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# Optimal functions with spectral constraints in hypercubes

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**Abstract**

The $n$-dimensional hypercube has $n+1$ distinct eigenvalues $n-2i$, $0 \leq i \leq n$, with corresponding eigenspaces $U_i(n)$. In 2021 it was proved by the author that if a function with non-empty support belongs to the direct sum $U_i(n) \oplus U_{i+1}(n) \oplus \ldots \oplus U_j(n)$, where $0 \leq i \leq j \leq n$, then it has at least $\max(2^i, 2^{n-j})$ non-zeros. In this work we give a characterization of functions achieving this bound.

*Keywords:* hypercube, eigenfunction, eigenfunctions of graphs, minimum support, trade, [$t$]-trade

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## 1. Introduction

There are the following extremal problems for eigenfunctions of graphs.

**Problem 1.** *Let $G$ be a graph and let $\lambda$ be an eigenvalue of $G$. Find the minimum cardinality of the support of a $\lambda$-eigenfunction of $G$.*

**Problem 2.** *Let $G$ be a graph and let $\lambda$ be an eigenvalue of $G$. Characterize $\lambda$-eigenfunctions of $G$ with the minimum cardinality of the support.*

During the last years, Problems 1 and 2 have been actively studied for various families of distance-regular graphs [1, 5, 6, 8, 11, 13, 14, 16, 17, 18, 19, 20] and Cayley graphs on the symmetric group [7]. In particular, Problem 1 is completely solved for all eigenvalues of the Hamming graph [8, 17, 18] and asymptotically solved for all eigenvalues of the Johnson graph [20]. In more details, Problems 1 and 2 are discussed in a recent survey [15].

The *Hamming graph* $H(n, q)$ is defined as follows. The vertex set of $H(n, q)$ is $\mathbb{Z}_q^n$, and two vertices are adjacent if they differ in exactly one coordinate. The adjacency matrix of $H(n, q)$ has $n+1$ distinct eigenvalues $n(q-1) - q \cdot i$, where $0 \leq i \leq n$. Let $U_{[i,j]}(n, q)$, where $0 \leq i \leq j \leq n$, denote the direct

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sum of eigenspaces of $H(n,q)$ corresponding to consecutive eigenvalues from $n(q-1)-q\cdot i$ to $n(q-1)-q\cdot j$. The support of a real-valued function $f$ is denoted by $S(f)$.

Let $0\leq i\leq j\leq n$. Denote

$$m_{i,j}(n,q)=\min_{f\in U_{[i,j]}(n,q),f\not\equiv 0}|S(f)|.$$

A function $f\in U_{[i,j]}(n,q)$ is called *optimal* in the space $U_{[i,j]}(n,q)$ if $|S(f)|=m_{i,j}(n,q)$. In this work we consider the following natural generalizations of Problems 1 and 2 for the Hamming graph.

**Problem 3.** *Let $n\geq 1$, $q\geq 2$ and $0\leq i\leq j\leq n$. Find $m_{i,j}(n,q)$.*

**Problem 4.** *Let $n\geq 1$, $q\geq 2$ and $0\leq i\leq j\leq n$. Characterize functions that are optimal in the space $U_{[i,j]}(n,q)$.*

Problem 3 is completely solved for all $n\geq 1$ and $q\geq 2$ in [17, 18]. Moreover, Problem 4 is solved for $q\geq 3$, $i+j\leq n$ and $q\geq 5$, $i=j$, $i>\frac{n}{2}$ in [17]. In this work we solve Problem 4 for $q=2$ and arbitrary $n$. The main ideas of the proof are the following. For $i+j\geq n$, we prove that functions that are optimal in the space $U_{[i,j]}(n,2)$ correspond to some $[i-1]$-trades in $H(n,2)$ (for more information on $[t]$-trades see [4, 10]). Then we apply a characterization of $[t]$-trades of size $2^{t+1}$ obtained by D. Krotov in [10]. Finally, using the bipartiteness of $H(n,2)$, we reduce the case $i+j\leq n$ to the case $i+j\geq n$.

The paper is organized as follows. In Section 2, we introduce basic definitions. In Section 3, we give preliminary results. In Section 4, we present constructions of functions that are optimal in the space $U_{[i,j]}(n,2)$. In Section 5, we characterize functions that are optimal in the space $U_{[i,j]}(n,2)$. In Section 6, we discuss the properties of the spectrum of optimal functions.

