{g \in G: A \cap p_{1}(g)^{-1} A \cap \cdots \cap p_{k}(g)^{-1} A \text { is a piecewise syndetic }\right\}
$$

is an $I P_{r}^{*}$ subset of $G$.
Proof. By Proposition 2.7, we can get $r \in \mathbb{N}$ such that if $S$ is a set with cardinality $\geq r$ and $q_{1}, \cdots, q_{k}: \mathcal{F}(S) \rightarrow H$ are polynomial mappings with $\operatorname{deg} q_{i} \leq d, q_{i}(\varnothing)=$ $e_{H}, 1 \leq i \leq k$, then there exists $\alpha \in \mathcal{F}^{*}(S)$ such that $A \cap q_{1}(\alpha)^{-1} A \cap \cdots \cap q_{k}(\alpha)^{-1} A$ is a piecewise syndetic subset of $H$.

Choose $g_{1}, \cdots, g_{r}$ from $G$ arbitrarily. For any $1 \leq i \leq k$, define polynomial mapping $\overline{p_{i}}: \mathcal{F}(\{1, \cdots, r\}) \rightarrow H$ by the rule $\overline{p_{i}}(\alpha)=p_{i}\left(\prod_{m \in \alpha} g_{i}\right)$ for any $\alpha \in$ $\mathcal{F}(\{1, \cdots, r\})$. Clearly, $\operatorname{deg} \overline{p_{i}} \leq d, \overline{p_{i}}(\varnothing)=e_{H}$ for any $1 \leq i \leq k$. So there exists $\beta \in \mathcal{F}^{*}(\{1, \cdots, r\})$ such that $A \cap \overline{p_{1}}(\beta)^{-1} A \cap \cdots \cap \overline{p_{1}}(\beta)^{-1} A$ is a piecewise syndetic subset of $H$. This finishes the proof.

## 3. Proof of Theorem 1.2

In this section, we prove our main result. Here, we only provide proof for (1) of Theorem 1.2 since the proof of the rest part is similar. During the process, the key points are Lemma 2.4, Proposition 2.9 and pigeonhole principle.

Proof of Theorem 1.2. There exists $d \in \mathbb{N}$ such that $\operatorname{deg} p_{i} \leq d, 1 \leq i \leq k$. Choose a finite coloring of $\mathbb{Q}$ arbitrarily and fix it. Then $\mathbb{Q} \backslash\{0\}$ can inherit a coloring from $\mathbb{Q}$. We write it as $\mathbb{Q} \backslash\{0\}=\bigcup_{m=1}^{n} C_{n}$. Clearly, 0 has new color $n+1$ or there exists $\omega \in\{1, \cdots, n\}$ such that 0 has color $\omega$.

By Lemma 2.4, There exist $M \in \mathbb{N}$, index sets $Y_{1}, \cdots, Y_{M} \subset\{1, \cdots, n\}$ and $H \in \mathcal{F}^{*}(\mathbb{Q} \backslash\{0\})$ such that
(1) for any $1 \leq l \leq M, \bigcup_{m \in Y_{l}} C_{m}$ is multiplicatively thick;
(2) for any $x \in \mathbb{Q} \backslash\{0\}$, there exists $1 \leq l \leq M$ such that for each $m \in Y_{l}$, one has $x \in H \cdot C_{m}$.

Let $s$ be a non-zero rational less than minimum of $H$. Let $F=H \cup\{s\}$. Then for any $x \in \mathbb{Q} \backslash\{0\}$, we can find minimal $1 \leq l_{x} \leq M$ such that for each $m \in Y_{l_{x}}$, one has $x \in f_{m, x} \cdot C_{m}$ where $f_{m, x}=\min \left\{f \in H: x \in f \cdot C_{m}\right\}$. If $m \in\{1, \cdots, n\} \backslash Y_{l_{x}}$, let $f_{m, x}$ be $s$. Then we define new finite coloring of $\mathbb{Q} \backslash\{0\}$. That is, for any $x \in \mathbb{Q} \backslash\{0\}$, it has color $\left(l_{x}, f_{1, x}, \cdots, f_{n, x}\right) \in\{1, \cdots, M\} \times F^{n}$.

