Full-homomorphisms to paths and cycles

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Abstract

A full-homomorphism between a pair of graphs is a vertex mapping that preserves adjacencies and non-adjacencies. For a fixed graph H, a full H-colouring is a full-homomorphism of G to H. A minimal H-obstruction is a graph that does not admit a full H-colouring, such that every proper induced subgraph of G admits a full H-colouring. Feder and Hell proved that for every graph H there is a finite number of minimal H-obstructions. We begin this work by describing all minimal obstructions of paths. Then, we study minimal obstructions of regular graphs to propose a description of minimal obstructions of cycles. As a consequence of these results, we observe that for each path P and each cycle C, the number of minimal P-obstructions and C-obstructions is $\mathcal{O}(|V(P)|^2)$ and $\mathcal{O}(|V(C)|^2)$, respectively. Finally, we propose some problems regarding the largest minimal H-obstructions, and the number of minimal H-obstructions.

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1 Introduction

All graphs considered in this work are finite graphs with no parallel edges. Later, we will further restrict ourselves to loopless graphs. For standard notions of Graph Theory we refer the reader to [2]. In particular, for $n \geq 3$, we denote by P_n (resp. C_n) the path (resp. cycle) on n vertices.

Given a pair of graphs G and H a *full-homomorphism* $\varphi\colon G \to H$ is a vertex mapping such that for each pair of vertices $x, y \in V(G)$ there is an edge $xy \in E(G)$ if and only if $\varphi(x)\varphi(y) \in E(G)$. In particular, if H is a simple graph, then adjacent vertices in G are mapped to different vertices in H. Moreover, if $\varphi(x) = \varphi(y)$, then x and y have the same neighbourhood in G.

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For a fixed graph H, a *full H-colouring* of a graph G is a full-homomorphism of G to H. A *minimal H-obstruction* is a graph that does not admit a full H-colouring, such that every proper induced subgraph of G admits a full H-colouring. We denote by $\mathrm{obs}(H)$ the set of minimal H-obstructions. In [3], Feder and Hell showed that for a graph H with l vertices with loops, and k vertices without loops, every graph in $\mathrm{obs}(H)$ has at most $(k+1)(l+1)$ vertices, and this bound is tight. Later, Hell and Hernández-Cruz showed that the same tight bound holds in the case of digraphs [5]. Independently and in a more general setting, Ball, Nešetřil, and Pultr [1], proved that for each relational structure A, there are a finite number of minimal A-obstructions. Each of these results imply that for every simple graph H there are finitely many minimal H-obstructions.

Proposition 1. *[1, 3, 5] For each graph H there is a finite number of minimal H-obstructions.*

Furthermore, Ball, Nešetřil, and Pultr [1] describe the connected minimal obstructions of paths and cycles, i.e., the connected graphs in $\mathrm{obs}(C_n)$ and in $\mathrm{obs}(P_n)$. They also propose a recursive description of disconnected minimal P_n-obstructions, but the "lists corresponding to the paths do not seem to be more transparent than those in the connected case" [1]. In this work, we propose a transparent description of the list of disconnected minimal obstructions of paths. We do so by means of positive solutions to integer equations. In particular, we list all minimal P_n-obstructions, and we build on this description to propose the complete list of minimal obstructions for cycles.

The rest of this work is structured as follows. First, in Section 2 we propose a description of minimal P_n-obstructions. In Section 3, we make some general observations regarding minimal obstructions of regular graphs, and use these to propose a description of minimal C_n-obstructions in terms of minimal P_{n-1}-obstructions. We conclude this work in Section 4 where we propose some problems that arise from observations in Section 3. The rest of this section contains some preliminary results needed for this work.

From this point onwards, we only consider loopless finite graphs. A pair of vertices x and y of a graph G are called *false twins* if $N(x) = N(y)$, and *true twins* of $N[x] = N[y]$. In particular, every pair of true twins are adjacent, while every pair of false twins are non-adjacent. In [6], Sumner defined a *point-determining* graph as a graph for which non adjacent vertices have distinct neighbourhoods, i.e., a graph G is point-determining if it has no pair of false twins.

Proposition 2. *[6] For every non trivial point-determining graph G there is a vertex $v \in V(G)$ such that $G - v$ is point-determining. Moreover, if G is connected, then there are two distinct vertices with that property.*

A pair of graphs G and H are *full-homomorphically equivalent* if G admits a full H-colouring and H admits a full G-colouring. A *core* in the category of graphs with full-homomorphisms, is a graph G such that every full-homomorphism $\varphi\colon G \to G$ is surjective. It is not hard to see that for each graph H, there is a unique (up to isomorphism) core G full-homomorphically equivalent to H. In this case, we say that G is the *full-core* of H.

Point-determining graphs play an important role in the category of graphs with full-homomorphisms. In particular, every core in the full-homomorphism category of graphs is a point-determining graph. Indeed, suppose that x and y are a pair of false twins in a graph G. By mapping x to y, we obtain a full-homomorphism of G onto a proper subgraph, which implies that G is not a core. Moreover, the same argument also implies that if G is a minimal H-obstruction for some graph H, then G is a point-determining graph. Finally, it is also straightforward to see that if G is a point-determining graph, then each full-homomorphism whose domain is G is an injective mapping. The following statement captures two of the facts argued in this paragraph.

