JACOBI POLYNOMIALS AND DESIGN THEORY II

HIMADRI SHEKHAR CHAKRABORTY, REINA ISHIKAWA, AND YUUHO TANAKA*

Abstract. In this paper, we introduce some new polynomials associated to linear codes over $\mathbb{F}_q$. In particular, we introduce the notion of split complete Jacobi polynomials attached to multiple sets of coordinate places of a linear code over $\mathbb{F}_q$, and give the MacWilliams type identity for it. We also give the notion of generalized q-colored t-designs. As an application of the generalized q-colored t-designs, we derive a formula that obtains the split complete Jacobi polynomials of a linear code over $\mathbb{F}_q$. Moreover, we define the concept of colored packing (resp. covering) designs. Finally, we give some coding theoretical applications of the colored designs for Type III and Type IV codes.

Key Words: Codes, Jacobi polynomials, designs, invariant theory.

2010 *Mathematics Subject Classification*. Primary 11T71; Secondary 94B05, 11F11.

1. Introduction

In 1997, Ozeki [35] introduced the notion of Jacobi polynomials of linear codes, an analogue to Jacobi forms [16] of lattices. Many authors studied the Jacobi polynomials in coding theory; for instance [6, 7, 8, 13, 14, 19]. Among these articles Bonnecaze et al. [6, 7, 8] pointed out some characterizations of the Jacobi polynomials with the codes supporting designs. Moreover, Bonnecaze, Rains and Solé [7] introduced the notion of colored t-designs and gave an application of these designs for $\mathbb{Z}_4$-codes in the evaluation of the Jacobi polynomials from the symmetrized weight enumerator of $\mathbb{Z}_4$-codes using the polarization operator. Later, Bonnecaze, Solé and Udaya [8] studied the 3-colored 3-designs in the case of Type III codes. Furthermore, Cameron [10] gave a new generalization of the combinatorial t-designs. In this paper, we would like to call these designs as the generalized t-designs. In a recent study, Chakraborty, Miezaki, Oura and Tanaka [15] introduced

*Corresponding author.

the notion of Jacobi polynomials of a linear code with multiple reference vectors and gave a design theoretical application of these Jacobi polynomials in the study of generalized t-designs.

In this paper, we introduced the notion of split complete weight enumerators for the linear codes over $\mathbb{F}_q$ which is independent from the sense of Bonnecaze et al. [8]. We also introduce the notion of split complete Jacobi polynomials of a linear code over $\mathbb{F}_q$ attached to multiple sets of coordinate places of the code. We show that both the code polynomials: split complete weight enumerators and split complete Jacobi polynomials satisfy the MacWilliams type identities. In particular, the complete Jacobi polynomials of linear codes over $\mathbb{F}_3$ in our sense are equivalent to the split complete weight enumerators of codes over $\mathbb{F}_3$ in the sense of Bonnecaze et al. [8]. Moreover, we define the concept of the generalized colored t-designs, and as an analogue to Bonnecaze et al. [8], we present a combinatorial interpretation of the polarization of the split complete Jacobi polynomials of a linear code over $\mathbb{F}_q$. In addition, we study the complete Jacobi polynomials of some Type III (resp. Type IV) codes of specific lengths through invariant theory to construct the colored packing (resp. covering) designs that correspond to the coefficients in the complete Jacobi polynomials. Bonnecaze et al. [8] investigated the 3-colored t-designs structure for the extremal Type III codes. In this paper, we study the Type IV codes of some specific lengths and obtain the 4-colored t-design structures.

This paper is organized as follows. In Section 2, we discuss the basic definitions and notations that we use in this paper. We also prove the MacWilliams type identity (Theorem 2.2) for the spilt complete weight enumerators of codes over $\mathbb{F}_q$. In Section 3, we introduced several colored designs, namely generalized colored t-designs, colored packing (resp. covering) designs and some of their properties. In Section 4, we give the MacWilliams type identity (Theorem 4.1) for the spilt complete Jacobi polynomials of linear codes over $\mathbb{F}_q$. We also observe (Theorem 4.3, Theorem 4.4) how polarization operator acts to obtain the split complete Jacobi polynomials attached to multiple sets of coordinate places of a code. In Section 5, we disclose some facts between a Type III (resp. Type IV) code of specific lengths and colored designs with the help of the complete Jacobi polynomials. We also show that the codewords of fixed composition in the Hermitian Type IV codes of length 6 hold 4-colored 2-designs (Theorem 5.12). Finally, we conclude the paper with some remarks in Section 6.

All computer calculations in this paper were done with the help of Magma [9].

2. Preliminaries

Let $\mathbb{F}_q$ be a finite field of order q, where q is a prime power. Then $\mathbb{F}_q^n$ denotes the vector space of dimension n over $\mathbb{F}_q$. The elements of $\mathbb{F}_q^n$ are known as *vectors*. The *Hamming weight* of a vector $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{F}_q^n$ is denoted by $\mathrm{wt}(\mathbf{u})$ and defined to be the number of i's such that $u_i \neq 0$. The *inner product* of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}_q^n$ is given by

$$\mathbf{u} \cdot \mathbf{v} := u_1 v_1 + \cdots + u_n v_n,$$

where $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$. If q is an even power of an arbitrary prime p, then it is convenient to consider another inner product known as the *Hermitian inner product* which can be defined as

$$\mathbf{u} \cdot \overline{\mathbf{v}} := u_1 \overline{v_1} + \cdots + u_n \overline{v_n},$$

where $\overline{v_i} := v_i^{\sqrt{q}}$. An $\mathbb{F}_q$-linear code of length n is a vector subspace of $\mathbb{F}_q^n$. The elements of an $\mathbb{F}_q$-linear code are called *codewords*. The *dual code* of an $\mathbb{F}_q$-linear code C of length n is defined by

$$C^{\perp} := \{\mathbf{v} \in \mathbb{F}_q^n \mid \mathbf{u} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{u} \in C\}.$$

An $\mathbb{F}_q$-linear code C is called *self-dual* if $C = C^{\perp}$. Let q be an even power of an arbitrary prime number. Then an $\mathbb{F}_q$-linear code C of length n is called *Hermitian self-dual* if $C = C^{\perp_H}$, where $C^{\perp_H}$ denotes the *Hermitian dual code* of C which is defined as

$$C^{\perp_H} := \{\mathbf{v} \in \mathbb{F}_q^n \mid \mathbf{u} \cdot \overline{\mathbf{v}} = 0 \text{ for all } \mathbf{u} \in C\}.$$

Most of the results in this paper are stated for $\mathbb{F}_q$-linear codes with usual inner product but it can be re-phrased with equal validity to the case of the codes with the Hermitian inner product.

Let $X \subseteq [n]$. The *composition* of an element $\mathbf{u} \in \mathbb{F}_q^n$ attached to X is the q-tuple:

$$\mathrm{comp}_X(\mathbf{u}) := (n_{a,X}(\mathbf{u}) : a \in \mathbb{F}_q),$$

where $n_{a,X}(\mathbf{u}) := \#\{i \in X \mid u_i = a\}$. Obviously, $\sum_{a \in \mathbb{F}_q} n_{a,X}(\mathbf{u}) = |X|$. If $X = [n]$, we prefer to write the composition of $\mathbf{u} \in \mathbb{F}_q^n$ as

$$\mathrm{comp}(\mathbf{u}) := (n_a(\mathbf{u}) : a \in \mathbb{F}_q),$$

where $n_a(\mathbf{u})$ denotes the number of coordinates of $\mathbf{u}$ that are equal to $a \in \mathbb{F}_q$.

It is well known that the length n of a self-dual code over $\mathbb{F}_q$ is even and the dimension is $n/2$. To study self-dual codes in detail, we refer the readers to [5, 17, 23, 34]. A self-dual code C over $\mathbb{F}_3$ of length $n \equiv 0 \pmod 4$ is called *Type* III if the weight of each codeword of C

is multiple of 3. Again a self-dual code C over $\mathbb{F}_4$ of length $n \equiv 0 \pmod 2$ having even weight is called *Type* IV.

Definition 2.1. Let C be and $\mathbb{F}_q$-linear code of length n. Then the *weight enumerator* of C is defined as

$$W_C(x,y) := \sum_{\mathbf{u}\in C} x^{n-\mathrm{wt}(\mathbf{u})} y^{\mathrm{wt}(\mathbf{u})}.$$

Definition 2.2. Let C be an $\mathbb{F}_q$-linear code of length n. Then the *complete weight enumerator* of C is defined as

$$\mathbf{cwe}_C(\{x_a\}_{a\in\mathbb{F}_q}) := \sum_{\mathbf{u}\in C} \prod_{a\in\mathbb{F}_q} x_a^{n_a(\mathbf{u})}.$$

Remark 2.1. $W_C(x,y) = \mathbf{cwe}_C(x_0 \leftarrow x, \{x_a \leftarrow y\}_{0\neq a\in\mathbb{F}_q})$.

Definition 2.3. Let C be an $\mathbb{F}_q$-linear code of length n. Then the *split complete weight enumerator* attached to ℓ mutually disjoint subset $X_1,\dots,X_\ell$ of coordinate places of the code C such that

$$X_1 \sqcup \cdots \sqcup X_\ell = [n]$$

is defined as follows:

$$\mathbf{scwe}_{C,X_1,\dots,X_\ell}(\{\{x_{X_i,a}\}_{a\in\mathbb{F}_q}\}_{1\leq i\leq \ell}) := \sum_{\mathbf{u}\in C} \prod_{i=1}^{\ell} \prod_{a\in\mathbb{F}_q} x_{X_i,a}^{n_{a,X_i}(\mathbf{u})}.$$

Note that when $\ell = 1$, the split complete weight enumerators of an $\mathbb{F}_q$-linear code C coincide with its complete weight enumerators.

Example 2.1. Let C_4 be an $\mathbb{F}_3$-linear code of length 4 with the generator matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

The elements of C_4 are listed as follows:

$$\begin{matrix} (0,0,0,0), & (0,1,1,2), & (0,2,2,1), \\ (1,0,1,1), & (1,1,2,0), & (1,2,0,2), \\ (2,0,2,2), & (2,1,0,1), & (2,2,1,0). \end{matrix}$$

Therefore the complete weight enumerator of C_4 is

$$\mathbf{cwe}_{C_4}(x_0,x_1,x_2) = x_0^4 + x_0^1x_1^3 + x_0^1x_2^3 + 3x_0^1x_1^2x_2^1 + 3x_0^1x_1^1x_2^2.$$

Let $X_1 = \{1,2\}$ and $X_2 = \{3,4\}$ be two disjoint sets such that $X_1 \sqcup X_2 = [4]$. Then the split complete weight enumerator of C_4 attached to X_1 and X_2 is

$$\begin{aligned}
&\mathbf{scwe}_{C_4,X_1,X_2}(x_{X_1,0},x_{X_1,1},x_{X_1,2},x_{X_2,0},x_{X_2,1},x_{X_2,2})\\
&\qquad = x_{X_1,0}^2x_{X_2,0}^2+x_{X_1,0}^1x_{X_1,1}^1x_{X_2,1}^2+x_{X_1,0}^1x_{X_1,2}^1x_{X_2,2}^2\\
&+x_{X_1,0}^1x_{X_1,1}^1x_{X_2,1}^1x_{X_2,2}^1+x_{X_1,1}^2x_{X_2,0}^1x_{X_2,2}^1+x_{X_1,1}^1x_{X_1,2}^1x_{X_2,0}^1x_{X_2,1}^1\\
&+x_{X_1,0}^1x_{X_1,2}^1x_{X_2,1}^1x_{X_2,2}^1+x_{X_1,1}^1x_{X_1,2}^1x_{X_2,0}^1x_{X_2,2}^1+x_{X_1,2}^2x_{X_2,0}^1x_{X_2,1}^1.
\end{aligned}$$

A *character* of $\mathbb{F}_q$, where $q=p^f$ for some prime number p, is a homomorphism from the additive group $\mathbb{F}_q$ to the multiplicative group of non-zero complex numbers. We review [13, 15, 24] to introduce some fixed non-trivial characters over $\mathbb{F}_q$. Now let $F(x)$ be a primitive irreducible polynomial of degree f over $\mathbb{F}_p$ and let λ be a root of $F(x)$. Then any element $a\in\mathbb{F}_q$ has a unique representation as:

$$a=a_0+a_1\lambda+a_2\lambda^2+\cdots+a_{f-1}\lambda^{f-1},$$

where $a_i\in\mathbb{F}_p$. For $b\in\mathbb{F}_q$, we define $\chi_b(a):=\zeta_p^{a_0b_0+\cdots+a_{f-1}b_{f-1}}$, where ζ_p is the p-th primitive root $e^{2\pi i/p}$ of unity. When $b\neq 0$, then χ_b is a non-trivial character of $\mathbb{F}_q$. Let χ be a non-trivial character of $\mathbb{F}_q$. Then for any $a\in\mathbb{F}_q$, we have the following property:

$$\sum_{b\in\mathbb{F}_q}\chi(ab):=\begin{cases} q & \text{if} \quad a=0,\\ 0 & \text{if} \quad a\neq 0.\end{cases}$$

Lemma 2.1 ([24])**.** *Let C be an $\mathbb{F}_q$-linear code of length n. For $\mathbf{v}\in\mathbb{F}_q^n$, define*

$$\delta_{C^\perp}(\mathbf{v}):=\begin{cases} 1 & \textit{if } \mathbf{v}\in C^\perp,\\ 0 & \textit{otherwise.}\end{cases}$$

Then we have the following identity:

$$\delta_{C^\perp}(\mathbf{v})=\frac{1}{|C|}\sum_{\mathbf{u}\in C}\chi(\mathbf{u}\cdot\mathbf{v}).$$

Now we have the following MacWilliams type identity for the split complete weight enumerators. The proof of the theorem is straightforward. So we leave it for the readers.

