ON THE q-BINOMIAL IDENTITIES INVOLVING THE LEGENDRE SYMBOL MODULO 3.

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ABSTRACT. I use polynomial analogue of the Jacobi triple product identity together with the Eisenstein formula for the Legendre symbol modulo 3 to prove six identities involving the q-binomial coefficients. These identities are then extended to the new infinite hierarchies of q-series identities by means of the special case of Bailey's lemma. Some of the identities of Ramanujan, Slater, McLaughlin and Sills are obtained this way.

1. INTRODUCTION

Let a and q be variables and define the q-Pochhammer symbol $(a;q)_n := (1-a)(1-aq)\ldots(1-aq^{n-1})$ for any non-negative integer n. For $|q| < 1$, we define $(a;q)_\infty := \lim_{n\to\infty}(a;q)_n$. We define the shorthand notation $(a_1, a_2, \ldots, a_k; q)_n := (a_1;q)_n(a_2;q)_n \ldots (a_k;q)_n$. Finally note that $1/(q;q)_n = 0$ for all negative n.

Next, we define the q-binomial coefficients

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q := \begin{cases} \frac{(q;q)_{m+n}}{(q;q)_m(q;q)_n}, & \text{for } m,n \geq 0, \\ 0, & \text{otherwise,} \end{cases},$$

where m, n are non-negative integers. We would also require the q-binomial recurrences [[7], I.45, p.353]

$$(1.1) \qquad \begin{bmatrix} m+n \\ m \end{bmatrix}_q = \begin{bmatrix} m+n-1 \\ m \end{bmatrix}_q + q^n \begin{bmatrix} m+n-1 \\ m-1 \end{bmatrix}_q.$$

The following two limits are well known. For any $j \in \mathbb{Z}_{\geq 0}$ and $a = 0$ or 1,

$$\lim_{L\to\infty} \begin{bmatrix} L \\ j \end{bmatrix}_q = \frac{1}{(q;q)_j},$$

$$\lim_{L\to\infty} \begin{bmatrix} 2L+a \\ L-j \end{bmatrix}_q = \frac{1}{(q;q)_\infty}.$$

In [4] we proved many q-binomial identities involving the Legendre symbol $\pmod 3$

$$(1.2) \qquad \left(\frac{j}{3}\right) = \begin{cases} 1, & \text{if } j \equiv 1 \pmod 3, \\ -1, & \text{if } j \equiv -1 \pmod 3, \\ 0, & \text{if } j \equiv 0 \pmod 3. \end{cases}$$

In particular, we established

Theorem 1.1. *[Berkovich, Uncu [4]] Let $L \in \mathbf{Z}_{\geq 0}$, then*

$$(1.3) \qquad \sum_{m,n\geq 0} \frac{q^{2m^2+6mn+6n^2}(q;q)_L}{(q;q)_{L-3n-2m}(q;q)_m(q^3;q^3)_n} = \sum_{j=-L}^{L} \left(\frac{j+1}{3}\right) q^{j^2} \begin{bmatrix} 2L \\ L-j \end{bmatrix}_q.$$

In the limit $L \to \infty$ we have

$$(1.4) \qquad \sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2}}{(q;q)_m (q^3;q^3)_n} = \frac{Q(q^6,-q)}{(q)_\infty},$$

where

$$(1.5) \qquad Q(q,z) := (q, -z, -\frac{q}{z}; q)_\infty (\frac{q}{z^2}, z^2 q; q^2)_\infty.$$

In deriving (1.4) we used

Theorem 1.2. *Quintuple product identity [[7], ex 5.6,p 147]*
For $0 < |q| < 1$ and $z \neq 0$,

$$(1.6) \qquad \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{3k^2-k}{2}} z^{3k} (1 + zq^k) = Q(q,z).$$

Identity (1.4) was independently found by Kanade–Russell [8] and Kurşungöz [9]. It represents the analytic version of Capparelli's Theorem [5], [6].