## 2. Basic definitions

The eigenvalues of a graph are the eigenvalues of its adjacency matrix. Let $G$ be a graph with vertex set $V$ and let $\lambda$ be an eigenvalue of $G$. The set of neighbors of a vertex $x$ is denoted by $N(x)$. A function $f:V\longrightarrow\mathbb{R}$ is called a $\lambda$-*eigenfunction* of $G$ if $f\not\equiv 0$ and the equality

$$\lambda\cdot f(x)=\sum_{y\in N(x)}f(y) \tag{1}$$

holds for any vertex $x\in V$. The set of functions $f:V\longrightarrow\mathbb{R}$ satisfying (1) for any vertex $x\in V$ is called a $\lambda$-*eigenspace* of $G$. The *support* of a function $f:V\longrightarrow\mathbb{R}$ is the set $S(f)=\{x\in V\mid f(x)\neq 0\}$. Denote $|f|=|S(f)|$.

Given a graph $G$, denote by $U(G)$ the set of all real-valued functions defined on the vertex set of $G$. Note that the set $U(G)$ forms a vector space over $\mathbb{R}$.

The *n-dimensional hypercube* $H(n)$ is defined as follows. The vertex set of $H(n)$ is $\mathbb{Z}_2^n$, and two vertices are adjacent if they differ in exactly one coordinate.

This graph has $n+1$ distinct eigenvalues $\lambda_i(n) = n-2i$, where $0 \le i \le n$. Denote by $U_i(n)$ the $\lambda_i(n)$-eigenspace of $H(n)$. The direct sum of subspaces

$$U_i(n) \oplus U_{i+1}(n) \oplus \ldots \oplus U_j(n)$$

for $0 \le i \le j \le n$ is denoted by $U_{[i,j]}(n)$. Denote $U(n) = U(H(n))$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *Cartesian product* $G_1 \square G_2$ of graphs $G_1$ and $G_2$ is defined as follows. The vertex set of $G_1 \square G_2$ is $V_1 \times V_2$; and any two vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if and only if either $x_1 = x_2$ and $y_1$ is adjacent to $y_2$ in $G_2$, or $y_1 = y_2$ and $x_1$ is adjacent to $x_2$ in $G_1$.

Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs. Let $f_1 : V_1 \longrightarrow \mathbb{R}$ and $f_2 : V_2 \longrightarrow \mathbb{R}$. Denote $G = G_1 \square G_2$. We define the *tensor product* $f_1 \otimes f_2$ on the vertices of $G$ by the following rule:

$$(f_1 \otimes f_2)(x, y) = f_1(x) f_2(y)$$

for $(x, y) \in V(G) = V_1 \times V_2$.

Let $f$ be a real-valued function defined on the vertices of $H(n)$ and let $k \in \{0, 1\}$, $r \in \{1, \ldots, n\}$. We define a function $f_k^r$ on the vertices of $H(n-1)$ as follows: for any vertex $y = (y_1, \ldots, y_{r-1}, y_{r+1}, \ldots, y_n)$ of $H(n-1)$

$$f_k^r(y) = f(y_1, \ldots, y_{r-1}, k, y_{r+1}, \ldots, y_n).$$

For a vector $u \in \mathbb{Z}_2^n$, where $u = (u_1, \ldots, u_n)$, we define a function $\chi_u$ on the vertices of $H(n)$ as follows:

$$\chi_u(x_1, \ldots, x_n) = (-1)^{u_1 x_1 + \ldots + u_n x_n}.$$

The functions $\chi_u$, where $u \in \mathbb{Z}_2^n$, are also known as the characters of the group $\mathbb{Z}_2^n$.

The *weight* of a vector $x \in \mathbb{Z}_2^n$, denoted by $\mathrm{wt}(x)$, is the number of its non-zero coordinates.

Let $A$ and $B$ be two finite subsets of $\mathbb{Z}$. Denote

$$A + B = \{c \in \mathbb{Z} \mid c = a + b, a \in A, b \in B\}.$$

Let $\{i_1, \ldots, i_m\}$ be an $m$-element subset of $\{1, 2, \ldots, n\}$ and let $a_i \in \{0, 1\}$ for all $1 \le i \le m$. Denote

$$\Gamma_{i_1, \ldots, i_m}^{a_1, \ldots, a_m} = \{(x_1, \ldots, x_n) \in \mathbb{Z}_2^n \mid x_{i_1} = a_1, \ldots, x_{i_m} = a_m\}.$$

For $m \in \{0, 1, \ldots, n\}$, a set $\Gamma \subseteq \mathbb{Z}_2^n$ is called an $(n-m)$-*face* if there exist an $m$-element subset $\{i_1, \ldots, i_m\}$ of $\{1, 2, \ldots, n\}$ and numbers $a_1, \ldots, a_m \in \{0, 1\}$ such that $\Gamma = \Gamma_{i_1, \ldots, i_m}^{a_1, \ldots, a_m}$.