Theorem 2.2. *Let C be an $\mathbb{F}_q$-linear code of length n. Again let χ be a non-trivial character of $\mathbb{F}_q$. Then*

$$\begin{aligned}&\mathbf{scwe}_{C^\perp,X_1,\dots,X_\ell}(\{\{x_{X_i,a}\}_{a\in\mathbb{F}_q}\}_{1\le i\le\ell})\\&\quad=\frac{1}{|C|}\mathbf{scwe}_{C,X_1,\dots,X_\ell}\left(\left\{\left\{\sum_{b\in\mathbb{F}_q}\chi(ab)x_{X_i,b}\right\}_{a\in\mathbb{F}_q}\right\}_{1\le i\le\ell}\right).\end{aligned}$$

3. Generalized colored designs

Let v be a positive integer, and let $\mathbf{v} := (v_1, \dots, v_\ell)$ such that $v = \sum_{i=1}^{\ell} v_i$. Let $\mathbf{X} := (X_1, \dots, X_\ell)$, where X_i's are pairwise disjoint sets such that $|X_i| = v_i$ for all i. Again let

$$\mathcal{B} \subseteq B_1 \times \cdots \times B_\ell,$$

where B_i is the set of blocks corresponding to X_i for all i.

Then the *generalized colored incidence structure* is a triple $\mathcal{D} := (\mathbf{X}, \mathcal{B}, \mathcal{C})$, where $\mathcal{C}$ is a set of *colors*, together with a function

$$\rho_i : X_i \times B_i \to \mathcal{C}$$

for all i. We will say that $\mathcal{B}$ has color $\rho_i(p, b)$ at p in the i-th component. For an element $\mathbf{K} := (K_1, \dots, K_\ell) \in \mathcal{B}$, we define a function $n_i : \mathcal{C} \to \mathbb{Z}$ called the *palette* on K_i for $1 \le i \le \ell$ that counts the number of occurrence of color $c \in \mathcal{C}$ in K_i. The generalized colored incident structure is said to be *uniform* if each color $c \in \mathcal{C}$ occurs $\sum_{i=1}^{\ell} n_i(c)$ times in every element $\mathbf{K} \in \mathcal{B}$.

Definition 3.1. The uniform colored incidence structure $\mathcal{D}$ is called the *generalized colored t-design* if $\mathbf{t} := (t_1, \dots, t_\ell)$ such that $t = \sum_{i=1}^{n} t_i$, then for each $\mathbf{C} := (C_1, \dots, C_\ell)$ such that C_i is the t_i-multiset of colors (repeated choice allowed) for all i, there is a number $\lambda \ge 0$ such that for any choice $\mathbf{T} := (T_1, \dots, T_\ell)$ with $T_i \in \binom{X_i}{t_i}$ for all i, there are precisely λ members $\mathbf{K} := (K_1, \dots, K_n) \in \mathcal{B}$ for which $T_i \subseteq K_i$ for all i that use the t_i-multiset of colors C_i for the points in T_i.

Let $\mathcal{D} = (\mathbf{X}, \mathcal{B}, \mathcal{C})$ be a generalized colored t-design with the set of colors $\mathcal{C} = \{1, 2, \dots, r\}$. It is immediate from the above definition that the λ's are not independent from the choices of colors. Therefore by $\lambda_{(j_1(1),\dots,j_1(r)),\dots,(j_\ell(1),\dots,j_\ell(r))}$, we denote the number of $\mathbf{K} = (K_1, \dots, K_\ell) \in \mathcal{B}$ that uses the color $c \in \mathcal{C}$ $j_i(c)$ times in K_i with $\sum_{c=1}^{r} j_i(c) = t_i$ for all $1 \le i \le \ell$. Obviously, the parameters of a generalized colored t-design $\mathcal{D} = (\mathbf{X}, \mathcal{B}, \mathcal{C})$ depend only on the number of points v_i in X_i for all i, the number of blocks $|\mathcal{B}|$ and palette

$((n_1(1), n_1(2), \ldots,), \ldots, (n_\ell(1), n_\ell(2), \ldots))$, therefore it is convenient to write a generalized colored t-design as a

$$t - (\mathbf{v}, ((n_1(1), n_1(2), \ldots,), \ldots, (n_\ell(1), n_\ell(2), \ldots)), |\mathcal{B}|)$$

design. Note that when $\mathbf{k} = (k)$ and $\mathbf{v} = (v)$, then the generalized colored t-design coincide with the colored *t-design*. For detail discussions on colored t-designs, we refer the readers to [7, 8].

To construct the generalized colored t-designs from a linear code C of length n over $\mathbb{F}_q$. Let $\mathbf{v} = (v_1, \ldots, v_\ell)$ such that $\sum_{i=1}^{\ell} v_i = n$ and $\mathbf{X} = (X_1, \ldots, X_\ell)$ of pairwise disjoint sets $X_i \subseteq [n]$ with $|X_i| = v_i$. We define the *split composition* $\mathbf{s} := (s_1, \ldots, s_\ell)$ such that $s_i = (s_{i1}, \ldots, s_{iq})$ for all i satisfying $\sum_{j=1}^{q} s_{ij} = v_i$. For any codeword $\mathbf{u} \in C$, let $\mathbf{K}(\mathbf{u}) := (K_{X_1}(\mathbf{u}), \ldots, K_{X_\ell}(\mathbf{u}))$ such that $K_{X_i}(\mathbf{u})$ for all i are the characteristic vectors of the supports of $\mathbf{u}$ with the coordinate place X_i. Let $C_\mathbf{s}$ be the set of codewords of $\mathbf{u} \in C$ such that $\mathrm{comp}_{X_i}(\mathbf{u}) = s_i$ for all i. We denote

$$\mathcal{B}(C_\mathbf{s}) := \{\mathbf{K}(\mathbf{u}) \mid \mathbf{u} \in C_\mathbf{s}\}.$$

In general, $\mathcal{B}(C_\mathbf{s})$ is a multi-set. Let $\mathcal{C}$ be a set of colors. Then we call $C_\mathbf{s}$ is a generalized colored t-design if the triple $(\mathbf{X}, \mathcal{B}(C_\mathbf{s}), \mathcal{C})$ together with the function $\rho_i(p, b)$ of u_p is a generalized colored t-design. We say the code C is *generalized colorwise t-homogeneous* if the set of codewords $C_\mathbf{s}$ for every given $\mathbf{s}$ holds a generalized colored t-design. The code is called *generalized colorwise homogeneous* when $t = 1$. In particular, for any $\mathbb{F}_q$-linear code one can choose the function $\rho_i(p, b) = a$ if $u_p = a$ where $a \in \mathbb{F}_q$.

A *colored design* with parameters t-$(v, (n(1), n(2), \ldots), (\lambda_1^{a_1}(P), \ldots, \lambda_N^{a_N}(P)))$ is a set of blocks $\mathcal{B}$ with palette $(n(1), n(2), \ldots)$ of a set of v points, called the varieties and a partition of the set of all t-tuples into N groups $G_1, \ldots, G_N$ satisfying that for each t-multiset of colors P (repeated choices allowed), there is a number λ_i such that for every t-set belonging to G_i (say a_i such t-set), there are exactly λ_i-blocks in $\mathcal{B}$ that use the t-multiset of colors P.

When $N = 1$, it is clearly a colored t-design. A *colored packing* (resp. *covering*) *design* with parameters t-$(v, (n(1), n(2), \ldots), \lambda(P))$ is a colored design with $\max(\lambda_i) = \lambda$ (resp. $\min(\lambda_i) = \lambda$). The minimum (resp. maximum) number of blocks of a covering (resp. packing) design is denoted by $C_{\lambda_{j(1),\ldots,j(r)}}(v, (n(1), \ldots, n(r)), t)$ (resp. $D_{\lambda_{j(1),\ldots,j(r)}}(v, (n(1), \ldots, n(r)), t)$). Note that the 2-colored packing (resp. covering) designs is the packing (resp. covering) designs in the sense of Bonnecaze et al. [6].

4. Jacobi polynomials and polarization

The MacWilliams type identity for the Jacobi polynomial of an $\mathbb{F}_q$-linear code with one reference vector was given in [35]. In this section, we give the MacWilliams type identity for the Jacobi polynomial of an $\mathbb{F}_q$-linear code attached to multiple sets of coordinate positions of the code. Bonnecaze et al. [7] defined the Aronhold polarization operator, and as an application of this operator, they obtained a formula to evaluate the Jacobi polynomial of a $\mathbb{Z}_4$-code from the symmetrized weight enumerator of the code. Later an analogue of the formula was given in [8] for complete weight enumerators of $\mathbb{F}_3$-linear codes. In this section, we give the generalizations of the polarization operation, and using these operators, we evaluate the complete Jacobi polynomial of an $\mathbb{F}_q$-linear code attached to the multiple reference sets.

Definition 4.1. Let C be an $\mathbb{F}_q$-linear code of length n. Then the *Jacobi polynomial* attached to a set T of coordinate places of the code C is defined as follows:

$$J_{C,T}(w,z,x,y) := \sum_{\mathbf{u}\in C} w^{m_{0,T}(\mathbf{u})} z^{m_{1,T}(\mathbf{u})} x^{m_{0,[n]\setminus T}(\mathbf{u})} y^{m_{1,[n]\setminus T}(\mathbf{u})},$$

where $T \subseteq [n]$, and for $\mathbf{u} \in C$,

$$\begin{aligned} m_{0,T}(\mathbf{u}) &:= \#\{i \in T \mid u_i = 0\}, \\ m_{1,T}(\mathbf{u}) &:= \#\{i \in T \mid u_i \neq 0\}, \\ m_{0,[n]\setminus T}(\mathbf{u}) &:= \#\{i \in [n]\setminus T \mid u_i = 0\}, \\ m_{1,[n]\setminus T}(\mathbf{u}) &:= \#\{i \in [n]\setminus T \mid u_i \neq 0\}. \end{aligned}$$

Remark 4.1. If $T \subseteq [n]$ is empty, then $J_{C,T}(w,z,x,y) = W_C(x,y)$.

Definition 4.2. Let C be an $\mathbb{F}_q$-linear code of length n. Then the *complete Jacobi polynomial* attached to a set T of coordinate places of the code C is defined as follows:

$$\mathrm{CJ}_{C,T}(\{x_a, y_a\}_{a\in\mathbb{F}_q}) := \sum_{\mathbf{u}\in C} \prod_{a\in\mathbb{F}_q} x_a^{n_{a,T}(\mathbf{u})} y_a^{n_{a,[n]\setminus T}(\mathbf{u})},$$

where $T \subseteq [n]$, and $n_{a,T}(\mathbf{u})$ is the composition of $\mathbf{u}$ on T and $n_{a,[n]\setminus T}(\mathbf{u})$ is the composition of $\mathbf{u}$ on $[n]\setminus T$.