Lemma 3. *The following statements hold for any pair of graphs G and H:*

1. *If G is point-determining, then every full-homomorphism $\varphi\colon G \to H$ is injective.*
2. *If $G \in \mathrm{obs}(H)$, then G is a point-determining graph.* $\square$

2 Path obstructions

In this section, we describe the minimal P-obstructions when P is a path. We begin by describing some particular minimal P-obstructions. To do so, we introduce the graphs A, B and E depicted in Fig. 1.

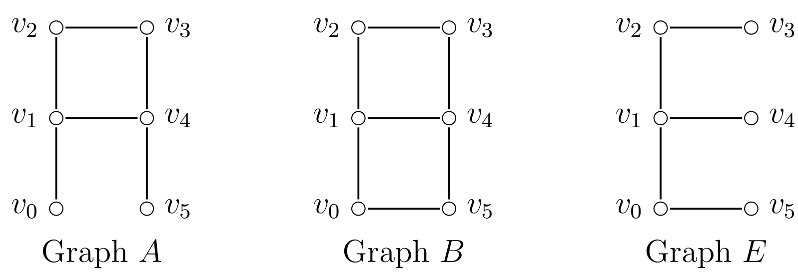

Figure 1: For a path P, every minimal P-obstruction that is neither a linear forest or a cycle, is one of these graphs (Lemma 5).

Recall that for $n \geq 3$, we denote by C_n (resp. by P_n) the cycle (resp. path) on n vertices; we denote by K_1 and K_2 the paths on one and two vertices, respectively. In general, we denote by K_n the complete graph on n vertices.

Lemma 4. *For every positive integer n, the following statements hold:*

1. *The graph A is a minimal P_n-obstruction if and only if $n \geq 6$.*
2. *The graph B is a minimal P_n-obstruction if and only if $n \geq 5$.*
3. *The graph E is a minimal P_n-obstruction if and only if $n \geq 7$.*
4. *The m-cycle is a minimal P_n-obstruction if and only if $m = 3$ or $5 \leq m \leq n + 1$.*

Proof. All graphs in statements 1–3 are point-determining graphs that do not admit a full P-colouring for any path P. By removing v_4 from A, we obtain $K_1 + P_4$ which is not full P_5-colourable, so A is not a minimal P_n obstruction for any $n \leq 5$. On the other hand, any induced subgraph of A admits a full P_6-colouring, and thus, it is a minimal P_n-obstruction for every $n \geq 6$. Similarly, $B - v_1$ is not full P_5-colourable, and $E - v_1$ is not full P_6-colourable. Also, every proper induced subgraph of B is full P_6-colourable, and every proper induced subgraph of E is full P_7-colourable. Hence, B is a minimal P_n-obstruction if and only if $n \geq 6$, and E is a minimal P_n-obstruction if and only if $n \geq 7$. The last statement is clearly true. □

Now, we observe that each path minimal obstructions is either a graph mentioned in Lemma 4 or a linear forest.

Lemma 5. *Consider a path P and a graph G. If $G \in \mathrm{obs}(P)$ then one of the following statements holds:*

1. *G is a cycle.*
2. *G is a linear forests.*
3. *G is one of the graphs A, B or E.*

Proof. Let n be a positive integer such that $G \in \mathrm{obs}(P_n)$. We show that if G is neither a cycle nor a linear forest, then is one of the graphs A, B or E. First, suppose that G is not a cycle or a forest. By minimality of G, and by the fourth statement of Lemma 4, we know that G does not contain a triangle nor a cycle of length m with $5 \leq m \leq n+1$. It is not hard to see that the path on $n+1$ vertices is not full P_n-colourable, thus G does not contain an induced path on $n+1$ vertices, and so, it does not contain a cycle of length $m \geq n+2$. Putting both of these observations together we conclude that G contains no triangle nor an induced cycle of length $m \geq 5$. Since G is not a forest, there is an induced 4 cycle C, $C = v_1, v_2, v_3, v_4$, in G. By the choice of G and by second part of Lemma 3, it is the case that G is a point-determining graph. In particular, $N(v_1) \neq N(v_3)$ and $N(v_2) \neq N(v_4)$ so, without loss of generality we assume that v_1 has a neighbour $v_0 \notin \{v_2, v_4\}$ and v_4 has a neighbour $v_5 \notin \{v_1, v_3\}$. Since G has no triangles, the unique neighbour of v_0 (resp. v_5) in C is v_1 (resp. v_4). Let H be the subgraph of G induced by $\{v_0, \ldots, v_5\}$. This graph is isomorphic to either A or B. Clearly, neither of A nor B admit a full P_n-colouring, and thus, by minimality of G we conclude that $G = H$.

