# Center of maximum-sum matchings of bichromatic points 

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#### Abstract

Let $R$ and $B$ be two disjoint point sets in the plane with $|R|=|B|=n$. Let $\mathcal{M}=\left\{\left(r_{i}, b_{i}\right), i=\right.$ $1,2, \ldots, n\}$ be a perfect matching that matches points of $R$ with points of $B$ and maximizes $\sum_{i=1}^{n}\left\|r_{i}-b_{i}\right\|$, the total Euclidean distance of the matched pairs. In this paper, we prove that there exists a point $o$ of the plane (the center of $\mathcal{M}$ ) such that $\left\|r_{i}-o\right\|+\left\|b_{i}-o\right\| \leq \sqrt{2}\left\|r_{i}-b_{i}\right\|$ for all $i \in\{1,2, \ldots, n\}$.


## 1 Introduction

Let $R$ and $B$ be two disjoint point sets in the plane with $|R|=|B|=n, n \geq 1$. The points in $R$ are red, and those in $B$ are blue. A matching of $R \cup B$ is a partition of $R \cup B$ into $n$ pairs such that each pair consists of a red and a blue point. A point $p \in R$ and a point $q \in B$ are matched if and only if the (unordered) pair $(p, q)$ is in the matching. For every $p, q \in \mathbb{R}^{2}$, we use $p q$ to denote the segment connecting $p$ and $q$, and $\|p-q\|$ to denote its length, which is the Euclidean norm of the vector $p-q$. Let $\mathcal{B}(p q)$ denote the disk with diameter equal to $\|p-q\|$, that is centered at the midpoint $\frac{p+q}{2}$ of the segment $p q$. For any matching $\mathcal{M}$, we use $\mathcal{B}_{\mathcal{M}}$ to denote the set of the disks associated with the matching, that is, $\mathcal{B}_{\mathcal{M}}=\{\mathcal{B}(p q):(p, q) \in \mathcal{M}\}$.
In this note, we consider the max-sum matching $\mathcal{M}$, as the matching that maximizes the total Euclidean distance of the matched points. As our main result, we prove the following theorem:

Theorem 1.1. There exists a point o of the plane such that for all $i \in\{1,2, \ldots, n\}$ we have:

$$
\left\|r_{i}-o\right\|+\left\|b_{i}-o\right\| \leq \sqrt{2}\left\|r_{i}-b_{i}\right\| .
$$

Fingerhut (see Eppstein [3]), motivated by a problem in designing communication networks (see Fingerhut et al. [4]), conjectured that given a set $P$ of $2 n$ uncolored points in the plane and a max-sum matching $\left\{\left(a_{i}, b_{i}\right), i=1, \ldots, n\right\}$ of $P$, there exists a point $o$ of the plane, not necessarily a point of $P$, such that

$$
\begin{equation*}
\left\|a_{i}-o\right\|+\left\|b_{i}-o\right\| \leq \frac{2}{\sqrt{3}}\left\|a_{i}-b_{i}\right\| \text { for all } i \in\{1, \ldots, n\}, \text { where } 2 / \sqrt{3} \approx 1.1547 \tag{1}
\end{equation*}
$$

[^0]Bereg et al. [2] obtained an approximation to this conjecture. They proved that for any point set $P$ of $2 n$ uncolored points in the plane and a max-sum matching $\mathcal{M}=\left\{\left(a_{i}, b_{i}\right), i=1, \ldots, n\right\}$ of $P$, all disks in $\mathcal{B}_{\mathcal{M}}$ have a common intersection, implying that any point $o$ in the common intersection satisfies

$$
\left\|a_{i}-o\right\|+\left\|b_{i}-o\right\| \leq \sqrt{2}\left\|a_{i}-b_{i}\right\|, \text { where } \sqrt{2} \approx 1.4142
$$

Recently, Barabanshchikova and Polyanskii [1] confirmed the conjecture of Fingerhut.
The statement of Equation (1) is equivalent to stating that the intersection $\mathcal{E}\left(a_{1} b_{1}\right) \cap \mathcal{E}\left(a_{2} b_{2}\right) \cap$ $\cdots \cap \mathcal{E}\left(a_{n} b_{n}\right)$ is not empty, where $\mathcal{E}(p q)$ is the region of the plane bounded by the ellipse with foci $p$ and $q$, and major axis length $(2 / \sqrt{3})\|p-q\|$ (see [3]).