Remark 4.2. $J_{C,T}(w,z,x,y) = \mathrm{CJ}_{C,T}(x_0 \leftarrow w, \{x_a \leftarrow z\}_{0\neq a\in\mathbb{F}_q}, y_0 \leftarrow x, \{y_a \leftarrow y\}_{0\neq a\in\mathbb{F}_q})$.

Definition 4.3. Let C be an $\mathbb{F}_q$-linear code of length n. Let $X_1, \ldots, X_\ell$ be ℓ mutually disjoint sets such that

$$[n] = X_1 \sqcup \cdots \sqcup X_\ell.$$

Then the *split complete Jacobi polynomial* of C attached to $T_1, \dots, T_\ell$ such that $T_i \subseteq X_i$ for all i is defined by

$$\mathrm{SCJ}_{C,X_1(T_1),\dots,X_\ell(T_\ell)}(\{\{x_{X_i,a}, y_{X_i,a}\}_{a\in\mathbb{F}_q}\}_{1\le i\le \ell}) := \sum_{\mathbf{u}\in C}\prod_{i=1}^{\ell}\prod_{a\in\mathbb{F}_q} x_{X_i,a}^{n_{a,T_i}(\mathbf{u})} y_{X_i,a}^{n_{a,X_i\setminus T_i}(\mathbf{u})}.$$

Note that if $\ell = 1$, the above definition is completely equivalent to the complete Jacobi polynomial with one reference vector (Definition 4.1).

Example 4.1. Let us consider the code C_4 from Example 2.1. Then the complete Jacobi polynomial of C_4 attached to a set of coordinate places $T = \{1, 3\}$ is

$$\begin{aligned}\mathrm{CJ}_{C_4,T}(x_0,x_1,x_2,y_0,y_1,y_2) &= x_0^2y_0^2 + x_1^2y_0^1y_1^1 + x_2^2y_0^1y_2^1 + x_0^1x_1^1y_1^1y_2^1 \\ &+ x_1^1x_2^1y_0^1y_1^1 + x_0^1x_2^1y_1^2 + x_0^1x_2^1y_1^1y_2^1 + x_0^1x_1^1y_2^2 + x_1^1x_2^1y_0^1y_2^1\end{aligned}$$

We consider the same X_1 and X_2 from Example 2.1. Let $T_1 = \{1\}$ and $T_2 = \{3\}$. Then the split complete Jacobi polynomials of C_4 attached to $T_1 \subseteq X_1$ and $T_2 \subseteq X_2$ is as follows:

$$\begin{aligned}&\mathrm{SCJ}_{C_4,X_1(T_1),X_2(T_2)}(\{\{x_{X_i,a}, y_{X_i,a}\}_{a\in\mathbb{F}_q}\}_{i=1,2})\\ &= x_{X_1,0}^1y_{X_1,0}^1x_{X_2,0}^1y_{X_2,0}^1 + x_{X_1,1}^1y_{X_1,0}^1x_{X_2,1}^1y_{X_2,1}^1 + x_{X_1,2}^1y_{X_1,0}^1x_{X_2,2}^1y_{X_2,2}^1\\ &+ x_{X_1,0}^1y_{X_1,1}^1x_{X_2,1}^1y_{X_2,2}^2 + x_{X_1,1}^1y_{X_1,1}^1x_{X_2,2}^1y_{X_2,0}^1 + x_{X_1,2}^1y_{X_1,1}^1x_{X_2,0}^1y_{X_2,1}^1\\ &+ x_{X_1,0}^1y_{X_1,2}^1x_{X_2,2}^1y_{X_2,1}^1 + x_{X_1,1}^1y_{X_1,2}^1x_{X_2,0}^1y_{X_2,2}^1 + x_{X_1,2}^1y_{X_1,2}^1x_{X_2,1}^1y_{X_2,0}^1\end{aligned}$$

The complete Jacobi polynomial of an $\mathbb{F}_q$-linear code attached to multiple sets satisfies the following MacWilliams type identity.

Theorem 4.1 (MacWilliams Identity)**.** *Let C be an $\mathbb{F}_q$-linear code of length n. Again let χ be a non-trivial character of $\mathbb{F}_q$. Then*

$$\begin{aligned}&\mathrm{SCJ}_{C^\perp,X_1(T_1),\dots,X_\ell(T_\ell)}(\{\{x_{X_i,a}, y_{X_i,a}\}_{a\in\mathbb{F}_q}\}_{1\le i\le\ell})\\ &\quad = \frac{1}{|C|}\mathrm{SCJ}_{C,X_1(T_1),\dots,X_\ell(T_\ell)}\left(\left\{\left\{\sum_{b\in\mathbb{F}_q}\chi(ab)x_{X_i,b}, \sum_{b\in\mathbb{F}_q}\chi(ab)y_{X_i,b}\right\}_{a\in\mathbb{F}_q}\right\}_{1\le i\le\ell}\right).\end{aligned}$$

Proof. By Lemma 2.1, we can write

$$
\begin{aligned}
&\mathrm{SCJ}_{C^\perp,X_1(T_1),\ldots,X_\ell(T_\ell)}(\{\{x_{X_i,a},y_{X_i,a}\}_{a\in\mathbb{F}_q}\}_{1\le i\le\ell})\\
&\quad=\sum_{\mathbf{u}\in C^\perp}\prod_{i=1}^{\ell}\prod_{a\in\mathbb{F}_q}x_{X_i,a}^{n_{a,T_i}(\mathbf{u})}y_{X_i,a}^{n_{a,X_i\setminus T_i}(\mathbf{u})}\\
&\quad=\sum_{\mathbf{v}\in\mathbb{F}_q^n}\delta_{C^\perp}(\mathbf{v})\prod_{i=1}^{\ell}\prod_{a\in\mathbb{F}_q}x_{X_i,a}^{n_{a,T_i}(\mathbf{v})}y_{X_i,a}^{n_{a,X_i\setminus T_i}(\mathbf{v})}\\
&\quad=\frac{1}{|C|}\sum_{\substack{\mathbf{u}\in C\\ \mathbf{v}\in\mathbb{F}_q^n}}\chi(\mathbf{u}\cdot\mathbf{v})\prod_{i=1}^{\ell}\prod_{a\in\mathbb{F}_q}x_{X_i,a}^{n_{a,T_i}(\mathbf{v})}y_{X_i,a}^{n_{a,X_i\setminus T_i}(\mathbf{v})}\\
&\quad=\frac{1}{|C|}\sum_{\substack{\mathbf{u}\in C\\ \mathbf{v}\in\mathbb{F}_q^n}}\chi(u_1v_1+\cdots+u_nv_n)\prod_{i=1}^{\ell}\left(\prod_{j\in T_i}x_{X_i,v_j}\right)\left(\prod_{j\in X_i\setminus T_i}y_{X_i,v_j}\right)\\
&\quad=\frac{1}{|C|}\sum_{\mathbf{u}\in C}\prod_{i=1}^{\ell}\left(\prod_{j\in T_i}\sum_{v_j\in\mathbb{F}_q}\chi(u_jv_j)x_{X_i,v_j}\right)\left(\prod_{j\in X_i\setminus T_i}\sum_{v_j\in\mathbb{F}_q}\chi(u_jv_j)y_{X_i,v_j}\right)\\
&\quad=\frac{1}{|C|}\sum_{\mathbf{u}\in C}\prod_{i=1}^{\ell}\prod_{a\in\mathbb{F}_q}\left(\sum_{b\in\mathbb{F}_q}\chi(ab)x_{X_i,b}\right)^{n_{a,T_i}(\mathbf{u})}\left(\sum_{b\in\mathbb{F}_q}\chi(ab)y_{X_i,b}\right)^{n_{a,X_i\setminus T_i}(\mathbf{u})}\\
&\quad=\frac{1}{|C|}\mathrm{SCJ}_{C,X_1(T_1),\ldots,X_\ell(T_\ell)}\left(\left\{\left\{\sum_{b\in\mathbb{F}_q}\chi(ab)x_{X_i,b},\sum_{b\in\mathbb{F}_q}\chi(ab)y_{X_i,b}\right\}_{a\in\mathbb{F}_q}\right\}_{1\le i\le\ell}\right).
\end{aligned}
$$

Hence the proof is completed. □

The following result reflects the basic motivation to introduce the concept of split complete Jacobi polynomials attached to multiple sets. We omit the proof of the theorem since it follows from the definitions.

Theorem 4.2. *Let C be a linear code of length n over $\mathbb{F}_q$. Let $\mathbf{v} := (v_1,\ldots,v_\ell)$ such that $\sum_i^\ell v_i = n$. Let $\mathbf{X} := (X_1,\ldots,X_\ell)$ of pairwise disjoint set $X_i \subseteq [n]$ with $|X_i| = v_i$ for all i. Then the set of codewords of C for every given split composition forms a generalized q-colored t-design with $\mathbf{t} = (t_1,\ldots,t_\ell)$ such that $\sum_{i=1}^\ell t_i = t$ if and only if the split complete Jacobi polynomial $\mathrm{SCJ}_{C,X_1(T_1),\ldots,X_\ell(T_\ell)}$ with $T_i \in \binom{X_i}{t_i}$ for all i is independent of the choices of the sets $T_1,\ldots,T_\ell$.*

Let C be a linear code of length n over $\mathbb{F}_q$. Then the code $C-i$ (resp. C/i) obtained from C by *puncturing* (resp. *shortening*) at coordinate place i. We denote by $C+i_a$ for all $a \in \mathbb{F}_q$ the subcodes of C where the i-th entry of each codeword takes the value a punctured at i.

Let ℓ, n be the positive integers. Let $\mathbf{v} := (v_1, \ldots, v_\ell)$ such that $\sum_{i=1}^{\ell} v_i = n$. Let $P(\{\{x_{X_i,a}, y_{X_i,a}\}_{a\in\mathbb{F}_q}\}_{1\le i\le \ell})$ be a polynomial of degree n in $2q\ell$ variables such that in its each term the sum of the powers of $x_{X_i,a}$ and $y_{X_i,a}$ for all $a \in \mathbb{F}_q$ is v_i for all i. Again let $P'_{k,a}$ denote the partial derivative with respect to $y_{X_k,a}$ for any integer $1 \le k \le \ell$ and $a \in \mathbb{F}_q$. Define the polarization operator $A_{2m\ell,k}$ for any integer $1 \le k \le \ell$ as follows:

$$A_{2q\ell,k} \cdot P := \frac{1}{v_k} \sum_{a\in\mathbb{F}_q} x_{X_k,a} P'_{k,a}.$$

For $\ell = 1$, it convenient to denote the polarization operator as A_{2q} instead of $A_{2q\ell,k}$. The detail of this particular case is as follows:

Let $P(\{x_a, y_a\}_{a\in\mathbb{F}_q})$ be a polynomial of degree n in $2q$ variables. Let P'_{y_a} be the partial derivative with respect to y_a for $a \in \mathbb{F}_q$. Then

$$A_{2q} \cdot P := \frac{1}{n} \sum_{a\in\mathbb{F}_q} x_a P'_{y_a}.$$

Now we have the following generalization of the $\mathbb{F}_q$-analogue of [7, Theorem 1].

Theorem 4.3. *Let C be a linear code of length n over $\mathbb{F}_q$. Let $\mathbf{v} := (v_1, \ldots, v_\ell)$ such that $\sum_{i=1}^{\ell} v_i = n$. We also let $X_1, \ldots, X_\ell$ be the mutually disjoint subsets of $[n]$ such that $X_1 \sqcup \cdots \sqcup X_\ell = [n]$ and $|X_i| = v_i$ for all i. Then for every coordinate place $i \in X_k$ for $1 \le k \le \ell$, we have*

$$\begin{aligned}
&\mathrm{SCJ}_{C,X_1(\emptyset),\ldots,X_{k-1}(\emptyset),X_k(\{i\}),X_{k+1}(\emptyset),\ldots,X_\ell(\emptyset)} \\
&\qquad = x_{X_k,0}\mathbf{scwe}_{C/i,X_1,\ldots,X_{k-1},X_k,X_{k+1},\ldots,X_\ell} \\
&\qquad\qquad + \sum_{a\in\mathbb{F}_q, a\neq 0} x_{X_k,a}\mathbf{scwe}_{C+i_a,X_1,\ldots,X_{k-1},X_k,X_{k+1},\ldots,X_\ell}.
\end{aligned}$$

If C contains no nonzero codewords of Hamming weight less than 1, we have

$$\begin{aligned} v_i(A_{2q\ell,k} \cdot \mathbf{scwe}_{C,X_1,\dots,X_{k-1},X_k,X_{k+1},\dots,X_\ell}) &= x_{X_k,0} \sum_{i \in X_k} \mathbf{scwe}_{C/i,X_1,\dots,X_{k-1},X_k,X_{k+1},\dots,X_\ell} \\ &\quad + \sum_{a \in \mathbb{F}_q, a \neq 0} x_{X_k,a} \sum_{i \in X_k} \mathbf{scwe}_{C+i_a,X_1,\dots,X_{k-1},X_k,X_{k+1},\dots,X_\ell}. \end{aligned}$$

If C is a generalized colorwise homogeneous code, then

$$\mathrm{SCJ}_{C,X_1(\emptyset),\dots,X_{k-1}(\emptyset),X_k(\{i\}),X_{k+1}(\emptyset),\dots,X_\ell(\emptyset)} = A_{2q\ell,k} \cdot \mathbf{scwe}_{C,X_1,\dots,X_{k-1},X_k,X_{k+1},\dots,X_\ell}.$$

Proof. The proof follows the similar arguments given in [7, Theorem 1]. So we omit the details. □

The following theorem is an analogue of the above theorem for $t > 1$. We leave the proof for the readers.