In the paragraph above, we showed that if $G \in \mathrm{obs}(G)$ and G is not a forest, then either G is a cycle or $G \in \{A, B\}$. To conclude the proof, suppose that G is a forest but not a linear forest. In this case, G contains an induced claw C. With a similar procedure to the paragraph above, we extend C to an induced subgraph H of G such that $H \cong E$. Since E does not admit a full P_n-colouring, we conclude that $G = H \cong E$, and the claim follows. □

In order to complete the characterization of $\mathrm{obs}(P_n)$, we study minimal P_n-obstructions that are linear forests. To do so, it will be convenient to introduce the following notation. First, notice that each linear forest L admits an injective full-homomorphism to any large enough path. So, we denote by $\mu(L)$ the minimum integer n such that there is an injective full-homomorphism from L to P_n. Also, since linear forests are disjoint unions of paths, we will denote a linear forests L as $\sum_{k=1}^{m} P_{n_k}$, where the k-th component of L is the path on n_k vertices. Finally, we denote by $c(G)$ the number of connected component of a graph G.

Lemma 6. *For a linear forest $L = \sum_{k=1}^{m} P_{n_k}$ the following equalities hold*

$$\mu(L) = |V(L)| + c(L) - 1 = (m-1) + \sum_{k=1}^{m} n_k.$$

Proof. One can soon notice that the rightmost equality holds. Now, notice that if $\varphi\colon L \to P_n$ is an injective full-homomorphism, then the image of each components of L must be at distance at least 2 in P_n, thus $n \geq |V(L)| + c(L) - 1$ and so, $\mu(L) \geq |V(L)| + c(L) - 1$. It is not hard to see that there is an injective full-homomorphism from L to the path on $|V(L)| + c(L) - 1$ vertices and hence, $\mu(L) = |V(L)| + c(L) - 1$. □

Consider a linear forest $L = \sum_{k=1}^{m} P_{n_k}$. In order to simplify our writing, we define m_i to be the number of components of length i in L. In other words, m_i is the cardinality of the set $\{k \in \{1, \dots, m\} : n_k = i\}$. In particular, $m_i = 0$ for all $i > |V(L)|$.

Notice that if a linear forest L contains two isolated vertices, then L is not a point-determining graph. Similarly, if L contains a component isomorphic to P_3, then it is also the case that L is not a point-determining graph.

Lemma 7. *Let $L = \sum_{k=1}^{m} P_{n_k}$ be a linear forest and n a positive integer. If $L \in \operatorname{obs}(P_n)$, then the following statements hold:*

1. $m_1 \leq 1$.
2. $n_k \in \{1, 2, 4, 6\}$ *for all* $k \in \{1, \dots, m\}$.
3. *If* $n_k \in \{4, 6\}$ *for some* k*, then* $m_1 = 1$.

Proof. By the second part of Lemma 3, L is a point-determining graph, so by the arguments in the paragraph above, we see that $m_1 \leq 1$ and there is no $k \in \{1, \dots, m\}$ such that $n_k = 3$. In particular, the first item holds, and to see that the second one is also true, we show that every component of L has at most 6 vertices but not 5. Anticipating a contradiction, suppose that there is a path $P_{n_k} = v_1, v_2, \dots, v_{n_k}$ with $n_k = 5$ or $n_k \geq 7$, for some $k \in \{1, \dots, m\}$. In such case, $L - v_3$ is a point-determining graph and $c(L - v_3) = c(L) + 1$ so, by applying Lemma 6 to $L - v_3$ and to L, we see that $\mu(L - v_3) = \mu(L)$. By the choice of L, there is a full P_n-colouring of $L - v_3$ which, by the first part of Lemma 3, must be injective. Thus, by definition of μ, it follows that $n \geq \mu(L - v_3) = \mu(L)$, contradicting the fact that L is not full P_n-colourable. Therefore, if $L \in \operatorname{obs}(P_n)$, then $n_k \in \{1, 2, 4, 6\}$ for every $k \in \{1, \dots, m\}$.

To prove the third statement, suppose that $P_{n_k} = v_1, \dots, v_{n_k}$ with $n_k \in \{4, 6\}$ for some $k \in \{1, 2, \dots, m\}$. In this case, $c(L - v_2) = c(L) + 1$. So, if $L - v_2$ is a point-determining graph, by using a similar arguments as in the first paragraph of this proof, we conclude that L admits a full P_n-colouring, contradicting the fact that $L \in \operatorname{obs}(P_n)$. Hence, $L - v_2$ is not a point-determining linear forest. Since every component of $L - v_2$ is either a component of L, or v_1, or the path $v_3, \dots, v_{n_k}$, it must be the case that there is an isolated vertex in $L - v_2$ other than v_1. Hence, L has at least one isolated vertex so $m_1 \geq 1$, and by the first statement of this lemma, we conclude that $m_1 = 1$.

□

It turns out the necessary conditions stated in Lemma 7 are almost sufficient conditions for a linear forest L to be a minimal P_n-obstruction.

Proposition 8. *Let $L = \sum_{k=1}^{m} P_{n_k}$ be a linear forest and n a positive integer. In this case, $L \in \mathrm{obs}(P_n)$ if and only if one of the following statements holds:*

1. $\mu(L) = n + 1$ *and* $n_k = 2$ *for all* $k \in \{1, \ldots, m\}$.
2. $\mu(L) = n + 1$ *and* $n_k \in \{1, 2, 4, 6\}$ *with* $m_1 = 1$.
3. $\mu(L) = n + 2$ *and* $n_k \in \{1, 2, 4\}$ *with* $m_1 = 1$.