In our context of bichromatic point sets, given $p \in R$ and $q \in B$, let $\mathcal{E}(p q)$ denote the region bounded by the ellipse with foci $p$ and $q$, and major axis length $\sqrt{2}\|p-q\|$. That is, $\mathcal{E}(p q)=\{x \in$ $\left.\mathbb{R}^{2}:\|p-x\|+\|q-x\| \leq \sqrt{2}\|p-q\|\right\}$. Then, the statement of Theorem 1.1 is equivalent to stating that the intersection $\mathcal{E}\left(r_{1} b_{1}\right) \cap \mathcal{E}\left(r_{2} b_{2}\right) \cap \cdots \cap \mathcal{E}\left(r_{n} b_{n}\right)$ is not empty, for any max-sum matching $\left\{\left(r_{i}, b_{i}\right), i=1,2, \ldots, n\right\}$ of $R \cup B$.
We note that the factor $\sqrt{2}$ is tight. It suffices to consider two red points and two blue points as vertices of a square, so that each diagonal has vertices of the same color. The center of the square is the only point in common of the two ellipses induced by any max-sum matching.
Hence, to prove Theorem 1.1 it suffices to consider $n \leq 3$, by Helly's Theorem. Let $X_{1}, X_{2}, \ldots, X_{n}$ be a collection of $n$ convex subsets of $\mathbb{R}^{d}$, with $n \geq d+1$. Helly's Theorem [5] asserts that if the intersection of every $d+1$ of these subsets is nonempty, then the whole collection has a nonempty intersection. That is why we prove our claim only for $n \leq 3$, since we are considering $n$ ellipses in $\mathbb{R}^{2}$. The arguments that we give in this paper are a simplification and adaptation of the arguments of Barabanshchikova and Polyanskii [1].

Huemer et al. [6] proved that if $\mathcal{M}^{\prime}$ is any perfect matching of $R$ and $B$ that maximizes the total squared Euclidean distance of the matched points, i.e., it maximizes $\sum_{(p, q) \in \mathcal{M}^{\prime}}\|p-q\|^{2}$, then all disks of $\mathcal{B}_{\mathcal{M}^{\prime}}$ have a point in common. As proved by Bereg et al. [2], the disks of our max-sum matching $\mathcal{M}$ of $R \cup B$ intersect pairwise, fact that will be used in this paper, but the common intersection is not always possible.

## 2 Proof of main result

Let $R$ and $B$ be two disjoint point sets defined as above, where $|R|=|B|=n, n \leq 3$, and let $\mathcal{M}$ be a max-sum matching of $R \cup B$. Note that for every pair $(p, q) \in \mathcal{M}$ the disk $\mathcal{B}(p q)$ is inscribed in the ellipse $\mathcal{E}(p q)$ (see Figure 1a), which implies $\mathcal{B}(p q) \subset \mathcal{E}(p q)$. Then, for $n=2$ Theorem 1.1 is true because the disks of $\mathcal{M}$ intersect pairwise [2, Proposition 2.1]. Trivially, the theorem is also true for $n=1$. Therefore, we will prove in the rest of the paper that the theorem is also true for $n=3$, which will require elaborated arguments.