Theorem 4.4. *Let C be a linear code of length n over $\mathbb{F}_q$. Let $\mathbf{v} := (v_1, \dots, v_\ell)$ such that $\sum_{i=1}^{\ell} v_i = n$. We also let $X_1, \dots, X_\ell$ be the mutually disjoint subsets of $[n]$ such that $X_1 \sqcup \cdots \sqcup X_\ell = [n]$ and $|X_k| = v_k$ for all k. If C is a generalized colorwise t-homogeneous and contains no codeword of Hamming weight less than t, then for $\mathbf{t} := (t_1, \dots, t_\ell)$ such that $\sum_{i=1}^{\ell} t_i = t$, we have*

$$\mathrm{SCJ}_{X_1(T_1),\dots,X_\ell(T_\ell)} = A_{2q\ell,\ell}^{t_\ell} \cdots A_{2q\ell,1}^{t_1} \cdot \mathbf{scwe}_{C,X_1,\dots,X_\ell},$$

for each $(T_1, \dots, T_\ell) \in \binom{X_1}{t_1} \times \cdots \times \binom{X_\ell}{t_\ell}$.

5. Application to colored design theory

Bonnecaze et al. [8] gave colored t-design structures using Type III codes. In this section, using their idea, we construct some (generalized) colored designs such as colored packing (resp. covering) designs using Type III and Type IV codes.

5.1. **Invariant theory.** Let G be a finite $n \times n$ matrix group that acts on a polynomial ring $\mathbb{C}[x_0, \dots, x_{n-1}]$; for $g \in G$ and $f(x_0, \dots, x_{n-1}) \in \mathbb{C}[x_0, \dots, x_{n-1}]$,

$$gf(x_0, \dots, x_{n-1}) = f(g(x_0, \dots, x_{n-1})^t).$$

Then

$$\widehat{G} = \left\{ \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \middle| \, g \in G \right\}$$

acts on a polynomial ring $\mathbb{C}[x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}]$ in a natural way. Let $M^{\widehat{G}}_{i,j} = \mathbb{C}[x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}]^{\widehat{G}}_{i,j}$ be the invariants of degree (i, j):

$$
\begin{aligned}
&\mathbb{C}[x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}]^{\widehat{G}}_{i,j} \\
&= \{f \in \mathbb{C}[x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}] \mid (g, h)f = f, \text{degree of } f \text{ in } \{x_k\} \text{ is } i, \\
&\qquad \text{degree of } f \text{ in } \{y_k\} \text{ is } j\}.
\end{aligned}
$$

In [37], Stanley defined the bivariate Molien series

$$f(u, v) = \sum_{u,v} \dim(M^{\widehat{G}}_{i,j}) u^i v^j,$$

and showed that $f(u, v)$ is written as follows:

$$f(u, v) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - ug) \det(1 - vg)}.$$

We denote the homogeneous part of degree d of $f(u, v)$ by $f[d]$. To obtain an invariant, the Reynolds operator is useful. For $f \in \mathbb{C}[x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}]$, the Reynolds operator of f and $\widehat{G}$ is defined as follows:

$$R(f, \widehat{G}) := \sum_{(g,g) \in \widehat{G}} (g, g) \cdot f.$$

Then it is easy to show that $R(f, \widehat{G})$ is an invariant of $\widehat{G}$.

5.2. **Type III codes.** The MacWilliams transform and some congruence conditions yield that the complete weight enumerator of a Type III code remains invariant under the action of group G_{III} of order 2592 which is generated by the following four matrices:

$$
\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & 1 \end{bmatrix},
$$
$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2\pi i/3} \end{bmatrix}, \quad \begin{bmatrix} e^{\pi i/6} & 0 & 0 \\ 0 & e^{\pi i/6} & 0 \\ 0 & 0 & e^{\pi i/6} \end{bmatrix}.
$$

It is easy to show that a split complete weight enumerator of a Type III code is an invariant of $\widehat{G_{\mathrm{III}}}$.

Now we assume that G is a 2×2 matrix group and $\widehat{G}$ acts on $\mathbb{C}[x_0, x_1, y_0, y_1]$. Let $P \in \mathbb{C}[x_0, x_1, y_0, y_1]$ be a polynomial of total degree n. Now define a polarization operator A_4 as follows:

$$A_4 \cdot P := \frac{x_0 P'_{y_0} + x_1 P'_{y_1}}{n}.$$

Definition 5.1. A linear code of length n over $\mathbb{F}_3$ is said to be t-homogeneous if the codewords of every given Hamming weight hold a t-design.

Definition 5.2. An $\mathbb{F}_3$-linear code of length n is said to be *colorwise t-homogeneous* if the codewords of every given composition hold a 3-colored t-design.

Lemma 5.1 ([8])**.** *Let C be a linear code of length n over $\mathbb{F}_3$. If C is t-homogeneous with no non-zero words of Hamming weight less than t, then for all T of size t we get*

$$J_{C,T} = A_4^t \cdot W_C.$$

Let P be a polynomial of total degree n in 6 variables $x_0, x_1, x_2, y_0, y_1, y_2$. Now define a polarization operator A_6 and specialization operator S as follows:

$$A_6 \cdot P := \frac{x_0 P'_{y_0} + x_1 P'_{y_1} + x_2 P'_{y_2}}{n},$$
$$S_6 \cdot P(x_0, x_1, x_2, y_0, y_1, y_2) := P(x_0, x_1, x_1, y_0, y_1, y_1).$$

Lemma 5.2 ([8])**.** *Let C be a linear code of length n over $\mathbb{F}_3$. If C is colorwise t-homogeneous with no non-zero words of Hamming weight less than t, then for all T of size t we get*

$$\mathrm{CJ}_{C,T} = A_6^t \cdot \mathbf{cwe}_C.$$

5.2.1. *Length* 12.

Example 5.1 (length 12)**.** Let C_{12}^{III} be the first ternary self-dual code of length 12 in [18].

$$f[12] = 2u^{12} + 2u^{11}v + 3u^{10}v^2 + 4u^3v^9 + \cdots .$$

In this case, it holds the following lemmas.

Lemma 5.3 ([8])**.** *A basis of $M_{3,9}^{\widehat{G_{\mathrm{III}}}}$ is obtained by applying $R(f, \widehat{G_{\mathrm{III}}})$ with f running over the monomials*

$$x_0^3 y_0^9,\ x_0^3 y_0^3 y_1^6,\ x_0^3 y_0^3 y_1^3 y_2^2,\ x_0^2 x_1 y_0^4 y_1^5.$$

Lemma 5.4 ([8])**.** *For $\ell = 1, 2, 3$ we have*

$$\dim(S_6 \cdot M_{\ell,12-\ell}^{\widehat{G_{\mathrm{III}}}}) = \dim(M_{\ell,12-\ell}^{\widehat{G_{\mathrm{III}}}}).$$

Combining the preceding lemmas and the bivariate Molien series, we obtain the following Theorem.

Theorem 5.5 ([8]). *The codewords of fixed composition in the ternary Golay hold* 3*-colored* 3*-designs.*

Since the codewords of fixed composition in C_{12}^{III} holds 3-colored 3-design, we assume that $|T| = 1, 2, 3$. Then

$$\begin{aligned}
\mathrm{CJ}_{C_{12}^{\mathrm{III}},1} &= x_0(y_0^{11} + 11y_0^5y_1^6 + 110y_0^5y_1^3y_2^3 + 11y_0^5y_2^6 + 55y_0^2y_1^6y_2^3 + 55y_0^2y_1^3y_2^6)\\
&\quad + x_1(y_1^{11} + 11y_0^6y_1^5 + 55y_0^6y_1^2y_2^3 + 110y_0^3y_1^5y_2^3 + 55y_0^3y_1^2y_2^6 + 11y_1^5y_2^6)\\
&\quad + x_2(y_2^{11} + 55y_0^6y_1^3y_2^2 + 11y_0^6y_2^5 + 55y_0^3y_1^6y_2^2 + 110y_0^3y_1^3y_2^5 + 11y_1^6y_2^5),\\
\mathrm{CJ}_{C_{12}^{\mathrm{III}},2} &= x_0^2(y_0^{10} + 5y_0^4y_1^6 + 50y_0^4y_1^3y_2^3 + 5y_0^4y_2^6 + 10y_0y_1^6y_2^3 + 10y_0y_1^3y_2^6)\\
&\quad + x_0x_1(12y_0^5y_1^5 + 60y_0^5y_1^2y_2^3 + 60y_0^2y_1^5y_2^3 + 30y_0^2y_1^2y_2^6)\\
&\quad + x_0x_2(60y_0^5y_1^3y_2^2 + 12y_0^5y_2^5 + 30y_0^2y_1^6y_2^2 + 60y_0^2y_1^3y_2^5)\\
&\quad + x_1^2(5y_0^6y_1^4 + 10y_0^6y_1y_2^3 + 50y_0^3y_1^4y_2^3 + 10y_0^3y_1y_2^6 + 5y_1^4y_2^6 + y_1^{10})\\
&\quad + x_1x_2(30y_0^6y_1^2y_2^2 + 60y_0^3y_1^5y_2^2 + 60y_0^3y_1^2y_2^5 + 12y_1^5y_2^5) + x_2^2(10y_0^6y_1^3y_2\\
&\quad + 5y_0^6y_2^4 + 10y_0^3y_1^6y_2 + 50y_0^3y_1^3y_2^4 + 5y_1^6y_2^4 + y_2^{10}),\\
\mathrm{CJ}_{C_{12}^{\mathrm{III}},3} &= x_0^3(y_0^9 + 2y_0^3y_1^6 + 20y_0^3y_1^3y_2^3 + 2y_0^3y_2^6 + y_1^6y_2^3 + y_1^3y_2^6) + x_0^2x_1(9y_0^4y_1^5\\
&\quad + 45y_0^4y_1^2y_2^3 + 18y_0y_1^5y_2^3 + 9y_0y_1^2y_2^6) + x_0^2x_2(45y_0^4y_1^3y_0^2 + 9y_0^4y_2^5\\
&\quad + 9y_0y_1^6y_2^2 + 18y_0y_1^3y_2^5) + x_0x_1^2(9y_0^5y_1^4 + 18y_0^5y_1y_2^3 + 45y_0^2y_1^4y_2^3\\
&\quad + 9y_0^2y_1y_2^6) + x_0x_2^2(18y_0^5y_1^3y_2 + 9y_0^5y_2^4 + 9y_0^2y_1^6y_2 + 45y_0^2y_1^3y_2^4)\\
&\quad + x_0x_1x_2(54y_0^5y_1^2y_2^2 + 54y_0^2y_1^5y_2^2 + 54y_0^2y_1^2y_2^5) + x_1^3(2y_0^6y_1^3 + y_1^9\\
&\quad + y_0^6y_2^3 + 20y_0^3y_1^3y_2^3 + y_0^3y_2^6 + 2y_1^3y_2^6) + x_1^2x_2(9y_0^6y_1y_2^2 + 45y_0^3y_1^4y_2^2\\
&\quad + 18y_0^3y_1y_2^5 + 9y_1^4y_2^5) + x_1x_2^2(9y_0^5y_1y_2^3 + 18y_0^3y_1^5y_2 + 45y_0^3y_1^2y_2^4\\
&\quad + 9y_1^5y_2^4) + x_2^3(y_0^6y_1^3 + 2y_0^6y_2^3 + y_0^3y_1^6 + y_2^9 + 20y_0^3y_1^3y_2^3 + 2y_1^6y_2^3).
\end{aligned}$$