In this paper we prove many new q-binomial identities involving the Legendre symbol (mod 3) such as

$$(1.7) \qquad \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} \left(\frac{j+1}{3} \right) \begin{bmatrix} 2L+1 \\ L+j \end{bmatrix}_{q^2} = \frac{(q^3;q^6)_L}{(q;q^2)_{L+1}} (1 - q^{2(1+2L)}).$$

To this end we will employ

Theorem 1.3. *Polynomial Analogue of the Jacobi Triple Product Identity [1] p. 49] For $n, m \in \mathbf{Z}_{\geq 0}$,*

$$(1.8) \qquad \sum_{i=-n}^{m} q^{i^2} x^i \begin{bmatrix} n+m \\ n+i \end{bmatrix}_{q^2} = (-q/x; q^2)_n (-qx; q^2)_m.$$

It is easy to check that (1.8) is a special case of the q-binomial theorem

$$(1.9) \qquad \sum_{i=0}^{L} q^{i^2} x^i \begin{bmatrix} L \\ i \end{bmatrix}_{q^2} = (-xq; q^2)_L,$$

with $L \in \mathbb{Z}_{\geq 0}$.
As $n, m \to \infty$, (1.8) becomes the Jacobi triple product identity

$$(1.10) \qquad \sum_{i=-\infty}^{\infty} q^{i^2} x^i = (-q/x, -qx, q^2; q^2)_\infty.$$

Eisenstein established the following formula for the Legendre symbol

$$(1.11) \qquad \left(\frac{j}{p} \right) = \prod_{n=1}^{\frac{p-1}{2}} \frac{\sin(\frac{2\pi}{p} nj)}{\sin(\frac{2\pi}{p} n)},$$

where p is an odd prime.
Hence,

$$(1.12) \qquad \left(\frac{j}{3} \right) = \frac{\sin(\frac{2\pi}{3} j)}{\sin(\frac{2\pi}{3})} = \frac{w^j - \bar{w}^j}{w - \bar{w}},$$

where $w = exp(\frac{2\pi}{3} I)$, $\bar{w} = w^{-1}$.

In Section 2, we will employ both (1.12) and Theorem 1.3. We would also need

Theorem 1.4. *Special Case of Bailey's Lemma* [12], [2]
For $a = 0, 1$, if

$$(1.13) \qquad F_a(L, q) = \sum_{j=-\infty}^{\infty} \alpha_j(q) \begin{bmatrix} 2L + a \\ L - j \end{bmatrix}_q,$$

then

$$(1.14) \qquad \sum_{r \geq 0} \frac{q^{r^2 + ar}(q; q)_{2L+a}}{(q; q)_{L-r}(q; q)_{2r+a}} F_a(r, q) = \sum_{j=-\infty}^{\infty} \alpha_j(q) q^{j^2 + aj} \begin{bmatrix} 2L + a \\ L - j \end{bmatrix}_q.$$

Observe that the right-hand side of the second equation in Theorem 1.4 is of the same form as the right-hand side of the first equation. Thus, we may iterate Theorem 1.4 as often as we desire by updating $\alpha_j(q)$'s in each step. This procedure gives rise to an infinite hierarchy of polynomial identities.

2. Six new q-hypergeometric hierarchies

In this section we will use polynomial analogue of the Jacobi triple identity to prove validity of six "seed" identities. These seeds are then extended to infinite hierarchies of q-series identities by means of Theorem 1.4.