Let $n=3$, with $R=\{a, b, c\}$ and $B=\left\{a^{\prime}, b^{\prime}, c^{\prime}\right\}$, and let $\mathcal{M}=\left\{\left(a, a^{\prime}\right),\left(b, b^{\prime}\right),\left(c, c^{\prime}\right)\right\}$ be a max-sum matching of $R \cup B$.
For two points $p, q \in \mathbb{R}^{2}$, let $r(p q)$ denote the ray with apex $p$ that goes through $q$, and for a real number $\lambda \geq 1$, let $\mathcal{E}_{\lambda}(p q)$ be the region bounded by the ellipse with foci $p$ and $q$ and major axis length $\lambda\|p-q\|$. That is, $\mathcal{E}_{\lambda}(p q)=\left\{x \in \mathbb{R}^{2}:\|p-x\|+\|q-x\| \leq \lambda\|p-q\|\right\}$. Note that in our
context $\mathcal{E}(p q)=\mathcal{E}_{\sqrt{2}}(p q)$, and $\mathcal{E}_{\lambda}(p q) \subset \mathcal{E}_{\lambda^{\prime}}(p q)$ for any $\lambda^{\prime}>\lambda$.
Assume by contradiction that $\mathcal{E}\left(a a^{\prime}\right) \cap \mathcal{E}\left(b b^{\prime}\right) \cap \mathcal{E}\left(c c^{\prime}\right)=\emptyset$. Then, we can "inflate uniformly" $\mathcal{E}\left(a a^{\prime}\right)$, $\mathcal{E}\left(b b^{\prime}\right)$, and $\mathcal{E}\left(c c^{\prime}\right)$ until they have a common intersection. Formally, we can take the minimum $\lambda>\sqrt{2}$ such that $\mathcal{E}_{\lambda}\left(a a^{\prime}\right) \cap \mathcal{E}_{\lambda}\left(b b^{\prime}\right) \cap \mathcal{E}_{\lambda}\left(c c^{\prime}\right)$ is not empty, case in which $\mathcal{E}_{\lambda}\left(a a^{\prime}\right) \cap \mathcal{E}_{\lambda}\left(b b^{\prime}\right) \cap \mathcal{E}_{\lambda}\left(c c^{\prime}\right)$ is singleton. Let $o$ denote the point of $\mathcal{E}_{\lambda}\left(a a^{\prime}\right) \cap \mathcal{E}_{\lambda}\left(b b^{\prime}\right) \cap \mathcal{E}_{\lambda}\left(c c^{\prime}\right)$.
Let $\ell\left(a a^{\prime}\right)$ denote the ray with apex $o$ that bisects $r(o a)$ and $r\left(o a^{\prime}\right)$. Similarly, we define $\ell\left(b b^{\prime}\right)$ and $\ell\left(c c^{\prime}\right)$. Let $t\left(a a^{\prime}\right)$ denote the line through $o$ tangent to $\mathcal{E}_{\lambda}\left(a a^{\prime}\right)$, oriented so that $\mathcal{E}_{\lambda}\left(a a^{\prime}\right)$ is to its right. Similarly, we define $t\left(b b^{\prime}\right)$ and $t\left(c c^{\prime}\right)$. It is well known that given an ellipse with foci $p$ and $q$, and a line tangent at it at some point $o$, the rays $r(o p)$ and $r(o q)$ form equal angles with the tangent line (see Figure 1b). This implies that rays $\ell\left(a a^{\prime}\right), \ell\left(b b^{\prime}\right)$, and $\ell\left(c c^{\prime}\right)$ are perpendicular to the tangent lines $t\left(a a^{\prime}\right), t\left(b b^{\prime}\right)$, and $t\left(c c^{\prime}\right)$, respectively. In other words, they are contained respectively in the normal lines at point $o$.
Since $\mathcal{E}\left(a a^{\prime}\right), \mathcal{E}\left(b b^{\prime}\right)$, and $\mathcal{E}\left(c c^{\prime}\right)$ intersect pairwise (and also none of them is contained inside other one), we have that $o$ belongs to the boundary of each of $\mathcal{E}_{\lambda}\left(a a^{\prime}\right), \mathcal{E}_{\lambda}\left(b b^{\prime}\right)$, and $\mathcal{E}_{\lambda}\left(c c^{\prime}\right)$. Then, $\mathcal{E}_{\lambda}\left(a a^{\prime}\right)$, $\mathcal{E}_{\lambda}\left(b b^{\prime}\right)$, and $\mathcal{E}_{\lambda}\left(c c^{\prime}\right)$ intersect pairwise, and each pairwise intersection contains interior points. This implies that no two lines of $t\left(a a^{\prime}\right), t\left(b b^{\prime}\right)$, and $t\left(c c^{\prime}\right)$ coincide. Furthermore, the six directions (positive and negative) of $t\left(a a^{\prime}\right), t\left(b b^{\prime}\right)$, and $t\left(c c^{\prime}\right)$ alternate around $o$, which implies that any two consecutive rays among $\ell\left(a a^{\prime}\right), \ell\left(b b^{\prime}\right)$, and $\ell\left(c c^{\prime}\right)$ counterclockwise around $o$, have rotation angle strictly less than $\pi$ (see Figure 1c).
Let $G=(R \cup B, E)$ be the bipartite graph such that $(p, q) \in E$ if and only if $p \in R, q \in B$, and either $(p, q) \in\left\{\left(a, a^{\prime}\right),\left(b, b^{\prime}\right),\left(c, c^{\prime}\right)\right\}$ or $o \in \mathcal{B}(p q)$. We color the edges into two colors: We say that edge $(p, q)$ is black if $(p, q) \in\left\{\left(a, a^{\prime}\right),\left(b, b^{\prime}\right),\left(c, c^{\prime}\right)\right\}$. Otherwise, we say that $(p, q)$ is white. Note that this color classification is consistent, since we have that $o \notin \mathcal{B}(p q)$ for all $(p, q) \in\left\{\left(a, a^{\prime}\right),\left(b, b^{\prime}\right),\left(c, c^{\prime}\right)\right\}$ because $\mathcal{B}(p q)$ is contained in the interior of $\mathcal{E}_{\lambda}(p q)$ and $o$ is in the boundary of $\mathcal{E}_{\lambda}(p q)$.
The proof of the next lemma is included for completeness.
Lemma 2.1 ([1]). If $G$ has a cycle whose edges are color alternating, then $\mathcal{M}$ is not a max-sum matching of $R \cup B$.