Corollary 5.6 ([8]). *There exist simple* 3*-colored* 3*-designs with the following parameters: Three designs with parameters* 3-$(12, (n(0), n(1), n(2)), 220)$ *where* $(n(0), n(1), n(2))$ *is equal to* $(6, 3, 3)$ *or any one of its three permutations. Three designs with parameters* 3-$(12, (n(0), n(1), n(2)), 22)$ *where* $(n(0), n(1), n(2))$ *is equal to* $(6, 6, 0)$ *or any one of its three permutations.*

The space of Jacobi polynomials $\mathrm{CJ}_{C_{12}^{\mathrm{III}},T}$ with $|T|=4$ may be generated by the two polynomials

$$
\begin{aligned}
C^1_{C_{12}^{\mathrm{III}},4} = {}& x_0^4(y_0^8+y_0^2y_1^6+6y_0^2y_1^3y_2^3+y_0^2y_2^6)+x_0^3x_1(4y_0^3y_1^5+28y_0^3y_1^2y_2^3\\
&+4y_1^5y_2^3)+x_0^3x_2(28y_0^3y_1^3y_2^2+4y_0^3y_2^5+4y_1^3y_2^5)+x_0^2x_1^2(12y_0^4y_1^4\\
&+18y_0^4y_1y_2^3+18y_0y_1^4y_2^3+6y_0y_1y_2^6)+x_0^2x_1x_2(60y_0^4y_1^2y_2^2+24y_0y_1^5y_2^2\\
&+24y_0y_1^2y_2^5)+x_0^2x_2^2(18y_0^4y_1^3y_2+12y_0^4y_2^4+6y_0y_1^6y_2+18y_0y_1^3y_2^4)\\
&+x_0x_1^3(4y_0^5y_1^3+4y_0^5y_2^3+28y_0^2y_1^3y_2^3)+x_0x_1^2x_2(24y_0^5y_1y_2^2+60y_0^2y_1^4y_2^2\\
&+24y_0^2y_1y_2^5)+x_0x_1x_2^2(24y_0^5y_1^2y_2+24y_0^2y_1^5y_2+60y_0^2y_1^2y_2^4)\\
&+x_0x_2^3(4y_0^5y_1^3+4y_0^5y_2^3+28y_0^2y_1^3y_2^3)+x_1^4(y_0^6y_1^2+6y_0^3y_1^2y_2^3+y_1^8\\
&+y_1^2y_2^6)+x_1^3x_2(28y_0^3y_1^3y_2^2+4y_0^3y_2^5+4y_1^3y_2^5)+x_1^2x_2^2(6y_0^6y_1y_2\\
&+18y_0^3y_1^4y_2+18y_0^3y_1y_2^4+12y_1^4y_2^4)+x_1x_2^3(4y_0^3y_1^5+28y_0^3y_1^2y_2^3\\
&+4y_1^5y_2^3)+x_2^4(y_0^6y_2^2+6y_0^3y_1^3y_2^2+y_1^6y_2^2+y_2^8),\\
C^2_{C_{12}^{\mathrm{III}},4} = {}& x_0^4(y_0^8+8y_0^2y_1^3y_2^3)+x_0^3x_1(8y_0^3y_1^5+24y_0^3y_1^2y_2^3+4y_1^2y_2^6)\\
&+x_0^3x_2(24y_0^3y_1^3y_2^2+8y_0^3y_2^5+4y_1^6y_2^2)+x_0^2x_1^2(6y_0^4y_1^4+24y_0^4y_1y_2^3\\
&+24y_0y_1^4y_2^3)+x_0^2x_1x_2(60y_0^4y_1^2y_2^2+24y_0y_1^5y_2^2+24y_0y_1^2y_2^5)\\
&+x_0^2x_2^2(6y_0^4y_2^4+24y_0^4y_1^3y_2+24y_0y_1^3y_2^4)+x_0x_1^3(8y_0^5y_1^3+24y_0^5y_2^3\\
&+4y_0^2y_2^6)+x_0x_1^2x_2(24y_0^5y_1y_2^2+60y_0^2y_1^4y_2^2+24y_0^2y_1y_2^5)\\
&+x_0x_1x_2^2(24y_0^5y_1^2y_2+24y_0^2y_1^5y_2+60y_0^2y_1^2y_2^4)\\
&+x_0x_2^3(8y_0^5y_2^3+4y_0^2y_1^6+24y_0^2y_1^3y_2^3)+x_1^4(8y_0^3y_1^2y_2^3+y_1^8)\\
&+x_1^3x_2(4y_0^6y_2^2+24y_0^3y_1^3y_2^2+8y_1^3y_2^5)+x_1^2x_2^2(24y_0^3y_1^4y_2+24y_0^3y_1y_2^4\\
&+6y_1^4y_2^4)+x_1x_2^3(4y_0^6y_1^2+24y_0^3y_1^2y_2^3+8y_1^5y_2^3)+x_2^4(8y_0^3y_1^3y_2^2+y_2^8).
\end{aligned}
$$

Combining these two equations we obtain 4-designs with parameters

$$4\text{-}(12,(6,6,0),(\lambda_1^1(P),\lambda_2^1(P))) \text{ and } 4\text{-}(12,(6,3,3),(\lambda_1^2(P),\lambda_2^2(P)),$$

where $\lambda(P)$'s are shown in Table 5.1. By the coefficient of the term $y_0^{n(0)}y_1^{n(1)}y_2^{n(2)}$ in the complete weight enumerator of the code, we obtain an upper (resp. lower) bound of $D_{\lambda_{\max}(P)}(12,(n(0),n(1),n(2)),4)$ (resp. $C_{\lambda_{\max}(P)}(12,(n(0),n(1),n(2)),4)$).

$$
\begin{aligned}
&D_{\lambda^1_{\max}(P)}(12,(6,6,0),4) \le 22 \le C_{\lambda^1_{\min}(P)}(12,(6,6,0),4),\\
&D_{\lambda^2_{\max}(P)}(12,(6,3,3),4) \le 220 \le C_{\lambda^2_{\min}(P)}(12,(6,3,3),4).
\end{aligned}
$$

The $\lambda_{\max}(P)$'s (resp. $\lambda_{\min}(P)$'s) are shown in Table 5.1.

TABLE 5.1. λ's in 3-colored 4-designs

P	0000	0001	0002	0011	0012	0022	0111	0112
$\lambda_1^1(P)$	1	4	0	12	0	0	4	0
$\lambda_2^1(P)$	0	8	0	6	0	0	8	0
$\lambda_{\max}^1(P)$	1	8	0	12	0	0	8	0
$\lambda_{\min}^1(P)$	0	4	0	6	0	0	4	0
$\lambda_1^2(P)$	6	28	28	18	60	18	4	24
$\lambda_2^2(P)$	8	24	24	24	60	0	24	24
$\lambda_{\max}^2(P)$	8	28	28	24	60	18	24	24
$\lambda_{\min}^2(P)$	6	24	24	18	60	0	4	24
P	0122	0222	1111	1112	1122	1222	2222	-
$\lambda_1^1(P)$	0	0	1	0	0	0	0	-
$\lambda_2^1(P)$	0	0	0	0	0	0	0	-
$\lambda_{\max}^1(P)$	0	0	1	0	0	0	0	-
$\lambda_{\min}^1(P)$	0	0	0	0	0	0	0	-
$\lambda_1^2(P)$	24	4	0	0	6	0	0	-
$\lambda_2^2(P)$	24	0	0	4	0	4	0	-
$\lambda_{\max}^2(P)$	24	4	0	4	6	4	0	-
$\lambda_{\min}^2(P)$	24	0	0	0	0	0	0	-

Similarly, we can obtain an upper (resp. lower) bound of D (resp. C) 3-colored 4-designs where $(n(0), n(1), n(2))$ is equal to anyone of the four choices: $(6, 0, 6)$, $(0, 6, 6)$, $(3, 3, 6)$, $(3, 6, 3)$.

5.3. **Type IV codes.** The MacWilliams transform and some congruence conditions yield that the complete weight enumerator of a Type IV code remains invariant under the action of group G_{IV} of order 576 which is generated by the following four matrices:

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

It is easy to show that a split complete weight enumerator of a Type IV code is an invariant of $\widehat{G_{\mathrm{IV}}}$.

Let the elements of $\mathbb{F}_4$ be $0, 1, s, s^2$. If P is a polynomial of total degree n in 8 variables $x_0, x_1, x_s, x_{s^2}, y_0, y_1, y_s, y_{s^2}$. Now define a polarization operator A_8 and specialization operator $\widetilde{S}$ as follows:

$$A_8 \cdot P := \frac{x_0 P'_{y_0} + x_1 P'_{y_1} + x_s P'_{y_s} + x_{s^2} P'_{y_{s^2}}}{n},$$

$$S_8 \cdot P(x_0, x_1, x_s, x_{s^2}, y_0, y_1, y_s, y_{s^2}) := P(x_0, x_1, x_1, x_1, y_0, y_1, y_1, y_1).$$

Definition 5.3. An $\mathbb{F}_4$-linear code of length n is said to be *colorwise t-homogeneous* if the codewords of every given composition hold a 4-colored t-design.

Now we have the following $\mathbb{F}_4$-code analogues of Lemma 5.1 and Lemma 5.2.

Lemma 5.7. *Let C be a linear code of length n over $\mathbb{F}_4$. If C is t-homogeneous with no non-zero words of Hamming weight less than t, then for all T of size t we get*

$$J_{C,T} = A_4^t \cdot W_C.$$

Lemma 5.8. *Let C be a linear code of length n over $\mathbb{F}_4$. If C is colorwise t-homogeneous with no non-zero words of Hamming weight less than t, then for all T of size t we get*

$$\mathrm{CJ}_{C,T} = A_8^t \cdot \mathbf{cwe}_C.$$

5.3.1. *Length* 4.

Example 5.2. Let C_4^{IV} be the Hermitian self-dual code over $\mathbb{F}_4$ of length 4 in [18].

$$f[4] = u^4 + u^3 v + 2u^2 + uv^3 + v^2.$$

Since the codewords of fixed composition in C_4^{IV} holds 4-colored 1-design, we assume that $|T| = 1$. Then,

$$\begin{aligned}\mathrm{CJ}_{C_4^{\mathrm{IV}},1} &= x_0(y_0^3 + y_0 y_1^2 + y_0 y_s^2 + y_0 y_{s^2}^2) + x_1(y_0^2 y_1 + y_1^3 + y_1 y_s^2 + y_1 y_{s^2}^2) \\ &\quad + x_s(y_0^2 y_s + y_1^2 y_s + y_s^3 + y_s y_{s^2}^2) + x_{s^2}(y_0^2 y_{s^2} + y_1^2 y_{s^2} + y_s^2 y_{s^2} + y_{s^2}^3).\end{aligned}$$

There exist simple 4-colored 1-designs with the following parameters: Six designs with parameters 1-$(4, (n(0), n(1), n(s), n(s^2)), 2)$ where $(n(0), n(1), n(s), n(s^2))$ is equal to $(2, 2, 0, 0)$ or any one of its four permutations. The space of Jacobi polynomials $\mathrm{CJ}_{C_4^{\mathrm{IV}},T}$ with $|T| = 2$ may be generated by the two polynomials

$$\mathrm{CJ}^1_{C_4^{\mathrm{IV}},2} = (x_0^2 + x_1^2 + x_s^2 + x_{s^2}^2)(y_0^2 + y_1^2 + y_s^2 + y_{s^2}^2),$$

$$\begin{aligned}\mathrm{CJ}^2_{C_4^{\mathrm{IV}},2} &= x_0^2 y_0^2 + 2x_0 x_1 y_0 y_1 + 2x_0 x_s y_0 y_s + 2x_0 x_{s^2} y_0 y_{s^2} + x_1^2 y_1^2 + 2x_1 x_s y_1 y_s \\ &\quad + 2x_1 x_{s^2} y_1 y_{s^2} + x_s^2 y_s^2 + 2x_s x_{s^2} y_s y_{s^2} + x_{s^2}^2 y_{s^2}^2.\end{aligned}$$

Combining these two equations we obtain 2-designs with parameters

$$2\text{-}(4,(2,2,0,0),(\lambda_1(P),(\lambda_2(P))),$$

where $\lambda(P)$'s are shown in Table 5.2. By the coefficient of the term $y_0^2y_1^2$ in the complete weight enumerator of the code, we obtain an upper (resp. lower) bound of $D_{\lambda_{\max}(P)}(4,(2,2,0,0),2)$ (resp. $C_{\lambda_{\max}(P)}(4,(2,2,0,0),2)$).

$$D_{\lambda_{\max}(P)}(4,(2,2,0,0),2) \leq 2 \leq C_{\lambda_{\min}(P)}(4,(2,2,0,0),2).$$

The $\lambda_{\max}(P)$'s (resp. $\lambda_{\min}(P)$'s) are shown in Table 5.2. Similarly, we can obtain an upper (resp. lower) bound of D (resp. C) 4-colored 2-designs where $(n(0),n(1),n(s),n(s^2))$ is equal to anyone of the four choices: $(2,0,2,0)$, $(2,0,0,2)$, $(0,2,2,0)$, $(0,0,2,2)$.