Proof. We prove the statement by case distinction depending on the components of L, and we begin by considering the case when $L = m_2K_2$. One can easily observe that for every vertex v of L, the equality $\mu(m_2K_2 - v) = \mu(m_2K_2) - 1$ holds. Also, L and $L - v$ are point-determining graphs so, L and $L - v$ admit a full-homomorphism to P_n if and only if they admit an injective full-homomorphism to P_n. Therefore, it follows from the definition of the parameter μ that $L \in \mathrm{obs}(P_n)$ if and only if $\mu(L) = n + 1$.

Now, suppose that $\mu(L) = n + 1$ but L is not a disjoint union of edges. By the second part of Lemma 7, it follows that $n_k \in \{1, 2, 4, 6\}$ for all $k \in \{1, \ldots, m\}$, and by the choice of L and the third part of the same lemma $m_1 = 1$. Now, we observe that in this case L is a minimal P_n-obstruction. Indeed, if v is an end vertex of any component of L, then $\mu(L - v) = \mu(L) - 1 = n$, so $L - v$ admits a full-homomorphism to P_n. Otherwise, if v is a middle vertex of a P_4 or a P_6, then $L - v$ is not point-determining: either $L - v$ has two isolated vertices, or $L - v$ has a component isomorphic to P_3. Thus, by identifying the isolated vertices, or the end vertices of P_3, we obtain a full-homomorphism of $L - v$ to P_n.

If neither of the previous cases holds, then L is not a disjoint union of edges, and $\mu(L) \leq n$ or $\mu(L) \geq n + 2$. In the former case, there is an injective full-homomorphism from L to P_n so L is not a minimal P_n-obstruction. So, if $L \in \mathrm{obs}(P_n)$, then $\mu(L) \geq n + 2$. Anticipating a contradiction suppose that $L \in \mathrm{obs}(P_n)$ and that $\mu(L) \geq n + 3$. Again, it must be the case that L has exactly one isolated vertex v. One can soon notice that $\mu(L-v) = \mu(L)-2 \geq n+1$, and that $L-v$ is a point-determining graph. Thus, by Lemma 3, we conclude that $L-v$ does not admit a full-homomorphism to P_n, contradicting the choice of L. Thus if $L \in \mathrm{obs}(P_n)$ and $\mu(L) \neq n + 1$, then $\mu(L) = n + 2$. One can easily notice that if L contains a component isomorphic to P_6, and v is an end vertex of this component, then $L-v$ is a point-determining graph, and $\mu(L - v) = \mu(L) - 1 = n - 1$. So, with similar arguments as before, we conclude that if $L \in \mathrm{obs}(P_n)$ and $\mu(L) = n + 2$, then $n_k \in \{1, 2, 4\}$ for all $k \in \{1, \ldots, m\}$, and $m_1 = 1$. We proceed to observe that if $\mu(L) = n+2$, and $n_k \in \{1, 2, 4\}$ for all $k \in \{1, \ldots, m\}$ with $m_1 = 1$, then $L \in \mathrm{obs}(P_n)$. Similar as before, if v is the isolated vertex of L, then $\mu(L-v) = \mu(L)-2 = n$, so L is full P_n-colourable. Otherwise, $L-v$ is not point determining: either $L-v$ has two isolated vertices, or a component isomorphic to P_3. Again, by identifying the isolated vertices, or the end vertices of P_3, we obtain a full-homomorphism from $L - v$ to P_n.

The claim now follows because on the one hand, in the first (resp. second and third) paragraph we observed that the first (resp. second and third) statement is a sufficient condition for L to be a minimal P_n-obstruction. On the other one, every linear forest L satisfies either the first assumption of the first paragraph, the first assumption of the second paragraph, or the first assumption of the third paragraph. Since in each of these paragraphs we showed that if L satisfies such assumption and $L \in \text{obs}(P_n)$, then L must satisfy one of the three items of this proposition; all together proving that if $L \in \text{obs}(P_n)$, then L satisfies one of 1–3. $\square$

We are ready to propose a description of all minimal P_n-obstructions. To do so, we introduce three sets $C(n)$, $LF(n)$ and $O(n)$, which depend on n — C stands for cycles, LF for linear forests, and O for other. We begin with the simplest,

$$C(n) := \{C_m : \; m = 3 \text{ or } 5 \leq m \leq n+1\}.$$

Secondly, we define $O(n)$ as follows

$$O(n) := \begin{cases} \varnothing \text{ if } n \leq 4, \\ \{B\} \text{ if } n = 5, \\ \{A, B\} \text{ if } n = 6, \\ \{A, B, E\} \text{ if } n \geq 7. \end{cases}$$

Finally, $LF(n)$ is the union $LF_1(n) \cup LF_2(n) \cup LF_3(n)$ where

$$LF_1(n) := \{m_2K_2 : \; 3m_2 = n+2\}$$

$$LF_2(n) := \{K_1 + m_2K_2 + m_4P_4 : \; 3m_3 + 5m_4 = n+1\}, \text{ and}$$

$$LF_3(n) := \{K_1 + m_2K_2 + m_4P_4 + m_6P_6 : \; 3m_2 + 5m_4 + 7m_6 = n\}.$$

We describe the set $\text{obs}(P_n)$ of minimal P_n-obstructions in terms of the previously defined sets.