Recall that the set $U(n)$ forms a vector space over $\mathbb{R}$. We define an inner product on this vector space as follows:

$$\langle f, g \rangle = \frac{1}{2^n} \sum_{x \in \mathbb{Z}_2^n} f(x) g(x)$$

Two functions $f \in U(n)$ and $g \in U(n)$ are called *orthogonal* if $\langle f, g \rangle = 0$.

A pair $\{T_0, T_1\}$ of two disjoint nonempty subsets of $\mathbb{Z}_2^n$ is called a $[t]$-*trade* in $H(n)$ if every $(n-t)$-face contains the same number of elements from $T_0$ and from $T_1$. For a subset $A$ of $\mathbb{Z}_2^n$, let $\mathbf{1}_A$ denote the characteristic function of $A$ in $\mathbb{Z}_2^n$.

For every non-negative integer $r$ and every positive integer $n \geq r$, the *Reed–Muller code* $\mathcal{RM}(r, n)$ of order $r$ is the set of all $n$-variable Boolean functions of algebraic degree at most $r$.

Let $G$ be a bipartite graph with parts $V_1$ and $V_2$. Suppose that $f$ is a real-valued function defined on the vertices of $G$. We define a function $f'$ on the vertices of $G$ by the following rule:

$$f'(x) = \begin{cases} f(x), & \text{if } x \in V_1; \\ -f(x), & \text{if } x \in V_2. \end{cases}$$

For a function $f \in U(n)$, we define a function $\widetilde{f}$ on the vertices of $H(n)$ as follows:

$$\widetilde{f}(x_1, \ldots, x_n) = (-1)^{x_1 + \ldots + x_n} \cdot f(x_1, \ldots, x_n).$$

Any function $f \in U(n)$ can be uniquely represented in the following form:

$$f = \sum_{i=0}^{n} f_i,$$

where $f_i \in U_i(n)$ for any $0 \leq i \leq n$. The *spectrum* of a function $f \in U(n)$ is the set

$$\mathrm{Spec}(f) = \{0 \leq i \leq n \mid f_i \not\equiv 0\}.$$

Two functions $f \in U(n)$ and $g \in U(n)$ are called *equivalent* if there exist an automorphism $\pi$ of $H(n)$ and a real non-zero constant $c$ such that the equality $g(x) = c \cdot f(\pi(x))$ holds for any vertex $x$ of $H(n)$. We denote this equivalence by $f \sim g$.

## 3. Preliminaries

In this section, we give preliminary results. The following lemma is a special case of Corollary 1 proved in [17].

**Lemma 1.** *Let $f_1 \in U_i(m)$ and $f_2 \in U_j(n)$. Then $f_1 \otimes f_2 \in U_{i+j}(m+n)$.*

The following result is a special case of Lemma 4 proved in [17].

**Lemma 2.** *Let $f \in U_{[i,j]}(n)$ and $r \in \{1, 2, \ldots, n\}$. Then the following statements are true:*

1. $f_0^r - f_1^r \in U_{[i-1,j-1]}(n-1)$.
2. $f_0^r + f_1^r \in U_{[i,j]}(n-1)$.
3. $f_k^r \in U_{[i-1,j]}(n-1)$ *for* $k \in \{0, 1\}$.

**Lemma 3.** *Let $f \in U_{[i,j]}(n)$ and $r \in \{1, 2, \ldots, n\}$. Then there are functions $g$ and $h$ such that $f^r_0 = g+h$, $f^r_1 = g-h$ and $g \in U_{[i,j]}(n-1)$, $h \in U_{[i-1,j-1]}(n-1)$.*

*Proof.* Denote $g = \frac{1}{2}(f^r_0 + f^r_1)$ and $h = \frac{1}{2}(f^r_0 - f^r_1)$. Then we have $f^r_0 = g + h$ and $f^r_1 = g - h$. In addition, by Lemma 2 we obtain that $g \in U_{[i,j]}(n-1)$ and $h \in U_{[i-1,j-1]}(n-1)$. □

We will use Lemma 3 in the proof of Lemma 17. The following two properties of the characters of $\mathbb{Z}_2^n$ are well-known.

**Lemma 4.** *The following statements hold:*

1. *The set $\{\chi_u \mid u \in \mathbb{Z}_2^n\}$ forms an orthonormal basis of the vector space $U(n)$.*
2. *For every $0 \leq i \leq n$, the set $\{\chi_u \mid u \in \mathbb{Z}_2^n, \mathrm{wt}(u) = i\}$ forms a basis of the vector space $U_i(n)$.*

We will use Lemma 4 for the proofs of Lemmas 6, 7, 8 and 15. The following result about the Cartesian product of graphs is well-known.