TABLE 5.2. λ's in 4-colored 2-designs

P	00	01	$0s$	$0s^2$	11	$1s$	$1s^2$	ss	ss^2	s^2s^2
$\lambda_1(P)$	1	0	0	0	1	0	0	0	0	0
$\lambda_2(P)$	0	2	0	0	0	0	0	0	0	0
$\lambda_{\max}(P)$	1	2	0	0	1	0	0	0	0	0
$\lambda_{\min}(P)$	0	0	0	0	0	0	0	0	0	0

5.3.2. *Length* 6.

Example 5.3. Let C_6^{IV} be the second Hermitian self-dual code over $\mathbb{F}_4$ of length 6 in [18].

$$f[6] = 2u^6 + 2u^5v + 3u^4v^2 + 4u^3v^3 + 3u^2v^4 + 2uv^5 + 2v^6.$$

In this case, it holds the following Lemmas and Theorem.

Lemma 5.9. *A basis of* $M_{2,4}^{\widehat{G_{\mathrm{IV}}}}$ *is obtained by applying* $R(f,\widehat{G_{\mathrm{IV}}})$ *with* f *running over the monomials*

$$x_0^2y_0^4,\ x_0^2y_1^2y_s^2,\ x_0x_1y_0y_1y_s^2.$$

Lemma 5.10. *A basis of* $M_{3,3}^{\widehat{G_{\mathrm{IV}}}}$ *is obtained by applying* $R(f,\widehat{G_{\mathrm{IV}}})$ *with* f *running over the monomials*

$$x_0^3y_0^3,\ x_0^2x_1y_1y_s^2,\ x_0x_1^2y_0y_s^2,\ x_0x_1x_sy_0y_1y_s.$$

We need to observe that that specialization is one-to-one on those spaces.

Lemma 5.11. *For* $\ell = 1,2$ *we have*

$$\dim(S_8 \cdot M_{\ell,6-\ell}^{\widehat{G_{\mathrm{IV}}}}) = \dim(M_{\ell,6-\ell}^{\widehat{G_{\mathrm{IV}}}}).$$

Proof. We obtain it by taking the image by S_8 of the preceding bases. ☐

Theorem 5.12. *The codewords of fixed composition in the Hermitian Type* IV *code of length* 6 *hold* 4*-colored* 2*-designs.*

Proof. We need to show that $\mathrm{CJ}_{C,T}$ does not depend on T for $|T| = 2$. Since it lives in $M_{2,4}^{\widehat{G_{\mathrm{IV}}}}$, we can expand it with indeterminate coefficients on the basis given in Lemma 5.9. The Hermitian code is 3-homogeneous. By specialization and Lemma 5.7 and Lemma 5.11, we determine these completely by solving a 3×3 linear system. ☐

Note that we can expand $\mathrm{CJ}_{C,T}$ for $|T| = 3$ with indeterminate coefficients on the basis given in Lemma 5.10 since it lives in $M_{3,3}^{\widehat{G_{\mathrm{IV}}}}$. But we can't determine these completely by solving a 3×3 linear system by specialization and Lemma 5.7 and Lemma 5.11.

Since the codewords of fixed composition in C_6^{IV} holds 4-colored 2-design, we assume that $|T| = 1, 2$. Then

$$\begin{aligned}
\mathrm{CJ}_{C_6^{\mathrm{IV}},1} &= x_0(y_0^5 + 5y_0y_1^2y_s^2 + 5y_0y_1^2y_{s^2}^2 + 5y_0y_s^2y_{s^2}^2) + x_1(5y_0^2y_1y_s^2 \\
&\quad + 5y_0^2y_1y_{s^2}^2 + y_1^5 + 5y_1y_s^2y_{s^2}^2) + x_s(5y_0^2y_1^2y_s + 5y_0^2y_sy_{s^2}^2 \\
&\quad + 5y_1^2y_sy_{s^2}^2 + y_s^5) + x_{s^2}(5y_0^2y_1^2y_{s^2} + 5y_0^2y_s^2y_{s^2} + 5y_1^2y_s^2y_{s^2} + y_{s^2}^5), \\
\mathrm{CJ}_{C_6^{\mathrm{IV}},2} &= x_0^2(y_0^4 + y_1^2y_s^2 + y_1^2y_{s^2}^2 + y_s^2y_{s^2}^2) + x_0x_1(4y_0y_1y_s^2 + 4y_0y_1y_{s^2}^2) \\
&\quad + x_0x_s(4y_0y_1^2y_s + 4y_0y_sy_{s^2}^2) + x_0x_{s^2}(4y_0y_1y_{s^2}^2 + 4y_0y_s^2y_{s^2}) \\
&\quad + x_1^2(y_0^2y_s^2 + y_0^2y_{s^2}^2 + y_1^4 + y_s^2y_{s^2}^2) + x_1x_s(4y_0^2y_1y_s + 4y_1y_sy_{s^2}^2) \\
&\quad + x_1x_{s^2}(4y_0^2y_1y_{s^2} + y_1y_s^2y_{s^2}) + x_s^2(y_0^2y_1^2 + y_0^2y_{s^2}^2 + y_1^2y_{s^2}^2 + y_s^4) \\
&\quad + x_sx_{s^2}(4y_0^2y_sy_{s^2} + 4y_1^2y_sy_{s^2}) + x_{s^2}^2(y_0^2y_1^2 + y_0^2y_s^2 + y_1^2y_s^2 + y_{s^2}^4).
\end{aligned}$$

Corollary 5.13. *Let* $\mathbb{F}_4 = \{0, 1, s, s^2\}$*. There exist simple* 4*-colored* 2*-designs with the following parameters: Four designs with parameters* 2-$(6, (n(0), n(1), n(s), n(s^2)), 15)$ *where* $(n(0), n(1), n(s), n(s^2))$ *is equal to* $(2, 2, 2, 0)$ *or any one of its four permutations.*

The space of Jacobi polynomials $\mathrm{CJ}_{C_6^{\mathrm{IV}},T}$ with $|T| = 3$ may be generated by the two polynomials

$$\begin{aligned}
\mathrm{CJ}^1_{C_6^{\mathrm{IV}},3} = {} & x_0^3y_0^3 + 3x_0^2x_1y_1y_s^2 + 3x_0^2x_sy_sy_{s^2}^2 + 3x_0^2x_{s^2}y_1^2y_{s^2} + 3x_0x_1^2y_0y_{s^2}^2 \\
& + 6x_0x_1x_sy_0y_1y_s + 6x_0x_1x_{s^2}y_0y_1y_{s^2} + 3x_0x_s^2y_0y_1^2 + 6x_0x_sx_{s^2}y_0y_sy_{s^2} \\
& + 3x_0x_{s^2}^2y_0y_s^2 + x_1^3y_1^3 + 3x_1^2x_sy_0^2y_s + 3x_1^2x_{s^2}y_s^2y_{s^2} + 3x_1x_s^2y_1y_{s^2}^2 \\
& + 6x_1x_sx_{s^2}y_1y_sy_{s^2} + 3x_1x_{s^2}^2y_0^2y_1 + x_s^3y_s^3 + 3x_s^2x_{s^2}y_0^2y_{s^2} \\
& + 3x_sx_{s^2}^2y_1^2y_s + x_{s^2}^3y_{s^2}^3, \\
\mathrm{CJ}^2_{C_6^{\mathrm{IV}},3} = {} & x_0^3y_0^3 + 3x_0^2x_1y_1y_{s^2}^2 + 3x_0^2x_sy_1^2y_s + 3x_0^2x_{s^2}y_s^2y_{s^2} + 3x_0x_1^2y_0y_s^2 \\
& + 6x_0x_1x_sy_0y_1y_s + 6x_0x_1x_{s^2}y_0y_1y_{s^2} + 3x_0x_s^2y_0y_{s^2}^2 + 6x_0x_sx_{s^2}y_0y_sy_{s^2} \\
& + 3x_0x_{s^2}^2y_0y_1^2 + x_1^3y_1^3 + 3x_1^2x_sy_sy_{s^2}^2 + 3x_1^2x_{s^2}y_0^2y_{s^2} + 3x_1x_s^2y_0^2y_1 \\
& + 6x_1x_sx_{s^2}y_1y_sy_{s^2} + 3x_1x_{s^2}^2y_1y_s^2 + x_s^3y_s^3 + 3x_s^2x_{s^2}y_1^2y_{s^2} + 3x_sx_{s^2}^2y_0^2y_s \\
& + x_{s^2}^3y_{s^2}^3.
\end{aligned}$$

Combining these two equations we obtain 3-designs with parameters

$$3\text{-}(6, (2, 2, 2, 0), (\lambda_1(P), (\lambda_2(P))),$$

where $\lambda(P)$'s are shown in Table 5.3. By the coefficient of the term $y_0^2y_1^2y_s^2$ in the complete weight enumerator of the code, we obtain an upper (resp. lower) bound of $D_{\lambda_{\max}(P)}(6, (2, 2, 2, 0), 3)$ (resp. $C_{\lambda_{\max}(P)}(6, (2, 2, 2, 0), 3)$).

$$D_{\lambda_{\max}(P)}(6, (2, 2, 2, 0), 3) \leq 15 \leq C_{\lambda_{\min}(P)}(6, (2, 2, 2, 0), 3).$$

The $\lambda_{\max}(P)$'s (resp. $\lambda_{\min}(P)$'s) are shown in Table 5.3. Similarly, we can obtain an upper (resp. lower) bound of D (resp. C) 4-colored 3-designs where $(n(0), n(1), n(s), n(s^2))$ is equal to anyone of the three choices: $(2, 2, 0, 2)$, $(2, 0, 2, 2)$, $(0, 2, 2, 2)$.

Table 5.3. λ's in 4-colored 3-designs

P	000	001	$00s$	$00s^2$	011	$01s$	$01s^2$	$0ss$	$0ss^2$	$0s^2s^2$
$\lambda_1(P)$	0	0	3	0	3	6	0	0	0	0
$\lambda_2(P)$	0	3	0	0	0	6	0	3	0	0
$\lambda_{\max}(P)$	0	3	3	0	3	6	0	3	0	0
$\lambda_{\min}(P)$	0	0	0	0	0	6	0	0	0	0
P	111	$11s$	$11s^2$	$1ss$	$1ss^2$	$1s^2s^2$	sss	sss^2	ss^2s^2	$s^2s^2s^2$
$\lambda_1(P)$	0	0	0	3	0	0	0	0	0	0
$\lambda_2(P)$	0	3	0	0	0	0	0	0	0	0
$\lambda_{\max}(P)$	0	3	0	3	0	0	0	0	0	0
$\lambda_{\min}(P)$	0	0	0	0	0	0	0	0	0	0

5.3.3. *Length* 8.