Proof. For a black edge $(p, q)$ we have that $\|p-o\|+\|q-o\|=\lambda\|p-q\|$. For a white edge $(p, q)$ we have that $\|p-o\|+\|q-o\|<\lambda\|p-q\|$, since $o \in \mathcal{B}(p q)$ and $\mathcal{B}(p q)$ is contained in the interior of $\mathcal{E}_{\lambda}(p q)$. Let $\left(r_{1}, b_{1}, r_{2}, b_{2}, \ldots, r_{m}, b_{m}, r_{m+1}=r_{1}\right)$ be a color alternating cycle of length $m$, where $r_{1}, \ldots, r_{m} \in R$ and $b_{1}, \ldots, b_{m} \in B$. Suppose w.l.o.g. that the edge $\left(r_{1}, b_{1}\right)$ is black, which means that the edges $\left(r_{1}, b_{1}\right), \ldots,\left(r_{m}, b_{m}\right) \in \mathcal{M}$ are all black, and the edges $\left(b_{1}, r_{2}\right), \ldots,\left(b_{m}, r_{m+1}\right) \in \mathcal{M}$ are all white. Then, we have that:

$$
\sum_{i=1}^{m}\left\|r_{i}-b_{i}\right\|=\frac{1}{\lambda} \sum_{i=1}^{m}\left(\left\|r_{i}-o\right\|+\left\|b_{i}-o\right\|\right)=\frac{1}{\lambda} \sum_{i=1}^{m}\left(\left\|b_{i}-o\right\|+\left\|r_{i+1}-o\right\|\right)<\sum_{i=1}^{m}\left\|b_{i}-r_{i+1}\right\| .
$$

Hence, by replacing in $\mathcal{M}$ the black edges of the cycle by the white edges, we will obtain a matching of larger total sum.

Lemma 2.2. Each vertex of $G$ has at least one white edge incident to it.
Proof. Consider the blue vertex $a^{\prime}$. Assume w.l.o.g. that $o$ is the origin of coordinates, and $a^{\prime}$ is in the positive direction of the $y$-axis. We have that $\angle a o a^{\prime}<\pi / 2$ because $o \notin \mathcal{B}\left(a a^{\prime}\right)$, then