**Lemma 5.** *Let $G_1$ and $G_2$ be graphs with $m$ and $n$ vertices. If $f_1, \ldots, f_m$ and $g_1, \ldots, g_n$ are orthogonal bases for the vector spaces $U(G_1)$ and $U(G_2)$, then the set*

$$\{f_i \otimes g_j \mid i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}\}$$

*forms an orthogonal basis of the vector space $U(G_1 \square G_2)$.*

Using Lemmas 4 and 5, we immediately obtain the following result.

**Lemma 6.** *The set $\{\chi_u \otimes \chi_v \mid u \in \mathbb{Z}_2^m, v \in \mathbb{Z}_2^n\}$ forms an orthogonal basis of the vector space $U(m+n)$.*

**Lemma 7.** *Let $f_1 \in U(m)$ and $f_2 \in U(n)$. Then*

$$\mathrm{Spec}(f_1 \otimes f_2) = \mathrm{Spec}(f_1) + \mathrm{Spec}(f_2).$$

*Proof.* It follows from Lemmas 4, 6 and 1. □

We will use Lemma 7 in the proof of Lemma 16. The following theorem is a combination of the results proved in [18] (see [18, Theorems 3 and 4]).

**Theorem 1.** *Let $f \in U_{[i,j]}(n)$ and $f \not\equiv 0$. Then the following statements hold:*

1. *If $i + j \geq n$, then $|f| \geq 2^i$ and this bound is sharp.*
2. *If $i + j \leq n$, then $|f| \geq 2^{n-j}$ and this bound is sharp.*

We will use Theorem 1 in the proof of Lemma 17.

**Lemma 8.** *Let $f \in U_i(n)$, where $1 \leq i \leq n$. Then for every $(n-i+1)$-face $\Gamma$ it holds $\sum_{x \in \Gamma} f(x) = 0$.*

*Proof.* Suppose that $\Gamma$ is an $(n-i+1)$-face. It is easy to check that the function $\mathbf{1}_\Gamma$ is orthogonal to $\chi_u$ for any $u \in \mathbb{Z}_2^n$ of weight $i$. Then by Lemma 4 we obtain that $\mathbf{1}_\Gamma$ is orthogonal to an arbitrary function from the space $U_i(n)$. So, the functions $\mathbf{1}_\Gamma$ and $f$ are orthogonal and we have $\sum_{x\in\Gamma} f(x) = 0$. $\square$

The following result was obtained in [10] (see the last paragraph in the proof of Theorem 1).

**Lemma 9.** *Let $\{T_0, T_1\}$ be a $[t]$-trade in $H(n)$. Then $\mathbf{1}_{T_0\cup T_1} \in \mathcal{RM}(n-t-1, n)$.*

The following fact is well known in coding theory (for example, see [12, Chapter 13, Theorem 5] or [3, Chapter 4, Theorem 8]).

**Lemma 10.** *Any Boolean function from $\mathcal{RM}(r, n)$ of weight $2^{n-r}$ is the characteristic function of an $(n-r)$-dimensional affine subspace of $\mathbb{Z}_2^n$.*

The following lemma was proved in [10].

**Lemma 11** ([10], Proposition 1)**.** *An affine subspace $T \subset \mathbb{Z}_2^n$ of dimension $t+1$ can be split into a $[t]$-trade $\{T_0, T_1\}$ if and only if it is a translation of the linear span of mutually disjoint base subsets.*

We will use Lemmas 8, 9, 10 and 11 in the proof of Theorem 2. The following fact is well known in spectral graph theory (for example, see [2, Section 1.3.6]).

**Lemma 12.** *Let $G$ be a bipartite graph. If $f$ is a $\lambda$-eigenfunction of $G$, then $f'$ is a (-$\lambda$)-eigenfunction of $G$.*

Since $H(n)$ is bipartite and $\lambda_i(n) = -\lambda_{n-i}(n)$, by Lemma 12 we immediately obtain the following result.

**Lemma 13.** *If $f \in U_i(n)$, then $\widetilde{f} \in U_{n-i}(n)$.*

Using the previous lemma for $U_k(n)$, where $i \le k \le j$, we obtain the following result.

**Lemma 14.** *If $f \in U_{[i,j]}(n)$, then $\widetilde{f} \in U_{[n-j,n-i]}(n)$.*

We will use Lemma 14 in the proof of Theorem 2.

## 4. Constructions of functions with the minimum cardinality of the support

In this section, we give constructions of functions that are optimal in the space $U_{[i,j]}(n)$. We also find the spectrum of these functions.