Theorem 9. *For every positive integer n the set $\text{obs}(P_n)$ of minimal P_n-obstructions is the union $C(n) \cup LF(n) \cup O(n)$.*

Proof. It follows from Lemma 4 that $C(n) \cup O(n) \subseteq \text{obs}(P_n)$, and from Lemma 5 that every $L \in \text{obs}(P_n) \setminus (C(n) \cup O(n))$ is a linear forest. The fact that the set of linear forests in $\text{obs}(P_n)$ equals $LF(n)$, follows from Proposition 8, and from the equality $\mu(L) = (m-1) + \sum_{k=1}^{m} n_k$ from Lemma 6. $\square$

To conclude this section, allow us to discuss an implication of Theorem 9. Since all paths are linear forests, any graph that admits a full-homomorphism to some path, admits a full-homomorphism to some linear forest. On the other hand, each linear forest admits a full-homomorphism to a large enough path. Thus, a graph G admits a full-homomorphism to a path if and only if it admits a full-homomorphism G to a linear forest.

A *blow-up* of a graph G is obtained by addition of false twins — intuitively, by "blowing up" some vertices of G to an independent set. Clearly, a graph G admits a full H-colouring if and only if G is a blow-up of some induced subgraph of H. Since the class of linear forest is closed under induced subgraphs, we use the observation in the paragraph above to prove the following statement.

Corollary 10. *A graph G is a blow-up of a linear forest if and only if G is an $\{A, B, E\}$-free graph such that all induced cycles have length four.*

Proof. If G is a blow-up of some linear forest, then G admits a full P_n-colouring for some large enough n. Thus, by Theorem 9, G is an $\{A, B, E\}$-free graph such that all induced cycles have length four. On the other hand, notice that the number of vertices of the smallest graph in $LF(n)$ increases with respect to n. Thus, for every a graph G there is a positive integer N such that G is an $LF(N)$-free graph. Hence, if G is an $\{A, B, E\}$-free graph such that all induced cycles have length four, then G is an $(O(N) \cup C(N) \cup LF(N))$-free graph. Therefore, G admits a full P_N-colouring, and so, G is a blow-up of a linear forest. $\square$

3 Cycle obstructions

The aim of this section is listing all minimal obstructions of cycles. To do so, we first make some general observations regarding minimal obstructions of regular graphs. Proposition 2 asserts that for each point-determining graph G, there is a vertex $x \in V(G)$ such that $G - x$ is point-determining. We begin by noticing that this can be strengthen in the case of regular graphs.

Proposition 11. *Let H be a point-determining graph. If H is a regular graph, then for each $x \in V(H)$ the graph $H - x$ is point-determining.*

Proof. Proceeding by contrapositive, suppose that there is a vertex $x \in V(H)$ such that $H-x$ is not point-determining. Let $r, s \in V(H - x)$ be a pair of false twins, i.e., $rs \notin E(H - x)$ and $N_{H-x}(r) = N_{H-x}(s)$. Since H is a point-determining graph and $rs \notin E(H)$, it must be the case that $xr \in E(H)$ and $xs \notin E(H)$ (or vice versa). Hence, $d_H(s) = d_{H-x}(s) = d_{H-x}(r) = d_H(r) - 1$. Thus, H is not a regular graph. $\square$

Consider a graph H and a minimal H-obstruction G. By the second part of Lemma 3, G is a point-determining graph so, by Proposition 2, there is a vertex $v \in V(G)$ such that $G-v$ is a point-determining graph, and $G-v$ admits a full H-colouring by minimality of G. Also, by the first part of Lemma 3, each full-homomorphism from $G - v$ to H is injective, and thus $|V(G - v)| \leq |V(H)|$. Therefore, every graph $G \in \mathrm{obs}(H)$ has at most $|V(H)| + 1$ vertices. We denote by $\mathrm{obs}^*(H)$ the set of minimal H-obstructions on $|V(H)| + 1$ vertices. The following statement was proved in [3].

Proposition 12. *[3] For any graph H, there are at most two non-isomorphic graphs in* $\mathrm{obs}^*(H)$.

By similar arguments as in the paragraph above, we observe that if $G \in \mathrm{obs}^*(H)$, then there is a vertex $v \in V(G)$ such that $G - v \cong H$.

Observation 13. *Consider a pair of graphs G and H. If $G \in \mathrm{obs}^*(H)$, then there is a vertex $v \in V(G)$ such that $G - v \cong H$.*

This observation about the structure of graphs in $\operatorname{obs}^*(H)$ can be strengthened when H is a regular graph. Recall that a pair of vertices u and v in a graph G are true twins if $N[u] = N[v]$.