Theorem 2.1.

$$(2.1) \qquad \sum_{j=-\infty}^{\infty} \left(\frac{j}{3}\right) q^{\binom{j-1}{2}} \begin{bmatrix} 2L \\ L + j \end{bmatrix}_q = \begin{cases} \frac{(-1; q^3)_{L-1}}{(-1; q)_{L-1}} q^{L-1} \frac{1-q^3}{1-q}(1 - q^L), & L > 0 \\ 0, & L = 0 \end{cases}.$$

Proof.

$$\sum_{j=-\infty}^{\infty} \left(\frac{j}{3}\right) q^{j^2 - 3j + 2} \begin{bmatrix} 2L \\ L + j \end{bmatrix}_{q^2} = q^2 \sum_{j=-\infty}^{\infty} q^{j^2} \frac{w^j - \bar{w}^j}{w - \bar{w}} q^{-3j} \begin{bmatrix} 2L \\ L + j \end{bmatrix}_{q^2} =$$

$$2\Re\left(\frac{q^2}{w - \bar{w}}(-q^4 \bar{w}; q^2)_L(-q^{-2}w; q^2)_L\right) =$$

$$(2.2) \qquad \begin{cases} (-w, -\bar{w}; q^2)_{L-1} q^{2(L-1)}(1 - q^{2L})(1 + q^2 + q^4), & L > 0 \\ 0, & L = 0 \end{cases} =$$

$$\begin{cases} \frac{(-1; q^6)_{L-1}}{(-1; q^2)_{L-1}} \frac{1-q^6}{1-q^2} q^{2(L-1)}(1 - q^{2L}), & L > 0 \\ 0, & L = 0 \end{cases}.$$

Replace $q^2 \to q$ to arrive at (2.1). $\qquad \square$

Applying Theorem 1.4 with $a = 0$ to (2.1) we derive

$$(2.3) \qquad \frac{1 - q^3}{1 - q} \sum_{n \geq 0} q^{n^2 + 3n} \frac{(q)_{2L}}{(q)_{2n+2}(q)_{L-1-n}} \frac{(-1; q^3)_n}{(-1; q)_n}(1 - q^{n+1}) = \sum_{j=-\infty}^{\infty} \left(\frac{j}{3}\right) q^{3\binom{j}{2}} \begin{bmatrix} 2L \\ L + j \end{bmatrix}_q.$$

Applying Theorem 1.4 with $a = 0$ repeatedly "v-times" to (2.1) we derive

$$\frac{1 - q^3}{1 - q} \sum_{n_1, \dots, n_v \geq 0} \frac{q^{\sum_{i=1}^v (N_i + 2)N_i + n_v}}{(q)_{n_1} \dots (q)_{n_{v-1}}(q)_{2n_v + 2}} \frac{(-1; q^3)_{n_v}}{(-1; q)_{n_v}}(1 - q^{1+n_v}) \frac{(q)_{2L}}{(q)_{L-1-N_1}} =$$

$$(2.4) \qquad \sum_{j=-\infty}^{\infty} \left(\frac{j}{3}\right) q^{\frac{(2v+1)j^2 - 3j}{2} - (v-1)} \begin{bmatrix} 2L \\ L + j \end{bmatrix}_q,$$

where, here and everywhere, for $j = 1, 2, \dots, v$

$$(2.5) \qquad N_i = n_i + n_{i+1} + \cdots + n_v.$$

Multiplying both sides of (2.4) by $\frac{1-q}{1-q^3}$ and letting $L \to \infty$, we have with the aid of (1.6)

(2.6)
$$\sum_{n_1,\dots,n_v \geq 0} \frac{q^{\sum_{i=1}^v (N_i+2)N_i + n_v}(1-q^{1+n_v})}{(q)_{n_1}\dots(q)_{n_{v-1}}(q)_{2n_v+2}} \frac{(-1;q^3)_{n_v}}{(-1;q)_{n_v}} = \frac{1-q}{1-q^3}$$

$$\sum_{j=-\infty}^{\infty} \frac{\left(\frac{j}{3}\right)q^{\frac{(2v+1)j^2-3j}{2}+1-v}}{(q)_\infty} = \frac{(q^{6v+3},q^{3v},q^{3v+3};q^{6v+3})_\infty (q^{12v+3},q^{12v+9};q^{12v+6})_\infty}{(q^2;q)_\infty}.$$

Next, we will prove

Theorem 2.2.