For $k \ge 1$, we define a function $\varphi_k$ on the vertices of $H(k)$ by the following rule:

$$\varphi_k(x) = \begin{cases} 1, & \text{if } x \text{ is the all-zeros vector;} \\ -1, & \text{if } x \text{ is the all-ones vector;} \\ 0, & \text{otherwise.} \end{cases}$$

For $k \geq 1$, we define a function $\psi_k$ on the vertices of $H(k)$ by the following rule:

$$\psi_k(x) = \begin{cases} 1, & \text{if } x \text{ is the all-zeros vector;} \\ 1, & \text{if } x \text{ is the all-ones vector;} \\ 0, & \text{otherwise.} \end{cases}$$

For $k \geq 1$, we define a function $I_k$ on the vertices of $H(k)$ by the following rule:

$$I_k(x) = \begin{cases} 1, & \text{if } x \text{ is the all-zeros vector;} \\ 0, & \text{otherwise.} \end{cases}$$

The functions $\varphi_3$, $\psi_3$ and $I_3$ are shown in Figure 1:

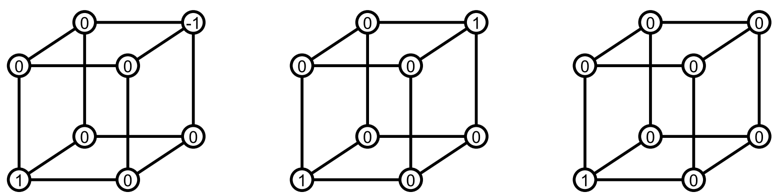


Figure 1: Functions $\varphi_3$, $\psi_3$ and $I_3$ in $H(3)$

**Lemma 15.** *The following statements are true:*

1. $\mathrm{Spec}(\varphi_{2k+1}) = \{1, 3, \dots, 2k+1\}$ *for* $k \geq 1$ *and* $\mathrm{Spec}(\varphi_1) = \{1\}$.
2. $\mathrm{Spec}(\varphi_{2k}) = \{1, 3, \dots, 2k-1\}$ *for* $k \geq 2$ *and* $\mathrm{Spec}(\varphi_2) = \{1\}$.
3. $\mathrm{Spec}(\psi_{2k+1}) = \{0, 2, \dots, 2k\}$ *for* $k \geq 1$ *and* $\mathrm{Spec}(\psi_1) = \{0\}$.
4. $\mathrm{Spec}(I_k) = \{0, 1, \dots, k\}$ *for* $k \geq 1$.

*Proof.* Let us consider the function $\varphi_n$. By Lemma 4, there exist the real numbers $c_u$, where $u \in \mathbb{Z}_2^n$, such that

$$\varphi_n = \sum_{u \in \mathbb{Z}_2^n} c_u \chi_u.$$

Then we have

$$\langle \varphi_n, \chi_u \rangle = \langle \sum_{u \in \mathbb{Z}_2^n} c_u \chi_u, \chi_u \rangle = c_u \langle \chi_u, \chi_u \rangle = c_u.$$

On the other hand,

$$\langle \varphi_n, \chi_u \rangle = \frac{1}{2^n} \sum_{x \in \mathbb{Z}_2^n} \varphi_n(x) \chi_u(x) = \frac{1}{2^n}(1 - (-1)^{u_1 + \ldots + u_n}).$$

Hence

$$c_u = \frac{1}{2^n}(1 - (-1)^{u_1 + \ldots + u_n})$$

for any $u \in \mathbb{Z}_2^n$. So, we have

$$c_u = \begin{cases} \frac{1}{2^{n-1}}, & \text{if } \mathrm{wt}(u) \text{ is odd;} \\ 0, & \text{if } \mathrm{wt}(u) \text{ is even.} \end{cases}$$

Using Lemma 4, we obtain that $\mathrm{Spec}(\varphi_n)$ consists of odd numbers belonging to the set $\{1, \dots, n\}$.

The proofs for the functions $\psi_n$ and $I_n$ are similar. $\square$

**Lemma 16.** *Let $n$ be a positive integer and $n = n_1 + \ldots + n_k + m_1 + \ldots + m_\ell + r$, where $n_1, \dots, n_k$ are odd positive integers, $m_1, \dots, m_\ell$ are even positive integers, $k$, $\ell$ and $r$ are nonnegative integers. Then the following statements hold:*