Figure 1: (a) The ellipse $\mathcal{E}(p q)$ and the disk $\mathcal{B}(p q)$. (b) A line tangent to an ellipse forms equal angles with the rays, whose apex is the tangency point, that go through the foci. (c) Point $o$ and the three ellipses.
assume w.l.o.g. that $a$ is in the interior of the first quadrant $Q_{1}$. Let $Q_{2}, Q_{3}$, and $Q_{4}$ be the second, third, and fourth quadrants, respectively. Further assume w.l.o.g. that rays $\ell\left(a a^{\prime}\right), \ell\left(b b^{\prime}\right)$, and $\ell\left(c c^{\prime}\right)$ appear in this order counterclockwise.
Assume by contradiction that there is no white edge incident to $a^{\prime}$. This implies that $b, c$ belong to the interior of $Q_{1} \cup Q_{2}$. If $c \in Q_{2}$, then the counterclockwise rotation angle from $\ell\left(c c^{\prime}\right)$ to $\ell\left(a a^{\prime}\right)$ is larger than $\pi$. Hence, $c \in Q_{1}$. If $b \in Q_{1}$, then the counterclockwise rotation angle from $\ell\left(a a^{\prime}\right)$ to $\ell\left(b b^{\prime}\right)$, or that from $\ell\left(b b^{\prime}\right)$ to $\ell\left(c c^{\prime}\right)$, is larger than $\pi$. Hence $b \in Q_{2}$. Furthermore, if both $b^{\prime}$ and $c^{\prime}$ belong to $Q_{1} \cup Q_{2}$, then the counterclockwise rotation angle from $\ell\left(b b^{\prime}\right)$ to $\ell\left(c c^{\prime}\right)$ is larger than $\pi$. Hence, at least one of $b^{\prime}, c^{\prime}$ belong to the interior of $Q_{3} \cup Q_{4}$. That is, $b^{\prime} \in Q_{3}$ and/or $c^{\prime} \in Q_{4}$. The proof is divided now into three cases:
Case 1: $b^{\prime} \in Q_{3}$ and $c^{\prime} \in Q_{4}$. Since $b \in Q_{2}$ and $c^{\prime} \in Q_{4}$, the angle $\angle b o c^{\prime} \geq \pi / 2$, which implies that $o \in \mathcal{B}\left(b c^{\prime}\right)$ (see Figure 2a). That is, edge ( $b, c^{\prime}$ ) is white. Similarly, edge $\left(b^{\prime}, c\right)$ is also white. The colors of the edges of the cycle ( $b, c^{\prime}, c, b^{\prime}, b$ ) alternate, then Lemma 2.1 implies a contradiction.
Case 2: $b^{\prime} \in Q_{3}$ and $c^{\prime} \notin Q_{4}$. Since the counterclockwise rotation angle $\theta$ from $\ell\left(b b^{\prime}\right)$ to $\ell\left(c c^{\prime}\right)$ is smaller than $\pi$, we must have that $c^{\prime} \in Q_{1}$. As in Case 1, we have that edge $\left(b^{\prime}, c\right)$ is white, given that $b^{\prime} \in Q_{3}$ and $c \in Q_{1}$. Let $\beta$ be the half of the angle between rays $r(o b)$ and $r\left(o b^{\prime}\right)$, and $\gamma$ the half of the angle between the rays $r(o c)$ and $r\left(o c^{\prime}\right)$ (see Figure 2b). We have that $\beta, \gamma<\pi / 4$, which


Figure 2: Proof of Lemma 2.2. Black edges are in normal line style, and white edges in dashed style.
implies that $\angle b o c^{\prime} \geq 2 \pi-\beta-\gamma-\theta \geq \pi / 2$. Hence, edge ( $b, c^{\prime}$ ) is also white. Again, the colors of the edges of the cycle ( $b, c^{\prime}, c, b^{\prime}, b$ ) alternate, and Lemma 2.1 implies a contradiction.
Case 3: $b^{\prime} \notin Q_{3}$ and $c^{\prime} \in Q_{4}$. The proof of this case is analogous to that of Case 2.
The lemma thus follows.
Lemma 2.2 implies that graph $G$ has always a cycle (of length four or six) whose edges are color alternating. Hence, Lemma 2.1 implies a contradiction, and we obtain that the max-sum matching $\mathcal{M}$ ensures that $\mathcal{E}\left(a a^{\prime}\right) \cap \mathcal{E}\left(b b^{\prime}\right) \cap \mathcal{E}\left(c c^{\prime}\right) \neq \emptyset$. Therefore, Theorem 1.1 holds.

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