(2.7)
$$\sum_{j=-\infty}^{\infty} q^{\binom{j-1}{2}}\left(\frac{j}{3}\right)\begin{bmatrix} 2L+1 \\ L+j \end{bmatrix}_q = q^L \frac{(-1;q^3)_L}{(-1;q)_L}(1+q-q^{L+1}).$$

Proof.

(2.8)
$$\sum_{j=-\infty}^{\infty} q^{j^2-3j+2}\left(\frac{j}{3}\right)\begin{bmatrix} 2L+1 \\ L+j \end{bmatrix}_{q^2} = q^2 \sum_{j=-\infty}^{\infty} q^{j^2}\frac{w^j-\bar{w}^j}{w-\bar{w}}q^{-3j}\begin{bmatrix} 2L+1 \\ L+j \end{bmatrix}_{q^2} = \quad by \ (1.8)$$

$$2\Re(\frac{1}{\bar{w}-w}(-\bar{w},-w;q^2)_L(1+\bar{w}q^{2L}(1+q^2)+\bar{w}^2q^{2(2L+1)})) =$$

$$q^{2L}(-\bar{w},-w;q^2)_L \cdot (1+q^2-q^{2L+2}) = q^{2L}\frac{(-1;q^6)_L}{(-1;q^2)_L}(1+q^2-q^{2(L+1)}).$$

Replace $q^2 \to q$ in the above to complete the proof.   □

Applying Theorem 1.4 with $a=1$ to (2.7) v-times we get

(2.9)
$$\sum_{n_1,\dots,n_v \geq 0} \frac{q^{\sum_{i=1}^v N_i(N_i+1)+n_v}}{(q)_{n_1}\dots(q)_{n_{v-1}}(q)_{1+2n_v}} \frac{(-1;q^3)_{n_v}}{(-1;q)_{n_v}} \frac{(q)_{2L+1}}{(q)_{L-N_1}}(1+q-q^{1+n_v}) =$$

$$\sum_{j=-\infty}^{\infty} \left(\frac{j}{3}\right)q^{\frac{(2v+1)j^2-(2v+3)j}{2}+1}\begin{bmatrix} 2L+1 \\ L+j \end{bmatrix}_q.$$

Letting $L \to \infty$ and using (1.6) we derive

(2.10)
$$\sum_{n_1,\dots,n_v \geq 0} \frac{q^{\sum_{i=1}^v N_i(N_i+1)+n_v}}{(q)_{n_1}\dots(q)_{n_{v-1}}(q)_{1+2n_v}} \frac{(-1;q^3)_{n_v}}{(-1;q)_{n_v}}(1+q-q^{1+n_v}) = \frac{Q(q^{6v+3},-q^{2v})}{(q)_\infty}.$$

It is instructive to compare (2.10) with the following formula

(2.11)
$$\sum_{n_1,\dots,n_v \geq 0} \frac{q^{\sum_{i=1}^v N_i^2}}{(q)_{n_1}\dots(q)_{n_{v-1}}(q)_{2n_v}} \frac{(-1;q^3)_{n_v}}{(-1;q)_{n_v}} = \frac{Q(q^{6v+3},-q^{v+1})}{(q)_\infty},$$

first proven by McLaughlin and Sills in [[10], (5.9)]. We remark that products on the right of (2.10) and (2.11) are identical when $v=1$.

Next, we will prove

Theorem 2.3.

(2.12)
$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j^2}\left(\frac{j+1}{3}\right)\begin{bmatrix} 2L+1 \\ L+j \end{bmatrix}_{q^2} = \frac{(q^3;q^6)_L}{(q;q^2)_{L+1}}(1-q^{2(1+2L)}).$$

Proof. Using (1.8) and (1.12) we have

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} \left(\frac{j+1}{3}\right) \begin{bmatrix} 2L+1 \\ L+j \end{bmatrix}_{q^2} = \sum_{j=-\infty}^{\infty} (-1)^j \frac{w^{1+j} - \bar{w}^{1+j}}{w - \bar{w}} \begin{bmatrix} 2L+1 \\ L+j \end{bmatrix}_{q^2} q^{j^2} =$$