1. *Let $f = \varphi_{n_1} \otimes \dots \otimes \varphi_{n_k} \otimes \varphi_{m_1} \otimes \dots \otimes \varphi_{m_\ell} \otimes I_r$. Then $f \in U_{[k+\ell, n-\ell]}(n)$ and $|f| = 2^{k+\ell}$. Moreover,*

$$\mathrm{Spec}(f) = \{k+\ell, k+\ell+1, \dots, n-\ell\}$$

   *for $r > 0$ and*

$$\mathrm{Spec}(f) = \{k+\ell, k+\ell+2, \dots, n-\ell\}$$

   *for $r = 0$.*

2. *Let $f = \psi_{n_1} \otimes \dots \otimes \psi_{n_k} \otimes \varphi_{m_1} \otimes \dots \otimes \varphi_{m_\ell} \otimes I_r$. Then $f \in U_{[\ell, n-k-\ell]}(n)$ and $|f| = 2^{k+\ell}$. Moreover,*

$$\mathrm{Spec}(f) = \{\ell, \ell+1, \dots, n-k-\ell\}$$

   *for $r > 0$ and*

$$\mathrm{Spec}(f) = \{\ell, \ell+2, \dots, n-k-\ell\}$$

   *for $r = 0$.*

*Proof.* Let us consider the first case. By Lemma 7 we have

$$\mathrm{Spec}(f) = \mathrm{Spec}(\varphi_{n_1}) + \ldots + \mathrm{Spec}(\varphi_{n_k}) + \ldots + \mathrm{Spec}(\varphi_{m_1}) + \ldots + \mathrm{Spec}(\varphi_{m_\ell}) + \mathrm{Spec}(I_r).$$

Then applying Lemma 15, we obtain that

$$\mathrm{Spec}(f) = \{k+\ell, k+\ell+1, \dots, n-\ell\}$$

for $r > 0$ and

$$\mathrm{Spec}(f) = \{k+\ell, k+\ell+2, \dots, n-\ell\}$$

for $r = 0$. Using the equality $|f_1 \otimes f_2| = |f_1| \cdot |f_2|$, we see that $|f| = 2^{k+\ell}$.

The proof for the second case is similar. $\square$

## 5. Main results

In this section, we prove the main theorem of this paper. Firstly, we prove the following result.

**Lemma 17.** *Let $f \in U_{[i,j]}(n)$, where $i+j \geq n$. If $|f| = 2^i$, then $f$ takes values from the set $\{-a, 0, a\}$, where $a$ is a positive real number.*

*Proof.* Let us prove this lemma by induction on $n$, $i$ and $j$. If $i = 0$, then $|f| = 1$ and the claim of the lemma holds. So, we can assume that $i \geq 1$. If $n = 1$, then $i = j = 1$. Then $f \in U_1(1)$ and the claim of the lemma holds.

Let us prove the induction step for $n \geq 2$ and $i \geq 1$. Let us consider the functions $f_0^n$ and $f_1^n$. Denote $f_k = f_k^n$ for $k \in \{0, 1\}$. Lemma 3 implies that there are functions $g$ and $h$ such that $f_0 = g+h$, $f_1 = g-h$ and $g \in U_{[i,j]}(n-1)$, $h \in U_{[i-1,j-1]}(n-1)$. Let us consider two cases.

In the first case we suppose that $g \equiv 0$. In this case we have $|h| = \frac{1}{2}|f| = 2^{i-1}$. Let us show that $i + j \geq n+1$. Indeed, if $i + j = n$, then $|h| \geq 2^i$ due to Theorem 1. Since $|h| = 2^{i-1}$, we get a contradiction. So, in this case we have $i + j \geq n+1$. Applying the induction assumption for $h$, we obtain that $h$ takes values from the set $\{-a, 0, a\}$, where $a$ is a positive real number. Therefore, $f$ also takes values from the set $\{-a, 0, a\}$.

In the second case we suppose that $g \not\equiv 0$. Since $g \in U_{[i,j]}(n-1)$, by Theorem 1 we obtain that $|g| \geq 2^i$. Then we have

$$|f| = |f_0| + |f_1| \geq |f_0 + f_1| = |g| \geq 2^i.$$

Therefore $|g| = |f| = 2^i$. Applying the induction assumption for $g$, we obtain that $g$ takes values from the set $\{-a', 0, a'\}$, where $a'$ is a positive real number. Since $|f| = |g|$, we have $h(x) \in \{-g(x), g(x)\}$ for every vertex $x$ of $H(n-1)$. Thus, $f$ takes values from the set $\{-2a', 0, 2a'\}$. $\square$

The main result of this paper is the following.