Lemma 14. *Let H be a non-complete regular connected graph. For every graph $G \in \operatorname{obs}^*(H)$ there is a pair of true twins $u, v \in V(G)$ $(u \neq v)$ such that $G - v \cong H$ and $G - u \cong H$.*

Proof. Since H is non-complete, it is not isomorphic to K_2, and since it is connected, it is not a matching. Thus, H is a k-regular graph with $k \geq 2$. Let $G \in \operatorname{obs}^*(H)$. By Observation 13, there is a vertex $x \in V(G)$ such that $G - x \cong H$. For this proof, it will be convenient to identify H with the subgraph of G induced by $V(G) - \{x\}$. We fix x and use this identification throughout the proof. We proceed to show x is not an isolated vertex. Since $k \geq 2$, there are no leaves in H. Consider a vertex $v \in V(G) - \{x\}$ and let $\varphi\colon G - v \to H$ be a full-homomorphism. Since H is a regular graph, by Proposition 11 we know that $H - v$ is point-determining so, by the first part of Lemma 3, the restriction of φ to $H - v$ is an injective mapping. Let L be the image $\varphi[H - v]$ of $H - v$. In particular, $|V(L)| = |V(H)| - 1$. Since H is connected, either $\varphi(x)$ has a neighbour in L or $\varphi(x)$ belongs to L. Recall, that $L \cong H - v$ and H has no leaves so, L has no isolated vertices. Therefore, if $\varphi(x)$ belongs to L, then $\varphi(x)$ has a neighbour in L, and since φ is a full-homomorphism, x cannot be an isolated vertex in G.

In the paragraph above, we proved that x is not an isolated vertex. Since $G - x$ is connected (recall that $G - x = H$), G is a connected graph. By Proposition 2, there is a vertex $y \in V(G) - \{x\}$ such that $G - y$ is point-determining, and so, $G - y \cong H$. We conclude the proof by showing that x and y are true twins in G. Since H is a k-regular graph, for each $v \in V(G - y)$ the equality $d_{G-y}(v) = k$ holds. Also, since $H = G - x$, the equality $d_{H-y}(v) = k - 1$ holds if and only if $v \in N_H(y) = N_{G-x}(y)$. On the other hand, $k - 1 = d_{(G-y)-x}(v) - 1$ if and only if $v \in N_{G-y}(x)$. Clearly, $(G - y) - x = H - y$, and so, $v \in N_{G-y}(x)$ if and only if $v \in N_{G-x}(y)$ for any $v \in V_{G-\{x,y\}}$. Thus, $N_G(x) - y = N_{G-y}(x) = N_{G-x}(y) = N_G(y) - x$ so in particular, $N_G(x) - y = N_G(y) - x$. Since G is point-determining, x and y are not false twins in G so, $xy \in E(G)$, and thus x and y are true twins. The claim follows. □

Proposition 12 asserts that $|\operatorname{obs}^*(H)| \leq 2$ for every graph H. Using Lemma 14, we show that in the case of regular non-complete graphs $\operatorname{obs}^*(H) = \varnothing$. Recall that a *universal* vertex in a graph G is a vertex $x \in V(G)$ adjacent to every $y \in V(G) \setminus \{x\}$.

Proposition 15. *For a connected regular graph H, the following equalities hold*

$$\operatorname{obs}^*(H) = \begin{cases} \{K_1 + K_2, K_3\} \text{ if } H \cong K_2, \\ \{K_{n+1}\} \text{ if } H \cong K_n \text{ and } n \neq 2, \\ \varnothing \text{ otherwise.} \end{cases}$$

Proof. Since the class of complete multipartite graphs is the class of $K_1 + K_2$-free graphs, the class of full K_n-colourable graphs is the class of $\{K_1 + K_2, K_{n+1}\}$-free graphs. Now, suppose that H is a regular non-complete connected graph and let $G \in \operatorname{obs}^*(H)$. By Lemma 14, there is a pair of true twins x and y of G, such that $G - x \cong H \cong G - y$. Again, we identify H with the subgraph of G induced by $V(G) - x$. Notice that if x and y are universal vertices in G, then y is a universal vertex in H and so, H is a complete graph (because H is a regular graph). So, by the choice of H, there is a vertex $z \in V(G)$ such that $zy \notin E(G)$, and since x and y are true twins, it is the case that $xz \notin E(G)$. By the choice of G, there is a full-homomorphism $\varphi\colon G - z \to H$. Let k be the degree of every vertex in H so, $d_G(x) = d_G(y) = k + 1$. Since $zx, zy \notin E(G)$, it is the case that $d_{G-z}(x) = d_{G-z}(y) = k + 1$. But $d_H(\varphi(y)) = k$ so, there are two vertices $r, s \in N_{G-z}(y)$ such that $\varphi(r) = \varphi(s)$. Hence $N_{G-z}(r) = N_{G-z}(s)$ and $rs \notin E(G - z)$. Recall that $H = G - x$, so $N_{H-z}(r) = N_{H-z}(s)$ and $rs \notin E(G - z)$, i.e., r and s are false twins in $H - z$. Thus, $H - z$ is not a point-determining graph which contradicts the fact that H is a regular graph and Proposition 11. $\square$

The following statement shows that if a graph G is a minimal H-obstruction of size $|V(H)| + 1$, then every minimal G-obstruction F is either a minimal H-obstruction or $|V(F)| = |V(G)| + 1$. Conversely, every minimal H-obstruction other than G is a minimal G-obstruction.