(2.13)
$$2\Re(\frac{w}{w-\bar{w}} \sum_{j=-\infty}^{\infty} q^{j^2} (-w)^j \begin{bmatrix} 2L+1 \\ L+j \end{bmatrix}_{q^2}) = 2\Re(\frac{w}{w-\bar{w}} (q\bar{w}, qw; q^2)_L (1 - wq^{2L+1})) =$$

$$(q\bar{w}, qw; q^2)_L (1 + q^{2L+1}) = (1 + q^{2L+1}) \prod_{j=0}^{L-1} (1 + q^{1+2j} + q^{2(1+2j)}) = \frac{(q^3; q^6)_L}{(q; q^2)_{L+1}} (1 - q^{2(1+2L)}).$$

□

Applying Theorem 1.4 with $a = 1$ and $q \to q^2$ v-times we obtain

(2.14)
$$\sum_{n_1,\ldots,n_v \geq 0} \frac{\tilde{q}^{\sum_{i=1}^{v} N_i(N_i+1)}}{(\tilde{q})_{n_1}\ldots(\tilde{q})_{n_{v-1}}(\tilde{q})_{2n_v}} \cdot \frac{(q^3; q^6)_{n_v}}{(q; q^2)_{1+n_v}} \cdot \frac{(\tilde{q})_{2L+1}}{(\tilde{q})_{L-N_1}} = \sum_{j=-\infty}^{\infty} q^{(2v+1)j^2 - 2vj} (-1)^j \begin{bmatrix} 2L+1 \\ L+j \end{bmatrix}_{\tilde{q}} \left(\frac{j+1}{3}\right),$$

where $\tilde{q} = q^2$.

Letting $L \to \infty$ and using (1.6) we arrive at

(2.15)
$$\sum_{n_1,\ldots,n_v \geq 0} \frac{\tilde{q}^{\sum_{i=1}^{v} N_i(N_i+1)}}{(\tilde{q})_{n_1}\ldots(\tilde{q})_{n_{v-1}}(\tilde{q})_{2n_v}} \cdot \frac{(q^3; q^6)_{n_v}}{(q; q^2)_{1+n_v}} = \frac{Q(q^{12v+6}, q)}{(\tilde{q})_\infty}.$$

We remark that $v = 1$ case of (2.15) was first proven by McLaughlin and Sills [[11], Thm 4.8].

Next, we will prove

Theorem 2.4.

(2.16)
$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} \left(\frac{j+1}{3}\right) \begin{bmatrix} 2L \\ L+j \end{bmatrix}_{q^2} = \frac{(q^3; q^6)_L}{(q; q^2)_L}.$$

Proof.

(2.17)
$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} \left(\frac{j+1}{3}\right) \begin{bmatrix} 2L \\ L+j \end{bmatrix}_{q^2} = \sum_{j=-\infty}^{\infty} (-1)^j \frac{w^{1+j} - \bar{w}^{1+j}}{w - \bar{w}} \begin{bmatrix} 2L \\ L+j \end{bmatrix}_{q^2} q^{j^2} =$$
$$2\Re(\frac{w}{w-\bar{w}} (q\bar{w}, qw; q^2)_L = \frac{(q^3; q^6)_L}{(q; q^2)_L}.$$

□

Applying Theorem 1.4 with $a = 0$ and $q \to q^2$ v-times to (2.16) we obtain

(2.18)
$$\sum_{n_1,\ldots,n_v \geq 0} \frac{\tilde{q}^{\sum_{i=1}^{v} N_i^2}}{(\tilde{q})_{n_1}\ldots(\tilde{q})_{n_{v-1}}(\tilde{q})_{2n_v}} \cdot \frac{(q^3; q^6)_{n_v}}{(q; q^2)_{n_v}} \cdot \frac{(\tilde{q})_{2L}}{(\tilde{q})_{L-N_1}} = \sum_{j=-\infty}^{\infty} (-1)^j \left(\frac{j+1}{3}\right) q^{(2v+1)j^2} \begin{bmatrix} 2L \\ L+j \end{bmatrix}_{\tilde{q}},$$

where $\tilde{q} = q^2$.