**Theorem 2.** *The following statements hold:*

1. *Let $f \in U_{[i,j]}(n)$, where $i + j \geq n$. The equality $|f| = 2^i$ holds if and only if $f$ is equivalent to*

$$\varphi_{n_1} \otimes \cdots \otimes \varphi_{n_k} \otimes \varphi_{m_1} \otimes \cdots \otimes \varphi_{m_\ell} \otimes I_r,$$

   *where $n = n_1 + \ldots + n_k + m_1 + \ldots + m_\ell + r$, $n_1, \ldots, n_k$ are odd positive integers, $m_1, \ldots, m_\ell$ are even positive integers, $k$, $\ell$ and $r$ are nonnegative integers, $k + \ell = i$ and $\ell \geq n - j$.*

2. *Let $f \in U_{[i,j]}(n)$, where $i + j \leq n$. The equality $|f| = 2^{n-j}$ holds if and only if $f$ is equivalent to*

$$\psi_{n_1} \otimes \cdots \otimes \psi_{n_k} \otimes \varphi_{m_1} \otimes \cdots \otimes \varphi_{m_\ell} \otimes I_r,$$

   *where $n = n_1 + \ldots + n_k + m_1 + \ldots + m_\ell + r$, $n_1, \ldots, n_k$ are odd positive integers, $m_1, \ldots, m_\ell$ are even positive integers, $k$, $\ell$ and $r$ are nonnegative integers, $k + \ell = n - j$ and $\ell \geq i$.*

*Proof.* 1. Suppose that $|f| = 2^i$. If $i = 0$, then $j = n$. In this case $|f| = 1$. Therefore, $f \sim I_n$ and the claim of the theorem holds. In what follows in the proof of Theorem 2 for $i + j \geq n$ we can assume that $i \geq 1$.

Let us consider a pair $\{T_0, T_1\}$, where $T_0 = \{x \in \mathbb{Z}_2^n \mid f(x) > 0\}$ and $T_1 = \{x \in \mathbb{Z}_2^n \mid f(x) < 0\}$. Lemmas 8 and 17 imply that every $(n-i+1)$-face contains the same number of elements from $T_0$ and from $T_1$. So, $\{T_0, T_1\}$ is an $[i-1]$-trade in $H(n)$. Lemma 9 implies that

$$\mathbf{1}_{T_0 \cup T_1} \in \mathcal{RM}(n-i, n).$$

Since $|\mathbf{1}_{T_0 \cup T_1}| = |f|$, we have $|\mathbf{1}_{T_0 \cup T_1}| = 2^i$. Then by Lemma 10 we have that $\mathbf{1}_{T_0 \cup T_1}$ is the characteristic function of an $i$-dimensional affine subspace of $\mathbb{Z}_2^n$. Applying Lemma 11, we obtain that

$$f \sim \varphi_{t_1} \otimes \cdots \otimes \varphi_{t_i} \otimes I_r,$$

where $n = t_1 + \ldots + t_i + r$, $t_1, \ldots, t_i$ are positive integers and $r$ is a nonnegative integer. Suppose that the set $\{t_1, \ldots, t_i\}$ consists of $k$ odd numbers $n_1, \ldots, n_k$ and $\ell$ even numbers $m_1, \ldots, m_\ell$. Then

$$f \sim \varphi_{n_1} \otimes \cdots \otimes \varphi_{n_k} \otimes \varphi_{m_1} \otimes \cdots \otimes \varphi_{m_\ell} \otimes I_r.$$

Using Lemma 16, we see that $f \in U_{[i,n-\ell]}(n)$ and $n - \ell \in \mathrm{Spec}(f)$. Since $f \in U_{[i,j]}(n)$, we obtain $\ell \geq n - j$.

Conversely, suppose that

$$f \sim \varphi_{n_1} \otimes \cdots \otimes \varphi_{n_k} \otimes \varphi_{m_1} \otimes \cdots \otimes \varphi_{m_\ell} \otimes I_r,$$

where $n = n_1 + \ldots + n_k + m_1 + \ldots + m_\ell + r$, $n_1, \ldots, n_k$ are odd positive integers, $m_1, \ldots, m_\ell$ are even positive integers, $k$, $\ell$ and $r$ are nonnegative integers, $k + \ell = i$ and $\ell \geq n - j$. Lemma 16 implies that $f \in U_{[k+\ell, n-\ell]}(n)$ and $|f| = 2^{k+\ell}$. Since $k + \ell = i$ and $\ell \geq n - j$, we have $f \in U_{[i,j]}(n)$ and $|f| = 2^i$.

2. Suppose that $|f| = 2^{n-j}$. Lemma 14 implies that $\widetilde{f} \in U_{[n-j, n-i]}(n)$. Note that $|\widetilde{f}| = |f| = 2^{n-j}$. By the first case of this theorem we obtain that $\widetilde{f} \sim v$, where

$$v = \varphi_{n_1} \otimes \cdots \otimes \varphi_{n_k} \otimes \varphi_{m_1} \otimes \cdots \otimes \varphi_{m_\ell} \otimes I_r,$$