Theorem 16. *Consider a pair of graphs H and G. If $G \in \mathrm{obs}^*(H)$, then*

$$\mathrm{obs}(G) = (\mathrm{obs}(H) \setminus \{G\}) \cup \mathrm{obs}^*(G).$$

Proof. To simplify notation, let $S = (\mathrm{obs}(H) \setminus \{G\}) \cup \mathrm{obs}^*(G)$. We need to prove that a graph F belongs to S if and only if it belongs to $\mathrm{obs}(G)$. Clearly, every graph in S has at most $|V(G)| + 1$ vertices and so does every graph in $\mathrm{obs}(G)$. Moreover, by definition of $\mathrm{obs}^*(G)$, a graph on $|V(G)|+1$ vertices belongs to $\mathrm{obs}(G)$ if and only if it belongs to $\mathrm{obs}^*(G)$. Thus, it suffices to prove the claim for graphs on at most $|V(G)|$ vertices, and by the second part of Lemma 3 it suffices to consider point-determining graphs. We begin by showing that the claim holds for graphs on at most $|V(G)| - 1$ vertices (recall that $|V(G)| = |V(H)| + 1$). Since G is a minimal H-obstruction, every proper induced subgraph of G admits a full-homomorphism to H. Thus, any graph that admits a non-surjective full homomorphism to G, admits a full H-colouring. Hence, a graph on at most $|V(G)| - 1$ vertices admits a full G-colouring if and only if it admits a full H-colouring. Therefore a graph on at most $|V(G) - 1|$ vertices belongs to S if and only if it belongs to $\mathrm{obs}(G)$.

Finally, consider a point-determining graph L on $|V(G)|$ vertices. By the first part of Lemma 3, every full-homomorphism from L to G is injective, thus L admits a full G-colouring if and only if $L \cong G$. By similar arguments as in the paragraph above, we conclude that L is a minimal G-obstruction if and only if it is a minimal H-obstruction. $\square$

Corollary 17. *Consider a pair of graphs H_1 and H_2. If $\mathrm{obs}^*(H_1) \cap \mathrm{obs}^*(H_2) \neq \varnothing$, then $H_1 \cong H_2$.*

Proof. One soon notices that if $\mathrm{obs}^*(H) \neq \varnothing$, then H is a point-determining graph. Let $G \in \mathrm{obs}^*(H_1) \cap \mathrm{obs}^*(H_2)$. By applying Theorem 16 to H_1 and G, and to H_2 and G, we conclude that $\mathrm{obs}(H_1) = \mathrm{obs}(H_2)$. Thus, H_1 admits a full-homomorphism to H_2, and vice versa. Since H_1 and H_2 are point-determining graphs, both full-homomorphisms are injective (Lemma 3), and thus, they are isomorphisms. $\square$

In other words, Corollary 17 asserts that if a graph G is a minimal obstruction of two smaller graphs, then these graphs are isomorphic. Another immediate implication of Theorem 16 is the following one.

Corollary 18. *Consider a pair of graphs H and G. If $G \in \operatorname{obs}^*(H)$, then there is at most one minimal G-obstruction on $|V(G)|$ vertices.*

Recall that the orbit of a vertex $y \in V(G)$ is the set of vertices $x \in V(G)$ such that there is an automorphism $\varphi\colon G \to G$ such that $\varphi(y) = x$. We denote the orbit of y by $o(y)$. Clearly, if $x \in o(y)$, then $G - x \cong G - y$.

Proposition 19. *Let H be a non-complete connected vertex transitive graph. For any vertex x of H, the following equalities hold*

$$obs(H) = obs(H - x) \setminus \{H\} \text{ and } obs(H - x) = obs(H) \cup \{H\}$$

Proof. Since H is vertex transitive, $H - x \cong H - y$ for any pair of vertices $x, y \in V(H)$. So, every proper induced subgraph of H admits a full $(H - x)$-colouring. Since H is a point-determining graph, H is not full $(H - x)$-colourable, so, $H \in \operatorname{obs}^*(H - x)$. By Theorem 16, we conclude that $\operatorname{obs}(H) = \operatorname{obs}(H - x) \setminus \{H\} \cup \operatorname{obs}^*(H)$ so, using Proposition 15 we observe that $\operatorname{obs}^*(H) = \varnothing$. ∎

By applying Proposition 19 to cycles, we see that minimal C_n-obstructions are determined by minimal P_{n-1}-obstructions, and vice versa.

Corollary 20. *For every positive integer n, $n \geq 5$, the following equalities hold*

$$\operatorname{obs}(C_n) = \operatorname{obs}(P_{n-1}) \setminus \{C_n\} \text{ and } \operatorname{obs}(P_{n-1}) = \operatorname{obs}(C_n) \cup \{C_n\}.$$

The following characterization of minimal C_n-obstructions follows from Corollary 20 and Theorem 9.