Letting $L \to \infty$ we obtain with the aid of (1.6)

(2.19)
$$\sum_{n_1,\ldots,n_v \geq 0} \frac{\tilde{q}^{\sum_{i=1}^{v} N_i^2}}{(\tilde{q})_{n_1}\ldots(\tilde{q})_{n_{v-1}}(\tilde{q})_{2n_v}} \cdot \frac{(q^3; q^6)_{n_v}}{(q; q^2)_{n_v}} = \frac{Q(q^{12v+6}, q^{2v+1})}{(\tilde{q})_\infty}.$$

Case $v = 1$

(2.20)
$$\sum_{n \geq 0} \frac{q^{2n^2}}{(\tilde{q})_{2n}} \frac{(q^3; q^6)_n}{(q; q^2)_n} = \frac{Q(q^{18}, q^3)}{(\tilde{q})_\infty},$$

was first recorded by Ramanujan and proven by Andrews and Berndt in [[3], Ent:5, 3.4].

Next, we will prove

Theorem 2.5.

$$(2.21) \qquad \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} \left(\frac{j+1}{3} \right) \begin{bmatrix} 2L+1 \\ L-j \end{bmatrix}_{q^2} = \frac{(q^3; q^6)_L}{(q; q^2)_L}.$$

Proof. Using q-binomial recurrences (1.1) we have

$$(2.22) \qquad \begin{aligned} \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} \left(\frac{j+1}{3} \right) \begin{bmatrix} 2L+1 \\ L+1+j \end{bmatrix}_{\tilde{q}} &= \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} \left(\frac{j+1}{3} \right) \begin{bmatrix} 2L \\ L+j \end{bmatrix}_{\tilde{q}} + \\ q^{2L+1} \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2+2j+1} \left(\frac{j+1}{3} \right) \begin{bmatrix} 2L \\ L+1+j \end{bmatrix}_{\tilde{q}}. & \end{aligned}$$

Note that

$$(2.23) \qquad \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2+2j+1} \left(\frac{j+1}{3} \right) \begin{bmatrix} 2L \\ L+1+j \end{bmatrix}_{\tilde{q}} = - \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} \left(\frac{j}{3} \right) \begin{bmatrix} 2L \\ L+j \end{bmatrix}_{\tilde{q}}.$$

Observe that the last sum on the right negates under $j \to -j$. Hence, it equals zero.
And so, with the aid of (2.16) we have

$$(2.24) \qquad \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} \left(\frac{j+1}{3} \right) \begin{bmatrix} 2L+1 \\ L-j \end{bmatrix}_{q^2} = \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} \left(\frac{j+1}{3} \right) \begin{bmatrix} 2L \\ L-j \end{bmatrix}_{q^2} = \frac{(q^3; q^6)_L}{(q; q^2)_L}.$$

$\square$

Using Theorem 1.4 with $a = 1$ and $q \to q^2$ to get

$$(2.25) \qquad \sum_{n_1,\dots,n_v \geq 0} \frac{\tilde{q}^{\sum_{i=1}^{v} N_i(N_i+1)}}{(\tilde{q})_{n_1}\dots(\tilde{q})_{n_{v-1}}(\tilde{q})_{2n_v+1}} \cdot \frac{(q^3; q^6)_{n_v}}{(q; q^2)_{n_v}} \cdot \frac{(\tilde{q})_{2L+1}}{(\tilde{q})_{L-N_1}} = \sum_{j=-\infty}^{\infty} (-1)^j q^{(2v+1)j^2+2vj} \left(\frac{j+1}{3} \right) \begin{bmatrix} 2L+1 \\ L-j \end{bmatrix}_{q^2}.$$