Example 5.4. Let C_8^{IV} be the third Hermitian self-dual code over $\mathbb{F}_4$ of length 8 in [18].

$$f[8] = 3u^8 + 5u^7v + 7u^6v^2 + 8u^5v^3 + 10u^4v^4 + \cdots .$$

Since the codewords of fixed composition in C_8^{IV} holds 4-colored 3-design, we assume that $|T| = 1, 2, 3$. Then,

$$\begin{aligned}
\mathrm{CJ}_{C_8^{\mathrm{IV}},1} &= x_0(y_0^7 + 7y_0^3y_1^4 + 7y_0^3y_s^4 + 7y_0^3y_{s^2}^4 + 42y_0y_1^2y_s^2y_{s^2}^2)\\
&\quad + x_1(7y_0^4y_1^3 + 42y_0^2y_1y_s^2y_{s^2}^2 + y_1^7 + 7y_1^3y_s^4 + 7y_1^3y_{s^2}^4)\\
&\quad + x_s(7y_0^4y_s^3 + 42y_0^2y_1^2y_sy_{s^2}^2 + 7y_1^4y_s^3 + y_s^7 + 7y_s^3y_{s^2}^4)\\
&\quad + x_{s^2}(7y_0^4y_{s^2}^3 + 42y_0^2y_1^2y_s^2y_{s^2} + 7y_1^4y_{s^2}^3 + 7y_s^4y_{s^2}^3 + y_{s^2}^7),\\
\mathrm{CJ}_{C_8^{\mathrm{IV}},2} &= x_0^2(y_0^6 + 3y_0^2y_1^4 + 3y_0^2y_s^4 + 3y_0^2y_{s^2}^4 + 6y_1^2y_s^2y_{s^2}^2)\\
&\quad + x_0x_1(8y_0^3y_1^3 + 24y_0y_1y_s^2y_{s^2}^2) + x_0x_s(8y_0^3y_s^3 + 24y_0y_1^2y_sy_{s^2}^2)\\
&\quad + x_0x_{s^2}(8y_0^3y_{s^2}^3 + 24y_0y_1^2y_s^2y_{s^2}) + x_1^2(3y_0^4y_1^2 + 6y_0^2y_s^2y_{s^2}^2\\
&\quad + y_1^6 + 3y_1^2y_s^4 + 3y_1^2y_{s^2}^4) + x_1x_s(24y_0^2y_1y_sy_{s^2}^2 + 8y_1^3y_s^3)\\
&\quad + x_1x_{s^2}(24y_0^2y_1y_s^2y_{s^2} + 8y_1^3y_{s^2}^3) + x_s^2(3y_0^4y_s^2 + 6y_0^2y_1^2y_{s^2}^2\\
&\quad + 3y_1^4y_s^2 + 3y_s^2y_{s^2}^4 + y_s^6) + x_sx_{s^2}(24y_0^2y_1^2y_sy_{s^2} + 8y_s^3y_{s^2}^3)\\
&\quad + x_{s^2}^2(3y_0^4y_{s^2}^2 + 6y_0^2y_1^2y_s^2 + 3y_1^4y_{s^2}^2 + 3y_s^4y_{s^2}^2 + y_{s^2}^6),\\
\mathrm{CJ}_{C_8^{\mathrm{IV}},3} &= x_0^3(y_0^5 + y_0y_s^4 + y_0y_s^4 + y_0y_{s^2}^4) + x_0^2x_1(6y_0^2y_1^3 + 6y_1y_s^2y_{s^2}^2)\\
&\quad + x_0^2x_s(6y_0^2y_s^3 + 6y_1^2y_sy_{s^2}^2) + x_0^2x_{s^2}(6y_0^2y_{s^2}^3 + 6y_1^2y_s^2y_{s^2})\\
&\quad + x_0x_1^2(6y_0^3y_1^2 + 6y_0y_s^2y_{s^2}^2) + 24x_0x_1x_sy_0y_1y_sy_{s^2}^2\\
&\quad + 24x_0x_1x_{s^2}y_0y_1y_s^2y_{s^2} + x_0x_s^2(6y_0^3y_s^2 + 6y_0y_1^2y_{s^2}^2)\\
&\quad + 24x_0x_sx_{s^2}y_0y_1^2y_sy_{s^2} + x_0x_{s^2}^2(6y_0^3y_{s^2}^2 + 6y_0y_1^2y_s^2)\\
&\quad + x_1^3(y_0^4y_1 + y_1^5 + y_1y_s^4 + y_1y_{s^2}^4) + x_1^2x_s(6y_0^2y_sy_{s^2}^2 + 6y_1^2y_s^3)\\
&\quad + x_1^2x_{s^2}(6y_0^2y_s^2y_{s^2} + 6y_1^2y_{s^2}^3) + x_1x_s^2(6y_0^2y_1y_{s^2}^2 + 6y_1^3y_s^2)\\
&\quad + x_1x_{s^2}^2(6y_0^2y_1y_s^2 + 6y_1^3y_{s^2}^2) + 24x_1x_sx_{s^2}y_0^2y_1y_sy_{s^2}\\
&\quad + x_s^3(y_0^4y_s + y_1^4y_s + y_s^5 + y_sy_{s^2}^4) + x_s^2x_{s^2}(6y_0^2y_1^2y_{s^2} + 6y_s^2y_{s^2}^3)\\
&\quad + x_sx_{s^2}^2(6y_0^2y_1^2y_s + 6y_s^3y_{s^2}^2) + x_{s^2}^3(y_0^4y_{s^2} + y_1^4y_{s^2} + y_s^4y_{s^2} + y_{s^2}^5).
\end{aligned}$$

There exist simple 4-colored 3-designs with the following parameters: Six designs with parameters 3-$(8, (n(0), n(1), n(s), n(s^2)), 14)$ where $(n(0), n(1), n(s), n(s^2))$ is equal to $(4, 4, 0, 0)$ or any one of its four permutations. The space of Jacobi polynomials $\mathrm{CJ}_{C_8^{\mathrm{IV}},T}$ with $|T| = 4$ may be generated by the two polynomials

$$
\begin{aligned}
\mathrm{CJ}^1_{C_8^{\mathrm{IV}},4} &= x_0^4(y_0^4+y_1^4+y_s^4+y_{s^2}^4)+x_0^2x_1^2(12y_0^2y_1^2+12y_s^2y_{s^2}^2)\\
&+x_0^2x_s^2(12y_0^2y_s^2+12y_1^2y_{s^2}^2)+x_0^2x_{s^2}^2(12y_0^2y_{s^2}^2+12y_1^2y_s^2)\\
&+96x_0x_1x_sx_{s^2}y_0y_1y_sy_{s^2}+x_1^4(y_0^4+y_1^4+y_s^4+y_{s^2}^4)+x_1^2x_s^2(12y_0^2y_{s^2}^2\\
&+12y_1^2y_s^2)+x_1^2x_{s^2}^2(12y_0^2y_s^2+12y_1^2y_{s^2}^2)+x_s^4(y_0^4+y_1^4+y_s^4+y_{s^2}^4)\\
&+x_s^2x_{s^2}^2(12y_0^2y_1^2+12y_s^2y_{s^2}^2)+x_{s^2}^4(y_0^4+y_1^4+y_s^4+y_{s^2}^4),\\
\mathrm{CJ}^2_{C_8^{\mathrm{IV}},4} &= x_0^4y_0^4+4x_0^3x_1y_0y_1^3+4x_0^3x_sy_0y_s^3+4x_0^3x_{s^2}y_0y_{s^2}^3+6x_0^2x_1^2y_0^2y_1^2\\
&+12x_0^2x_1x_sy_1y_sy_{s^2}^2+12x_0^2x_1x_{s^2}y_1y_s^2y_{s^2}+6x_0^2x_s^2y_0^2y_s^2\\
&+12x_0^2x_sx_{s^2}y_1^2y_sy_{s^2}+6x_0^2x_{s^2}^2y_0^2y_{s^2}^2+4x_0x_1^3y_0^3y_1+12x_0x_1^2x_sy_0y_sy_{s^2}^2\\
&+12x_0x_1^2x_{s^2}y_0y_s^2y_{s^2}+12x_0x_1x_s^2y_0y_1y_{s^2}^2+24x_0x_1x_sx_{s^2}y_0y_1y_sy_{s^2}\\
&+12x_0x_1x_{s^2}^2y_0y_1y_s^2+4x_0x_s^3y_0^3y_s+12x_0x_s^2x_{s^2}y_0y_1^2y_{s^2}\\
&+12x_0x_sx_{s^2}^2y_0y_1^2y_s+4x_0x_{s^2}^3y_0^3y_{s^2}+x_1^4y_1^4+4x_1^3x_sy_1y_s^3\\
&+4x_1^3x_{s^2}y_1y_{s^2}^3+6x_1^2x_s^2y_1^2y_s^2+12x_1^2x_sx_{s^2}y_0^2y_sy_{s^2}+6x_1^2x_{s^2}^2y_1^2y_{s^2}^2\\
&+4x_1x_s^3y_1^3y_s+12x_1x_s^2x_{s^2}y_0^2y_1y_{s^2}+12x_1x_sx_{s^2}^2y_0^2y_1y_s+4x_1x_{s^2}^3y_1^3y_{s^2}\\
&+x_s^4y_s^4+4x_s^3x_{s^2}y_sy_{s^2}^3+6x_s^2x_{s^2}^2y_s^2y_{s^2}^2+4x_sx_{s^2}^3y_s^3y_{s^2}+x_{s^2}^4y_{s^2}^4.
\end{aligned}
$$

Combining these two equations we obtain 4-designs with parameters

$$4\text{-}(8,(4,4,0,0),(\lambda_1^1(P),\lambda_2^1(P))) \text{ and } 4\text{-}(8,(2,2,2,2),(\lambda_1^2(P),\lambda_2^2(P)),$$

where $\lambda(P)$'s are shown in Table 5.4. By the coefficient of the term $y_0^{n(0)}y_1^{n(1)}y_s^{n(s)}y_{s^2}^{n(s^2)}$ in the complete weight enumerator of the code, we obtain an upper (resp. lower) bound of $D_{\lambda_{\max}(P)}(8,(n(0),n(1),n(s),n(s^2)),4)$ (resp. $C_{\lambda_{\max}(P)}(8,(n(0),n(1),n(s),n(s^2)),4)$).

$$
\begin{aligned}
&D_{\lambda^1_{\max}(P)}(8,(4,4,0,0),4)\le 14\le C_{\lambda^1_{\min}(P)}(8,(4,4,0,0),4),\\
&D_{\lambda^2_{\max}(P)}(8,(2,2,2,2),4)\le 168\le C_{\lambda^2_{\min}(P)}(8,(2,2,2,2),4).
\end{aligned}
$$

The $\lambda_{\max}(P)$'s (resp. $\lambda_{\min}(P)$'s) are shown in Table 5.4. Similarly, we can obtain an upper (resp. lower) bound of D (resp. C) 4-colored 3-designs where $(n(0),n(1),n(s),n(s^2))$ is equal to anyone of the five choices: $(4,0,4,0)$, $(4,0,0,4)$, $(0,4,4,0)$, $(0,4,0,4)$, $(0,0,4,4)$.

In the case the extremal Type III (resp. Type IV) code of length n containing the all-one vector, it holds 3- (resp. 4-) colored t-design as in Table 5.5 (resp. Table 5.6).

6. Concluding remarks

Let D_w be the support design of a code C for weight w and

$$\delta(C) := \max\{t \in \mathbb{N} \mid \forall w, D_w \text{ is a } t\text{-design}\},$$
$$s(C) := \max\{t \in \mathbb{N} \mid \exists w \text{ s.t. } D_w \text{ is a } t\text{-design}\}.$$

We note that $\delta(C) \leq s(C)$. In our previous papers [3, 20, 29, 30, 31, 32, 33], we considered the possible occurrence of $\delta(C) < s(C)$. This was motivated by Lehmer's conjecture, which is an analogue of $\delta(C) < s(C)$ in the theory of lattices and vertex operator algebras. For the details, see [1, 2, 4, 22, 26, 27, 29, 38, 39].

Let CD_w be the support colored design of a code C for weight w and

$$\delta_c(C) := \max\{t \in \mathbb{N} \mid \forall w, CD_w \text{ is a colored } t\text{-design}\},$$
$$s_c(C) := \max\{t \in \mathbb{N} \mid \exists w \text{ s.t. } CD_w \text{ is a colored } t\text{-design}\}.$$

It is natural to give upper and lower bounds of $\delta_c(C)$ and $s_c(C)$ for all extremal Type II, III, and IV codes.

We will continue the study of this paper in [11] to the case of $\mathbb{Z}_k$-codes as a generalization of the works done by Bonnecaze et al. [6]. Moreover, we investigate the colored designs to the case of Kleinian codes in [12].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors thank Tsuyoshi Miezaki and Manabu Oura for their helpful discussions and comments to this research.