Theorem 21. *For every positive integer n the set $\operatorname{obs}(C_n)$ of minimal C_n-obstructions is the union $C(n-2) \cup LF(n-1) \cup O(n-1)$.*

Proof. By Corollary 20, the equality $\operatorname{obs}(C_n) = \operatorname{obs}(P_{n-1}) \setminus \{C_n\}$ holds. By Theorem 9, the set of minimal P_n-obstructions is $C(n-1) \cup LF(n-1) \cup O(n-1)$. Finally, by definition of $C(n)$, the equality $C(n-2) = C(n-1) \setminus \{C_n\}$ holds, and so, the claim follows. ∎

Corollary 22. *A graph G is full C_5-colourable if and only if it is $\{C_3, K_1 + P_4, 2K_2\}$-free.*

To conclude this section, we list all C_n-minimal obstructions for small integers n in Table 1.

n	Linear forests in $\mathrm{obs}(C_n)$	Other minimal C_n-obstructions
5	$K_1 + P_4$ and $2K_2$	C_3
6	$K_1 + P_4$ and $K_1 + 2K_2$	$C_3,\ C_5$ and B
7	$K_1 + 2K_2$	$C_3,\ C_5,\ C_6,\ A$ and B
8	$3K_2,\ K_1 + K_2 + P_4$, and $K_1 + P_6$	$C_3,\ C_5,\ C_6,\ C_7,\ A,\ B$ and E
9	$K_1 + 3K_2$ and $K_1 + K_2 + P_4$	$C_3,\ C_5,\ C_6,\ C_8,\ A,\ B$ and E
10	$K_1 + 2P_4$ and $K_1 + 3K_2$	$C_3,\ C_5,\ C_6,\ C_7,\ C_8,\ C_9,\ A,\ B$ and E

Table 1: To the left, the number of vertices in a cycle C. In the middle, the linear forests which are minimal C-obstructions. To the right, all minimal C-obstructions that are not linear forests.

4 Conclusions

Proposition 15 asserts that for a connected regular graph H the set $\mathrm{obs}^*(H)$ is empty if and only if H is not a complete graph. Also, if H is obtained from a vertex-transitive graph G by removing one vertex, then $G \in \mathrm{obs}^*(H)$ so, $\mathrm{obs}^*(H) \neq \varnothing$. A possible interesting question to investigate is the following one.

Question. *Is there a meaningful characterization of those graphs H for which* $\mathrm{obs}^*(H) \neq \varnothing$*?*

Theorem 16 suggests that there is a close relation between a graph H such that $\mathrm{obs}^*(H) \neq \varnothing$ and a graph $G \in \mathrm{obs}^*(H)$. For this reason, we believe that another possible interesting problem is determining which graphs G are a minimal H-obstruction of size $|V(H)| + 1$ for some graph H.

Question. *For which graphs G there is a graph H such that G is a minimal H-obstruction in* $\mathrm{obs}^*(H)$*?*

We briefly observe that this problem is not interesting if we remove the restriction that $|V(G)| = |V(H)| + 1$.

Proposition 23. *For every point-determining connected graph G, there is a graph H such that G is a minimal H-obstruction.*

Proof. Let G be as in the hypothesis, and for each vertex $x \in V(G)$ let H_x be the full-core of $G - x$. Finally, let H be the disjoint union $\sum_{x \in V(G)} H_x$. Since every connected component of H has less than $|V(G)|$ vertices, there is no injective full-homomorphism from G to H. By the first part of Lemma 3, and since G is a point-determining graph, we conclude that G does not admit a full H-colouring. It is not hard to see that, by the choice of H_x, for every vertex $x \in V(G)$ there is a full H-colouring of $G - x$. The claim follows. $\square$

The following observation shows that, in the case of regular graphs, Section 4 has a meaningful answer.

Proposition 24. *Let G be a point-determining regular graph. There is a graph H such that $G \in \mathrm{obs}^*(H)$ if and only if G is a vertex transitive graph.*

Proof. By Proposition 11, if G is a point-determining regular graph, then for each $x \in V(G)$ the induced subgraph $G - x$ is point-determining. So, if $|V(G)| = |V(H)| + 1$, then by the first part of Lemma 3, for each $x \in V(G)$, every full-homomorphism from $G - x$ to H is an isomorphism. Hence, all vertex-deleted subgraphs of G are isomorphic, and thus G is a vertex transitive graph. $\square$

As a final implication of this work, notice that for every positive integer n, there are at most three graphs in $O(n)$, at most $n - 2$ graphs in $C(n)$, and as many graphs in $LF(n)$ as non-negative solutions to the diophantine equations, $3x = n + 2$, $3x + 5y = n + 1$, and $3x + 5y + 7z = n$. It is not hard to observe that there are $O(n^{k-1})$ solutions to each of these equations, where k is the number of variables in the corresponding equation. Hence, there are quadratically many linear forests in $LF(n)$. These arguments, together with Theorems 9 and 21, imply that the following statement holds.

Corollary 25. *For every positive integer n, there are quadratically many (with respect to n) minimal P_n-obstructions and minimal C_n-obstructions.*

The well-defined and simple structure of paths and cycles might be the reason why their number of minimal obstructions is polynomially bounded (with respect to n). Nonetheless, having made this observation, it is natural to ask about the cardinality of $\mathrm{obs}(H)$ in terms of the cardinality of the vertex set of H.

Question. *Is there a polynomial $p(n)$ such that the size of* $\mathrm{obs}(H)$ *is bounded by* $p(|V(H)|)$ *for each graph H?*

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