As $L \to \infty$, we obtain with the aid of (1.6)

$$(2.26) \qquad \sum_{n_1,\dots,n_v \geq 0} \frac{\tilde{q}^{\sum_{i=1}^{v} N_i(N_i+1)}}{(\tilde{q})_{n_1}\dots(\tilde{q})_{n_{v-1}}(\tilde{q})_{2n_v+1}} \cdot \frac{(q^3; q^6)_{n_v}}{(q; q^2)_{n_v}} = \frac{Q(q^{12v+6}, q^{4v+1})}{(\tilde{q})_\infty},$$

where $\tilde{q} = q^2$. Case $v = 1$ yields

$$(2.27) \qquad \sum_{n \geq 0} \frac{q^{2n(n+1)}}{(\tilde{q})_{2n+1}} \frac{(q^3; q^6)_n}{(q; q^2)_n} = \frac{Q(q^{18}, q^5)}{(\tilde{q})_\infty},$$

which is item (124) in Slater's compendium of Rogers–Ramanujan type identities [13].

Next, we will prove

Theorem 2.6.

$$(2.28) \qquad \sum_{j=-\infty}^{\infty} (-1)^j \left(\frac{j}{3} \right) \begin{bmatrix} 2L+1 \\ L-j \end{bmatrix}_{\tilde{q}} q^{j^2} = q^{1+2L} \frac{(q^3; q^6)_L}{(q; q^2)_L}.$$

Proof.

$$(2.29) \qquad \begin{aligned} \sum_{j=-\infty}^{\infty} (-1)^j \left(\frac{j}{3} \right) \begin{bmatrix} 2L+1 \\ L-j \end{bmatrix}_{\tilde{q}} q^{j^2} &= \sum_{j=-\infty}^{\infty} (-1)^j \frac{w^j - \bar{w}^j}{w - \bar{w}} \begin{bmatrix} 2L+1 \\ L+1+j \end{bmatrix}_{q^2} q^{j^2} = \\ 2\Re(\frac{1}{w - \bar{w}} (q\bar{w}; q^2)_{L+1}(qw; q^2)_L) &= 2\Re(\frac{(q\bar{w}, qw; q^2)_L}{w - \bar{w}} (1 - q\bar{w}q^{2L})) = \\ (q\bar{w}, qw; q^2)_L q^{1+2L} &= \frac{(q^3; q^6)_L}{(q; q^2)_L} q^{1+2L}, \end{aligned}$$

where $\tilde{q} = q^2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Applying Theorem 1.4 with $a = 1$ and $q \to q^2$ to (2.28) v-times we obtain

$$(2.30) \qquad \sum_{n_1,\ldots,n_v \geq 0} \frac{\tilde{q}^{\sum_{i=1}^v N_i(N_i+1)+n_v}}{(\tilde{q})_{n_1}\ldots(\tilde{q})_{n_{v-1}}(\tilde{q})_{2n_v+1}} \frac{(q^3;q^6)_{n_v}}{(q;q^2)_{n_v}} \frac{(\tilde{q})_{2L+1}}{(\tilde{q})_{L-N_1}} = \sum_{j=-\infty}^{\infty} (-1)^j q^{(2v+1)j^2+2vj-1} \left(\frac{j}{3}\right) \begin{bmatrix} 2L+1 \\ L-j \end{bmatrix}_{q^2}.$$

As $L \to \infty$ we get with the aid of (1.6)

$$(2.31) \qquad \sum_{n_1,\ldots,n_v \geq 0} \frac{\tilde{q}^{\sum_{i=1}^v N_i(N_i+1)+n_v}}{(\tilde{q})_{n_1}\ldots(\tilde{q})_{n_{v-1}}(\tilde{q})_{2n_v+1}} \frac{(q^3;q^6)_{n_v}}{(q;q^2)_{n_v}} = \frac{Q(q^{12v+6},q^{4v+3})}{(\tilde{q})_\infty}.$$

Case $v = 1$ yields item (125) on Slater's list in [13]

$$(2.32) \qquad \sum_{n \geq 0} \frac{\tilde{q}^{n(n+2)}}{(\tilde{q})_{2n+1}} \frac{(q^3;q^6)_n}{(q;q^2)_n} = \frac{Q(q^{18},q^7)}{(\tilde{q})_\infty},$$

where $\tilde{q} = q^2$.