Data availability statement

The data that support the findings of this study are available from the corresponding author.

References

[1] E. Bannai, and T. Miezaki, Toy models for D. H. Lehmer's conjecture. *J. Math. Soc. Japan* **62** (2010), no. 3, 687–705.

[2] E. Bannai, and T. Miezaki, Toy models for D. H. Lehmer's conjecture II. *Quadratic and higher degree forms*, 1–27, Dev. Math., **31**, Springer, New York, 2013.

[3] E. Bannai, T. Miezaki, and H. Nakasora, A note on the Assmus–Mattson theorem for some binary codes II, submitted.

[4] E. Bannai, T. Miezaki, and V.A. Yudin, An elementary approach to toy models for Lehmer's conjecture. (Russian) *Izv. Ross. Akad. Nauk Ser. Mat.* **75** (2011), no. 6, 3–16; *translation in Izv. Math.* **75** (2011), no. 6, 1093–1106.

[5] E.R. Berlekamp, F.J. MacWilliams, and N.J.A. Sloane, Gleason's theorem on self-dual codes, *IEEE Trans. Inform. Theory* **IT-18** (1972), 409–414.

[6] A. Bonnecaze, B. Mourrain, and P. Solé, Jacobi polynomials, type II codes, and designs, *Des. Codes Cryptogr.* **16** (1999), no. 3, 215–234.

[7] A. Bonnecaze, E. Rains, and P. Solé, 3-colored 5-designs and $\mathbb{Z}_4$-codes, *J. Stat. Plan. Inference* **86** (2000), 349–368.

[8] A. Bonnecaze, P. Solé, and P. Udaya, Tricolore 3-designs in Type III codes, *Discrete Math.* **241** (2001), 129–138.

[9] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, *J. Symb. Comp.* **24** (1997), 235–265.

[10] P.J. Cameron, A generalisation of t-designs, *Discrete Math.*, **309**(2009), 4835–4842.

[11] H.S. Chakraborty, R. Ishikawa, T. Miezaki, and Y. Tanaka, Jacobi polynomials and design theory III, in preparation.

[12] H.S. Chakraborty, R. Ishikawa, T. Miezaki, and Y. Tanaka, Jacobi polynomials and design theory IV, in preparation.

[13] H.S. Chakraborty, and T. Miezaki, Variants of Jacobi polynomials in coding theory, *Des. Codes Cryptogr.* **90** (2022), 2583–2597.

[14] H.S. Chakraborty, T. Miezaki, and M. Oura, Weight enumerators, intersection enumerators and Jacobi polynomials II, *Discrete Math.* **345**(12) (2022), Paper No. 113098.

[15] H.S. Chakraborty, T. Miezaki, M. Oura, and Y. Tanaka, Jacobi polynomials and design theory I, *Discrete Math.* **346**(6) (2023), Paper No. 113339.

[16] M. Eichler, and D. Zagier, *The Theory of Jacobi Forms*, Progress in Mathematics, vol. 55, Birkhauser Boston, Inc., Boston, MA, 1985.

[17] A.M. Gleason, Weight polynomials of self-dual codes and the MacWilliams identities, in: Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 3, Gauthier-Villars, Paris, 1971, pp. 211–215.

[18] M. Harada, and A. Munemasa, Database of self-dual codes, https://www.math.is.tohoku.ac.jp/~munemasa/selfdualcodes.htm.

[19] K. Honma, T. Okabe, and M. Oura, Weight enumerators, intersection enumerators, and Jacobi polynomials, *Discrete Math.* **343** (2020), no. 6, 111815.

[20] N. Horiguchi, T. Miezaki, and H. Nakasora, On the support designs of extremal binary doubly even self-dual codes, *Des. Codes Cryptogr.*, **72** (2014), 529–537.

[21] W. C. Huffman, and V. Pless, *Fundamentals of Error-Correcting Codes*, first edition, Cambridge University Press, United Kingdom, 2003.

[22] D.H. Lehmer, The vanishing of Ramanujan's $\tau(n)$, *Duke Math. J.* **14** (1947), 429–433.

[23] F.J. MacWilliams, C.L. Mallows, and N.J.A. Sloane, Generalizations of Gleason's theorem on weight enumerators of self-dual codes, *IEEE Trans. Inform. Theory* **IT-18** (1972) 794–805.

[24] F.J. MacWilliams, and N.J.A. Sloane, *The Theory of Error-Correcting Codes*, first edition, Elsevier/North Holland, New York, 1977.

[25] T. Miezaki, Design-theoretic analogies between codes, lattices, and vertex operator algebras, *Des. Codes Cryptogr.*, **89** (2021), 763–780.
[26] T. Miezaki, Conformal designs and D.H. Lehmer's conjecture, *J. Algebra* **374** (2013), 59–65.
[27] T. Miezaki, Design-theoretic analogies between codes, lattices, and vertex operator algebras., *Des. Codes Cryptogr.* **89** (2021), no. 5, 763–780.
[28] T. Miezaki, A. Munemasa, and H. Nakasora, An note on Assmus–Mattson theorems, *Des. Codes Cryptogr.*, **89** (2021), 843–858.
[29] T. Miezaki, A. Munemasa, and H. Nakasora, A note on Assmus–Mattson type theorems, *Des. Codes Cryptogr.*, **89** (2021), 843–858.
[30] T. Miezaki, and H. Nakasora, An upper bound of the value of t of the support t-designs of extremal binary doubly even self-dual codes, *Des. Codes Cryptogr.*, **79** (2016), 37–46.
[31] T. Miezaki, and H. Nakasora, The support designs of the triply even binary codes of length 48, *J. Combin. Designs*, **27** (2019), 673–681.
[32] T. Miezaki, and H. Nakasora, A note on the Assmus–Mattson theorem for some binary codes, *Des. Codes Cryptogr.*, **90** (2022), no. 6, 1485–1502.
[33] T. Miezaki, and H. Nakasora, On the Assmus–Mattson type theorem for Type I and even formally self-dual codes, submitted.
[34] G. Nebe, E.M. Rains, and N.J.A. Sloane, *Self-Dual Codes and Invariant Theory*, Algorithms and Computation in Mathematics, vol. **17**, Springer-Verlag, Berlin, 2006.
[35] M. Ozeki, On the notion of Jacobi polynomials for codes, *Math. Proc. Cambridge Philos. Soc.* **121** (1) (1997) 15–30.
[36] N.J.A. Sloane, Error-correcting codes and invariant theory: New applications of a nineteenth-century technique, *Amer. Math. Monthly* **84** (1977), 82–107.
[37] R.P. Stanley, Invariants of finite groups and their applications to combinatorics, *Bull. (New Series) Amer. Math. Soc.* **1**(3) (1979), 475–511.
[38] B.B. Venkov, Even unimodular extremal lattices (Russian), *Algebraic geometry and its applications. Trudy Mat. Inst. Steklov.* **165** (1984), 43–48; translation in *Proc. Steklov Inst. Math.* **165** (1985) 47–52.
[39] B.B. Venkov, Réseaux et designs sphériques, (French) [Lattices and spherical designs] *Réseaux euclidiens, designs sphériques et formes modulaires*, 10–86, Monogr. Enseign. Math., 37, Enseignement Math., Geneva, 2001.

Department of Mathematics, Shahjalal University of Science and Technology, Sylhet 3114, BANGLADESH
Email address: himadri-mat@sust.edu

Graduate School of Science and Engineering, Waseda University, Tokyo 169-8555, Japan
Email address: reina.i@suou.waseda.jp

Graduate School of Science and Engineering, Waseda University, Tokyo, 169-8555, Japan
Email address: tanaka_yuuho_dc@akane.waseda.jp

TABLE 5.4. λ's in 4-colored 4-designs

P	0000	0001	$000s$	$000s^2$	0011	$001s$	$001s^2$
$\lambda_1^1(P)$	1	0	0	0	12	0	0
$\lambda_2^1(P)$	0	4	0	0	6	0	0
$\lambda_{\max}^1(P)$	1	4	0	0	12	0	0
$\lambda_{\min}^1(P)$	0	0	0	0	6	0	0
$\lambda_1^2(P)$	0	0	0	0	12	0	0
$\lambda_2^2(P)$	0	0	0	0	0	12	12
$\lambda_{\max}^2(P)$	0	0	0	0	12	12	12
$\lambda_{\min}^2(P)$	0	0	0	0	0	0	0
P	$00ss$	$00ss^2$	$00s^2s^2$	0111	$011s$	$011s^2$	$01ss$
$\lambda_1^1(P)$	0	0	0	0	0	0	0
$\lambda_2^1(P)$	0	0	0	4	0	0	0
$\lambda_{\max}^1(P)$	0	0	0	4	0	0	0
$\lambda_{\min}^1(P)$	0	0	0	0	0	0	0
$\lambda_1^2(P)$	12	0	12	0	0	0	0
$\lambda_2^2(P)$	0	12	0	0	12	12	12
$\lambda_{\max}^2(P)$	12	12	12	0	12	12	12
$\lambda_{\min}^2(P)$	0	0	0	0	0	0	0
P	$01ss^2$	$01s^2s^2$	$0sss$	$0sss^2$	$0ss^2s^2$	$0s^2s^2s^2$	1111
$\lambda_1^1(P)$	0	0	0	0	0	0	1
$\lambda_2^1(P)$	0	0	0	0	0	0	0
$\lambda_{\max}^1(P)$	0	0	0	0	0	0	1
$\lambda_{\min}^1(P)$	0	0	0	0	0	0	0
$\lambda_1^2(P)$	96	0	0	0	0	0	0
$\lambda_2^2(P)$	24	12	0	12	12	0	0
$\lambda_{\max}^2(P)$	96	12	0	12	12	0	0
$\lambda_{\min}^2(P)$	24	0	0	0	0	0	0
P	$111s$	$111s^2$	$11ss$	$11ss^2$	$11s^2s^2$	$1sss$	$1sss^2$
$\lambda_1^1(P)$	0	0	0	0	0	0	0
$\lambda_2^1(P)$	0	0	0	0	0	0	0
$\lambda_{\max}^1(P)$	0	0	0	0	0	0	0
$\lambda_{\min}^1(P)$	0	0	0	0	0	0	0
$\lambda_1^2(P)$	0	0	12	0	12	0	0
$\lambda_2^2(P)$	12	0	0	12	0	0	12
$\lambda_{\max}^2(P)$	12	0	12	12	12	0	12
$\lambda_{\min}^2(P)$	0	0	0	0	0	0	0
P	$1ss^2s^2$	$1s^2s^2s^2$	$ssss$	$ssss^2$	sss^2s^2	$ss^2s^2s^2$	$s^2s^2s^2s^2$
$\lambda_1^1(P)$	0	0	0	0	0	0	0
$\lambda_2^1(P)$	0	0	0	0	0	0	0
$\lambda_{\max}^1(P)$	0	0	0	0	0	0	0
$\lambda_{\min}^1(P)$	0	0	0	0	0	0	0
$\lambda_1^2(P)$	0	0	0	0	12	0	0
$\lambda_2^2(P)$	12	0	0	0	0	0	0
$\lambda_{\max}^2(P)$	12	0	0	0	12	0	0
$\lambda_{\min}^2(P)$	0	0	0	0	0	0	0

TABLE 5.5. 3-colored t-$(n,(n_0,n_1,n_2),|\mathcal{B}|)$ design in Type III code

n	t	Blocks in the cwe (n_0,n_1,n_2) up to permutation	Number of blocks
12	1	$(6,3,3)$	220
	1	$(6,6,0)$	22
	2	$(6,3,3)$	220
	2	$(6,6,0)$	22
	3	$(6,3,3)$	220
	3	$(6,6,0)$	22

TABLE 5.6. 4-colored t-$(n,(n_0,n_1,n_s,n_{s^2}),|\mathcal{B}|)$ design in Type IV code

n	t	Blocks in the cwe (n_0,n_1,n_s,n_{s^2}) up to permutation	Number of blocks
4	1	$(2,2,0,0)$	2
6	1	$(2,2,2,0)$	15
8	1	$(4,4,0,0)$	14
8	1	$(2,2,2,2)$	168
6	2	$(2,2,2,0)$	15
8	2	$(4,4,0,0)$	14
8	2	$(2,2,2,2)$	168
8	3	$(4,4,0,0)$	14
8	3	$(2,2,2,2)$	168