$n = n_1 + \ldots + n_k + m_1 + \ldots + m_\ell + r$, $n_1, \ldots, n_k$ are odd positive integers, $m_1, \ldots, m_\ell$ are even positive integers, $k$, $\ell$ and $r$ are nonnegative integers, $k + \ell = n - j$ and $\ell \geq i$. Using the equality $\widetilde{f_1 \otimes f_2} = \widetilde{f}_1 \otimes \widetilde{f}_2$, we obtain that

$$\widetilde{v} = \psi_{n_1} \otimes \cdots \otimes \psi_{n_k} \otimes \varphi_{m_1} \otimes \cdots \otimes \varphi_{m_\ell} \otimes I_r.$$

Therefore, we have

$$f \sim \psi_{n_1} \otimes \cdots \otimes \psi_{n_k} \otimes \varphi_{m_1} \otimes \cdots \otimes \varphi_{m_\ell} \otimes I_r.$$

Conversely, suppose that

$$f \sim \psi_{n_1} \otimes \cdots \otimes \psi_{n_k} \otimes \varphi_{m_1} \otimes \cdots \otimes \varphi_{m_\ell} \otimes I_r,$$

where $n = n_1 + \ldots + n_k + m_1 + \ldots + m_\ell + r$, $n_1, \ldots, n_k$ are odd positive integers, $m_1, \ldots, m_\ell$ are even positive integers, $k$, $\ell$ and $r$ are nonnegative integers, $k+\ell = n-j$ and $\ell \geq i$. Lemma 16 implies that $f \in U_{[\ell,n-k-\ell]}(n)$ and $|f| = 2^{k+\ell}$. Since $k + \ell = n - j$ and $\ell \geq i$, we have $f \in U_{[i,j]}(n)$ and $|f| = 2^{n-j}$. □

Applying Theorem 2 for $i = j$, we obtain the following result.

**Corollary 1.** *The following statements hold:*

1. *Let $f \in U_i(n)$, where $i \geq \frac{n}{2}$. The equality $|f| = 2^i$ holds if and only if $f$ is equivalent to $\varphi_1^{2i-n} \otimes \varphi_2^{n-i}$.*
2. *Let $f \in U_i(n)$, where $i \leq \frac{n}{2}$. The equality $|f| = 2^{n-i}$ holds if and only if $f$ is equivalent to $\psi_1^{n-2i} \otimes \varphi_2^{i}$.*

Finally, we illustrate Theorem 2 in the following examples:

**Example 1.** *Let $n = 4$, $i = 2$ and $j = 3$. There are exactly two partitions of 4 such that $k + \ell = 2$ and $\ell \geq 1$: $4 = 2 + 2$ and $4 = 1 + 2 + 1$. These partitions correspond to the functions $\varphi_2 \otimes \varphi_2$ and $\varphi_1 \otimes \varphi_2 \otimes I_1$ respectively.*

**Example 2.** *Let $n = 3$, $i = 0$ and $j = 2$. There are exactly three partitions of 3 such that $k + \ell = 1$ and $\ell \geq 0$: $3 = 2 + 1$, $3 = 3$ and $3 = 1 + 2$. These partitions correspond to the functions $\varphi_2 \otimes I_1$, $\psi_3$ and $\psi_1 \otimes I_2$ respectively.*

## 6. Spectrum of optimal functions

In this section, we discuss the spectrum of functions that are optimal in the space $U_{[i,j]}(n)$. Theorem 2 and Lemma 16 imply that the spectrum of such functions forms an arithmetic progression with common difference 1 or 2. More precisely, we have the following result.

**Corollary 2.** *The following statements hold:*

1. *Let $f \in U_{[i,j]}(n)$, where $i + j \geq n$. If $|f| = 2^i$, then*
$$\mathrm{Spec}(f) = \{i, i + d, \ldots, i + kd\},$$
*where $d \in \{1, 2\}$, $k$ is non-negative integer and $i + kd \leq j$.*
2. *Let $f \in U_{[i,j]}(n)$, where $i + j \leq n$. If $|f| = 2^{n-j}$, then*
$$\mathrm{Spec}(f) = \{j - kd, j - (k - 1)d, \ldots, j\},$$
*where $d \in \{1, 2\}$, $k$ is non-negative integer and $j - kd \geq i$.*

Corollary 2 implies that if $f \in U_{[i,j]}(n)$ and $\mathrm{Spec}(f)$ is not an arithmetic progression of a special kind, then $|f| > \max(2^i, 2^{n-j})$. For example, if $f \in U(n)$ and $\mathrm{Spec}(f) = \{0, 3\}$, where $n \geq 3$, then $|f| > 2^{n-3}$. In view of these observations, it seems natural to consider the following question.

**Problem 5.** *Let $n \geq 3$. Find*
$$\min_{f \in U(n), \mathrm{Spec}(f) = \{0,3\}} |f|.$$

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