3. Some additional identities

In this section, we present three new polynomial identities. The proof of these identities is left as an exercise for a motivated reader.

$$(3.1) \qquad \sum_{j=-\infty}^{\infty} (-1)^j \left(\frac{j+1}{3}\right) q^{\binom{j}{2}} \begin{bmatrix} 2L+1 \\ L-j \end{bmatrix}_q = \begin{cases} \frac{(q^3;q^3)_{L-1}}{(q)_{L-1}}(2+q^L(1+q)-q^{2L+1}), & L > 0 \\ 1, & L = 0 \end{cases}.$$

Using Theorem 1.4 with $a = 1$ and $L = \infty$, we get with the aid of Theorem 1.2

$$(3.2) \qquad 1 + \sum_{r \geq 1} \frac{q^{r^2+r}}{(q^2;q)_{2r}} \frac{(q^3;q^3)_{r-1}}{(q)_{r-1}}(2+q^r(1+q)-q^{1+2r}) = \frac{(q^9,-q^2,-q^7;q^9)_\infty(q^5,q^{13};q^{18})_\infty}{(q^2;q)_\infty}.$$

Next,

$$(3.3) \qquad \sum_{j=-\infty}^{\infty} (-1)^j \left(\frac{j}{3}\right) q^{\binom{j}{2}} \begin{bmatrix} 2L+1 \\ L-j \end{bmatrix}_q = \begin{cases} \frac{(q^3;q^3)_{L-1}}{(q)_{L-1}}(-1+q^L(1+q)+2q^{2L+1}), & L > 0 \\ q, & L = 0 \end{cases}.$$

Using Theorem 1.4 with $a = 1$ and $L = \infty$, we obtain with the aid of Theorem 1.2

$$(3.4) \qquad 1 + \sum_{r \geq 1} \frac{q^{r^2+r-1}}{(q^2;q)_{2r}} \cdot \frac{(q^3;q^3)_{r-1}}{(q)_{r-1}} \cdot (-1+q^r(1+q)+2q^{2r+1}) = \frac{(q^9,-q^4,-q^5;q^9)_\infty(q,q^{17};q^{18})_\infty}{(q^2;q)_\infty}.$$

Lastly, adding (3.1) and (3.3), and observing that

$$\left(\frac{j}{3}\right) + \left(\frac{j+1}{3}\right) + \left(\frac{j+2}{3}\right) = 0,$$

we get

$$(3.5) \qquad \sum_{j=-\infty}^{\infty} (-1)^{j+1} \left(\frac{j+2}{3}\right) q^{\binom{j}{2}} \begin{bmatrix} 2L+1 \\ L-j \end{bmatrix}_q = \begin{cases} \frac{(q^3;q^3)_{L-1}}{(q)_{L-1}}(1+q^{2L+1}+2q^L(1+q)), & L > 0 \\ 1+q, & L = 0 \end{cases}.$$

We divide (3.5) by $(1+q)$, and then use Theorem 1.4 with $a = 1$ to obtain, as $L \to \infty$

$$(3.6) \qquad 1 + \sum_{r \geq 1} \frac{q^{r^2+r}}{(q^2;q)_{2r}} \frac{(q^3;q^3)_{r-1}}{(q)_{r-1}} \left(\frac{1+q^{1+2r}}{1+q}+2q^r\right) = \frac{(q^9,-q^8,-q^{10};q^9)_\infty(q^7,q^{11};q^{18})_\infty}{(q^2;q)_\infty}.$$

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