By Theorem 2.2 and Proposition 2.3, we know there exists $\left(l_{1}, f_{1,1}, \cdots, f_{n, 1}\right) \in$ $\{1, \cdots, M\} \times F^{n}$ such that the set

$$
A_{1}=\left\{x \in \mathbb{Q} \backslash\{0\}: x \text { has color }\left(l_{1}, f_{1,1}, \cdots, f_{n, 1}\right)\right\}
$$

is a piecewise syndetic subset of $(\mathbb{Q},+)$. Let $N=36^{100 M|F|^{n}}, T=36^{100 N|F|} k$. Apply Proposition 2.9 to $T, d, A_{1}$, then we get a natural number $r_{1}$. We can construct $I P_{r_{1}}$ subset $S_{1}$ of $(\mathbb{Q},+)$ such that $S_{1} \subset \bigcup_{m \in Y_{l_{1}}} C_{m}$. Let

$$
Q_{1}=\left\{f \cdot p_{i}(t): 1 \leq i \leq k, f \in F\right\}
$$

Clearly, $\left|Q_{1}\right|<T$. Then there exists $y_{1} \in S_{1}$ such that

$$
\tilde{A}_{1}=A_{1} \bigcap_{q \in Q_{1}}\left(A_{1}-q\left(y_{1}\right)\right)
$$

is a piecewise syndetic subset of $(\mathbb{Q},+)$.
Next, we construct $r_{j}, A_{j}, \tilde{A}_{j}, Q_{j}, y_{j}, S_{j},\left(l_{j}, f_{1, j}, \cdots, f_{n, j}\right)$ by induction until $j=N$ under the following requirements: for any $1 \leq j \leq N$, we have
(a) $\left(l_{j}, f_{1, j}, \cdots, f_{n, j}\right) \in\{1, \cdots, M\} \times F^{n}, r_{j} \in \mathbb{N}$;
(b) $S_{j}$ is an $I P_{r_{j}}$ subset of $(\mathbb{Q},+)$ and $S_{j} \subset \bigcup_{m \in Y_{l_{j}}} C_{m}$;
(c) $y_{j} \in S_{j}$;
(d) $Q_{j}=\left\{\left(y_{1} \cdots y_{c-1}\right)^{a} \cdot f \cdot p_{i}\left(t \cdot y_{c} \cdots y_{j-1}\right): f \in F, 1 \leq i \leq k, 1 \leq c<j\right\}$ where we put $y_{1} \cdots y_{0}=1$ and $\left|Q_{j}\right|<T$;
(e) $A_{j}$ and $\tilde{A}_{j}=A_{j} \bigcap_{q \in Q_{j}}\left(A_{j}-q\left(y_{j}\right)\right)$ are two piecewise syndetic subsets of $(\mathbb{Q},+)$.
and for any $1 \leq j<N$, we have
(f) $A_{j+1} \subset A_{j} \bigcap_{q \in Q_{j}}\left(A_{j}-q\left(y_{j}\right)\right)$;
(g) $A_{j+1}=\left\{x \in \tilde{A}_{j}: x \cdot\left(\prod_{b=1}^{j} y_{b}\right)^{-a}\right.$ has color $\left.\left(l_{j+1}, f_{1, j+1}, \cdots, f_{n, j+1}\right)\right\}$.

Clearly, we have finished construction for $j=1$. Let $j \geq 1$ and assume that $r_{j}, A_{j}, \tilde{A}_{j}, Q_{j}, y_{j}, S_{j},\left(l_{j}, f_{1, j}, \cdots, f_{n, j}\right)$ have been constructed.

By Theorem 2.2, there exist a subset $A_{j+1}$ of $\tilde{A}_{j}$ which is a piecewise syndetic subset of $(\mathbb{Q},+)$ and $\left(l_{j+1}, f_{1, j+1}, \cdots, f_{n, j+1}\right) \in\{1, \cdots, M\} \times F^{n}$ such that

$$
A_{j+1}=\left\{x \in \tilde{A}_{j}: x \cdot\left(\prod_{b=1}^{j} y_{b}\right)^{-a} \text { has color }\left(l_{j+1}, f_{1, j+1}, \cdots, f_{n, j+1}\right)\right\}
$$

Let

$$
Q_{j+1}=\left\{\left(y_{1} \cdots y_{c-1}\right)^{a} \cdot f \cdot p_{i}\left(t \cdot y_{c} \cdots y_{j}\right): f \in F, 1 \leq i \leq k, 1 \leq c<j+1\right\}
$$

Clearly, $\left|Q_{j+1}\right|<T$. Apply Proposition 2.9 to $T, d, A_{j+1}$, then we get a natural number $r_{j+1}$. We can construct $I P_{r_{j+1}}$ subset $S_{j+1}$ of $(\mathbb{Q},+)$ such that $S_{j+1} \subset$ $\bigcup_{m \in Y_{l_{j+1}}} C_{m}$. And there exists $y_{j+1} \in S_{j+1}$ such that

$$
\tilde{A}_{j+1}=A_{j+1} \bigcap_{q \in Q_{j+1}}\left(A_{2}-q\left(y_{j+1}\right)\right)
$$

is a piecewise syndetic subset of $(\mathbb{Q},+)$.
Obviously, there exist $2<\eta<j<N$ and $\left(l, f_{1}, \cdots, f_{n}\right) \in\{1, \cdots, M\} \times F^{n}$ such that

$$
\left(l, f_{1}, \cdots, f_{n}\right)=\left(l_{j}, f_{1, j}, \cdots, f_{n, j}\right)=\left(l_{\eta}, f_{1, \eta}, \cdots, f_{n, \eta}\right)
$$

and $j-\eta>2$. Let $y=y_{\eta} \cdots y_{j-1}$. Let $x^{\prime} \in A_{j}$ and set $x=\left(f_{m}\right)^{-1} \cdot x^{\prime} \cdot\left(y_{1} \cdots y_{\eta-1}\right)^{-a}$ where $m \in Y_{l}$. So $x, \frac{x}{y^{a}} \in C_{m}$. Moreover, for any $q \in Q_{j-1}, x^{\prime}+q\left(y_{j-1}\right) \in A_{\eta}$. Then for any $q \in Q_{j-1}$, we have

$$
x^{\prime} \cdot\left(y_{1} \cdots y_{\eta-1}\right)^{-a}+q\left(y_{j-1}\right) \cdot\left(y_{1} \cdots y_{\eta-1}\right)^{-a} \in f_{m} \cdot C_{m} .
$$

Therefore, for each $1 \leq i \leq k$, we have $x+p_{i}(y) \in C_{m}$ by definition of $Q_{j-1}$. This finishes the proof.

In the above proof, we can not determine the color of $y$. For linear polynomials with zero constant term, V. Bergelson and D. Glasscock gave the upper Banach density version of Proposition 2.9(see [2, Theorem 7.5]). By combining [6, Proof of Theorem 4.3] and the above proof, we have the following result.
Proposition 3.1. For any $k, n \in \mathbb{N}$, the families $\{(x, y) \mapsto x,(x, y) \mapsto y,(x, y) \mapsto$ $\left.x \cdot y^{n},(x, y) \mapsto x+y, \cdots,(x, y) \mapsto x+k y\right\}$ and $\{(x, y) \mapsto x,(x, y) \mapsto y,(x, y) \mapsto$ $\left.x \cdot y^{-n},(x, y) \mapsto x+y, \cdots,(x, y) \mapsto x+k y\right\}$ are Ramsey on $\mathbb{Q}$.

Likely, for general polynomials with zero constant term, if one can build the upper Banach density version of Proposition 2.9 , it is possible to confirm the color of $y$.

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